A REMARK ON THE PRINCIPLE OF ZERO UTILITY

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Let $u(x)$ be a utility function, i.e., a function with $u'(x) > 0$, $u''(x) < 0$ for all x. If S is a risk to be insured (a random variable), the premium $P = P(x)$ is obtained as the solution of the equation

$$u(x) = E[u(x+P-S)] \tag{1}$$

which is the condition that the premium is fair in terms of utility. It is clear that an affine transformation of u generates the same principle of premium calculation. To avoid this ambiguity, one can standardize the utility function in the sense that

$$u(y) = 0, \quad u'(y) = 1 \tag{2}$$

for an arbitrarily chosen point y. Alternatively, one can consider the risk aversion

$$r(x) = -u''(x)/u'(x) \tag{3}$$

which is the same for all affine transformations of a utility function.

Given the risk aversion $r(x)$, the standardized utility function can be retrieved from the formula

$$u(x) = \int_y^x \exp\left(-\int_y^z r(u)\,du\right) dz. \tag{4}$$

It is easily verified that this expression satisfies (2) and (3).

The following lemma states that the greater the risk aversion the greater the premium, a result that does not surprise.

LEMMA. *Let $u_1(x)$ and $u_2(x)$ be two utility functions with corresponding risk aversions $r_1(x)$, $r_2(x)$. Let P_i denote the premium that is generated by $u_i (i = 1, 2)$. If $r_1(x) \geq r_2(x)$ for all x, it follows that $P_1(w) \geq P_2(w)$ for any risk S and all w.*

PROOF. $P_i = P_i(w)$ is obtained as the solution of the equation

$$u_i(w) = E[u_i(w+P_i-S)], \qquad i = 1, 2. \tag{5}$$

We standardize u_1 and u_2 such that

$$u_i(w) = 0, \; u_i'(w) = 1. \tag{6}$$

Using (4), with $y = w$, we can express u_i in terms of r_i. Since $r_1(x) \geq r_2(x)$ for all x, it follows that

$$u_1(x) \leq u_2(x) \quad \text{for all } x. \tag{7}$$

Using (5), (6), (7) we see that

(8) $$E[u_2(w+P_2-S)]=E[u_1(w+P_1-S)]\leqslant E[u_2(w+P_1-S)]$$

Since u_2 is an increasing function, the inequality between the first term and the last term means that $P_1 \geqslant P_2$. Q.E.D.

The lemma has some immediate consequences:

APPLICATION 1. *The exponential premium, $P=(1/a)\log E[e^{aS}]$, is an increasing function of the parameter a.*

PROOF. Let $a_1 > a_2$. Use the lemma in the special case $r_i(x)=a_i$ (constant) to see that the exponential premium (parameter a_1) exceeds the exponential premium (parameter a_2). Q.E.D.

APPLICATION 2. *Suppose that $r(x)$ is a nonincreasing function. Then $P=P(x)$ as determined from* (1) *is a nonincreasing function of x for any risk S.*

PROOF. Let $h>0$. Use the lemma with $r_1(x)=r(x)$, $r_2(x)=r(x+h)$ to see that $P(x)\geqslant P(x+h)$. Q.E.D.

REMARKS. (1) The last two proofs are simpler than the original proofs given by Gerber (1974, p. 216) for the first application and by Leepin (1975, pp. 31–35) for the second application.

(2) For a small risk S (i.e., a random variable S with a narrow range) $P(x)$ is approximately $E[S]+r(x)\operatorname{var}[S]/2$. Thus the converse of the Lemma ($P_1(w)\geqslant P_2(w)$ for all S implies that $r_1(x)\geqslant r_2(x)$) is trivial.

(3) In Pratt's terminology (1964) the premium P is a (negative) *bid price*. However, Pratt's discussion focusses essentially on (what he calls) the *insurance premium Q*, which is defined as the solution of the equation

(9) $$u(x-Q)=E[u(x-S)]$$

and which should be interpreted as the largest premium someone with fortune x and liability S is willing to pay for full coverage. The counterpart of the Lemma (with $P_i(w)$ replaced by $Q_i(w)$) has been discussed by Pratt (1964, p. 128). A short proof of this counterpart is obtained if one standardizes u_1 and u_2 such that

(10) $$u_i(w-Q_1)=0, \qquad u_i'(w-Q_1)=1.$$

Details are left to the reader.

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