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Methoden

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Parametric Model Order Reduction of Port-Hamiltonian Systems by Matrix Interpolation

Parametrische Modellordnungsreduktion von Port-Hamilton Systemen mittels Matrixinterpolation

Abstract: In this paper, parametric model order reduction of linear time-invariant systems by matrix interpolation is adapted to large-scale systems in port-Hamiltonian form. A new weighted matrix interpolation of locally reduced models is introduced in order to preserve the port-Hamiltonian structure, which guarantees the passivity and stability of the interpolated system. The performance of the new method is demonstrated by technical examples.

Keywords: Parametric model order reduction, matrix interpolation, port-Hamiltonian systems.

Zusammenfassung: In diesem Beitrag wird eine Methode zur parametrischen Modellordnungsreduktion linearer, zeitinvarianter Systeme mittels Matrixinterpolation auf Originalsysteme in Port-Hamilton Form angepasst. Ein Vorgehen zur Matrixinterpolation, welches die Port-Hamilton Struktur im interpolierten System erhält, wird vorgestellt. Dies garantiert Passivität und Stabilität bei der parametrischen Modellordnungsreduktion. Zwei technische Beispiele zeigen die Leistungsfähigkeit der neuen Methode.

Schlüsselwörter: Parametrische Modellordnungsreduktion, Matrixinterpolation, Port-Hamilton Systeme.

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1 Introduction

The precise mathematical modeling of complex dynamical systems may result in systems of differential equations of such a high order that their processing can become very time-consuming or even unfeasible due to shortage of RAM, which is therefore neither reasonable nor profitable. *Model Order Reduction* (MOR) addresses that problem and seeks to approximate the original data of a large-scale model to a much smaller, reduced system. *Parametric Model Order Reduction* (pMOR) additionally tries to preserve the system's dependency on one or more parameters – for example geometric variables or material properties – within the reduced model.

One approach to describe dynamical systems, alongside the well-known first-order state-space representation, Equation (1), is the *port-Hamiltonian* representation. This is an energy-oriented form of system description with high physical interpretability, which additionally ensures passivity. In recent years, the port-Hamiltonian representation has gradually advanced and gained importance for automatic control purposes [1]. Likewise, efforts for developing passivity-preserving model reduction techniques for port-Hamiltonian systems have been undertaken. For example, structure-preserving MOR approaches using moment matching were presented in [2, 3]. An $\mathcal{H}_2$-optimal technique including tangential interpolation was suggested in [4].

In this paper, the framework for parametric model order reduction of linear time-invariant systems as presented by Panzer et al. in [5] is extended to linear time-invariant (LTI) systems that are originally modeled in port-Hamiltonian form. Merging pMOR by matrix interpolation with port-Hamiltonian representation results in the advantage that stability is preserved in the interpolated system.

In Section 2, moment matching and parametric model order reduction by matrix interpolation are reviewed. The port-Hamiltonian representation is summarized in Sec-

tion 3 and the employed structure-preserving MOR technique is presented. In Section 4, methods for structure-preserving pMOR for two different representations of port-Hamiltonian systems are introduced and discussed. Two simulation examples follow in Section 5.

2 Review of Projection-based MOR and pMOR

In this section, we briefly review projection-based model order reduction and parametric model order reduction by matrix interpolation.

2.1 Order Reduction by Moment Matching and Krylov Subspaces

Consider a linear time-invariant Multi-Input/Multi-Output (MIMO) state-space system in the form

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned} \tag{1}$$

with the state vector $\mathbf{x} \in \mathbb{R}^n$, the input vector $\mathbf{u} \in \mathbb{R}^m$ and the output vector $\mathbf{y} \in \mathbb{R}^l$. We have $\mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}$ and assume that $\det \mathbf{E} \neq 0$ and that $\mathbf{E}^{-1}\mathbf{A}$ is Hurwitz. Furthermore we have $\mathbf{B} \in \mathbb{R}^{n\times m}$ and $\mathbf{C} \in \mathbb{R}^{l\times n}$. It is assumed that system (1) is both controllable and observable. The order n is considered to be large. Let the reduced system be

$$\begin{aligned} \mathbf{E}_\mathrm{r}\dot{\mathbf{x}}_\mathrm{r}(t) &= \mathbf{A}_\mathrm{r}\mathbf{x}_\mathrm{r}(t) + \mathbf{B}_\mathrm{r}\mathbf{u}(t) \\ \mathbf{y}_\mathrm{r}(t) &= \mathbf{C}_\mathrm{r}\mathbf{x}_\mathrm{r}(t) \end{aligned} \tag{2}$$

with model order $q \ll n$, $\mathbf{x}_\mathrm{r} \in \mathbb{R}^q$, $\mathbf{E}_\mathrm{r}, \mathbf{A}_\mathrm{r} \in \mathbb{R}^{q\times q}$, $\mathbf{B}_\mathrm{r} \in \mathbb{R}^{q\times m}$ and $\mathbf{C}_\mathrm{r} \in \mathbb{R}^{l\times q}$.

The main idea of moment matching is to expand the frequency-domain transfer functions of the original- and the reduced system in Taylor series about an expansion point $s_0 \in \mathbb{C}$ and match the first q coefficients by means of a suitable projection: the original state vector $\mathbf{x}$ is approximated by $\mathbf{x} \approx \mathbf{V}\mathbf{x}_\mathrm{r}$ with $\mathbf{V} \in \mathbb{R}^{n\times q}$ and the state equation is multiplied with a suitably chosen matrix $\mathbf{W}^\top \in \mathbb{R}^{q\times n}$ from the left.

A block Krylov subspace of order p with respect to matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{R} \in \mathbb{R}^{n\times m}$ is defined as

$$\mathcal{K}_p(\mathbf{A}, \mathbf{R}) = \operatorname{colspan}\left\{\mathbf{R}, \mathbf{A}\mathbf{R}, \mathbf{A}^2\mathbf{R}, \ldots, \mathbf{A}^{p-1}\mathbf{R}\right\} \tag{3}$$

Remark: For the ease of presentation it is assumed that all directions defining a block Krylov subspace (3) are linearly independent. This implies that all block Krylov subspaces which will be employed in this work have full column rank $p \cdot m$. If the assumption does not hold, deflated block Krylov subspaces should be employed [7], which can be incorporated into the presented framework in a straightforward way.

Lemma 1. [8]: If $\mathbf{V}$ is chosen as a set of basis vectors of the block Krylov subspace

$$\mathcal{K}_p^{s_0} := \mathcal{K}_p((\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{E}, (\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{B}) \tag{4}$$

and $\mathbf{W}$ is arbitrary but such that $\mathbf{A}_\mathrm{r}$ and $\mathbf{E}_\mathrm{r}$ are non-singular, the first q block moments of the reduced system with

$$\begin{aligned} \mathbf{E}_\mathrm{r} &= \mathbf{W}^\top\mathbf{E}\mathbf{V} & \mathbf{A}_\mathrm{r} &= \mathbf{W}^\top\mathbf{A}\mathbf{V} \\ \mathbf{B}_\mathrm{r} &= \mathbf{W}^\top\mathbf{B} & \mathbf{C}_\mathrm{r} &= \mathbf{C}\mathbf{V} \end{aligned} \tag{5}$$

match.

In this work, we calculate an orthogonal basis $\mathbf{V}$ of the union of different Krylov subspaces

$$\bigcup_{i\in\{1,2,\ldots,k\}} \mathcal{K}_{p_i}^{s_i}, \qquad \text{with} \quad \sum_{i=1}^{k} p_i \cdot m = q \tag{6}$$

by an *Arnoldi*-like approach, [9]. Krylov subspace methods for model order reduction are computationally fast, as the main effort is an LU-decomposition during the Arnoldi-procedure. Furthermore, a figurative interpretation of the reduction process is available: we can think of it as a projection from the original state space to the particular subspace which is spanned by $\mathbf{EV}$, using the projector $\mathbf{EV}(\mathbf{W}^\top\mathbf{EV})^{-1}\mathbf{W}^\top$. We will denote the tall and skinny matrices $\mathbf{V}$ and $\mathbf{W}$ as *projection matrices* in the remainder of this paper. Drawbacks of the method are that no general error bound is known and that stability might not be preserved in the reduced model.

2.2 Parametric Model Order Reduction by Weighted Matrix Interpolation

If the large-scale system's behavior additionally depends on a parameter vector $\mathbf{p} \in \Pi \subseteq \mathbb{R}^d$, the system matrices become $\mathbf{E} \to \mathbf{E}(\mathbf{p})$, $\mathbf{A} \to \mathbf{A}(\mathbf{p})$, $\mathbf{B} \to \mathbf{B}(\mathbf{p})$ and $\mathbf{C} \to \mathbf{C}(\mathbf{p})$. An exact computation of the reduced system at numerous points in the parameter space may be inefficient. The general idea of parametric model order reduction by matrix interpolation is therefore the following: given a large-scale system at k sampling points $\mathbf{p}_1, \mathbf{p}_2 \ldots \mathbf{p}_k$ in the parameter

space, first reduce each system to desired order q. Then, in order to approximate the parameter-dependency, the idea is to interpolate between the resulting low-order systems by

$$\mathbf{E}_{\mathrm{r,int}} = \sum_{i=1}^{k} \omega(\mathbf{p}_i)\mathbf{E}_{\mathrm{r},i} \qquad \mathbf{A}_{\mathrm{r,int}} = \sum_{i=1}^{k} \omega(\mathbf{p}_i)\mathbf{A}_{\mathrm{r},i}$$
$$\mathbf{B}_{\mathrm{r,int}} = \sum_{i=1}^{k} \omega(\mathbf{p}_i)\mathbf{B}_{\mathrm{r},i} \qquad \mathbf{C}_{\mathrm{r,int}} = \sum_{i=1}^{k} \omega(\mathbf{p}_i)\mathbf{C}_{\mathrm{r},i} \tag{7}$$

with weighting factors $\omega(\mathbf{p}_i) \in \mathbb{R}$ and $\sum_{i=1}^{k}\omega(\mathbf{p}_i) = 1$. The interpolation could also take place in nonlinear form, but for simplicity, we restrict ourselves to linear interpolation in this work.

However, a direct interpolation of reduced matrices (5) according to (7) is not meaningful since the subspaces are themselves parameter-dependent (see e. g. Equation (4)). Two state vectors $\mathbf{x}_{\mathrm{r},1}$ and $\mathbf{x}_{\mathrm{r},2}$ from different reduced systems are not comparable to each other as they are located in different subspaces – except for the improbable case of matching projection matrices $\mathbf{V}_1 = \mathbf{V}_2$.

The direct interpolation (7) of systems that are not comparable to each other can yield e. g. singular $\mathbf{E}_{\mathrm{r,int}}$ or other unwanted effects. Therefore, it is mandatory to make the reduced system matrices compatible in some sense. As shown in [5], this can be achieved by embedding the locally reduced systems into the original coordinates of the large-scale system and subsequently orthogonally reprojecting them into a common subspace $\mathbf{U}_q$. We review this approach in the remainder of this section.

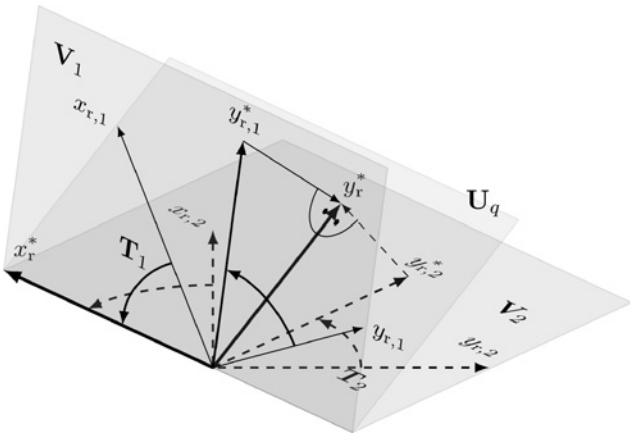

Figure 1: The process of making two pairs of basis vectors of locally reduced systems $\{x_{\mathrm{r},1}, y_{\mathrm{r},1}\}$ and $\{x_{\mathrm{r},2}, y_{\mathrm{r},2}\}$ compatible with respect to a common subspace $\mathbf{U}_q$. The subspace $\mathbf{U}_q$ unites the dominant directions of $\mathbf{V}_1$ and $\mathbf{V}_2$. The matrices $\mathbf{T}_i$ are chosen such that the images of corresponding pairs of basis vectors coincide after embedding them into the coordinates of the large-scale systems and a subsequent orthogonal projection into $\mathbf{U}_q$. In this example, $x_{\mathrm{r},1}$ and $x_{\mathrm{r},2}$ are rotated such that x^*_{r} is located in in the intersection of $\mathbf{V}_1$ and $\mathbf{V}_2$. Accordingly, the images of $y^*_{\mathrm{r},1}$ and $y^*_{\mathrm{r},2}$ coincide after orthogonal projection into $\mathbf{U}_q$.

The subspace $\mathbf{U}_q$ is found using a *Singular Value Decomposition* (SVD) $\mathbf{V}_{\mathrm{all}} = \mathbf{U}\mathbf{S}\mathbf{N}^\top$ with $\mathbf{V}_{\mathrm{all}} := [\mathbf{V}_1\ \mathbf{V}_2 \dots \mathbf{V}_k]$ being the total of all k projection matrices and every $\mathbf{V}_i$ being orthogonal. By choosing the first q columns of $\mathbf{U}$ as basis for $\mathbf{U}_q$, the new combined subspace unites the most dominant directions of all involved reduced systems' subspaces. In this work, we assume that the SVD is solvable in a large-scale setting, given that the number of sampling points in the parameter space is small enough.

A regular state transformation

$$\mathbf{x}_{\mathrm{r},i} = \mathbf{T}_i^{-1}\mathbf{x}^*_{\mathrm{r},i} \tag{8}$$

is applied to each of the k reduced systems (where $\mathbf{x}^*_{\mathrm{r},i}$ denotes a state vector in reduced and re-adjusted representation) and each of these k systems is multiplied with a regular matrix $\mathbf{M}_i \in \mathbb{R}^{q\times q}$ from the left, which leaves the input-output behavior unchanged:

$$\overbrace{\mathbf{M}_i\mathbf{E}_{\mathrm{r},i}\mathbf{T}_i^{-1}}^{\mathbf{E}^*_{\mathrm{r},i}}\dot{\mathbf{x}}^*_{\mathrm{r},i} = \overbrace{\mathbf{M}_i\mathbf{A}_{\mathrm{r},i}\mathbf{T}_i^{-1}}^{\mathbf{A}^*_{\mathrm{r},i}}\mathbf{x}^*_{\mathrm{r},i} + \overbrace{\mathbf{M}_i\mathbf{B}_{\mathrm{r},i}}^{\mathbf{B}^*_{\mathrm{r},i}}\mathbf{u}$$
$$\mathbf{y}_{\mathrm{r},i} = \underbrace{\mathbf{C}_{\mathrm{r},i}\mathbf{T}_i^{-1}}_{\mathbf{C}^*_{\mathrm{r},i}}\mathbf{x}^*_{\mathrm{r},i} \quad . \tag{9}$$

In the following, we review how the matrices $\mathbf{M}_i$ and $\mathbf{T}_i$ are chosen in [5] such that a weighted interpolation of the locally reduced systems becomes meaningful.

The reasoning for choosing $\mathbf{T}_i$ is as follows: consider k locally reduced systems with state vectors $\mathbf{x}^*_{\mathrm{r},i}$. First, reapply the transformation (8). Second, write every $\mathbf{x}_{\mathrm{r},i}$ in coordinates of the large-scale system by multiplying it with $\mathbf{V}_i$ from the left:

$$\hat{\mathbf{x}}_i = \mathbf{V}_i\mathbf{T}_i^{-1}\mathbf{x}^*_{\mathrm{r},i} \tag{10}$$

In the following, we will refer to this as 'embedding'-step. Third, every system is orthogonally projected into the subspace spanned by the columns of $\mathbf{U}_q$, which we will subsequently call 'reprojection'-step:

$$\mathbf{U}_q^\top\hat{\mathbf{x}}_i = \mathbf{U}_q^\top\mathbf{V}_i\mathbf{T}_i^{-1}\mathbf{x}^*_{\mathrm{r},i}\,. \tag{11}$$

Definition 1 [5]. *The coordinate systems of the state vectors $\mathbf{x}^*_{r,i}$ are called compatible w. r. t. a matrix $\mathbf{U}_q$, if the images of their basis vectors under a transformation $\mathbf{T}_i$, embedding using the matrices $\mathbf{V}_i$ and reprojection into the subspace spanned by the columns of $\mathbf{U}_q$ are identical.*

Figure 1 illustrates the process of making two different, locally reduced systems compatible. From Definition 1, it is

clear that $\mathbf{T}_i$ in (11) has to be chosen as

$$\mathbf{T}_i = \mathbf{U}_q^\top \mathbf{V}_i \tag{12}$$

in order to make the coordinate systems of all $\mathbf{x}_{\mathrm{r},i}^*$, $i = 1, 2, \ldots, k$ compatible w. r. t. a subspace spanned by $\mathbf{U}_q$. In [5], $\mathbf{M}_i$ is chosen as $\mathbf{M}_i = (\mathbf{W}_i^\top \mathbf{U}_q)^{-1}$ in order to represent the state equations of all reduced models with respect to the same basis. Subsequently, the re-adjusted system matrices $\mathbf{E}_{\mathrm{r},i}^*$, $\mathbf{A}_{\mathrm{r},i}^*$, $\mathbf{B}_{\mathrm{r},i}^*$ and $\mathbf{C}_{\mathrm{r},i}^*$ from Equation (9) can be interpolated according to (7).

This approach to parametric model order reduction exhibits several advantageous properties. An evaluation of the reduced system for any parameter vector $\mathbf{p}^* \notin \{\mathbf{p}_1, \mathbf{p}_2 \ldots \mathbf{p}_k\}$ is possible by weighted interpolation without having to repeat the computationally more expensive reduction steps, which is especially useful for large-scale systems. Therefore, the original system has to be evaluated and reduced at k sampling points only. At the same time, the reduced system order is independent of k. Furthermore, no analytic dependency of the original system on parameter-vector $\mathbf{p}$ is necessary – an advantage if the large-scale systems are to be obtained by system identification techniques.

3 Review of the Port-Hamiltonian Representation

3.1 (Linear) Time-Invariant Port-Hamiltonian Systems

A time-invariant port-Hamiltonian system is defined as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= (\mathbf{J} - \mathbf{R})\nabla H(\mathbf{x}(t)) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{B}^\top \nabla H(\mathbf{x}(t)) \,. \end{aligned} \tag{13}$$

$\mathbf{J} \in \mathbb{R}^{n\times n}$ is the skew-symmetric interconnection matrix. It captures the interconnection structure and is responsible for a proceeding, dynamical redistribution of energy between connected parts of the system. $\mathbf{R} \in \mathbb{R}^{n\times n}$ is the symmetric, positive semi-definite dissipation matrix. $H(\mathbf{x}(t))$ is the Hamiltonian, the energy function. By deriving the Hamiltonian with respect to time, it can be shown that port-Hamiltonian systems are passive, $\dot{H}(\mathbf{x}) \leq \mathbf{y}^\top \mathbf{u}$, which ensures stability according to Lyapunov's second method.

For a linear system, the energy function is quadratic, $H(\mathbf{x}(t)) = \frac{1}{2}\mathbf{x}^\top \mathbf{Q}\mathbf{x}$. Then, with $\mathbf{Q} \in \mathbb{R}^{n\times n}$ being symmetric and positive-definite, system (13) becomes

$$\begin{aligned} \dot{\mathbf{x}}(t) &= (\mathbf{J} - \mathbf{R})\mathbf{Q}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{B}^\top \mathbf{Q}\mathbf{x}(t) \end{aligned} \tag{14}$$

with constant system matrices, which we call a standard port-Hamiltonian system in the following.

Depending on the modeling process, a port-Hamiltonian system may either arise in standard form (14) or as

$$\begin{aligned} \mathbf{Q}^{-1}\dot{\mathbf{e}}(t) &= (\mathbf{J} - \mathbf{R})\mathbf{e}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{B}^\top \mathbf{e}(t) \end{aligned} \tag{15}$$

which is called *co-energy representation* [6]. Both are connected through a regular state transformation $\mathbf{Q}\mathbf{x}(t) = \mathbf{e}(t)$ with $\mathbf{e}(t)$ being called the *effort vector*. However, a numerical inversion of $\mathbf{Q}$, which would be required in order to transform a port-Hamiltonian system from one representation into the other, may be unfeasible due to large model order or bad numerical conditioning. For mechanical systems, for instance, $\mathbf{Q}$ often contains the mass and the stiffness matrices on its diagonal, which typically leads to a numerically ill-conditioned problem. Consequently, it is important to distinguish between two cases where the system arises either in explicit form (14) or implicit form (15). Therefore, two different methods for the pMOR of port-Hamiltonian systems – one for each representation – are required.

In the following, $\mathbf{Q}^{-1}$ in system (15) is assumed to be obtained directly from a modeling process that results in co-energy representation, not by inversion of a large-scale matrix. However, in order to emphasize its character as inverse of the system's energy matrix, we maintain the notation $\mathbf{Q}^{-1}$.

3.2 Structure-Preserving Model Reduction of Port-Hamiltonian Systems using Krylov Subspaces

In order to benefit from the port-Hamiltonian representation's advantages for reduced-order models, the symmetry and definiteness properties of the system matrices $\mathbf{J}$, $\mathbf{R}$ and $\mathbf{Q}$ must be preserved for the reduced models. In the following, structure- and stability-preserving MOR is reviewed for standard port-Hamiltonian systems and systems in co-energy representation.

3.2.1 Systems in Co-Energy Representation

Consider a large-scale system given in co-energy representation (15). It is emphasized again that $\mathbf{Q}^{-1}$ is assumed to be directly obtained from the modeling process and not by inversion of a large-scale matrix. For order reduction,

we follow Section 2 and approximate $\mathbf{e} \approx \mathbf{V}\mathbf{e}_\mathrm{r}$ with $\mathbf{V}$ being computed as a basis of the input Krylov subspace according to (4) with $\mathbf{A} = \mathbf{J} - \mathbf{R}$ and $\mathbf{E} = \mathbf{Q}^{-1}$. By choosing $\mathbf{W} = \mathbf{V}$, the preservation of definiteness and symmetry properties is guaranteed for the reduced system

$$\begin{aligned} \overbrace{\mathbf{V}^\top \mathbf{Q}^{-1} \mathbf{V}}^{\mathbf{Q}_\mathrm{r}^{-1}} \dot{\mathbf{e}}_\mathrm{r} &= \overbrace{\mathbf{V}^\top (\mathbf{J} - \mathbf{R}) \mathbf{V}}^{\mathbf{J}_\mathrm{r} - \mathbf{R}_\mathrm{r}} \mathbf{e}_\mathrm{r} + \overbrace{\mathbf{V}^\top \mathbf{B}}^{\mathbf{B}_\mathrm{r}} \mathbf{u} \\ \mathbf{y} &= \underbrace{\mathbf{B}^\top \mathbf{V}}_{\mathbf{B}_\mathrm{r}^\top} \mathbf{e}_\mathrm{r} \,. \end{aligned} \tag{16}$$

$\mathbf{V}^\top \mathbf{Q}^{-1} \mathbf{V}$ is symmetric and positive definite. As $(\mathbf{K}^\top \mathbf{J} \mathbf{K})^\top = -\mathbf{K}^\top \mathbf{J} \mathbf{K}$ for every skew-symmetric matrix $\mathbf{J}$ and arbitrary matrix $\mathbf{K}$, the matrix $\mathbf{J}_\mathbf{r} = \mathbf{V}^\top \mathbf{J} \mathbf{V}$ is skew-symmetric again. Moreover, $\mathbf{R}_\mathbf{r} = \mathbf{V}^\top \mathbf{R} \mathbf{V}$ is at least positive semidefinite. Thus, passivity is preserved.

3.2.2 Systems in Standard Port-Hamiltonian Form

Now consider an identical large-scale system, but given in standard port-Hamiltonian representation (14). The techniques from Section 2.1 are not directly applicable without losing structural properties and possibly passivity. A structure-preserving model order reduction approach for LTI port-Hamiltonian systems using Krylov-subspaces was presented in [2] and is briefly summarized in the following:

An approximation $\mathbf{x} \approx \widetilde{\mathbf{V}}\mathbf{e}_\mathbf{r}$ of the original state vector is introduced; $\widetilde{\mathbf{V}}$ is chosen as the basis of a Krylov subspace according to Equation (4) with $\mathbf{E} = \mathbf{I}$ and $\mathbf{A} = (\mathbf{J} - \mathbf{R})\mathbf{Q}$. The state equations are multiplied with a matrix $\widetilde{\mathbf{W}}^\top$ from the left. $\widetilde{\mathbf{W}}$ is a degree of freedom which may be chosen in any suitable way as long as $\det(\widetilde{\mathbf{W}}^\top \widetilde{\mathbf{V}}) \neq 0$ and $\det(\widetilde{\mathbf{W}}^\top (\mathbf{J} - \mathbf{R})\mathbf{Q}\widetilde{\mathbf{V}}) \neq 0$. Now, consider the particular choice $\widetilde{\mathbf{W}} = \mathbf{Q}\widetilde{\mathbf{V}}$, which leads to

$$\begin{aligned} \overbrace{\widetilde{\mathbf{V}}^\top \mathbf{Q} \widetilde{\mathbf{V}}}^{\mathbf{Q}_\mathrm{r}^{-1}} \dot{\mathbf{e}}_\mathrm{r} &= \overbrace{\widetilde{\mathbf{V}}^\top \mathbf{Q} (\mathbf{J} - \mathbf{R}) \mathbf{Q} \widetilde{\mathbf{V}}}^{\mathbf{J}_\mathrm{r} - \mathbf{R}_\mathrm{r}} \mathbf{e}_\mathrm{r} + \overbrace{\widetilde{\mathbf{V}}^\top \mathbf{Q} \mathbf{B}}^{\mathbf{B}_\mathrm{r}} \mathbf{u} \\ \mathbf{y} &= \underbrace{\mathbf{B}^\top \mathbf{Q} \widetilde{\mathbf{V}}}_{\mathbf{B}_\mathrm{r}^\top} \mathbf{e}_\mathrm{r} \,. \end{aligned} \tag{17}$$

We therefore obtain the reduced system in co-energy representation.

Note that evaluating the Krylov subspace in (4) for standard port-Hamiltonian form and co-energy representation shows that

$$\mathrm{span}\{\mathbf{V}\} = \mathrm{span}\{\mathbf{Q}\widetilde{\mathbf{V}}\} \tag{18}$$

where $\mathrm{span}\{\mathbf{V}\}$ denotes the subspace spanned by the columns of $\mathbf{V}$. As the transfer behavior of the reduced model only depends on the subspace (18), we can substitute $\mathbf{V}$ in (16) by $\mathbf{Q}\widetilde{\mathbf{V}}$, which shows that both reduced systems (16) and (17) are in fact identical w. r. t. input-output behavior.

4 Parametric Model Order Reduction of Port-Hamiltonian Systems

By applying the steps from Section 2.2 directly to a port-Hamiltonian system, it becomes obvious that the standard pMOR procedure destroys the system's definiteness and symmetry properties. For example, consider two port-Hamiltonian systems to be interpolated linearly with a weighting factor $\omega = 0.5$. Recombining the system matrices to $(\mathbf{J}_i - \mathbf{R}_i)\mathbf{Q}_i = \mathbf{A}_i$ and interpolating $\mathbf{A}_i$ shows that:

$$\begin{aligned} \tfrac{1}{2}\mathbf{A}_1 + \tfrac{1}{2}\mathbf{A}_2 &= \tfrac{1}{2}(\mathbf{J}_1 - \mathbf{R}_1)\mathbf{Q}_1 + \tfrac{1}{2}(\mathbf{J}_2 - \mathbf{R}_2)\mathbf{Q}_2 \\ &\neq \left(\left(\tfrac{1}{2}\mathbf{J}_1 + \tfrac{1}{2}\mathbf{J}_2\right) - \left(\tfrac{1}{2}\mathbf{R}_1 + \tfrac{1}{2}\mathbf{R}_2\right)\right)\left(\tfrac{1}{2}\mathbf{Q}_1 + \tfrac{1}{2}\mathbf{Q}_2\right). \end{aligned}$$

An implicit assumption for this interpolation is that the system and output equation are linear in the interpolation variables. This is obviously not true for standard port-Hamiltonian systems, but for systems in co-energy representation.

In the following, parametric model order reduction by matrix interpolation is extended to port-Hamiltonian systems. These systems may be available to the user in either of the representations given in Section 3, therefore two different methods are required. In Section 4.1, we adapt the pMOR from Section 2.2 to port-Hamiltonian systems being modeled in co-energy representation (15). In Section 4.2, we do the same for systems in standard port-Hamiltonian representation (14). In Section 4.3, it is shown that both methods result in interpolated systems with identical input-output behavior.

4.1 Systems in Co-Energy Representation

Consider k large-scale systems given in co-energy representation (15) at different sampling points $\mathbf{p}_\mathrm{i}$, $i = 1, 2, \ldots, k$, in the parameter space. For order reduction, we follow Section 3.2.1 and obtain k reduced systems

$$\begin{aligned} \mathbf{V}_i^\top \mathbf{Q}_i^{-1} \mathbf{V}_i \dot{\mathbf{e}}_{\mathrm{r},i} &= \mathbf{V}_i^\top (\mathbf{J}_i - \mathbf{R}_i) \mathbf{V}_i \mathbf{e}_{\mathrm{r},i} + \mathbf{V}_i^\top \mathbf{B}_i \mathbf{u} \\ \mathbf{y}_i &= \mathbf{B}_i^\top \mathbf{V}_i \mathbf{e}_{\mathrm{r},i} \quad . \end{aligned} \tag{19}$$

We note that a port-Hamiltonian system in co-energy representation strongly resembles a standard state-space sys-

tem in implicit form (Equation (1)). Therefore, in this particular case, all steps from Section 2.2 can analogously be applied to Equation (19). It is clear that an analysis of the required transformation, embedding and reprojection steps for the reduced systems in order to make them compatible w. r. t. a subspace $\mathbf{U}_q$ also yields a transformation matrix $\mathbf{T}_i = \mathbf{U}_q^\top \mathbf{V}_i$. Thus, in analogy to Equation (9), we get:

$$\begin{aligned}&\mathbf{M}_i \mathbf{V}_i^\top \mathbf{Q}_i^{-1} \mathbf{V}_i (\mathbf{U}_q^\top \mathbf{V}_i)^{-1} \dot{\mathbf{e}}_{\mathrm{r},i}^* \\ &\quad = \mathbf{M}_i \mathbf{V}_i^\top (\mathbf{J}_i - \mathbf{R}_i) \mathbf{V}_i (\mathbf{U}_q^\top \mathbf{V}_i)^{-1} \mathbf{e}_{\mathrm{r},i}^* + \mathbf{M}_i \mathbf{V}_i^\top \mathbf{B}_i \mathbf{u} \\ &\mathbf{y}_{\mathrm{r},i} = \mathbf{B}_i^\top \mathbf{V} (\mathbf{U}_q^\top \mathbf{V}_i)^{-1} \mathbf{e}_{\mathrm{r},i}^* \quad .\end{aligned}$$

where $\mathbf{U}_q$ is chosen as the first q columns of $\mathbf{U}$ from the SVD $\mathbf{V}_{\mathrm{all}} = \mathbf{U}\mathbf{S}\mathbf{N}^\top$ with

$$\mathbf{V}_{\mathrm{all}} := [\mathbf{V}_1 \; \mathbf{V}_2 \ldots \mathbf{V}_k] \tag{20}$$

and every $\mathbf{V}_i$ being orthogonal.

The choice

$$\mathbf{M}_i := \mathbf{T}_i^{-\top} = (\mathbf{U}_q^\top \mathbf{V}_i)^{-\top} = (\mathbf{V}_i^\top \mathbf{U}_q)^{-1} \tag{21}$$

preserves symmetry and definiteness properties. The reduced system matrices in co-energy-representation for every sampling point i then result as

$$\begin{aligned}\mathbf{Q}_{\mathrm{r},i}^{-1*} &:= (\mathbf{V}_i^\top \mathbf{U}_q)^{-1} \mathbf{V}_i^\top \mathbf{Q}_i^{-1} \mathbf{V}_i (\mathbf{U}_q^\top \mathbf{V}_i)^{-1} \\ \mathbf{J}_{\mathrm{r},i}^* &:= (\mathbf{V}_i^\top \mathbf{U}_q)^{-1} \mathbf{V}_i^\top \mathbf{J}_i \mathbf{V}_i (\mathbf{U}_q^\top \mathbf{V}_i)^{-1} \\ \mathbf{R}_{\mathrm{r},i}^* &:= (\mathbf{V}_i^\top \mathbf{U}_q)^{-1} \mathbf{V}_i^\top \mathbf{R}_i \mathbf{V}_i (\mathbf{U}_q^\top \mathbf{V}_i)^{-1} \\ \mathbf{B}_{\mathrm{r},i}^* &:= (\mathbf{V}_i^\top \mathbf{U}_q)^{-1} \mathbf{V}_i^\top \mathbf{B}_i\end{aligned} \tag{22}$$

which are ready for interpolation by

$$\begin{aligned}\mathbf{Q}_{\mathrm{r,int}}^{-1} &:= \sum_{i=1}^{k} \omega(\mathbf{p}_i) \mathbf{Q}_{\mathrm{r},i}^{-1*} \qquad \mathbf{J}_{\mathrm{r,int}} := \sum_{i=1}^{k} \omega(\mathbf{p}_i) \mathbf{J}_{\mathrm{r},i}^* \\ \mathbf{R}_{\mathrm{r,int}} &:= \sum_{i=1}^{k} \omega(\mathbf{p}_i) \mathbf{R}_{\mathrm{r},i}^* \qquad \mathbf{B}_{\mathrm{r,int}} := \sum_{i=1}^{k} \omega(\mathbf{p}_i) \mathbf{B}_{\mathrm{r},i}^* \, .\end{aligned} \tag{23}$$

If desired, the interpolated system can afterwards be transformed into standard port-Hamiltonian form by a state transformation $\mathbf{e}_{\mathrm{r,int}} = \mathbf{Q}_{\mathrm{r,int}} \cdot \mathbf{x}_{\mathrm{r,int}}$.

Remark: Note that $\mathbf{M}_i = (\mathbf{V}_i^\top \mathbf{U}_q)^{-1}$ and $\mathbf{T}_i = \mathbf{U}_q^\top \mathbf{V}_i$ are the same as in Section 2.2, if $\mathbf{W}_i = \mathbf{V}_i$. Therefore, the pMOR approach presented in [5] includes port-Hamiltonian systems in co-energy representation as special case, which was already highlighted in [10].

4.2 Systems in Standard Port-Hamiltonian Representation

Now, assume that k large-scale models in standard port-Hamiltonian form

$$\begin{aligned}\dot{\mathbf{x}}_i &= (\mathbf{J}_i - \mathbf{R}_i) \mathbf{Q}_i \mathbf{x}_i + \mathbf{B}_i \mathbf{u} \\ \mathbf{y}_i &= \mathbf{B}_i^\top \mathbf{Q}_i \mathbf{x}_i\end{aligned} \tag{24}$$

are given at sampling points $\mathbf{p}_i$, $i = 1, 2, \ldots, k$, in the parameter space. An inversion of $\mathbf{Q}_i$, which would be required to bring the system into co-energy representation, might be a tedious or unfeasible task if the order n of the original system is large, thus the procedure from Section 4.1 is not applicable. For order reduction, we follow Section 3.2.2 instead. We approximate

$$\mathbf{x}_i \approx \widetilde{\mathbf{V}}_i \mathbf{e}_{\mathrm{r},i}, \tag{25}$$

and eventually get k locally reduced systems

$$\begin{aligned}\overbrace{\widetilde{\mathbf{V}}_i^\top \mathbf{Q}_i \widetilde{\mathbf{V}}_i}^{\mathbf{Q}_{\mathrm{r},i}^{-1}} \dot{\mathbf{e}}_{\mathrm{r},i} &= \overbrace{\widetilde{\mathbf{V}}_i^\top \mathbf{Q}_i (\mathbf{J}_i - \mathbf{R}_i) \mathbf{Q}_i \widetilde{\mathbf{V}}_i}^{\mathbf{J}_{\mathrm{r},i} - \mathbf{R}_{\mathrm{r},i}} \mathbf{e}_{\mathrm{r},i} + \overbrace{\widetilde{\mathbf{V}}_i^\top \mathbf{Q}_i \mathbf{B}_i}^{\mathbf{B}_{\mathrm{r},i}} \mathbf{u} \\ \mathbf{y}_{\mathrm{r},i} &= \underbrace{\mathbf{B}_i^\top \mathbf{Q}_i \widetilde{\mathbf{V}}_i}_{\mathbf{B}_{\mathrm{r},i}^\top} \mathbf{e}_{\mathrm{r},i}\end{aligned} \tag{26}$$

in co-energy representation. Since matrix interpolation must take place in co-energy representation, a conflict with the pMOR procedure for standard state-space systems as in Section 2.2 is encountered: not the reduced state vectors in the sense of $\mathbf{x}_{\mathrm{r},i} = \mathbf{Q}_{\mathrm{r},i}^{-1} \mathbf{e}_{\mathrm{r},i}$, but the reduced effort-vectors $\mathbf{e}_{\mathrm{r,i}}$ have to be rewritten in coordinates of the original systems and then be reprojected into a combined subspace $\widetilde{\mathbf{U}}_q$ as follows: To make an interpolation meaningful, we readjust the locally reduced systems by applying a state transformation

$$\mathbf{e}_{\mathrm{r},i} = \widetilde{\mathbf{T}}_i^{-1} \mathbf{e}_{\mathrm{r},i}^* \tag{27}$$

to (26) and by multiplying the state equations with regular matrices $\widetilde{\mathbf{M}}_i$ from the left, which leads to

$$\begin{aligned}\overbrace{\widetilde{\mathbf{M}}_i \mathbf{Q}_{\mathrm{r},i}^{-1} \widetilde{\mathbf{T}}_i^{-1}}^{\widetilde{\mathbf{Q}}_{\mathrm{r},i}^{-1*}} \dot{\mathbf{e}}_{\mathrm{r},i}^* &= \overbrace{\widetilde{\mathbf{M}}_i (\mathbf{J}_{\mathrm{r},i} - \mathbf{R}_{\mathrm{r},i}) \widetilde{\mathbf{T}}_i^{-1}}^{\widetilde{\mathbf{J}}_{\mathrm{r},i}^* - \widetilde{\mathbf{R}}_{\mathrm{r},i}^*} \mathbf{e}_{\mathrm{r},i}^* + \overbrace{\widetilde{\mathbf{M}}_i \mathbf{B}_{\mathrm{r},i}}^{\widetilde{\mathbf{B}}_{\mathrm{r},i}^*} \mathbf{u} \\ \mathbf{y}_{\mathrm{r},i} &= \underbrace{\mathbf{B}_{\mathrm{r},i}^\top \widetilde{\mathbf{T}}_i^{-1}}_{\widetilde{\mathbf{B}}_{\mathrm{r},i}^{\top *}} \mathbf{e}_{\mathrm{r},i}^* \, .\end{aligned} \tag{28}$$

Appropriate matrices $\widetilde{\mathbf{M}}_i$ and $\widetilde{\mathbf{T}}_i$ are found as follows:

1. A locally reduced system can be expressed in coordinates of the original systems by writing $\hat{\mathbf{x}}_i = \widetilde{\mathbf{V}}_i \mathbf{e}_{\mathrm{r},i}$, cf. Equation (25).
2. The state vector $\hat{\mathbf{x}}_i$ is linked to its corresponding effort vector in large-scale coordinates $\hat{\mathbf{e}}_i$ by $\hat{\mathbf{e}}_i = \mathbf{Q}_i \hat{\mathbf{x}}_i$. Thus,

$$\hat{\mathbf{e}}_i = \mathbf{Q}_i \widetilde{\mathbf{V}}_i \mathbf{e}_{\mathrm{r},i}. \tag{29}$$

3. To find the dominant directions of all involved subspaces, the matrix $\mathbf{V}_{\text{all}}$ must be chosen as the total of the following k orthogonal matrices

$$\widetilde{\mathbf{P}}_i = \operatorname{orth}(\mathbf{Q}_i \widetilde{\mathbf{V}}_i) \tag{30}$$

where every matrix $\widetilde{\mathbf{P}}_i$ contains a set of mutually orthonormal basis vectors for the subspace spanned by $\mathbf{Q}_i \widetilde{\mathbf{V}}_i$. Hence, we define

$$\mathbf{V}_{\text{all}} := [\widetilde{\mathbf{P}}_1 \quad \widetilde{\mathbf{P}}_2 \ \dots \ \widetilde{\mathbf{P}}_k]. \tag{31}$$

4. Inserting (27) in (29) results in

$$\hat{\mathbf{e}}_i = \mathbf{Q}_i \widetilde{\mathbf{V}}_i \widetilde{\mathbf{T}}_i^{-1} \mathbf{e}^*_{\mathrm{r},i} \,. \tag{32}$$

5. Subsequently, every system written in coordinates of the large-scale co-energy representation is projected orthogonally into a common subspace $\widetilde{\mathbf{U}}_q$.

$$\widetilde{\mathbf{U}}_q^\top \hat{\mathbf{e}}_i = \widetilde{\mathbf{U}}_q^\top \mathbf{Q}_i \widetilde{\mathbf{V}}_i \widetilde{\mathbf{T}}_i^{-1} \mathbf{e}^*_{\mathrm{r},i} \,. \tag{33}$$

The basis of $\widetilde{\mathbf{U}}_q$ is chosen as the first q columns of $\widetilde{\mathbf{U}}$ from the SVD $\widetilde{\mathbf{V}}_{\text{all}} = \widetilde{\mathbf{U}}\widetilde{\mathbf{S}}\widetilde{\mathbf{N}}^\top$. Recalling (30) and (31), we see that $\widetilde{\mathbf{U}}_q$ captures the most dominant directions of the subspaces of all involved reduced systems in co-energy representation.

6. The coordinate systems of the reduced and adjusted effort vectors $\mathbf{e}^*_{\mathrm{r},i}$ are compatible w. r. t. the matrix $\widetilde{\mathbf{U}}_q$ if the images of their basis vectors under the transformation $\widetilde{\mathbf{T}}_i$, embedding using the matrices $\mathbf{Q}_i \widetilde{\mathbf{V}}_i$ and reprojection into the subspace spanned by the columns of $\widetilde{\mathbf{U}}_q$ are identical. Therefore, choose

$$\widetilde{\mathbf{T}}_i = \widetilde{\mathbf{U}}_q^\top \mathbf{Q}_i \widetilde{\mathbf{V}}_i \,. \tag{34}$$

7. $\widetilde{\mathbf{M}}_i$ is chosen such that definiteness and symmetry properties of the reduced and re-adjusted matrices are preserved:

$$\widetilde{\mathbf{M}}_i = \widetilde{\mathbf{T}}_i^{-\top} = (\widetilde{\mathbf{V}}_i^\top \mathbf{Q}_i \widetilde{\mathbf{U}}_q)^{-1} \,. \tag{35}$$

Finally, the interpolation-ready matrices read as

$$\begin{aligned}
\widetilde{\mathbf{Q}}_{\mathrm{r},i}^{-1*} &:= (\widetilde{\mathbf{V}}_i^\top \mathbf{Q}_i \widetilde{\mathbf{U}}_q)^{-1} \mathbf{Q}_{\mathrm{r},i}^{-1} (\widetilde{\mathbf{U}}_q^\top \mathbf{Q}_i \widetilde{\mathbf{V}}_i)^{-1} \\
\widetilde{\mathbf{J}}^*_{\mathrm{r},i} &:= (\widetilde{\mathbf{V}}_i^\top \mathbf{Q}_i \widetilde{\mathbf{U}}_q)^{-1} \mathbf{J}_{\mathrm{r},i} (\widetilde{\mathbf{U}}_q^\top \mathbf{Q}_i \widetilde{\mathbf{V}}_i)^{-1} \\
\widetilde{\mathbf{R}}^*_{\mathrm{r},i} &:= (\widetilde{\mathbf{V}}_i^\top \mathbf{Q}_i \widetilde{\mathbf{U}}_q)^{-1} \mathbf{R}_{\mathrm{r},i} (\widetilde{\mathbf{U}}_q^\top \mathbf{Q}_i \widetilde{\mathbf{V}}_i)^{-1} \\
\widetilde{\mathbf{B}}^*_{\mathrm{r},i} &:= (\widetilde{\mathbf{V}}_i^\top \mathbf{Q}_i \widetilde{\mathbf{U}}_q)^{-1} \mathbf{B}_{\mathrm{r},i} \quad ,
\end{aligned} \tag{36}$$

they can now be interpolated according to

$$\begin{aligned}
\widetilde{\mathbf{Q}}_{\mathrm{r,int}}^{-1} &:= \sum_{i=1}^{k} \omega(\mathbf{p}_i) \widetilde{\mathbf{Q}}_{\mathrm{r},i}^{-1*} \qquad \widetilde{\mathbf{J}}_{\mathrm{r,int}} := \sum_{i=1}^{k} \omega(\mathbf{p}_i) \widetilde{\mathbf{J}}^*_{\mathrm{r},i} \\
\widetilde{\mathbf{R}}_{\mathrm{r,int}} &:= \sum_{i=1}^{k} \omega(\mathbf{p}_i) \mathbf{R}^*_{\mathrm{r},i} \qquad \widetilde{\mathbf{B}}_{\mathrm{r,int}} := \sum_{i=1}^{k} \omega(\mathbf{p}_i) \widetilde{\mathbf{B}}^*_{\mathrm{r},i} \,.
\end{aligned} \tag{37}$$

It should be noted that identical results can be achieved in the following way: By interpreting the left-hand side of Equation (17) as

$$\begin{aligned}
\widetilde{\mathbf{V}}_i^\top \mathbf{Q}_i \widetilde{\mathbf{V}}_i \dot{\mathbf{e}}_{\mathrm{r},i} &= \widetilde{\mathbf{V}}_i^\top \mathbf{Q}_i \left(\mathbf{Q}_i^{-1} \mathbf{Q}_i \right) \widetilde{\mathbf{V}}_i \dot{\mathbf{e}}_{\mathrm{r},i} \\
&= \left(\mathbf{Q}_i \widetilde{\mathbf{V}}_i \right)^\top \mathbf{Q}_i^{-1} \left(\mathbf{Q}_i \widetilde{\mathbf{V}}_i \right) \dot{\mathbf{e}}_{\mathrm{r},i} \quad ,
\end{aligned} \tag{38}$$

it can be seen that (17) can be considered as projection of (15) using the projection matrix $\mathbf{Q}_i \widetilde{\mathbf{V}}_i$ (with $\widetilde{\mathbf{V}}_i$ being chosen as basis of the block Krylov subspace (4) with $\mathbf{E}_i = \mathbf{I}$ and $\mathbf{A}_i = (\mathbf{J}_i - \mathbf{R}_i)\mathbf{Q}_i$). Since the resulting system is in co-energy representation, the theory from Section 2.2 can be applied analogously to Section 4.1, which also leads to the result (35) for systems that are originally modeled in standard port-Hamiltonian form (14).

4.3 Comparison

With above presented methods, stability-preserving pMOR by matrix interpolation is possible for systems modeled in co-energy representation and standard port-Hamiltonian form without inversion of a large-scale matrix. In the following, it is shown that both methods result in interpolated systems with identical input-output behavior.

Theorem 1. *The interpolation* (23) *of the reduced systems* (22)*, which were originally given in co-energy representation, results in a system with the same input-output behavior as the interpolation of reduced systems* (36)*, which were originally modeled in standard port-Hamiltonian form.*

Proof. Given two systems in standard port-Hamiltonian form and co-energy representation at a sampling point $\mathbf{p} \in \Pi$, we showed in Section 3.2 that the locally reduced systems are identical for both presented MOR methods. From (18) it is known that $\operatorname{span}\{\mathbf{V}_i\} = \operatorname{span}\{\mathbf{Q}_i \widetilde{\mathbf{V}}_i\}$ and we can state that both matrices are linked by

$$\mathbf{V}_i = \mathbf{Q}_i \widetilde{\mathbf{V}}_i \mathbf{S}_i \,. \tag{39}$$

with a regular matrix $\mathbf{S}_i \in \mathbb{R}^{q \times q}$. It follows from (20) and (31) that $\operatorname{span}\{\mathbf{U}_q\} = \operatorname{span}\{\widetilde{\mathbf{U}}_q\}$ and

$$\mathbf{U}_q = \widetilde{\mathbf{U}}_q \mathbf{L} \tag{40}$$

with $\mathbf{L} \in \mathbb{R}^{q \times q}$ and $\mathbf{L}\mathbf{L}^\top = \mathbf{I}$, because $\mathbf{U}_q$ and $\widetilde{\mathbf{U}}_q$ are orthogonal. By substituting (39) and (40) in (22) and using $\mathbf{J}^*_{\mathrm{r},i}$ as an example it follows that

$$\begin{aligned}
\mathbf{J}^*_{\mathrm{r},i} &= (\mathbf{V}_i^\top \mathbf{U}_q)^{-1} \mathbf{V}_i^\top \mathbf{J}_i \mathbf{V}_i (\mathbf{U}_q^\top \mathbf{V}_i)^{-1} \\
&= \mathbf{L}^\top (\widetilde{\mathbf{V}}_i^\top \mathbf{Q}_i \widetilde{\mathbf{U}}_q)^{-1} \widetilde{\mathbf{V}}_i^\top \mathbf{Q}_i \mathbf{J}_i \mathbf{Q}_i \widetilde{\mathbf{V}}_i (\widetilde{\mathbf{U}}_q^\top \mathbf{Q}_i \widetilde{\mathbf{V}}_i)^{-1} \mathbf{L} \\
&= \mathbf{L}^\top (\widetilde{\mathbf{V}}_i^\top \mathbf{Q}_i \widetilde{\mathbf{U}}_q)^{-1} \mathbf{J}_{\mathrm{r},i} (\widetilde{\mathbf{U}}_q^\top \mathbf{Q}_i \widetilde{\mathbf{V}}_i)^{-1} \mathbf{L} \qquad \text{(by (26))} \\
&= \mathbf{L}^\top \widetilde{\mathbf{J}}^*_{\mathrm{r},i} \mathbf{L} \qquad \text{(by (36)),}
\end{aligned}$$

where all $\mathbf{S}_i$ cancel themselves. The same steps hold analogously for $\mathbf{Q}_{\mathrm{r},i}^{-1*}$, $\mathbf{R}_{\mathrm{r},i}^{*}$ and $\mathbf{b}_{\mathrm{r},i}^{*}$. Referring to the interpolation steps in (23) and (37), it results that

$$\begin{aligned} \mathbf{Q}_{\mathrm{r,int}}^{-1} &= \mathbf{L}^\top \widetilde{\mathbf{Q}}_{\mathrm{r,int}}^{-1} \mathbf{L} \qquad & \mathbf{J}_{\mathrm{r,int}} &= \mathbf{L}^\top \widetilde{\mathbf{J}}_{\mathrm{r,int}} \mathbf{L} \\ \mathbf{R}_{\mathrm{r,int}} &= \mathbf{L}^\top \widetilde{\mathbf{R}}_{\mathrm{r,int}} \mathbf{L} \qquad & \mathbf{B}_{\mathrm{r,int}} &= \mathbf{L}^\top \widetilde{\mathbf{B}}_{\mathrm{r,int}} \,. \end{aligned} \tag{41}$$

Therefore, the interpolated systems result in different representations which are linked by an orthogonal state transformation $\widetilde{\mathbf{e}}_{\mathrm{r},i} = \mathbf{L}\mathbf{e}_{\mathrm{r},i}$. Consequently, for both presented methods, the resulting interpolated systems have the same input-output behavior. □

4.4 Remarks

It is emphasized that the presented methods are suitable for models where no analytic dependency of the system matrices w. r. t. the parameters is given. Both methods are independent of the parameter space. Additionally, as no symbolic computations are necessary and only matrix-vector products (and LU decompositions during the Arnoldi procedure, if using Krylov subspace methods) must be executed, pMOR with matrix interpolation is well-suited for large-scale systems.

Note that a broad class of second order systems can be transformed into both co-energy and standard port-Hamiltonian representation, which makes it interesting for applications arising from classic modeling techniques such as FEM.

5 Simulation Results

In this section, we give two numerical examples for the presented parametric order reduction methods for port-Hamiltonian systems. The first is an example from structural dynamics: a clamped plate modeled in co-energy representation. The second is a large electrical network, which is modeled in standard port-Hamiltonian form.

5.1 Plate

Figure 2 shows a clamped steel plate with given dimensions which is excited by a point force $F(t)$ [5]. The plate is 0.2 mm thick. The system output is the displacement of the point where $F(t)$ attacks. The plate length L is a free variable. The model was derived in [5] using a parametric Ansys model with 225 shell elements and 1452 degrees of freedom. After evaluating it for a desired length L, it can be written as port-Hamiltonian system in co-energy representation. We consider two plates with $L = 450$ mm and $L = 550$ mm which are to be interpolated with $\omega = 0.5$. A third model has been generated for $x = 500$ mm and serves as reference. We chose the reduced model order $q = 12$ and expansion point $s_0 = 0$.

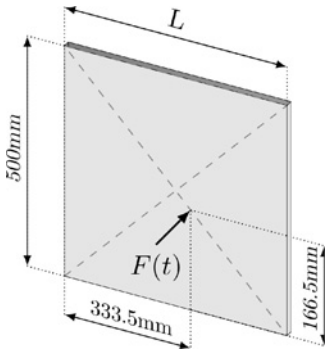

Figure 2: A steel plate, being excited by a force $F(t)$ in normal direction. The plate is assumed to be clamped at the edges.

Figure 3 shows the magnitude plots of the three directly reduced systems and the interpolated system. It can be seen that the linear interpolation approximates the reference system well.

Note that matrix interpolation can handle eigenvalue crossing: due to its quadratic shape, the reference system with $L = 500$ mm has a double eigenvalue at a resonance frequency close to 10^2 rad/sec. In contrast, the non-quadratic plates with $L_1 = 450$ mm and $L_2 = 550$ mm both have two distinct eigenvalues at this frequency range. The interpolation method captures this behavior and approximates the square plate system correctly.

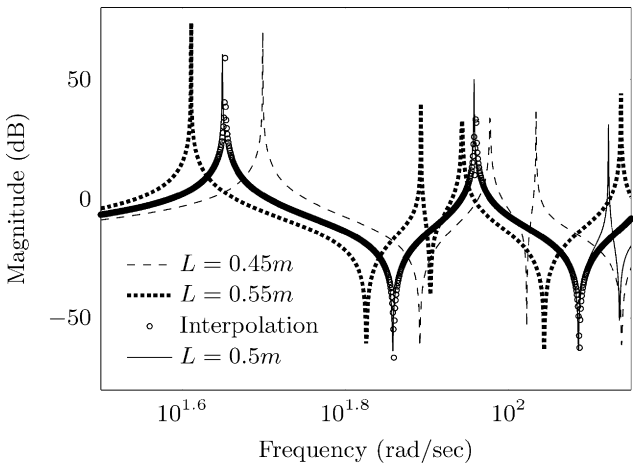

Figure 3: Amplitude of test point displacement for the reduced plate models with $L = 0.45$ m and $L = 0.55$ m as well as the interpolated system and the reference model with $L = 0.5$ m.

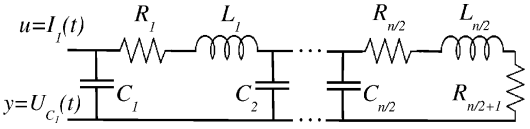

Figure 4: Structure of the ladder network as presented in [11]. The input is the current $I_1(t)$, the output is the voltage $U_{C_1}(t)$ across the first capacitor.

5.2 Electric Circuit Network

A ladder network is a system consisting of an arbitrary number of coupled RLC-circuits as shown in Figure 4. In [11], Matlab code for the generation of an electrical ladder network model in standard port-Hamiltonian form is provided. The free parameters are resistance $R_1 = R_2 = \ldots = R$, inductance $L_1 = L_2 = \ldots = L$ and capacity $C_1 = C_2 = \ldots = C$. The input is the current $I_1(t)$ from an external current source, the output is the voltage $U_{C_1}(t)$ across the first capacitor. In this example, we show a linear interpolation of two network models in three parameters, with $\omega = 0.5$.

System 1 has the parameters $R = 0.01\ \Omega$, $L = 1$ mH and $C = 1\ \mu$F. System 2 has the parameters $R = 0.012\ \Omega$, $L = 3$ mH and $C = 2\ \mu$F. Both were modeled with 250 circuits, resulting in $n = 1000$. Additionally, a reference system for linearly interpolated parameters was generated. The *RK-ICOP*-algorithm [12] was used to find a single expansion point at $s_0^{(1)} = 33565$ rad/sec for system 1 and $s_0^{(2)} = 13681$ rad/sec for system 2.

Figure 5 shows the magnitude plots of reduced ($q = 10$) and interpolated ladder network systems as well as the reference system with $R = 0.011\ \Omega\Omega$, $L = 2$ mH and $C = 1.5\ \mu$F. The interpolation fits the reference system well. In fact, the new pMOR method for port-Hamiltonian systems in standard form shows good performance in interpolating three independent parameters, using models at only two sampling points in the parameter space.

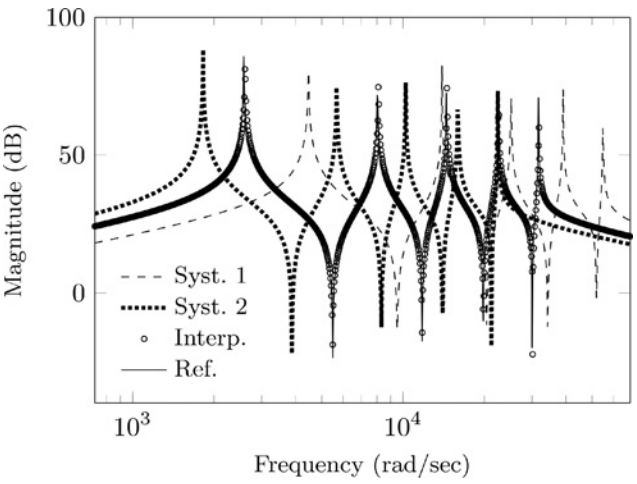

Figure 5: Magnitude plots of two ladder networks of order $n = 1000$ which differ in three parameters, their linear interpolation and a corresponding, separately reduced reference system.

6 Conclusion

This paper demonstrates that the advantages of port-Hamiltonian systems and their structure-preserving order reduction can be combined with parametric order reduction. A procedure for parametric model order reduction by matrix interpolation was suggested for both co-energy representation and standard port-Hamiltonian form. The method which is to be used is determined by in which representation the system is given. For each option, there is a method which does not require the inversion of a large-scale matrix.

It was shown that an interpolation of the port-Hamiltonian system matrices has to take place in co-energy representation in order to preserve the port-Hamiltonian structure. Adjustments to the locally reduced systems were outlined, which make matrix interpolation meaningful. Both presented methods preserve the symmetry and definiteness properties, stability, and result in interpolated systems with identical input-output behavior. For a parametric system which is defined by a finite number of sampling points, stability is obtained for all inter-adjacent, reduced systems. Simulation examples were given for both methods, for a one- and three-dimensional parameter space.

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