SPECTRUM OF A SELF-ADJOINT OPERATOR AND PALAIS–SMALE CONDITIONS

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Abstract

The spectrum and essential spectrum of a self-adjoint operator in a real Hilbert space are characterized in terms of Palais–Smale conditions on its quadratic form and Rayleigh quotient respectively.

1. *Introduction*

The purpose of this paper is to point out a precise and useful relationship between two important notions in operator theory and variational analysis. In the theory of linear operators, the spectrum and its refinements, particularly the essential spectrum, are fundamental concepts. In the study of critical points of real-valued functionals, with or without constraints, the Palais–Smale condition and its variants play an essential role. For a self-adjoint operator, S, the spectrum and essential spectrum of S can be characterized in terms of (P–S) conditions for the associated quadratic form, J, and its Rayleigh quotient, j, respectively. In the case of a bounded operator the results can be stated very simply.

Consider a bounded self-adjoint operator $S: H \longrightarrow H$ acting on a real Hilbert space $(H, \langle \cdot, \cdot \rangle)$ and let $J(u) = \langle Su, u \rangle$ for $u \in H$. Let σ and σ_e denote the spectrum and essential spectrum of S, respectively. Set $M = \{u \in H: \langle u, u \rangle = 1\}$ and let j denote the restriction of J to M. Then,

$$\sigma = \{\lambda \in \mathbb{R}: \text{the functional } J_\lambda \text{ does not satisfy (P–S) on } H\}$$

where $J_\lambda(u) = J(u) - \lambda \langle u, u \rangle$ for $u \in H$ and

$$\sigma_e = \{\lambda \in \mathbb{R}: \text{the functional } j \text{ does not satisfy (P–S) at level } \lambda \text{ on } M\}.$$

There are analogous results for unbounded operators but their statement requires a little more care since J is not differentiable with respect to the norm of H even at points in the domain of S. However there is a natural domain, H_1, and norm associated with the form J and, once these have been introduced, similar relations hold.

The main definitions are recalled in Section 2. The auxiliary space H_1, required to deal with unbounded operators, is introduced in Section 3 where the main step involves showing that S is equivalent to a bounded self-adjoint operator on the Hilbert space H_1. This construction is studied in some detail because of its use in nonlinear analysis. (See [**1**] and [**8**], for example, where it is used in the study of nonlinear Schrödinger operators and Hamiltonian systems.) Our results give a simpler and more complete description of the relationship between the operator $S - \lambda I$ on H and its representation $A - \lambda L$ in H_1 than was previously available. The extensions to H_1 of the quadratic form for S and its Rayleigh quotient are discussed in Section 4. The results relating the spectrum of S to the (P–S) conditions are stated

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and proved in full generality in Section 5. When S is bounded the discussion can be abridged by setting $H_1 = H$, $A = S$ and $L = I$. Then Section 3 can be ignored and some parts of the proofs in Sections 4 and 5 can be simplified.

2. *Preliminaries*

Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space with norm $\| \cdot \|$ and consider a self-adjoint operator, $S: D(S) \subset H \longrightarrow H$, acting in H where $D(S)$ is a dense subspace of H. The following definitions are standard. See [**3**, **4**, **6**, **9**].

$$\rho(S) = \{\lambda \in \mathbb{R}: \text{the operator } S - \lambda I: D(S) \longrightarrow H \text{ is an isomorphism}\}$$
$$\sigma(S) = \mathbb{R} \backslash \rho(S)$$
$$\sigma_{\mathrm{d}}(S) = \{\lambda \in \mathbb{R}: \text{the operator } S - \lambda I: D(S) \longrightarrow H \text{ is Fredholm}\}$$
$$\sigma_{\mathrm{e}}(S) = \sigma(S) \backslash \sigma_{\mathrm{d}}(S).$$

Here $\rho(S)$ is usually called the resolvent set and $\sigma_{\mathrm{d}}(S)$ the discrete spectrum. The spectrum, $\sigma(S)$, is always a closed non-empty set whereas the essential spectrum, $\sigma_{\mathrm{e}}(S)$, is a closed subset of $\sigma(S)$ which may be empty. Since a self-adjoint operator is always closed, the resolvent set consists of those $\lambda \in \mathbb{R}$ such that $S - \lambda I: D(S) \subset H \longrightarrow H$ has a bounded inverse defined on all of H. The discrete spectrum consists of the eigenvalues of S which have finite multiplicity and which are isolated points of $\sigma(S)$. The splitting of $\sigma(S)$ into $\sigma_{\mathrm{d}}(S) \cup \sigma_{\mathrm{e}}(S)$ is also important because $\sigma_{\mathrm{e}}(S)$ is invariant under compact perturbation of S.

For general closed linear operators, several different notions of what is meant by the essential spectrum are used. However, in the case of self-adjoint operators, they all coincide with the above definition. See [**3**] or [**6**]. This is one reason for restricting our discussion to the self-adjoint case.

Since it was first formulated by Palais and Smale under the name Condition (C), variants of their idea have become a standard part of variational methods. See [**2**, **5**, **10**]. Consider a smooth H-manifold V and a functional $f \in C^1(V, \mathbb{R})$. (In fact we shall only use two trivial cases, namely $V = H$ and $V = \{u \in H: g(u) = 0\}$ where $g \in C^\infty(H, \mathbb{R})$ with $g'(u) \neq 0$ for all $u \in V$.) See [**2**, §27.4] or [**10**, Chapter 43]. For $c \in \mathbb{R}$, the functional f satisfies condition (P–S) at level c on V if every sequence $\{u_n\} \subset V$ such that

$$f(u_n) \to c \quad \text{and} \quad f'(u_n) \to 0$$

has a subsequence which converges in H. If f satisfies condition (P–S) at level c on V for every $c \in \mathbb{R}$, then f is said to satisfy condition (P–S) on V. In these definitions $f'(u)$ is a bounded linear functional on the tangent space to V at u, and $f'(u_n) \to 0$ in the sense that $\|f'(u_n)\|_* \to 0$ where $\| \cdot \|_*$ denotes the usual norm on the dual space $T_u(V)^*$.

The kernel and range of a linear operator T will be denoted by $\ker T$ and $\operatorname{rge} T$, respectively.

3. *The form space of a self-adjoint operator*

Let $(H, \langle \cdot, \cdot \rangle)$ be a real Hilbert space with norm $\| \cdot \|$ and consider a self-adjoint operator, $S: D(S) \subset H \longrightarrow H$, acting in H where $D(S)$ is a dense subspace of H. There is a unique right-continuous resolution of the identity (or spectral family [**7**, **9**]) $\{E(\lambda): \lambda \in \mathbb{R}\}$ such that

$$D(S) = \left\{ u \in H: \int \lambda^2 d\langle E(\lambda) u, u \rangle < \infty \right\} \tag{1}$$

and

$$\langle Su, v\rangle = \int \lambda d\langle E(\lambda)u, v\rangle \quad \text{for all } u \in D(S) \text{ and } v \in H. \tag{2}$$

When no domain of integration is indicated it is understood that the integral is over $\mathbb{R}$. For any continuous function, $f:\mathbb{R} \longrightarrow \mathbb{R}$, a self-adjoint operator, $f(S): D(f(S)) \subset H \longrightarrow H$, is defined by

$$D(f(S)) = \left\{u \in H: \int f(\lambda)^2\, d\langle E(\lambda)u, u\rangle < \infty\right\} \tag{3}$$

and

$$\langle f(S)u, v\rangle = \int f(\lambda)\, d\langle E(\lambda)u, v\rangle \quad \text{for all } u \in D(f(S)) \text{ and } v \in H. \tag{4}$$

In particular, $D(f(S))$ is a dense subspace of H. For all $u \in D(f(S))$,

$$\|f(S)u\|^2 = \int f(\lambda)^2\, d\langle E(\lambda)u, u\rangle \tag{5}$$

and

$$E(\lambda)u \in D(f(S)) \text{ with } f(S)E(\lambda)u = E(\lambda)f(S)u \quad \text{for all } \lambda \in \mathbb{R}. \tag{6}$$

If $u \in D(f(S))$, the element $f(S)u \in D(f(S))$ if and only if $u \in D(f^2(S))$, and when $u \in D(f(S)) \cap D(f^2(S))$,

$$f(S)^2 u = f^2(S)u. \tag{7}$$

If $u \in D(S) \cap D(f(S))$ and $f(S)u \in D(S)$, then $Su \in D(f(S))$ and

$$Sf(S)u = f(S)Su. \tag{8}$$

Statements (1) to (5) are standard. For the properties (6), (7) and (8), see [**7**, Theorem 6.1, parts (4), (6) and (7)].

The *form space* of $S: D(S) \subset H \longrightarrow H$ is now defined as the domain of the operator $|S|^{1/2}$ equipped with its graph norm. (See [**3**, p. 183].) More explicitly, we set

$$H_1 = D(|S|^{1/2}) \text{ with } \langle u, v\rangle_1 = \langle u, v\rangle + \langle |S|^{1/2}u, |S|^{1/2}v\rangle$$

and

$$\|u\|_1^2 = \|u\|^2 + \||S|^{1/2}u\|^2$$

for all $u, v \in H_1$. It is well known that $(H_1, \langle\cdot,\cdot\rangle_1)$ is a Hilbert space and that $D(S)$ is a dense subspace of $(H_1, \langle\cdot,\cdot\rangle_1)$. Furthermore,

$$H_1 = \left\{u \in H: \int |\lambda|\, d\langle E(\lambda)u, u\rangle < \infty\right\} \tag{9}$$

and

$$\|u\|_1^2 = \int (1+|\lambda|)\, d\langle E(\lambda)u, u\rangle. \tag{10}$$

There are two settings in which S can be extended to a bounded linear operator on H_1. One consists of identifying H_1 with its dual $(H_1)^*$ and then representing S as a bounded self-adjoint operator from H_1 into itself. This is the procedure we adopt below because it is frequently used when critical point theory is applied to nonlinear problems. The alternative (equivalent) method is to identify H with its dual H^* (hence $H_1 \subset H = H^* \subset (H_1)^*$), and then to extend S as a bounded symmetric operator from

H_1 into $(H_1)^*$. This is what is done in [**3**], for example, and our results could easily be reformulated in this context.

LEMMA 3.1. *Let $S: D(S) \subset H \longrightarrow H$ be a self-adjoint operator on H with form space $(H_1, \langle \cdot, \cdot \rangle_1)$. Consider the self-adjoint operator $T: D(T) \subset H \longrightarrow H$ on H defined by $T = (I+|S|)^{1/2}$. Then*

(a) *$D(T) = H_1$ and T is an isometric isomorphism of the Hilbert space $(H_1, \langle \cdot, \cdot \rangle_1)$ onto $(H, \langle \cdot, \cdot \rangle)$. Also,*

(b) *$D(S) = D(T^2) = \{u \in H_1 : Tu \in H_1\}$ and*

$$\|u\|^2 + \|Su\|^2 \leqslant \|Tu\|_1^2 \leqslant 2(\|u\|^2 + \|Su\|^2) \quad \textit{for all } u \in D(S).$$

Finally,

(c) *if $u \in D(S)$ we have*

$$T^{-1}u \in D(S) \quad \textit{and} \quad ST^{-1}u = T^{-1}Su.$$

REMARK 3.2. It follows from (a) that

$$\langle T^{-1}u, T^{-1}v \rangle_1 = \langle u, v \rangle \quad \text{for all } u, v \in H. \tag{11}$$

Proof of Lemma 3.1. Let $f(t) = (1+|t|)^{1/2}$ for $t \in \mathbb{R}$ and consider the operator $T = f(S)$ defined by (3) and (4).

(a) By definition,

$$D(T) = \left\{ u \in H : \int (1+|\lambda|)\, d\langle E(\lambda)\, u, u \rangle \right\}$$

and so by (9) and (10), $H_1 = D(T)$ with

$$\|Tu\|^2 = \int (1+|\lambda|)\, d\langle E(\lambda)\, u, u \rangle = \|u\|_1^2 \quad \text{for all } u \in H_1$$

by (5) and (10). We have already shown that the operator $T: H_1 \longrightarrow H$ is isometric. This implies that $\operatorname{rge} T = \{Tu : u \in H_1\}$ is a closed subset of H. However, the self-adjointness of $T: D(T) \subset H \longrightarrow H$ now yields

$$\operatorname{rge} T = [\ker T]^{\perp} = \{0\}^{\perp} = H$$

and so $T: H_1 \longrightarrow H$ is an isometric isomorphism.

(b) Suppose first that $u \in D(S)$. Then $u \in H_1 = D(f(S))$ and to prove that $Tu \in H_1$ we need only show that $u \in D(f^2(S))$. Since $u \in D(S)$ we have

$$\begin{aligned} \|u\|^2 + \|Su\|^2 &= \int (1+\lambda^2)\, d\langle E(\lambda)\, u, u \rangle \\ &\geqslant \frac{1}{2} \int (1+|\lambda|)^2\, d\langle E(\lambda)\, u, u \rangle = \frac{1}{2} \int [f^2(\lambda)]^2\, d\langle E(\lambda)\, u, u \rangle. \end{aligned}$$

Hence $u \in D(f^2(S))$ and

$$\|f^2(S)\, u\|^2 \leqslant 2\{\|u\|^2 + \|Su\|^2\}.$$

It follows from (7) that $Tu = f(S)\, u \in D(f(S)) = H_1$ and

$$\|T^2 u\|^2 = \|f^2(S)\, u\|^2 \leqslant 2\{\|u\|^2 + \|Su\|^2\}.$$

Thus we see that $D(S) \subset D(T^2)$ and, using part (a),

$$\|Tu\|_1^2 = \|T^2 u\|^2 \leqslant 2\{\|u\|^2 + \|Su\|^2\}.$$

Conversely, if $u \in H_1 = D(f(S))$ and $Tu \in H_1 = D(f(S))$, it follows from (7) that $u \in D(f^2(S))$ and that

$$\|f^2(S)\,u\| = \|[f(S)]^2\,u\| = \|T^2 u\|.$$

However

$$\|f^2(S)\,u\|^2 = \int (1+|\lambda|)^2\, d\langle E(\lambda)\,u, u\rangle \geqslant \int |\lambda|^2\, d\langle E(\lambda)\,u, u\rangle$$

and so, by (1) and (5), $u \in D(S)$ with

$$\|Su\|^2 = \int |\lambda|^2\, d\langle E(\lambda)\,u, u\rangle.$$

Hence

$$\|u\|^2 + \|Su\|^2 = \int (1+|\lambda|^2)\, d\langle E(\lambda)\,u, u\rangle \leqslant \int (1+|\lambda|)^2\, d\langle E(\lambda)\,u, u\rangle$$

$$= \|f^2(S)\,u\|^2 = \|T^2 u\|^2 = \|Tu\|_1^2.$$

Thus $D(T^2) \subset D(S)$ and

$$\|Tu\|_1^2 \geqslant \|u\|^2 + \|Su\|^2 \quad \text{for all } u \in D(T^2).$$

(c) Consider $u \in D(S)$. Clearly $T^{-1}u \in H_1$ by part (a) and $T(T^{-1}u) = u \in D(S) \subset H_1$. Hence $T^{-1}u \in D(S)$ by part (b). Now we set $v = T^{-1}u$ and use the result (8). In fact, as we have just shown, $v \in D(S) \cap D(T)$ and $Tv = u \in D(S)$, so we can conclude that $Sv \in D(T)$ and $STv = TSv$. Hence, by part (a),

$$T^{-1}Su = T^{-1}STv = Sv = ST^{-1}u$$

as required. □

COROLLARY 3.3. *In the context of Lemma* 3.1, *let* $(H_2, \langle\cdot,\cdot\rangle_2)$ *denote the graph space of* S:

$$H_2 = D(S) \quad \text{with} \quad \|u\|_2 = \{\|u\|^2 + \|Su\|^2\}^{1/2} \quad \text{for } u \in H_2.$$

Then

(a) *for all* $u \in H_2$,

$$Tu \in H_1 \quad \text{and} \quad \|u\|_2 \leqslant \|Tu\|_1 \leqslant \sqrt{2}\|u\|_2; \tag{12}$$

(b) $T: H_2 \longrightarrow H_1$ *is a homeomorphism;*

(c) $T^{-1}ST^{-1}: H_1 \longrightarrow H_1$ *is a bounded linear operator and*

$$\langle T^{-1}ST^{-1}u, v\rangle_1 = \langle Su, v\rangle \quad \text{for all } u \in H_2 \text{ and } v \in H_1;$$

(d) $T^{-1}T^{-1}: H \longrightarrow H_2$ *is a bounded linear operator and*

$$\langle T^{-1}T^{-1}u, v\rangle_1 = \langle u, v\rangle \quad \text{for all } u \in H \text{ and } v \in H_1.$$

Proof. Part (a) follows immediately from Lemma 3.1(b). It also shows that T is a bounded operator from $(H_2, \langle\cdot,\cdot\rangle_2)$ into $(H_1, \langle\cdot,\cdot\rangle_1)$ which is one-to-one. Suppose that $w \in H_1$. Since $T: H_1 \longrightarrow H$ is an isomorphism, there is an element $u \in H_1$ such that $Tu = w$. Lemma 3.1(b) now shows that $u \in D(S) = H_2$ and so $T(H_2) = H_1$. Thus $T: H_2 \longrightarrow H_1$ is onto and the first inequality in part (a) completes the proof that $T: H_2 \longrightarrow H_1$ is a homeomorphism.

(c) For $u \in H_1$, we have $T^{-1}u \in H_2$ by part (b) and so $T^{-1}ST^{-1}u \in H_1$ with

$$\|T^{-1}ST^{-1}u\|_1 = \|ST^{-1}u\| \leqslant \|T^{-1}u\|_2 \leqslant \|u\|_1$$

by (11) and (12). Thus $T^{-1}ST^{-1}$ is a bounded operator from $(H_1, \langle\cdot,\cdot\rangle_1)$ into $(H_1, \langle\cdot,\cdot\rangle_1)$. Finally, for $u \in H_2$ and $v \in H_1$, we have

$$\begin{aligned}\langle T^{-1}ST^{-1}u, v\rangle_1 &= \langle ST^{-1}u, Tv\rangle \quad \text{by (11)}\\ &= \langle T^{-1}Su, Tv\rangle \quad \text{by Lemma 3.1(c)}\\ &= \langle Su, v\rangle\end{aligned}$$

since $T^{-1}: H \longrightarrow H$ is a bounded self-adjoint operator.

(d) For $u \in H$, we have $T^{-1}u \in H_1$ by Lemma 3.1(a) and so $T^{-1}T^{-1}u \in H_2$ by part (b) with

$$\|T^{-1}T^{-1}u\|_2 \leqslant \|T^{-1}u\|_1 = \|u\|$$

by (12) and (11). Thus $T^{-1}T^{-1}$ is a bounded operator from $(H, \langle \cdot, \cdot \rangle)$ into $(H_2, \langle \cdot, \cdot \rangle_2)$. Finally for $u \in H$ and $v \in H_1$, we have

$$\begin{aligned}\langle T^{-1}T^{-1}u, v\rangle_1 &= \langle T^{-1}u, Tv\rangle \quad \text{by (11)}\\ &= \langle u, v\rangle\end{aligned}$$

since $T^{-1}: H \longrightarrow H$ is a bounded self-adjoint operator. □

We now introduce the representation of S as a bounded self-adjoint operator acting on $(H_1, \langle \cdot, \cdot \rangle_1)$.

THEOREM 3.4. *Let $S: D(S) \subset H \longrightarrow H$ be a self-adjoint operator on H with form space $(H_1, \langle \cdot, \cdot \rangle_1)$.*

(i) *There is a unique bounded linear operator A from $(H_1, \langle \cdot, \cdot \rangle_1)$ into itself such that*

$$\langle Au, v\rangle_1 = \langle Su, v\rangle \quad \text{for all } u \in H_2 \text{ and } v \in H_1. \tag{13}$$

Furthermore,

$$\langle Au, v\rangle_1 = \langle u, Av\rangle_1 = \int \lambda d\langle E(\lambda)u, v\rangle \tag{14}$$

for all $u, v \in H_1$ where $\{E(\lambda): \lambda \in \mathbb{R}\}$ is the resolution of the identity associated with S.

(ii) *There is a unique bounded linear operator L from $(H_1, \langle \cdot, \cdot \rangle_1)$ into itself such that*

$$\langle Lu, v\rangle_1 = \langle u, v\rangle \quad \text{for all } u, v \in H_1. \tag{15}$$

Furthermore, $\langle Lu, v\rangle_1 = \langle u, Lv\rangle_1$ for all $u, v \in H_1$.

Let $T = (I + |S|)^{1/2}: D(T) = H_1 \subset H \longrightarrow H$ be the operator introduced in Lemma 3.1.

(iii) *Then $A = T^{-1}ST^{-1}$ and $L = T^{-1}T^{-1}$.*

(iv) $\sigma(S) = \{\lambda \in \mathbb{R}$: the bounded operator $A - \lambda L: H_1 \longrightarrow H_1$ is not an isomorphism$\}$.

(v) $\sigma_e(S) = \{\lambda \in \mathbb{R}$: the bounded operator $A - \lambda L: H_1 \longrightarrow H_1$ is not Fredholm$\}$.

Proof. (i) Setting $A = T^{-1}ST^{-1}$, Corollary 3.3 shows that this operator is bounded from $(H_1, \langle \cdot, \cdot \rangle_1)$ into itself and has the property (13). The symmetry and uniqueness of A follow from the fact that $D(S)$ is a dense subset of H_1.

For all $\lambda \in \mathbb{R}$, $E(\lambda)$ and T^{-1} are bounded self-adjoint operators on H which commute. Hence

$$\langle E(\lambda)T^{-1}u, Tv\rangle = \langle T^{-1}E(\lambda)u, Tv\rangle = \langle E(\lambda)u, v\rangle$$

for all $u, v \in H_1$. Thus, by (11),

$$\begin{aligned}\langle Au, v\rangle_1 = \langle T^{-1}ST^{-1}u, v\rangle_1 &= \langle ST^{-1}u, Tv\rangle\\ &= \int \lambda d\langle E(\lambda)T^{-1}u, Tv\rangle = \int \lambda d\langle E(\lambda)u, v\rangle\end{aligned}$$

for all $u, v \in H_1$ since $T^{-1}u \in D(S)$.

(ii) The proof of part (ii) is similar to that for part (i).

(iv), (v) For all $\lambda \in \mathbb{R}$, $A-\lambda L = T^{-1}(S-\lambda I)T^{-1} = U(S-\lambda I)V$ where $U = T^{-1}: H \longrightarrow H_1$ and $V = T^{-1}: H_1 \longrightarrow H_2$ are linear homeomorphisms and $S-\lambda I: H_2 \longrightarrow H$ is a bounded linear operator. Hence

$$\ker(A-\lambda L) = V^{-1}\ker(S-\lambda I)$$

and

$$\operatorname{rge}(A-\lambda L) = U\operatorname{rge}(S-\lambda I).$$

It follows that $\dim\ker(A-\lambda L) = \dim\ker(S-\lambda I)$ and that $\operatorname{rge}(A-\lambda L)$ is a closed subspace of H_1 if and only if $\operatorname{rge}(S-\lambda I)$ is a closed subspace of H. From these observations we see that $S-\lambda I: D(S) \longrightarrow H$ is an isomorphism (respectively a Fredholm operator) if and only if $A-\lambda L: H_1 \longrightarrow H_1$ is an isomorphism (respectively a Fredholm operator). Statements (iv) and (v) now follow from the definitions of $\sigma(S)$ and $\sigma_e(S)$ given in Section 2. □

Finally we relate A to the polar decomposition of S.

LEMMA 3.5. *Let $S: D(S) \subset H \longrightarrow H$ be a self-adjoint operator on H with form space $(H_1, \langle\cdot,\cdot\rangle_1)$ and let $A: H_1 \longrightarrow H_1$ be the operator introduced in Theorem* 3.4. *Set*

$$Ru = [I-P]u - Pu = u - 2Pu \quad \text{for } u \in H \tag{16}$$

where $P = E(0)$ and $\{E(\lambda): \lambda \in \mathbb{R}\}$ is the resolution of the identity associated with S.

(i) *$R^2 = I$ and R is a self-adjoint isometric isomorphism of $(H, \langle\cdot,\cdot\rangle)$ onto itself.*

(ii) *R is also a self-adjoint isometric isomorphism of $(H_1, \langle\cdot,\cdot\rangle_1)$ onto itself.*

(iii) *$AR = RA$ and*

$$\langle RAu, v\rangle_1 = \langle Au, Rv\rangle_1 = \int |\lambda|\, d\langle E(\lambda)u, v\rangle = \langle |S|^{1/2}u, |S|^{1/2}v\rangle \tag{17}$$

for all $u, v \in H_1$. Thus

$$\langle RAu, v\rangle_1 = \langle ARu, v\rangle_1 = \langle |S|u, v\rangle$$

for all $u \in D(|S|) = D(S)$ and $v \in H_1$.

(iv) *For all $u \in D(S) = D(|S|)$,*

$$Ru \in D(S) \quad \text{and} \quad SRu = RSu = |S|u. \tag{18}$$

REMARK 3.6. It follows that $S = \tilde{R}|S|$ is the usual polar decomposition of S ([**3**, Chapter IV, §3], for example) where $\tilde{R} = R(I-N)$ and N denotes the orthogonal projection of H onto $\ker S$.

Proof of Lemma 3.5. (i) $R^2 = I - 4P + 4P^2 = I$ since $P^2 = P$ and, for any $u \in H$,

$$\|[I-P]u \pm Pu\|^2 = \|[I-P]u\|^2 + \|Pu\|^2.$$

(ii) By (6), $Pu \in H_1$ for all $u \in H_1$ and so $RH_1 \subset H_1$. Then $H_1 = R^2H_1 \subset RH_1$ by part (i) and so $RH_1 = H_1$. Furthermore, by (10) and (11), for $u \in H_1$,

$$\|Ru\|_1^2 = \int (1+|\lambda|)\, d\langle E(\lambda)Ru, Ru\rangle$$

$$= \int (1+|\lambda|)\, d\langle E(\lambda)u, u\rangle = \|u\|_1^2$$

since $\langle E(\lambda)Ru, Ru\rangle = \langle RE(\lambda)u, Ru\rangle = \langle E(\lambda)u, u\rangle$ by part (i). Hence

$$\langle Ru, v\rangle_1 = \langle R^2u, Rv\rangle_1 = \langle u, Rv\rangle_1 \quad \text{for all } u, v \in H_1.$$

(iii) For $u, v \in H_1$,

$$\langle Au, Rv\rangle_1 = \int \lambda d\langle E(\lambda)\, u, Rv\rangle$$

by (14) where

$$\langle E(\lambda)\, u, Rv\rangle = \begin{cases} -\langle E(\lambda)\, u, v\rangle & \text{if } \lambda \leqslant 0 \\ \langle E(\lambda)\, u, v\rangle - 2\langle E(0)\, u, v\rangle & \text{if } \lambda > 0 \end{cases}$$

and so

$$\langle Au, Rv\rangle_1 = \int |\lambda|\, d\langle E(\lambda)\, u, v\rangle.$$

If $u \in D(S) = D(|S|)$ and $v \in H_1$,

$$\langle |S|^{1/2} u, |S|^{1/2} v\rangle = \langle |S|\, u, v\rangle = \int |\lambda|\, d\langle E(\lambda)\, u, v\rangle. \tag{19}$$

Since $D(|S|) = D(S)$ is a dense subspace of H_1, the continuity of $|S|^{1/2}: H_1 \longrightarrow H$ and of A, $R: H_1 \longrightarrow H_1$ implies that

$$\langle Au, Rv\rangle_1 = \langle |S|^{1/2} u, |S|^{1/2} v\rangle$$

for all $u, v \in H_1$. Hence $\langle Au, Rv\rangle_1 = \langle Av, Ru\rangle_1 = \langle v, ARu\rangle_1$ for all $u, v \in H_1$. However $\langle Au, Rv\rangle_1 = \langle RAu, R^2 v\rangle_1 = \langle RAu, v\rangle_1$ and so

$$\langle ARu, v\rangle_1 = \langle RAu, v\rangle_1 \quad \text{for all } u, v \in H_1,$$

showing that $AR = RA$.

(iv) By (6), $Pu \in D(S)$ and $SPu = PSu$ for all $u \in D(S)$. Hence $Ru \in D(S)$ and $SRu = RSu$ for all $u \in D(S)$. By part (iii),

$$\langle |S|\, u, v\rangle = \langle RAu, v\rangle_1 = \langle ARu, v\rangle_1 = \langle SRu, v\rangle$$

for all $u \in D(S) = D(|S|)$ and $v \in H_1$. Thus $|S|\, u = SRu = RSu$ for all $u \in D(S) = D(|S|)$ since H_1 is dense in H. □

4. *The quadratic form*

Let $S: D(S) \subset H \longrightarrow H$ be a self-adjoint operator on H with form space $(H_1, \langle \cdot, \cdot \rangle_1)$ and let A and $L: H_1 \longrightarrow H_1$ be the operators introduced in Theorem 3.4. The set

$$M = \{u \in H_1 : \langle Lu, u\rangle_1 = 1\} = \{u \in H_1 : \|u\| = 1\}$$

is a smooth manifold of codimension 1 in H_1 and the tangent space, $T_u(M)$, to M at u is given by

$$T_u(M) = \{v \in H_1 : \langle Lu, v\rangle_1 = 0\}. \tag{20}$$

(See [**2**, Example 27.2].)

The *quadratic form*, $J: H_1 \longrightarrow \mathbb{R}$, associated with S is defined by

$$J(u) = \langle Au, u\rangle_1 \quad \text{for all } u \in H_1.$$

Since $A: H_1 \longrightarrow H_1$ is a bounded self-adjoint operator, $J \in C^\infty(H_1, \mathbb{R})$ and

$$J'(u)\, v = 2\langle Au, v\rangle_1 \quad \text{for all } u, v \in H_1.$$

For $u \in H_2$, $J(u) = \langle Su, u\rangle$ and J is the unique continuous extension of $\langle Su, u\rangle$ to H_1.

The restriction of J to the manifold M is denoted by j and will be referred to as the *Rayleigh quotient* for S since

$$\frac{\langle Su, u\rangle}{\langle u, u\rangle} = j\left(\frac{u}{\|u\|}\right) \quad \text{for all } u \in D(S)\backslash\{0\}.$$

Thus $j \in C^\infty(M, \mathbb{R})$ and

$$j'(u)\,v = 2\langle Au, v\rangle_1 \quad \text{for all } u \in M \text{ and } v \in T_u(M). \tag{21}$$

(See [**2**, Example 27.3].)

LEMMA 4.1. *Let $S: D(S) \subset H \longrightarrow H$ be a self-adjoint operator on H with form space $(H_1, \langle \cdot, \cdot \rangle_1)$ and let j denote its Rayleigh quotient. For all $u \in M$ and $\lambda \in \mathbb{R}$,*

$$\| j'(u)\|_* \leqslant 2\|(A-\lambda L)\,u\|_1 \leqslant \| j'(u)\|_*(1+\|u\|_1)+2|j(u)-\lambda|$$

where

$$\| j'(u)\|_* = \sup\left\{\frac{|j'(u)\,v|}{\|v\|_1} : v \in T_u(M) \text{ and } v \neq 0\right\}$$

denotes the norm of $j'(u)$ in $T_u(M)^*$.

Proof. For $u \in M$ and $v \in T_u(M)$ with $v \neq 0$,

$$\frac{|j'(u)\,v|}{\|v\|_1} = \frac{2|\langle (A-\lambda L)\,u, v\rangle_1|}{\|v\|_1} \leqslant 2\|(A-\lambda L)\,u\|_1$$

for all $\lambda \in \mathbb{R}$, by (20) and (21). Thus $\| j'(u)\|_* \leqslant 2\|(A-\lambda L)\,u\|_1$.

For $u \in M$, define $P_u: H_1 \longrightarrow H_1$ by

$$P_u\,v : v - \langle Lu, v\rangle_1\,u \quad \text{for all } v \in H_1.$$

Clearly $P_u\,v \in T_u(M)$ for all $u \in M$ and $v \in H_1$ and

$$\|P_u\,v\|_1 \leqslant \|v\|_1 + |\langle u, v\rangle|\,\|u\|_1 \leqslant (1+\|u\|_1)\,\|v\|_1$$

by (15), since $\|u\| = 1$ and $\|v\| \leqslant \|v\|_1$.

Consider $u \in M$ and $v \in H_1$. Then, for any $\lambda \in \mathbb{R}$,

$$\begin{aligned}\langle (A-\lambda L)\,u, v\rangle_1 &= \langle Au, P_u\,v + \langle Lu, v\rangle_1\,u\rangle_1 - \lambda\langle Lu, v\rangle_1\\ &= \langle Au, P_u\,v\rangle_1 + \langle Lu, v\rangle_1\{\langle Au, u\rangle_1 - \lambda\}\\ &= \langle Au, P_u\,v\rangle_1 + \langle u, v\rangle\{j(u)-\lambda\}\end{aligned}$$

and so

$$\begin{aligned}|\langle (A-\lambda L)\,u, v\rangle_1| &\leqslant |\langle Au, P_u\,v\rangle_1| + \|u\|\,\|v\|\,|j(u)-\lambda|\\ &\leqslant \frac{1}{2}|j'(u)\,P_u\,v| + \|v\|_1\,|j(u)-\lambda|\\ &\leqslant \frac{1}{2}\| j'(u)\|_*\,\|P_u\,v\|_1 + \|v\|_1\,|j(u)-\lambda|\\ &\leqslant \left\{\frac{1}{2}\| j'(u)\|_*(1+\|u\|_1) + |j(u)-\lambda|\right\}\|v\|_1.\end{aligned}$$

Hence

$$\|(A-\lambda L)\,u\|_1 \leqslant \frac{1}{2}\| j'(u)\|_*(1+\|u\|_1) + |j(u)-\lambda|. \qquad \square$$

5. *The main results*

Using the notions introduced in Sections 3 and 4 we can now state the results mentioned in the introduction in full generality.

THEOREM 5.1. *Let $S: D(S) \subset H \longrightarrow H$ be a self-adjoint operator on H with form space $(H_1, \langle \cdot, \cdot \rangle_1)$ and let J and $j = J|_M$ denote the associated quadratic form and Rayleigh quotient as defined in Section* 4. *Then,*

$$\sigma(S) = \{\lambda \in \mathbb{R}: \text{the functional } J_\lambda \text{ does not satisfy (P–S) on } H_1\}$$

where $J_\lambda(u) = J(u) - \lambda\langle u, u\rangle = \langle (A - \lambda L)\, u, u\rangle_1$ *for* $u \in H_1$ *and*

$$\sigma_e(S) = \{\lambda \in \mathbb{R}: \text{the functional } j \text{ does not satisfy (P–S) at level } \lambda \text{ on } M\}$$

where $\sigma(S)$ and $\sigma_e(S)$ denote the spectrum and essential spectrum of S as defined in Section 2.

REMARK 5.2. If $S: H \longrightarrow H$ is a bounded self-adjoint operator, $H = H_1$ (up to equivalence of norms) and we obtain the results in the simple form stated in the introduction.

Proof of Theorem 5.1. Suppose first that $\lambda \in \sigma(S)$. Then $S - \lambda I: D(S) \longrightarrow H$ is not an isomorphism. Since S is self-adjoint (and hence closed), it follows (see [**4**, Theorem 5.2], for example) that there is a sequence $\{u_n\} \subset D(S)$ such that

$$\|u_n\| = 1 \text{ and } \|(S - \lambda I)\, u_n\| \to 0.$$

Set

$$\alpha_n = \begin{cases} \|(S - \lambda I)\, u_n\|^{-1/4} & \text{if } \|(S - \lambda I)\, u_n\| \neq 0 \\ n & \text{if } \|(S - \lambda I)\, u_n\| = 0. \end{cases}$$

and

$$u_n = \alpha_n\, Tu_n$$

where $T = (I + |S|)^{1/2}: H_1 \longrightarrow H$ is the operator introduced in Lemma 3.1. Then $\alpha_n \to \infty$ and

$$v_n \in H_1 \text{ with } \|v_n\|_1 = \alpha_n \|Tu_n\|_1 \geqslant \alpha_n \|u_n\| = \alpha_n$$

by Lemma 3.1(b). Hence the sequence $\{v_n\}$ has no subsequence which converges strongly in H_1.

However

$$J_\lambda(v_n) = \alpha_n^2 \langle (A - \lambda L)\, Tu_n, Tu_n\rangle_1 = \alpha_n^2 \langle (S - \lambda I)\, u_n, T^2 u_n\rangle$$

by (11) since $u_n \in D(S)$. Hence

$$|J_\lambda(v_n)| \leqslant \alpha_n^2 \|(S - \lambda I)\, u_n\|\, \|T^2 u_n\|$$

where

$$\|T^2 u_n\| = \|Tu_n\|_1 \leqslant \sqrt{2}\|u_n\|_2$$

by (11) and (12). Now the sequence $\{\|T^2 u_n\|\}$ is bounded since

$$\begin{aligned} \|u_n\|_2^2 &= \|u_n\|^2 + \|Su_n\|^2 \\ &\leqslant 1 + \{\|(S - \lambda I)\, u_n\| + |\lambda|\, \|u_n\|\}^2 \\ &\leqslant 1 + \{\|(S - \lambda I)\, u_n\| + |\lambda|\}^2. \end{aligned}$$

On the other hand

$$\alpha_n^2 \|(S-\lambda I)\, u_n\| \to 0 \text{ as } n \to \infty$$

by the definition of α_n. Thus $J_\lambda(v_n) \to 0$ as $n \to \infty$.

Also, for all $w \in H_1$,

$$J'_\lambda(v_n)\, w = 2\langle (A-\lambda L)\, v_n, w\rangle_1 = 2\alpha_n \langle (S-\lambda I)\, u_n, Tw\rangle$$

by (11) and so

$$|J'_\lambda(v_n)\, w| \leqslant 2\alpha_n \|(S-\lambda I)\, u_n\|\, \|Tw\| = 2\alpha_n \|(S-\lambda I)\, u_n\|\, \|w\|_1.$$

Hence

$$\|J'_\lambda(v_n)\|_* \leqslant 2\alpha_n \|(S-\lambda I)\, u_n\|$$

where $\alpha_n \|(S-\lambda I)\, u_n\| \to 0$ as $n \to \infty$ by the definition of α_n.

This shows that J_λ does not satisfy condition (P–S) at level 0 on H_1.

Conversely, suppose that $\lambda \in \rho(S)$ and choose any $c \in \mathbb{R}$. Consider a sequence $\{u_n\} \subset H_1$ such that $J_\lambda(u_n) \to c$ and $\|J'_\lambda(u_n)\|_* \to 0$. It follows from Theorem 3.4(iv) that $A-\lambda L : H_1 \longrightarrow H_1$ is an isomorphism and so there exists a constant $k > 0$ such that

$$\|(A-\lambda L)\, u\|_1 \geqslant k \|u\|_1 \quad \text{for all } u \in H_1.$$

However

$$\begin{aligned}\|J'_\lambda(u)\|_* &= \sup\left\{\frac{2|\langle (A-\lambda L)\, u, v\rangle_1|}{\|v\|_1} : v \in H_1 \text{ with } v \neq 0\right\} \\ &= 2\|(A-\lambda L)\, u\|_1 \quad \text{for all } u \in H_1.\end{aligned}$$

Thus $\|u_n\|_1 \to 0$ and J_λ satisfies condition (P–S) at level c on H_1. (In fact, if $c \neq 0$ there is no sequence in H_1 such that $J_\lambda(u_n) \to c$ and $\|J'_\lambda(u_n)\|_* \to 0$.)

We now turn to the Rayleigh quotient j and the essential spectrum of S. Suppose first that $\lambda \in \sigma_e(S)$. Then there exists a sequence (called a Weyl sequence, see [**4**, Theorem 7.2], for example) $\{u_n\} \subset D(S) \cap M$ such that $\|(S-\lambda I)\, u_n\| \to 0$ and $u_n \rightharpoonup 0$ weakly in H as $n \to \infty$. Thus

$$j(u_n) - \lambda = \langle Au_n, u_n\rangle_1 - \lambda \langle Lu_n, u_n\rangle_1 = \langle (S-\lambda I)\, u_n, u_n\rangle,$$

so

$$|j(u_n) - \lambda| \leqslant \|(S-\lambda I)\, u_n\|\, \|u_n\| = \|(S-\lambda I)\, u_n\|$$

and hence $j(u_n) \to \lambda$ as $n \to \infty$. Furthermore, by Lemma 4.1,

$$\begin{aligned}\|j'(u_n)\|_* &\leqslant 2\|(A-\lambda L)\, u_n\|_1 = 2\|(S-\lambda I)\, T^{-1} u_n\| && \text{by (11)} \\ &= 2\|T^{-1}(S-\lambda I)\, u_n\| && \text{by Lemma 3.1(c), since } u_n \in D(S) \\ &\leqslant 2\|T^{-1}(S-\lambda I)\, u_n\|_1 = 2\|(S-\lambda I)\, u_n\| && \text{by (11)}\end{aligned}$$

and so $\|j'(u_n)\|_* \to 0$ as $n \to \infty$. However, since M is a closed subset of H_1 and $u_n \rightharpoonup 0 \notin M$ weakly in H, the sequence $\{u_n\}$ cannot have a subsequence which converges strongly in H_1 and hence in H. This shows that j does not satisfy condition (P–S) at the level λ on M.

Conversely, let $\lambda \in \mathbb{R} \backslash \sigma_e(S)$ and consider a sequence $\{u_n\} \subset M$ such that $j(u_n) \to \lambda$ and $\|j'(u_n)\|_* \to 0$ as $n \to \infty$. We begin by showing that the sequence $\{u_n\}$ is bounded in H_1. In fact, for any $u \in M$,

$$\begin{aligned}\|u\|_1^2 &= \|u\|^2 + \langle |S|^{1/2} u, |S|^{1/2} u\rangle = 1 + \langle Au, Ru\rangle_1 && \text{by (17)} \\ &\leqslant 1 + \|Au\|_1 \|Ru\|_1 = 1 + \|Au\|_1 \|u\|_1 && \text{by Lemma 3.5(ii)} \\ &\leqslant 1 + \left\{\frac{1}{2}\|j'(u)\|_* (1 + \|u\|_1) + |j(u)|\right\} \|u\|_1 && \text{by Lemma 4.1.}\end{aligned}$$

Thus

$$\left\{1-\frac{1}{2}\|j'(u)\|_*\right\}\|u\|_1^2 \leqslant 1+\left\{\frac{1}{2}\|j'(u)\|_*+|j(u)|\right\}\|u\|_1$$

for all $u \in M$. Since $j(u_n) \to \lambda$ and $\|j'(u_n)\|_* \to 0$ as $n \to \infty$, it follows that $\{\|u_n\|_1\}$ is bounded.

Next we note that, since $A-\lambda L: H_1 \longrightarrow H_1$ is a bounded Fredholm operator (of index 0) by Theorem 3.4(v), it follows from [**3**, Chapter 1, Theorem 3.15] that there exist a bounded linear operator $W: H_1 \longrightarrow H_1$ and a compact linear operator $K: H_1 \longrightarrow H_1$ such that

$$W(A-\lambda L) = I-K.$$

Thus

$$u_n = W(A-\lambda L)\, u_n + Ku_n \tag{22}$$

where

$$\|(A-\lambda L)\, u_n\|_1 \to 0 \quad \text{as } n \to \infty$$

since

$$\|(A-\lambda L)\, u_n\|_1 \leqslant \frac{1}{2}\|j'(u_n)\|_*(1+\|u_n\|_1)+|j(u_n)-\lambda|$$

by Lemma 4.1 and the sequence $\{\|u_n\|_1\}$ is bounded. The boundedness of $\{\|u_n\|_1\}$ together with the compactness of K mean that there exist $z \in H_1$ and a subsequence $\{u_{n_i}\}$ such that $\|z-Ku_{n_i}\|_1 \to 0$ as $n_i \to \infty$. It follows from (22) that $u_{n_i} \to z$ in H_1 as $n_i \to \infty$. Since M is a closed subset of H_1 we have $z \in M$.

Thus $\{u_n\}$ has a strongly convergent subsequence in M and we have shown that the functional j satisfies condition (P–S) at the level λ on M. □

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