FACTORING ABSOLUTELY SUMMING OPERATORS THROUGH HILBERT-SCHMIDT OPERATORS

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Introduction. Let K be a compact Hausdorff space, and let $C(K)$ be the corresponding Banach space of continuous functions on K. It is well-known that every 1-summing operator $S:C(K)\to l_2$ is also nuclear, and therefore factors $S=S_1S_2$, with $S_1:l_2\to l_2$ a Hilbert–Schmidt operator and $S_1:C(K)\to l_2$ a bounded operator. It is easily seen that this latter property is preserved when $C(K)$ is replaced by any quotient, and that a Banach space X enjoys this property if and only if its second dual, X^{**}, does. This led A. Pełczyński [15] to ask if the second dual of a Banach space X must be isomorphic to a quotient of a $C(K)$-space if X has the property that every 1-summing operator $X\to l_2$ factors through a Hilbert-Schmidt operator. In this paper, we shall first of all reformulate the question in an appropriate manner and then show that counter-examples are available among super-reflexive Tsirelson-like spaces as well as among quasi-reflexive Banach spaces.

Preliminaries. We shall use standard notation and terminology for Banach space theory. In particular, by a subspace of a Banach space we always mean a closed linear submanifold, and an operator will always be a continuous linear mapping between Banach spaces. Given a Banach space X, we shall write as usual B_X for its closed unit ball, X^* for its dual, and X^{**} for its second dual. Further unexplained notation and termnology is as in [12] and [13].

As for the theory of operator ideals, we shall follow A. Pietsch [16] with only minor changes in notation. By Γ_p we denote the ideal of all L_p-factorable operators $(1\le p\le\infty)$ and by Π_r the ideal of all (absolutely) r-summing operators $(1\le r<\infty)$. We say that a Banach space X has the *property GL* if $\Pi_1(X,Y)$ is contained in $\Gamma_1(X,Y)$ for every Banach space Y. Every Banach lattice and, in particular, every Banach space with an unconditional basis has this property. By trace duality, X has the property GL if and only if X^* has it. The concept goes back to Y. Gordon and D. R. Lewis [5]. For more details, see G. Pisier [18, Ch. 8], [17] and [20].

Let (g_n) be a sequence of independent standard Gaussian random variables. Following W. Linde and A. Pietsch [11] we say that an operator $S:X\to Y$ between Banach spaces X and Y is *γ-summing* if there is a constant C such that $(\mathbb{E}\,\|\sum_{i=1}^{n} g_iSx_i\|^2)^{1/2}\le C\cdot\sup\{(\sum_{i=1}^{n}|\langle u,x_i\rangle|^2)^{1/2}\mid u\in B_{X^*}\}$ for all finite sequences $(x_i)_{i\le n}$ in X. These operators form an ideal Π_γ which contains all the ideals $\Pi_p(1\le p<\infty)$.

The concept of a γ-summing operator is closely related to the notion of type and cotype. Recall that $S:X\to Y$ is said to be of *type* p $(1<p\le 2)$, respectively *cotype* q

$(2 \le q < \infty)$, if there is a constant C such that

$$\left(\mathbb{E}\left\|\sum_{i=1}^{n} g_i S x_i\right\|^2\right)^{1/2} \le C \cdot \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p},$$

respectively

$$\left(\sum_{i=1}^{n} \|Sx_i\|^q\right)^{1/q} \le C \cdot \left(\mathbb{E}\left\|\sum_{i=1}^{n} g_i x_i\right\|^2\right)^{1/2},$$

again for all finite sequences $(x_i)_{i \le n}$ in X. If the identity map of a Banach space X has this property then we say that X has *type p*, respectively *cotype q*. It is known that X has cotype 2 if and only if $\Pi_\gamma(Z, X) = \Pi_2(Z, X)$ holds for every Banach space Z (or only for $Z = l_2$). Similarly, X has type 2 if and only if $\Gamma_1(Z, X)$ is contained in $\Pi_\gamma(Z, X)$ for all Z. Instead of using Gaussian variables, we could also have worked with independent Bernoulli variables, alias Rademacher functions. It is well-known, however, that this would lead to an equivalent definition of type and cotype, and also of a γ-summing operator (result by T. Figiel, communicated by A. Pietsch). We refer to B. Maurey and G. Pisier [**14**], A. Pietsch [**16**], and G. Pisier [**18**] for further information.

Factorization through Hilbert–Schmidt operators. We say that an operator $S: X \to Y$ *factors through a Hilbert–Schmidt operator* if $S = S_1 S_2 S_3$ with operators $S_1: l_2 \to Y$, $S_3: X \to l_2$, and a Hilbert–Schmidt operator $S_2: l_2 \to l_2$. These operators form an ideal which we denote by $\mathfrak{X}$. It consists of course of 1-summing operators only. As was mentioned before, we wish to characterize the Banach spaces X such that $\mathfrak{X}(X, l_2) = \Pi_1(X, l_2)$. We have the following result.

PROPOSITION 1. *For every Banach space X, the following are equivalent.*

(a) $\mathfrak{X}(X, l_2) = \Pi_1(X, l_2)$.

(b) *Every* 1-*summing operator with domain X has a* 2-*summing adjoint.*

(c) *Every* 2-*summing operator with domain* X^* *is* L_1-*factorable.*

(d) X^* *has cotype* 2 *and the property GL.*

Proof. (a)$\Leftrightarrow$(b) is due to the known fact that an operator $S: Y \to Z$ has a 2-summing adjoint iff US is in $\mathfrak{X}(Y, l_2)$ for every $U \in \mathscr{L}(Z, l_2)$.

(b)$\Leftrightarrow$(c) By trace duality, (b) is equivalent to saying that every operator from Y into X with 2-summing adjoint belongs to $\Gamma_\infty(Y, X)$, for any Banach space Y. But this is (c), by transposition and localization.

Since (d)$\Rightarrow$(c) is obvious, all we need to show is (c)$\Rightarrow$(d). Note that (c) implies that X^* has the property GL and satisfies $\Pi_1(X^*, Y) = \Pi_2(X^*, Y)$ for all Y. Every Banach space Z of cotype 2 is known to satisfy $\Pi_1(Z, Y) = \Pi_2(Z, Y)$, but it is an open question if conversely this property implies that Z has cotype 2. No problems arise, however, if Z has in addition the property GL; therefore the proof is completed by the following known result.

LEMMA 2. *If a Banach space Z has the property GL and satisfies* $\Pi_1(Z, l_2) = \Pi_2(Z, l_2)$, *then Z has cotype* 2.

A proof is given in [**18**, Proposition 8.16]. For the sake of completeness, we include another proof which is apparently new. Suppose Z has the announced properties. We wish to show that every $U \in \Pi_\gamma(l_2, Z)$ is 2-summing; equivalently, SU is nuclear for every $S \in \Pi_2(Z, l_2)$. By hypothesis, $S = BA$ for some $A: Z \rightarrow L_1(\mu)$, $B: L_1(\mu) \rightarrow l_2$, where μ is an appropriate measure. Since $L_1(\mu)$ has cotype 2, AU is 2-summing. By Grothendieck's theorem, B is 1-summing, and so $SU = BAU$ is a trace class operator on l_2. Since $S \in \Pi_2(Z, l_2)$ was arbitrary, $U \in \Pi_2(l_2, Z)$ follows by trace duality.

Some examples. It is clear that the equivalent properties listed in the preceding proposition are shared by all L_∞-spaces and that they are preserved under the formation of quotients. It is also clear that a Banach space X has this property if and only if X^{**} does. There are, however, Banach spaces X satisfying $\mathfrak{X}(X, l_2) = \Pi_1(X, l_2)$ such that X^{**} is not isomorphic to a quotient of an L_∞-space. We prove first of all the following result.

PROPOSITION 3. *Given $2 \leq q < \infty$, let X be an infinite-dimensional Banach space which is of cotype $q + \varepsilon$ for all $\varepsilon > 0$ but no quotient of which is isomorphic to l_q. Then X^{**} cannot be isomorphic to a quotient of any L_∞-space.*

Proof. Suppose we are wrong: X^{**} is isomorphic to a quotient space of an $\mathscr{L}_\infty$-space, say Z. Let $Q: Z \rightarrow X^{**}$ be the quotient map. We shall now use known results relating p-summing operators and cotype; cf. [**14**] and [**16**] for details. By the local reflexivity principle, X^{**} has also cotype $q + \varepsilon$ for all $\varepsilon > 0$. Fix $r > q$. Since Z is an $\mathscr{L}_\infty$-space, Q is r-summing and even r-integral. Consequently, X^{**} appears as a quotient of some $L_r(\mu)$. In particular, we get $X = X^{**}$, and this space is even super-reflexive. If follows that X^* has type $q^* - \varepsilon$ for all $\varepsilon > 0$, with $q^* := q \cdot (q-1)^{-1}$. Being isomorphic to a subspace of the $\mathscr{L}_1$-space Z^* (via Q^*), X^* contains an isomorphic copy of l_{q^*}, by the result of S. Guerre and M. Lévy [**6**]. Equivalently, l_q must be isomorphic to a quotient of X, a contradiction.

REMARK. In case $q = 2$ we could have applied as well the forerunner, due to D. Aldous [**2**], of the Guerre–Lévy result.

EXAMPLE 1. In [**9**], W. B. Johnson constructed a super-reflexive Banach space, $\mathscr{T}$ say, with monotone unconditional basis, which is of type 2 and of cotype $2 + \varepsilon$ for all $\varepsilon > 0$, and which has the property that every subspace of each of its quotients has a basis but fails to be isomorphic to l_2. $\mathscr{T}$ belongs to the class of "Tsirelson-like" Banach spaces; detailed information on these spaces may be found in the forthcoming lecture notes [**3**] by P. G. Casazza and T. J. Shura.

By the preceding two propositions, $\mathscr{T}$ satisfies $\mathfrak{X}(\mathscr{T}, l_2) = \Pi_1(\mathscr{T}, l_2)$ and is not isomorphic to a quotient of any L_∞-space.

EXAMPLE 2. A variant of Johnson's construction appears as Example 5.3 in [**4**]. The spaces $\tilde{X} = \tilde{X}(q, \eta)$ $(2 \leq q < \infty, \eta > 1)$ constructed there have again a monotone unconditional basis, are of type 2 and of cotype $q + \varepsilon$ for all $\varepsilon > 0$, and have the property that no quotient of $\tilde{X}$ (in fact, of any subspace of $\tilde{X}$) can contain a copy of l_q. I am grateful to P.

G. Casazza for pointing out to me this latter fact. Again the preceding two propositions show that $\mathfrak{X}(\tilde{X}, l_2) = \Pi_1(\tilde{X}, l_2)$ holds but $\tilde{X}$ cannot be the range of any L_∞-operator.

REMARK. We take this opportunity to point out that each of the above spaces $\tilde{X}$ and $\mathcal{T}$ provides a negative answer to the following question raised by T. Kühn [10]. Let X be a Banach space of type 2 such that $\Pi_\gamma(l_2, X) = \Pi_r(l_2, X)$ for $2 < r < \infty$. Is X then isomorphic to a subspace of L_r? Certainly $X = \tilde{X}(q, \eta)$ cannot be a subspace of any L_r, for $q < r < \infty$, but we have $S \in \Pi_\gamma(l_2, X) \Leftrightarrow S^* \in \Pi_1(X^*, l_2) \Leftrightarrow S^* \in \Gamma_1(X^*, l_2) \Leftrightarrow S \in \Gamma_\infty(l_2, X) \Leftrightarrow S \in \Pi_r(l_2, X)$.

EXAMPLE 3. Recently G. Pisier [20] has shown that the classical James space $\mathcal{J}$ (of codimension one in $\mathcal{J}^{**}$, cf. R. C. James [7]) has the properties discussed in Proposition 1: $\mathcal{J}^*$ has cotype 2 and the property GL. Actually the same holds for the analogously defined spaces v_p^0, $2 \le p < \infty$, obtained from completing $\mathbb{R}^{(\mathbb{N})}$ (or $\mathbb{C}^{(\mathbb{N})}$) with respect to the norm $\|(x_i)\| = \sup(\sum_i |x_{n_i} - x_{n_{i-1}}|^p)^{1/p}$, the supremum being extended over all increasing sequences (n_i) of positive integers. $\mathcal{J}$ is then just the space v_2^0. By Proposition 1, we have $\mathfrak{X}(v_p^0, l_2) = \Pi_1(v_p^0, l_2)$. It is easy to see that $(v_p^0)^{**}$ does not contain a copy of c_0. Hence it cannot appear as a quotient even of any C*-algebra since otherwise it would be reflexive [1] (and even super-reflexive [8]).

Of course, this last statement prompts us to ask the following questions.

QUESTION 1. Can any of the spaces $\mathcal{T}$ or $\tilde{X}(q, \eta)$ $(2 \le q < \infty, \eta > 1)$ mentioned before be isomorphic to a quotient of a C*-algebra?

$\mathcal{T}$ is the first genuine example of what is now called a weak Hilbert space [19].

QUESTION 2. Suppose that X is a weak Hilbert space and isomorphic to a quotient of an L_∞-space (or a C*-algebra). Is X isomorphic to a Hilbert space?

The spaces $\mathcal{T}$ and $\tilde{X}(q, \eta)$ are super-reflexive, whereas the spaces v_p^0 are non-reflexive but quasi-reflexive of order one. This leads to our final question.

QUESTION 3. Are there Banach spaces X which are (a) reflexive but not super-reflexive, or (b) not of finite codimension in X^{**}, such that $\mathfrak{X}(X, l_2) = \Pi_1(X, l_2)$ and X^{**} is not isomorphic to a quotient of any L_∞-space?

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