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THE JOURNAL OF SYMBOLIC LOGIC
Volume 60, Number 3, September 1995

GENERIC TREES

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Abstract. We continue the investigation of the Laver ideal ℓ^0 and Miller ideal m^0 started in [GJSp] and [GRShSp]; these are the ideals on the Baire space associated with Laver forcing and Miller forcing. We solve several open problems from these papers. The main result is the construction of models for $\mathfrak{t} < \operatorname{add}(\ell^0)$, $\mathfrak{p} < \operatorname{add}(m^0)$, where add denotes the additivity coefficient of an ideal. For this we construct amoeba forcings for these forcings which do not add Cohen reals. We show that $\mathfrak{c} = \omega_2$ implies $\operatorname{add}(m^0) \leq \mathfrak{h}$. We show that $\mathfrak{b} = \mathfrak{c}$, $\mathfrak{d} = \mathfrak{c}$ implies $\operatorname{cov}(\ell^0) \leq \mathfrak{h}^+$, $\operatorname{cov}(m^0) \leq \mathfrak{h}^+$ respectively. Here cov denotes the covering coefficient. We also show that in the Cohen model $\operatorname{cov}(m^0) < \mathfrak{d}$ holds. Finally we prove that Cohen forcing does not add a superperfect tree of Cohen reals.

Introduction. Marczewski's ideal s^0 (see [M]) is the set of all $X \subseteq {}^\omega 2$ with the property that every perfect tree on ${}^{<\omega}2$ has a perfect subtree such that all its branches are in ${}^\omega 2 \setminus X$. As for any ideal, the additivity of s^0, denoted $\operatorname{add}(s^0)$, is the least number of sets from s^0 whose union does not belong to s^0. The covering number of s^0, denoted $\operatorname{cov}(s^0)$, is the least number of sets in s^0 whose union is ${}^\omega 2$. Much work has been done to understand the structure of s^0. For an extensive bibliography on this topic we refer the reader to [JMSh]. One of the most remarkable properties of s^0 is that Martin's axiom does not imply $\operatorname{add}(s^0) = \mathfrak{c}$. This was proved independently in [JMSh] and [V]. If in the definition of s^0 we replace the Cantor space by the Baire space and the notion of perfect tree by Laver tree, or Miller tree we obtain the Laver ideal ℓ^0, or the Miller ideal m^0, respectively. The investigation of ℓ^0 and m^0 has been started in [GJSp] and [GRShSp]. In [GRShSp] we proved the following chains of inequalities:

$$\begin{aligned} \mathfrak{t} \leq \ \operatorname{add}(\ell^0) &\leq \ \operatorname{cov}(\ell^0) \leq \mathfrak{b} \\ \mathfrak{p} \leq \ \operatorname{add}(m^0) &\leq \ \operatorname{cov}(m^0) \leq \mathfrak{d}. \end{aligned}$$

Here $\mathfrak{p}, \mathfrak{t}, \mathfrak{b}, \mathfrak{d}$ are well-known instances of cardinal invariants of the continuum: $\mathfrak{t}$ is the tower number, i.e. the least size of a descending chain in $\mathscr{P}(\omega)/\mathrm{fin}\setminus\{0\}$ without a lower bound; $\mathfrak{b}$ is the bounding number, i.e. the least size of an unbounded family in $({}^\omega\omega, \leq^*)$, where ${}^\omega\omega$ is the set of functions from ω to ω and $\leq^*$ is eventual dominance; $\mathfrak{d}$ is the dominating number, i.e. the least size of a cofinal family in $({}^\omega\omega, \leq^*)$; and $\mathfrak{p}$ is the least size of a filter on $\mathscr{P}(\omega)/\mathrm{fin}\setminus\{0\}$ without a lower bound. By Bell's theorem (see [B]), equivalently $\mathfrak{p}$ is the least cardinal κ for which Martin's axiom for σ-centered forcings fails. This definition of $\mathfrak{p}$ is used to prove the inequality above which involves $\mathfrak{p}$. An obvious consequence is that

Received February 1, 1994; revised January 4, 1995.
The author is supported by the Schweizer Nationalfonds.

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0022-4812/95/6003-0001/$03.20

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Martin's axiom implies $\mathrm{add}(\ell^0) = \mathrm{add}(m^0) = \mathfrak{c}$. See [vD] for a survey on cardinal invariants.

The main result in [GRShSp] is the consistency of $\mathrm{add}(\ell^0) < \mathrm{cov}(\ell^0)$ and $\mathrm{add}(m^0) < \mathrm{cov}(m^0)$. This contrasts with the situation for the ideal of nowhere Ramsey sets r^0, which is similarly defined as s^0, ℓ^0, m^0 but using Mathias forcing conditions (considered as trees), and for which $\mathrm{add}(r^0) = \mathrm{cov}(r^0) = \mathfrak{h}$ holds (see [Pl]). Here $\mathfrak{h}$ is the distributivity number of $\mathscr{P}(\omega)/\mathrm{fin}$.

The main problems left open in [GRShSp] were whether any of $\mathfrak{t} < \mathrm{add}(\ell^0)$, $\mathfrak{p} < \mathrm{add}(m^0)$, $\mathrm{cov}(\ell^0) < \mathfrak{b}$, $\mathrm{cov}(m^0) < \mathfrak{d}$ are consistent. In §1 we give a positive answer to the first two of them:

THEOREM 1. (1) $\mathrm{Con}(ZFC) \Rightarrow \mathrm{Con}(ZFC + \mathfrak{t} < \mathrm{add}(\ell^0))$.

(2) $\mathrm{Con}(ZFC) \Rightarrow \mathrm{Con}(ZFC + \mathfrak{p} < \mathrm{add}(m^0))$.

For the proof we construct amoeba forcings $\mathbb{A}(\mathbb{L}), \mathbb{A}(\mathbb{M})$ for Laver forcing $\mathbb{L}$ and Miller forcing $\mathbb{M}$, i.e. forcings adding a generic Laver, superperfect tree with the property that every branch is generic for $\mathbb{L}$, $\mathbb{M}$ respectively. Such tree will have the property that its branches are in the complement of the union of all ℓ^0-sets and m^0-sets from the ground model. The crucial part of our construction is to ensure that $\mathbb{A}(\mathbb{L}), \mathbb{A}(\mathbb{M})$ do not add Cohen reals. The existence of such forcings is not trivial, as the natural amoeba forcings for Sacks forcing, $\mathbb{L}$ and $\mathbb{M}$, are easily seen to add hosts of Cohen reals. We even construct $\mathbb{A}(\mathbb{L}), \mathbb{A}(\mathbb{M})$ with the Laver property, i.e. a combinatorial property ruling out in a strong way that Cohen reals are added. By a result of Shelah, the Laver property is preserved under countable support iterations. Hence we can increase $\mathrm{add}(\ell^0)$, $\mathrm{add}(m^0)$ without adding Cohen reals. By a result of Szymanski, $\mathfrak{t} \leq \mathrm{add}(\mathscr{M})$, where $\mathscr{M}$ is the ideal of meagre subsets of the real line. But $\mathrm{cov}(\mathscr{M}) = \omega_1$ holds in our models. Since trivially $\mathfrak{p} \leq \mathfrak{t}$, we are done.

The consistency proofs in [GRShSp] of $\mathrm{add}(\ell^0) < \mathrm{cov}(\ell^0)$ and $\mathrm{add}(m^0) < \mathrm{cov}(m^0)$ are rather indirect. The model which works in both cases is obtained by a countable support iteration of length ω_2 where the iterands are stationarily often $\mathbb{L}$ and $\mathbb{M}$. A Löwenheim-Skolem argument shows $\mathrm{cov}(\ell^0) = \mathrm{cov}(m^0) = \omega_2$ in this model. To show that $\mathrm{add}(\ell^0) = \mathrm{add}(m^0) = \omega_1$ hold is more difficult. First we proved that $\mathfrak{b} = \mathfrak{c}$ implies $\mathrm{add}(\ell^0) \leq \kappa(\mathbb{L})$ and $\mathrm{add}(m^0) \leq \kappa(\mathbb{M})$, where $\kappa(\mathbb{L}), \kappa(\mathbb{M})$ is the least cardinal κ such that forcing with $\mathbb{L}$, $\mathbb{M}$ adds a bijection $\kappa \to \mathfrak{c}$. Moreover we proved $\kappa(\mathbb{L}) \leq \mathfrak{h}, \kappa(\mathbb{M}) \leq \mathfrak{h}$ in ZFC. Now by a result of [BJSh], $\mathfrak{h} = \omega_1$ holds in our model; moreover $\mathfrak{b} = \mathfrak{c}$ since we added a (dominating) Laver real cofinally often. Hence we are done. The same method of proof shows that $\mathrm{add}(\ell^0) < \mathrm{cov}(\ell^0)$ holds in the iterated Laver forcing model. Moreover in [JMSh] it was shown that in the iterated Sacks forcing model $\mathrm{add}(s^0) < \mathrm{cov}(s^0)$ holds. So it was natural to expect that the iterated Miller forcing model satisfies $\mathrm{add}(m^0) < \mathrm{cov}(m^0)$. However this model has $\mathfrak{b} = \omega_1$ (see [Mi]), so the above argument does not apply. In §2 we show that in fact $\mathrm{add}(m^0) < \mathrm{cov}(m^0)$ holds in this model. The main technical device is the following lemma:

LEMMA 2. *Suppose $\mathfrak{d} = \mathfrak{c}$. For every dense and open $D \subseteq \mathbb{M}$ there exists a maximal antichain $A \subseteq D$ such that no two members of A have a common branch.*

Combining this lemma with the techniques from [GRShSp], we obtain two corollaries:

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COROLLARY 3. $\mathrm{Con}(ZFC) \Rightarrow \mathrm{Con}(ZFC + \mathrm{cov}(\ell^0) < \mathrm{cov}(m^0))$.

COROLLARY 4. $\mathfrak{c} = \omega_2 \Rightarrow \mathrm{add}(m^0) \leq \mathfrak{h}$.

Lemma 2 is also used to obtain a better upper bound for $\mathrm{cov}(\ell^0)$ and $\mathrm{cov}(m^0)$. Inspired by the Kulpa-Szymanski theorem from topology, we prove the following.

THEOREM 5. (1) *If* $\mathfrak{b} = \mathfrak{c}$, *then* $\mathrm{cov}(\ell^0) \leq \mathfrak{h}^+$.

(2) *If* $\mathfrak{d} = \mathfrak{c}$, *then* $\mathrm{cov}(m^0) \leq \mathfrak{h}^+$.

In two recent papers [Bl], [E], Blass and Eisworth showed that in a model obtained by adding $\aleph_1$ random reals to a model of Martin's axiom $\mathrm{cov}(\ell^0) < \mathfrak{b}$ and $\mathrm{cov}(m^0) < \mathfrak{b}$ hold. Theorem 5 suggests another model where these inequalities hold. Force with a finite-support iteration of Hechler forcing (the natural *ccc* forcing which adds a dominating function) of length $\aleph_3$. By a result of Baumgartner and Dordal in [BD], then $\mathfrak{h}$ is ω_1. But clearly $\mathfrak{b} = \omega_3$. So by Theorem 5 we are done.

Some time ago Jörg Brendle wrote me that he knew that in the Cohen model $\mathrm{cov}(m^0) = \omega_1$ holds, and hence that $\mathrm{cov}(m^0) < \mathfrak{d}$ is consistent. He did not tell me the proof, so I thought about it and proved the following:

LEMMA 6. *Suppose that $\mathcal{T}$ is a $\mathbb{C}$-name for a superperfect tree and π is a $\mathbb{C}$-name for a member of ${}^{\omega}\omega$. Then in V there exists $S \in \mathbb{M}$ such that*

$$\Vdash_{\mathbb{C}} \text{“}S \text{ and } \mathcal{T} \text{ are compatible and } \pi \notin [S]\text{”}.$$

Now working under CH, using Lemma 6 we easily construct $\aleph_1$ predense sets in $\mathbb{M}$, say $\langle D_\alpha : \alpha < \omega_1 \rangle$, such that after adding a Cohen real they are still predense and moreover for every real x there is an $\alpha < \omega_1$ such that x is not a branch of any $p \in D_\alpha$. Dually this means that in V we have a definition of $\aleph_1$ sets in $(m^0)^{V^{\mathbb{C}}}$ whose union covers $({}^{\omega}\omega)^{V^{\mathbb{C}}}$. When I told Brendle about this he told me that his proof is essentially different—he constructs the predense sets step by step, after each Cohen real they have to be extended.

In the spirit of Lemma 6 we also prove the following.

THEOREM 7. *Cohen forcing does not add a superperfect tree of Cohen reals.*

Notation. Trees: By “$\subset$” we mean strict inclusion. A set $p \subseteq {}^{<\omega}\omega$ is called a *tree* if for every $\sigma \in p$ and $\tau \subset \sigma$, $\tau \in p$. We say “p is downward closed”. Given a tree p, for $\sigma \in p$ let $\mathrm{succ}_\sigma(p) = \{n \in \omega : \sigma^\frown\langle n \rangle \in p\}$, $\mathrm{split}(p) = \{\sigma \in p : |\mathrm{succ}_\sigma(p)| \geq 2\}$, $\mathrm{Split}(p) = \{\sigma \in p : |\mathrm{succ}_\sigma(p)| = \omega\}$. A member of $\mathrm{split}(p)$ we call a *split-node*. For $\sigma \in \mathrm{split}(p)$ let $\mathrm{Succ}_\sigma(p) = \{\tau \in p : \sigma \subset \tau \wedge \tau \in \mathrm{split}(p) \wedge \forall \rho(\sigma \subset \rho \subset \tau \Rightarrow \rho \notin \mathrm{split}(p))\}$. For $n \in \omega$, the nth *splitting level*, denoted $\mathrm{Lev}_n(p)$, is the set of those $\sigma \in \mathrm{split}(p)$ with the property that if $\tau_0 \subset \cdots \subset \tau_m = \sigma$ is the maximal chain such that each τ_i belongs to $\mathrm{split}(p)$, then $m = n$. For $\sigma \in p$ we say that σ is the *stem* of p and write $\sigma = \mathrm{stem}(p)$ if $\sigma \in \mathrm{split}(p)$, $\forall \tau \subset \sigma(\tau \notin \mathrm{split}(p))$ and $\forall \tau \in p(\tau \subset \sigma \vee \sigma \subset \tau)$. So if $\mathrm{stem}(p)$ exists it is uniquely determined. Given a tree p with a stem, then we let $p^- = \{\sigma \in p : \mathrm{stem}(p) \subseteq \sigma\}$. When we speak of a subtree of p^- we just mean a downward closed subset of p^-. If $\sigma \in p$, then $(p)_\sigma = \{\tau \in p : \tau \subseteq \sigma \vee \sigma \subseteq \tau\}$. By $[p]$ we denote the set of all branches of p, i.e. of all its maximal chains. A subset $F \subseteq p$ is called a *front* if it is an antichain and moreover every branch of p has an initial segment which belongs to F.

A tree p is called a *Laver tree* (see [L]), if $\mathrm{stem}(p)$ exists and for every $\sigma \in p$ extending $\mathrm{stem}(p)$, $\sigma \in \mathrm{Split}(p)$. The set of all Laver trees will be denoted by $\mathbb{L}$.

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A tree p is called *superperfect* if every $\sigma \in p$ has an extension in $\mathrm{Split}(p)$. The set of superperfect trees p which have a stem and the property $\mathrm{split}(p) = \mathrm{Split}(p)$ will be denoted by $\mathbb{M}$. Forcing with $\mathbb{M}$ is usually called *Miller forcing*.

Let $\prec$ be the following wellordering of ${}^{<\omega}\omega$ in type $\omega : s \prec t$ if and only if

$$\begin{aligned}\max\{\ell h(s), \max(\mathrm{ran}\,(s))\} &< \max\{\ell h(t), \max(\mathrm{ran}\,(t))\} \\ &\vee [\max\{\ell h(s), \max(\mathrm{ran}\,(s))\} = \max\{\ell h(t), \max(\mathrm{ran}\,(t))\} \wedge \ell h(s) < \ell h(t)] \\ &\vee [\max\{\ell h(s), \max(\mathrm{ran}\,(s))\} = \max\{\ell h(t), \max(\mathrm{ran}\,(t))\} \wedge \ell h(s) = \ell h(t) \\ &\qquad \wedge s \text{ precedes } t \text{ lexicographically}].\end{aligned}$$

Let $T : \omega \to {}^{<\omega}\omega$ be the order preserving enumeration of $\langle {}^{<\omega}\omega, \prec \rangle$. So clearly for every $n \in \omega$, $\mathrm{ran}(T{\restriction}n)$ is a subtree of $\langle {}^{<\omega}\omega, \subseteq \rangle$.

Ideals: With most natural forcings an ideal can be associated. Here we are mainly interested in the ones associated with Laver and Miller forcing. The *Laver ideal* ℓ^0 is defined by $X \in \ell^0$ if and only if $\forall p \in \mathbb{L} \exists q \in \mathbb{L}(q \subseteq p \wedge [q] \cap X = \emptyset)$. The *Miller ideal* m^0 is defined similarly, replacing $\mathbb{L}$ by $\mathbb{M}$. For any ideal $\mathscr{I}$ on ${}^{\omega}\omega$ (or on an arbitrary set in general), its additivity, covering and uniformity coefficients are defined as follows:

$$\mathrm{add}(\mathscr{I}) = \min\{|\mathscr{F}| : \mathscr{F} \subseteq \mathscr{I} \wedge \bigcup \mathscr{F} \notin \mathscr{I}\},$$

$$\mathrm{cov}(\mathscr{I}) = \min\{|\mathscr{F}| : \mathscr{F} \subseteq \mathscr{I} \wedge \bigcup \mathscr{F} = {}^{\omega}\omega\},$$

$$\mathrm{non}(\mathscr{I}) = \min\{|X| : X \subseteq {}^{\omega}\omega \wedge X \notin \mathscr{I}\}.$$

Standard fusion arguments show that $\mathrm{add}(\ell^0)$ and $\mathrm{add}(m^0)$, and hence also the covering coefficients for these ideals, are uncountable; trivially they are at most $\mathfrak{c}$, the cardinality of the continuum. Moreover $\mathrm{non}(\ell^0) = \mathrm{non}(m^0) = \mathfrak{c}$. This is true since every Laver (Miller) tree can be decomposed into $\mathfrak{c}$ Laver (Miller) subtrees such that no two of them have a common branch.

Cardinal invariants: By $A \subseteq^* B$ we mean that $A \setminus B$ is finite. For $f, g \in {}^{\omega}\omega$, let $f \leq^* g$ mean that $f(n) \leq g(n)$ for all but finitely many n. A *tower* $\mathscr{T}$ is a decreasing chain in $([\omega]^{\omega}, \subseteq^*)$ without an infinite almost intersection, i.e. for no $A \in [\omega]^{\omega}$ do we have $A \subseteq^* B$ for every $B \in \mathscr{T}$. Then the tower number $\mathfrak{t}$ is the least size of a tower. The cardinal invariant $\mathfrak{p}$ is the least cardinality of a filter on $([\omega]^{\omega}, \subseteq^*)$ without an infinite almost intersection. The bounding number $\mathfrak{b}$ is the least size of an unbounded family in $({}^{\omega}\omega, \leq^*)$. The dominating number $\mathfrak{d}$ is the least size of a cofinal family in $({}^{\omega}\omega, \leq^*)$. The distributivity number $\mathfrak{h}$ is the least number of infinite partitions of the Boolean algebra $\mathscr{P}(\omega)/\mathrm{fin}$ which do not have a common refinement. The inequalities $\omega_1 \leq \mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{h} \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$ are either trivial or well known. Except for $\mathfrak{p} \leq \mathfrak{t}$, it is also well known that each of these inequalities can consistently be either strict or an equality. See [vD] and [BS] for details and proofs.

Forcing: A forcing P is said to have the *Laver property* if there exists $f \in {}^{\omega}\omega \cap V$ such that for every family $\langle \tau_n : n \in \omega \rangle$ of P-names, $p \in P$ and $H \in {}^{\omega}V \cap V$

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such that for every n, $H(n)$ is finite and $\Vdash_P \forall n \in \omega(\tau_n \in H(n))$, there exists $F \in {}^{\omega}V \cap V$ and $q \in P$, $q \leq p$, such that, for every n, $|F(n)| = f(n)$ and

$$q \Vdash_P \forall n \in \omega(\tau_n \in F(n)).$$

A forcing with the Laver property does not add Cohen reals. In fact, if P added a Cohen real then it added one for the space $\prod_{n<\omega}(f(n)+2)$. But by genericity it is clear that this Cohen real escapes every f-belt F from the ground model.

§1. Models for $\mathfrak{t} < \mathrm{add}(\ell^0)$ and $\mathfrak{p} < \mathrm{add}(m^0)$.

1.1. Forcing generic Laver trees without adding Cohen reals. In order to motivate the definition to follow we first sketch why the natural amoeba forcing for $\mathbb{L}$ adds Cohen reals. The natural amoeba for $\mathbb{L}$ looks as follows. Its conditions are pairs (s,p), where $p \in \mathbb{L}$ and $s \subseteq p$ is a finite subtree. The ordering is: $(s,p) \leq$ (extends) (t,q) if $s \supseteq t$ and $p \subseteq q$. If G is generic for this forcing over V, then easily $p_G = \bigcup\{s : (s,p) \in G\}$ is a Laver tree. Let $\sigma, \tau \in \mathrm{Split}(p_G)$, let f_σ, f_τ be the increasing enumeration of $\mathrm{succ}_\sigma(p_G)$, $\mathrm{succ}_\tau(p_G)$ respectively. Define $c \in {}^{\omega}2$ by $c(n) = \mathrm{signum}(f_\sigma(n) - f_\tau(n))$. Then c is a Cohen real.

The idea for avoiding these Cohen reals is to prescribe them from the beginning so that what was a Cohen real with the natural amoeba becomes a recursive function. Surprisingly it turns out that then there are no Cohen reals at all.

We define an amoeba forcing for Laver forcing, $\mathbb{A}(\mathbb{L})$, as follows: Conditions in $\mathbb{A}(\mathbb{L})$ are pairs (s,p) such that

(1) $p \in \mathbb{L}$ and $s \in {}^{n}(p^-)$ for some $n \in \omega \setminus \{0\}$;
(2) $\mathrm{ran}(s)$ is a subtree of p^-;
(3) the map sending $s(i)$ to $T(i)$ is an isomorphism between $(\mathrm{ran}(s), \subseteq)$ and $(\mathrm{ran}(T\restriction n), \subseteq)$;
(4) the sequence $\langle s(i)(\ell h(s(i))-1) : 0 < i < n\rangle$ is strictly increasing.

The ordering on $\mathbb{A}(\mathbb{L})$ is defined as follows:

$$(s,p) \leq (t,q) \quad \text{if and only if} \quad p \subseteq q \text{ and } s \supseteq t.$$

Note that then $\mathrm{stem}(p) = \mathrm{stem}(q)$, as $t \neq \emptyset$ and $\mathrm{ran}(s) \subseteq p^-$.

LEMMA 1.1.1. *Suppose $(s,p) \in \mathbb{A}(\mathbb{L})$ and τ is an $\mathbb{A}(\mathbb{L})$-name such that*

$$(s,p) \Vdash_{\mathbb{A}(\mathbb{L})} \tau \in \{0,1\}.$$

Then there exists $(s,q) \in \mathbb{A}(\mathbb{L})$ with $(s,q) \leq (s,p)$ and $(s,q)||_{\mathbb{A}(\mathbb{L})} \tau$, i.e $\mathbb{A}(\mathbb{L})$ has the pure decision property.

PROOF. First, inductively we construct $(s,q) \leq (s,p)$ such that for every $t \in {}^{<\omega}(q)$, if for some $q' \in \mathbb{L}$ we have $(t,q') \in \mathbb{A}(\mathbb{L})$, $(t,q') \leq (s,q)$ and $(t,q')||_{\mathbb{A}(\mathbb{L})} \tau$, then $(t,q)||_{\mathbb{A}(\mathbb{L})} \tau$.

For the construction, if there exists $(s,q') \leq (s,p)$ which decides τ, let $q = q'$, and we are done.

Otherwise, clearly we may choose $\sigma_0 \in p$ such that $(s\hat{}\langle\sigma_0\rangle, p) \in \mathbb{A}(\mathbb{L})$. Let $s_0 = s\hat{}\langle\sigma_0\rangle$. If there exists $(s_0,q') \in \mathbb{A}(\mathbb{L})$ with $(s_0,q') \leq (s_0,p)$ and $(s_0,q')||_{\mathbb{A}(\mathbb{L})} \tau$, let $q_0 = q'$. Otherwise let $q_0 = p$.

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Suppose now that $(s_n, q_n) \in \mathbb{A}(\mathbb{L})$ with $(s_n, q_n) \leq (s, p)$ has been constructed for some n. Choose $\sigma_{n+1} \in q_n$ such that $(s_n{}^\frown\langle\sigma_{n+1}\rangle, q_n) \in \mathbb{A}(\mathbb{L})$. Let $s_{n+1} = s_n{}^\frown\langle\sigma_{n+1}\rangle$.

Claim 1. *There exists* $q_{n+1} \in \mathbb{L}$ *such that* $(s_{n+1}, q_{n+1}) \in \mathbb{A}(\mathbb{L})$, $(s_{n+1}, q_{n+1}) \leq (s_{n+1}, q_n)$ *and for every* $t \in {}^{<\omega}(\mathrm{ran}(s_{n+1}))$ *with* $s \subset t$ *and* $t(\ell h(t) - 1) = \sigma_{n+1}$, *if there exists* q' *with* $(t, q') \in \mathbb{A}(\mathbb{L})$, $(t, q') \leq (t, q_{n+1})$ *and* $(t, q')||_{\mathbb{A}(\mathbb{L})}\ \tau$, *then* $(t, q_{n+1})||_{\mathbb{A}(\mathbb{L})}\ \tau$.

PROOF. The crucial observation is that if (t, q') is as in the claim (so in particular $(t, q') \leq (t, q_{n+1})$ and $(t, q')||_{\mathbb{A}(\mathbb{L})}\ \tau$), then letting

$$q'' = q' \cup \bigcup\{(q_{n+1})_\sigma : \sigma \in \mathrm{ran}(s_{n+1}) \setminus \mathrm{ran}(t)\},$$

we have $(t, q'') \in \mathbb{A}(\mathbb{L})$ (note that $q'' \in \mathbb{L}$ since $\mathrm{ran}(t) \subseteq (q'')^-$ by definition), and $(t, q'')||_{\mathbb{A}(\mathbb{L})}\ \tau$. This is true since, by definition of $\mathbb{A}(\mathbb{L})$, if $(\bar{t}, \bar{q}) \leq (t, q'')$, then for every $i \in \mathrm{dom}(\bar{t}) \setminus \mathrm{dom}(t)$ we have

$$\begin{aligned}\bar{t}(i)(\ell h(\bar{t}(i)) - 1) &> \sigma_{n+1}(\ell h(\sigma_{n+1}) - 1) \\ &> \max\{\sigma(\ell h(\sigma) - 1) : \sigma \in \mathrm{ran}(s_{n+1}) \setminus \mathrm{ran}(t)\},\end{aligned}$$

and hence $\mathrm{ran}(\bar{t}) \subseteq q'$, and hence $(\bar{t}, \bar{q})$ and (t, q') are compatible.

Using the crucial observation, we can easily go through all the finitely many t as in Claim 1, shrinking q_n to obtain q' as in the claim if possible, and such that $\mathrm{ran}(s_{n+1}) \subseteq q'$. Finally we obtain q_{n+1}, as desired. □

Finally, if $\langle (s_n, q_n) : n \in \omega\rangle$ has been constructed, let q such that $q^- = \bigcup\{\mathrm{ran}(s_n) : n \in \omega\}$. It is not difficult to see that (s, q) is as desired at the beginning of the present proof.

Claim 2. $(s, q)||_{\mathbb{A}(\mathbb{L})}\ \tau$.

PROOF. Suppose that Claim 2 is false. We will construct $(s, \bar{q}) \in \mathbb{A}(\mathbb{L})$ extending (s, q) such that no extension of it decides τ, which will be the final contradiction.

First note that there are only finitely many $\sigma \in q$ with $(s{}^\frown\langle\sigma\rangle, q) \in \mathbb{A}(\mathbb{L})$ and $(s{}^\frown\langle\sigma\rangle, q)||_{\mathbb{A}(\mathbb{L})}\ \tau$, as otherwise we would have infinitely many such σ, say $\{\sigma_n : n \in \omega\}$, and $i \in \{0, 1\}$ such that, for all n,

$$(s{}^\frown\langle\sigma_n\rangle, q) \Vdash_{\mathbb{A}(\mathbb{L})} \tau = i.$$

Note that then there exists $\nu \in \mathrm{ran}(s)$ such that for all n we have $\nu \subset \sigma_n$ and $\ell h(\sigma_n) = \ell h(\nu) + 1$. Let

$$q' = \bigcup\{(q)_{\sigma_n} : n \in \omega\} \cup \{\sigma \in q : \nu \not\subseteq \sigma \vee \exists \mu \in \mathrm{ran}(s)(\nu \subset \mu \subseteq \sigma)\}.$$

It is not difficult to see that if $(t, q'') \in \mathbb{A}(\mathbb{L})$ extends (s, q'), then (t, q'') is compatible with $(s{}^\frown\langle\sigma_n\rangle, q)$ for some n, and hence $(s, q') \Vdash_{\mathbb{A}(\mathbb{L})} \tau = i$. But then, by construction, (s, q) decides τ.

Fix $\sigma_0 \in q$ such that $(s{}^\frown\langle\sigma_0\rangle, q) \in \mathbb{A}(\mathbb{L})$ and $(s{}^\frown\langle\sigma_0\rangle, q)$ does not decide τ.

Suppose that $s_n = s{}^\frown\langle\sigma_0, \dots, \sigma_n\rangle$ has been constructed such that $(s_n, q) \in \mathbb{A}(\mathbb{L})$ and for every $t \in {}^{<\omega}(\mathrm{ran}(s_n))$ with $(t, q) \in \mathbb{A}(\mathbb{L})$ and $(t, q) \leq (s, q)$, (t, q) does not decide τ.

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Claim 3. *There are only finitely many* $\sigma \in q$ *with* $(s_n{}^\frown\langle\sigma\rangle, q) \in \mathbb{A}(\mathbb{L})$ *such that there exists* $t \in {}^{<\omega}(\mathrm{ran}(s_n))$ *with* $(t^\frown\langle\sigma\rangle, q) \in \mathbb{A}(\mathbb{L})$, $(t^\frown\langle\sigma\rangle, q) \leq (s, q)$ *and* $(t^\frown\langle\sigma\rangle, q)||_{\mathbb{A}(\mathbb{L})}\, \tau$.

PROOF. Otherwise, there exist $t \in {}^{<\omega}(\mathrm{ran}(s_n))$, $i \in \{0, 1\}$ and $A \in [q]^\omega$ such that for all $\sigma \in A$ we have $(s_n{}^\frown\langle\sigma\rangle, q), (t^\frown\langle\sigma\rangle, q) \in \mathbb{A}(\mathbb{L})$, $(t^\frown\langle\sigma\rangle, q) \leq (s, q)$. Let $\nu \in \mathrm{ran}(t)$ be the unique member such that, for all $\sigma \in A$, $\nu \subset \sigma$ and $\ell h(\sigma) = \ell h(\nu) + 1$. Then set

$$q' = \bigcup\{(q)_\sigma : \sigma \in A\} \cup \{\rho \in q : \nu \not\subseteq \rho\}.$$

But now $(t, q') \Vdash_{\mathbb{A}(\mathbb{L})} \tau = i$, and hence, by construction, (t, q) decides τ, a contradiction. □

By Claim 3 it is clear that we can pick $\sigma_{n+1} \in q$ such that $(s_n{}^\frown\langle\sigma_{n+1}\rangle, q) \in \mathbb{A}(\mathbb{L})$ and for no $t \in {}^{<\omega}q$ with $(t^\frown\langle\sigma_{n+1}\rangle, q) \in \mathbb{A}(\mathbb{L})$ and $(t^\frown\langle\sigma_{n+1}\rangle, q) \leq (s, q)$ is it true that $(t^\frown\langle\sigma_{n+1}\rangle, q)$ decides τ. Let $s_{n+1} = s_n{}^\frown\langle\sigma_{n+1}\rangle$.

Finally, if $\langle s_n : n \in \omega\rangle$ has been constructed, let $\bar{q} \in \mathbb{L}$ be such that $\bar{q}^- = \bigcup\{\mathrm{ran}(s_n) : n \in \omega\}$. By construction, for no $t \in {}^{<\omega}\bar{q}$ with $(t, \bar{q}) \in \mathbb{A}(\mathbb{L})$ and $(t, \bar{q}) \leq (s, \bar{q})$ is it true that $(t, \bar{q})$ decides τ. But as $\bar{q} \subseteq q$ this means that no extension of $(s, \bar{q})$ at all decides τ. □□

Now we are ready to prove that $\mathbb{A}(\mathbb{L})$ has the Laver property, i.e. we will show that for every family $\langle\tau_n : n \in \omega\rangle$ of $\mathbb{A}(\mathbb{L})$-names, $(s, p) \in \mathbb{A}(\mathbb{L})$ and $H \in {}^\omega V \cap V$ such that for every n, $H(n)$ is finite and $\Vdash_{\mathbb{A}(\mathbb{L})} \forall n \in \omega(\tau_n \in H(n))$, there exist $\langle B_n : n < \omega\rangle \in {}^\omega V \cap V$ and $(s, q) \in \mathbb{A}(\mathbb{L})$, $(s, q) \leq (s, p)$ such that, for every n, $|B(n)| = 2^n$ and

$$(s, q) \Vdash_{\mathbb{A}(\mathbb{L})} \forall n \in \omega(\tau_n \in B(n)).$$

For simplicity we will assume $s = \emptyset$. The general case is analogous.

PROPOSITION 1.1.2. *The forcing* $\mathbb{A}(\mathbb{L})$ *has the Laver property.*

PROOF. We first prove the following lemma, from which the proposition will follow easily.

LEMMA 1.1.3. *Suppose that* $(s, p), (t, p) \in \mathbb{A}(\mathbb{L})$ *with* $(t, p) \leq (s, p)$ *and* $\ell h(t) - \ell h(s) = n$. *Moreover let* τ *be an* $\mathbb{A}(\mathbb{L})$*-name and let* $A \subseteq V$ *be finite such that*

$$(s, p) \Vdash_{\mathbb{A}(\mathbb{L})} \tau \in A.$$

Then there exist $(t, q) \in \mathbb{A}(\mathbb{L})$ *and* $B \subseteq V$ *such that* $(t, q) \leq (t, p)$, $|B| = 2^n$ *and* $(s, q) \Vdash_{\mathbb{A}(\mathbb{L})} \tau \in B$.

PROOF.

DEFINITION 1.1.4. Suppose $(t, q), (t', q) \in \mathbb{A}(\mathbb{L})$ with $t' \in {}^{<\omega}(\mathrm{ran}(t))$. Define

$$\begin{aligned} q(t, t') = \{\rho \in q \colon & \exists \nu \in \mathrm{ran}(t')(\rho \subseteq \nu) \\ & \vee (\exists \nu \in \mathrm{ran}(t')(\nu \subseteq \rho) \wedge \forall \nu \in \mathrm{ran}(t) \setminus \mathrm{ran}(t')(\nu \not\subseteq \rho))\}. \end{aligned}$$

We will show that there exists $(t, q) \in \mathbb{A}(\mathbb{L})$ with $(t, q) \leq (t, p)$ such that for every $t' \in {}^{<\omega}(\mathrm{ran}(t))$ with $(t', q) \in \mathbb{A}(\mathbb{L})$ and $(t', q) \leq (s, p)$ we have $(t', q(t, t'))||_{\mathbb{A}(\mathbb{L})}\, \tau$. Note that there are at most 2^n such t'. Since every extension of (s, q) extends one such (t', q), this will suffice.

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Let $\langle u_i : i < N \rangle$ list all $t' \in {}^{<\omega}(\mathrm{ran}(t))$ with $(t', p) \in \mathbb{A}(\mathbb{L})$ and $(t', p) \leq (s, p)$. So we have $N \leq 2^n$. By Lemma 1.1.1 there exists $(u_0, q') \in \mathbb{A}(\mathbb{L})$ with $(u_0, q') \leq (u_0, p(t, u_0))$ and $(u_0, q') ||_{\mathbb{A}(\mathbb{L})} \tau$. Define

$$q_0 = q' \cup \bigcup \{p_\sigma : \sigma \in \mathrm{ran}(t) \setminus \mathrm{ran}(u_0)\}.$$

It is not difficult to see that $(t, q_0) \in \mathbb{A}(\mathbb{L})$.

Suppose now that $q_i \in \mathbb{L}$ has been constructed such that $(t, q_i) \in \mathbb{A}(\mathbb{L})$. By Lemma 1.1.1 we can choose $(u_{i+1}, q') \in \mathbb{A}(\mathbb{L})$ with $(u_{i+1}, q') \leq (u_{i+1}, q_i(t, u_{i+1}))$ such that (u_{i+1}, q') decides τ. Define

$$q_{i+1} = q' \cup \bigcup \{(q_i)_\sigma : \sigma \in \mathrm{ran}(t) \setminus \mathrm{ran}(u_{i+1})\}.$$

Note that $(t, q_{i+1}) \in \mathbb{A}(\mathbb{L})$. Finally let $q \in \mathbb{L}$ be obtained by pruning q_{N-1} such that in the well-ordering induced on q^- by $\prec$ (defined in the notation section), t is an initial segment, and moreover, for every $\sigma \in q^- \setminus \mathrm{ran}(t)$, $\tau \in \mathrm{ran}(t)$, if σ' is the maximal node of $\mathrm{ran}(t)$ contained in σ, $n \in \mathrm{dom}(\sigma) \setminus \mathrm{dom}(\mathrm{stem}(p))$ and $m \in \mathrm{dom}(\tau) \setminus \mathrm{dom}(\mathrm{stem}(q))$, then we have $\sigma(n) > \tau(m)$.

Now define

$$B = \{x : \exists i < N((u_i, q_i(t, u_i)) \Vdash_{\mathbb{A}(\mathbb{L})} \tau = x)\}.$$

Now suppose $(u, q') \in \mathbb{A}(\mathbb{L})$ with $(u, q') \leq (s, q)$. Choose $t' \subseteq u$ maximal such that $\mathrm{ran}(t') \subseteq \mathrm{ran}(t)$; hence clearly $t' = u_i$ for some $i < N$. But now, by construction, every extension of (u, q') is compatible with $(u_i, q_i(t, u_i))$ and

$$(u_i, q_i(t, u_i)) \Vdash_{\mathbb{A}(\mathbb{L})} \tau \in B.$$

Consequently, $(s, q) \Vdash_{\mathbb{A}(\mathbb{L})} \tau \in B$. □

Now suppose that we are given a family $\langle \tau_n : n \in \omega \rangle$ of $\mathbb{A}(\mathbb{L})$-names and $H \in {}^\omega\omega \cap V$ such that

$$\Vdash_{\mathbb{A}(\mathbb{L})} \forall n \in \omega(\tau_n \in H(n)).$$

Using Lemma 1.1.3 it is straightforward to construct $\langle (s_n, q_n) : n \in \omega \rangle$ and $\langle B_n : n \in \omega \rangle$ in V such that for every $n \in \omega$ the following hold:

(1) $s_0 = \langle \emptyset \rangle$;
(2) $(s_n, q_n) \in \mathbb{A}(\mathbb{L})$, $(s_{n+1}, q_{n+1}) \leq (s_n, q_n)$, $\ell h(s_{n+1}) - \ell h(s_n) = 1$;
(3) $(s_n, q_n) \Vdash_{\mathbb{A}(\mathbb{L})} \tau_n \in B_n$;
(4) $|B_n| = 2^n$.

Then, letting $q = \bigcup \{\mathrm{ran}(s_n) : n \in \omega\}$, we conclude that

$$(\emptyset, q) \Vdash_{\mathbb{A}(\mathbb{L})} \forall n \in \omega(\tau_n \in B_n). \qquad \square$$

Combining standard arguments with the proof of Proposition 1.1.2, we can easily derive the following:

LEMMA 1.1.5. *Suppose that $\langle \tau_n : n < \omega \rangle$ is a family of $\mathbb{A}(\mathbb{L})$-names of ordinals and $(s, p) \in \mathbb{A}(\mathbb{L})$. Then there exists $(s, q) \in \mathbb{A}(\mathbb{L})$, $(s, q) \leq (s, p)$, such that for every n and for every $t \in {}^{<\omega}q$ of length at least n with $(t, q) \leq (s, q)$, if there exists $(t, q') \in \mathbb{A}(\mathbb{L})$, $(t, q') \leq (t, q)$, which decides τ_n, then (t, q) decides τ_n.*

Given a countable substructure $(N, \in)$ of some $(H(\chi), \in)$, where χ is large enough and regular, and letting $\langle \tau_n : n < \omega \rangle$ enumerate all $\mathbb{A}(\mathbb{L})$-names of ordinals

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which belong to N, then doing the fusion which is used for Lemma 1.1.5 such that every initial segment of it belongs to N, we get the following corollary.

COROLLARY 1.1.6. *The forcing $\mathbb{A}(\mathbb{L})$ is proper.*

Suppose that G is $\mathbb{A}(\mathbb{L})$-generic over V. Then clearly there exists a unique $p_G \in \mathbb{L}$ with

$$(p_G)^- = \bigcup\{\operatorname{ran}(s) : \exists p \in \mathbb{L}((s,p) \in G)\}.$$

For short, we will say that p_G is $\mathbb{A}(\mathbb{L})$-generic over V.

If P is a forcing adding a generic real, then by an amoeba forcing for P we mean a forcing which adds a large tree, where "large" depends on P, such that every branch is P-generic. Lemma 1.1.8, below, will justify the name "amoeba forcing" for $\mathbb{A}(\mathbb{L})$.

LEMMA 1.1.7 (JSh, Fact 1.9). *Suppose $D \subseteq \mathbb{L}$ is dense and open. Then the set*

$$D^0 = \{p \in \mathbb{L} : \exists F \subseteq p(F \text{ is a front in } p \wedge \forall \sigma \in F((p)_\sigma \in D))\}$$

is $\leq^0$-dense, i.e. for every $p \in \mathbb{L}$ there exists $q \in D^0$ with $q \subseteq p$ and $\operatorname{stem}(q) = \operatorname{stem}(p)$.

LEMMA 1.1.8. *Suppose that $D \subseteq \mathbb{L}$ is open and dense $(D \in V)$ and G is $\mathbb{A}(\mathbb{L})$-generic over V. Then, in $V[G]$, $[p_G] \subseteq \bigcup\{[p] : p \in D\}$. Hence every branch through p_G is $\mathbb{L}$-generic over V.*

PROOF.

DEFINITION 1.1.9. For $(s,p) \in \mathbb{A}(\mathbb{L})$ and $\sigma \in \operatorname{ran}(s)$ let

$$p(s,\sigma) = \bigcup\{(p)_{\sigma^\frown\langle k\rangle} : k \in \omega \wedge \forall \tau \in \operatorname{ran}(s)(\sigma^\frown\langle k\rangle \not\subseteq \tau)\}.$$

By Lemma 1.1.7, for given $(s,p) \in \mathbb{A}(\mathbb{L})$ it is easy to find $(s,q) \in \mathbb{A}(\mathbb{L})$ such that $(s,q) \leq (s,p)$ and, for every $\sigma \in \operatorname{ran}(s)$, $q(s,\sigma) \in D^0$ (D^0 is defined as in 1.1.7). By genericity, there exists $(s,q) \in G$ which has these properties. Then clearly every branch of p_G is a branch of $q(s,\sigma)$ for some $\sigma \in \operatorname{ran}(s)$. Since being a front is a Π^1_1-property, a front in $q(s,\sigma)$ in V is still a front in $V[G]$, and hence every branch of p_G is a branch of some member of D.

So we have to prove that if $x \in {}^\omega\omega$ has the property that for every dense set $D \subseteq \mathbb{L}$ which belongs to V there exists $p \in D$ such that $x \in [p]$, then x is a Laver real over V, i.e. the set $H = \{p \in \mathbb{L}^V : x \in [p]\}$ is an $\mathbb{L}$-generic filter. It suffices to show that H is a filter. Suppose $p_1, p_2 \in H$ were incompatible, hence by Shoenfield absoluteness incompatible in V, i.e. $[p_1] \cap [p_2]$ does not contain the branches of a Laver tree. Since $[p_1] \cap [p_2]$ is closed, by [GRShSp, Lemma 2.3] $[p_1] \cap [p_2]$ is not strongly dominating, i.e. there exists $f \in {}^\omega\omega \cap V$ such that for all $y \in [p_1] \cap [p_2]$ there exist infinitely many $k \in \omega$ such that $f(y(k-1)) \geq y(k)$. Moreover it is easy to see that the set

$$E = \{q \in \mathbb{L} : \forall y \in [q] \forall^\infty k \in \omega(f(y(k-1)) < y(k))\}$$

is dense in $\mathbb{L}$. By assumption, x meets a member of E. But this is impossible since $x \in [p_1] \cap [p_2]$. □

REMARK 1.1.10. The absoluteness argument from the proof of Lemma 1.1.8 shows that in every ZFC-model containing $V[G]$, every branch of p_G is a branch of a member of D.

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REMARK 1.1.11. As a corollary of the proof of Lemma 1.1.8 we obtain that Laver forcing is proper in the following stronger sense: For *every* countable transitive model $(N, \in)$ of ZF^- (so $(N, \in)$ need not be elementarily embeddable into $(H(\chi), \in)$ for some cardinal χ) and every $p \in \mathbb{L} \cap N$ there exists an $(\mathbb{L}, N)$-generic condition q below p. In fact, in V choose $q \leq p$ $\mathbb{A}(\mathbb{L})$-generic over N (since N is countable this is no problem). Then if $(N, \in)$ is clever enough to prove Lemma 1.1.8, every branch of q is a Laver real over N. This is certainly enough to ensure that q is $(\mathbb{L}, N)$-generic.

1.2. Forcing generic superperfect trees without adding Cohen reals. The natural amoeba forcing for $\mathbb{M}$ has conditions (s, p), where $p \in \mathbb{M}$ and s is a finite subtree of p, and the ordering is $(s, p) \leq (t, q)$ if $s \supseteq t$ and $p \subseteq q$. A generic filter G for this forcing determines a tree $p_G \in \mathbb{M}$ as in the case of $\mathbb{L}$. Then one type of Cohen reals coded by p_G is similar to that in the case of $\mathbb{L}$. But there are two more types. Let x, y be two branches of p_G which can be described in V, say letting σ, τ be distinct members of $\mathrm{Succ}_{\mathrm{stem}(p_G)}(p_G)$ let x be the left-most branch in $(p_G)_\sigma$ and let y be the left-most branch in $(p_G)_\tau$. Let g_x, g_y increasingly enumerate $\{n < \omega : x\restriction n \in \mathrm{Split}(p_G)\}$, $\{n < \omega : y\restriction n \in \mathrm{Split}(p_G)\}$ respectively. Then $\mathrm{signum}(g_x - g_y)$ is a Cohen real. Moreover, if $\rho \in \mathrm{Split}(p_G)$ and f_ρ increasingly enumerates $\mathrm{succ}_\rho(p_G)$, then also $\mathrm{signum}(g_x - f_\rho)$ is a Cohen real.

We define an amoeba forcing for Miller forcing, $\mathbb{A}(\mathbb{M})$, as follows: Conditions in $\mathbb{A}(\mathbb{M})$ are pairs (s, p) such that

(1) $p \in \mathbb{M}$ and $s \in {}^n(\mathrm{Split}(p))$ for some $n \in \omega \setminus \{0\}$;
(2) if S is the downward closure of $\mathrm{ran}(s)$, then $\mathrm{split}(S) \subseteq \mathrm{ran}(s)$;
(3) the map sending $s(i)$ to $T(i)$ is an isomorphism between $(\mathrm{ran}(s), \subseteq)$ and $(\mathrm{ran}(T\restriction n), \subseteq)$;
(4) if, for $0 < i < n$, ℓ_i^s is the length of the longest member of $\mathrm{ran}(s)$ which is strictly contained in $s(i)$, then for every $0 < i < j < n$ we have

$$\max\{\ell h(s(i)), s(i)(\ell_i^s)\} < \min\{\ell h(s(j)), s(j)(\ell_j^s)\}.$$

The ordering on $\mathbb{A}(\mathbb{M})$ is defined by

$$(s, p) \leq (t, q) \quad \text{if and only if } p \subseteq q \quad \text{and } s \supseteq t.$$

Note that if S, T is the downward closure of $\mathrm{ran}(s)$, $\mathrm{ran}(t)$ respectively, and $\sigma \in T \setminus \mathrm{split}(T)$ but $\sigma \in \mathrm{split}(S)$, then σ must be a terminal node of T.

Now the analogues of Lemma 1.1.1 and Proposition 1.1.2 hold:

LEMMA 1.2.1. *Suppose $(s, p) \in \mathbb{A}(\mathbb{M})$ and τ is an $\mathbb{A}(\mathbb{M})$-name such that*

$$(s, p) \Vdash_{\mathbb{A}(\mathbb{M})} \tau \in \{0, 1\}.$$

Then there exists $(s, q) \in \mathbb{A}(\mathbb{M})$ with $(s, q) \leq (s, p)$ and $(s, q) ||_{\mathbb{A}(\mathbb{M})} \tau$, i.e $\mathbb{A}(\mathbb{M})$ has the pure decision property.

PROOF. As in the proof of 1.1.1, first we construct $(s, q) \leq (s, p)$ such that, for every $t \in {}^{<\omega}(\mathrm{Split}(q))$, if for some $(t, q') \in \mathbb{A}(\mathbb{M})$ we have $(t, q') \leq (s, q)$ and $(t, q') ||_{\mathbb{A}(\mathbb{M})} \tau$, then $(t, q) ||_{\mathbb{A}(\mathbb{M})} \tau$.

Suppose that $(s_n, q_n) \in \mathbb{A}(\mathbb{M})$ with $(s_n, q_n) \leq (s, p)$ has been constructed. Since $q_n \in \mathbb{M}$ and $\mathrm{ran}(s_n) \subseteq \mathrm{Split}(q_n)$, we may certainly choose $\sigma_{n+1} \in q_n$ such that $(s_n {}^\frown \langle \sigma_{n+1} \rangle, q_n) \in \mathbb{A}(\mathbb{M})$. Let $s_{n+1} = s_n {}^\frown \langle \sigma_{n+1} \rangle$.

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Claim 1′. *There exists* $q_{n+1} \in \mathbb{M}$ *such that* $(s_{n+1}, q_{n+1}) \in \mathbb{A}(\mathbb{M})$, $(s_{n+1}, q_{n+1}) \leq (s_{n+1}, q_n)$ *and for every* $t \in {}^{<\omega}(\operatorname{ran}(s_{n+1}))$ *with* $s \subset t$ *and* $t(\ell h(t) - 1) = \sigma_{n+1}$, *if there exists* q' *with* $(t, q') \in \mathbb{A}(\mathbb{M})$, $(t, q') \leq (t, q_{n+1})$ *and* $(t, q')||_{\mathbb{A}(\mathbb{M})}\ \tau$, *then* $(t, q_{n+1})||_{\mathbb{A}(\mathbb{M})}\ \tau$.

PROOF. Let $\langle t_i : i < N \rangle$ list all t as in Claim 1′. Let $q_n^0 = q_n$. Suppose that, for some $i < N$, q_n^i has been constructed such that $(s_{n+1}, q_n^i) \in \mathbb{A}(\mathbb{M})$. Suppose there exists $(t_i, q') \in \mathbb{A}(\mathbb{M})$ with $(t_i, q') \leq (t_i, q_n^i)$ and $(t_i, q')||_{\mathbb{A}(\mathbb{M})}\ \tau$. Let T_i denote the downward closure of $\operatorname{ran}(t_i)$. Define

$$r_1 = \bigcup\{(q_n^i)_\sigma : \sigma \in \operatorname{ran}(s_{n+1}) \setminus T_i\},$$

$$r_2 = \bigcup\{(q_n^i)_{\sigma^\frown\langle k\rangle} : \sigma \in (\operatorname{ran}(s_{n+1}) \setminus \operatorname{ran}(t_i)) \cap T_i \wedge \sigma^\frown\langle k\rangle \notin T_i\},$$

$$q_n^{i+1} = q' \cup r_1 \cup r_2.$$

It is not difficult to check that $(s_{n+1}, q_n^{i+1}) \in \mathbb{A}(\mathbb{M})$ and $(s_{n+1}, q_n^{i+1}) \leq (s_{n+1}, q_n^i)$.

Suppose now $(\bar{t}, \bar{q}) \in \mathbb{A}(\mathbb{M})$ and $(\bar{t}, \bar{q}) \leq (t_i, q_n^{i+1})$. We claim that $\operatorname{ran}(\bar{t}) \subseteq q'$ and hence $(\bar{t}, \bar{q}), (t_i, q')$ are compatible, and hence $(t_i, q_n^{i+1})||_{\mathbb{A}(\mathbb{M})}\ \tau$. First remember $t_i(\ell h(t_i) - 1) = \sigma_{n+1}$.

Let $j \in \operatorname{dom}(\bar{t}) \setminus \operatorname{dom}(t_i)$ be such that $\bar{t}(j)$ is a minimal node in $\operatorname{ran}(\bar{t}) \setminus \operatorname{ran}(t_i)$. Then clearly $j > 0$. Let $\rho = \bar{t}(j){\restriction}\ell_j^{\bar{t}}$ (see clause (4) in the definition of $\mathbb{A}(\mathbb{M})$). Now by definition of $\mathbb{A}(\mathbb{M})$ we have

$$\bar{t}(j)(\ell_j^{\bar{t}}) > \ell h(\sigma_{n+1}) \geq \max\{s(k)(\ell_k^s) : k \in \operatorname{dom}(s) \setminus \{0\}\}.$$

Hence $\rho \in \operatorname{ran}(s_{n+1}) \cap \operatorname{ran}(t)$, but $\rho^\frown\langle \bar{t}(j)(\ell_j^{\bar{t}})\rangle \notin S$, where S is the downward closure of $\operatorname{ran}(s_{n+1})$. Consequently $\bar{t}(j) \notin r_1 \cup r_2$, and hence $\bar{t}(j) \in q'$.

Finally let $q_{n+1} = q_n^{N-1}$. Then q_{n+1} is as desired in Claim 1′. □

If $\langle (s_n, q_n) : n \in \omega \rangle$ has been constructed let $q \in \mathbb{M}$ be determined by

$$\operatorname{Split}(q) = \bigcup\{\operatorname{ran}(s_n) : n \in \omega\}.$$

Claim 2′. $(s, q)||_{\mathbb{A}(\mathbb{M})}\ \tau$.

PROOF. Otherwise we could construct $(s, \bar{q}) \in \mathbb{A}(\mathbb{M})$ extending (s, q) such that no extension of it decides τ, as follows.

Let $\nu \in \operatorname{ran}(s)$ be the maximal member such that whenever $(s^\frown\langle\sigma\rangle, q) \in \mathbb{A}(\mathbb{M})$ extends (s, q), then $\nu \subset \sigma$. Define

$$q' = \{\rho \in q : \rho \subseteq \nu \vee [\nu \subset \rho \wedge \forall \sigma \in \operatorname{ran}(s) \setminus \{\nu\}(\sigma \not\subseteq \nu \Rightarrow (\rho \not\subseteq \sigma \wedge \sigma \not\subseteq \rho))]\}.$$

So $q' \in \mathbb{M}$ and $\operatorname{stem}(q') = \nu$. Let $\rho \in \operatorname{Succ}_\nu(q')$. Note that $(q')_\rho = (q)_\rho$. Define

$$A_0^\rho = \{\sigma \in \operatorname{Split}((q)_\rho) : (s^\frown\langle\sigma\rangle, q) \Vdash_{\mathbb{A}(\mathbb{M})} \tau = 0\},$$

$$A_1^\rho = \{\sigma \in \operatorname{Split}((q)_\rho) : (s^\frown\langle\sigma\rangle, q) \Vdash_{\mathbb{A}(\mathbb{M})} \tau = 1\},$$

$$A_2^\rho = \{\sigma \in \operatorname{Split}((q)_\rho) : \neg((s^\frown\langle\sigma\rangle, q)||_{\mathbb{A}(\mathbb{M})}\ \tau)\}.$$

Then clearly we have $\operatorname{Split}((q)_\rho) = A_0^\rho \cup A_1^\rho \cup A_2^\rho$. It is not difficult to see (see [Mi, Claim 2.4]) that there exist $q^\rho \in \mathbb{M}$ and $i_\rho \in \{0, 1, 2\}$ such that $q^\rho \subseteq (q)_\rho$

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and $\mathrm{Split}(q^\rho) \subseteq A^\rho_{i_\rho}$. If, for some infinite $A \subseteq \mathrm{Succ}_\nu(q')$ and $i \in \{0,1\}$, for every $\rho \in A$ we had $i_\rho = i$, then, letting

$$q'' = \bigcup\{q^\rho : \rho \in A\} \cup \{\sigma \in q : \nu \not\subseteq \sigma \vee [\exists \mu \in \mathrm{ran}(s)(\nu \subset \mu \subseteq \sigma \vee \sigma \subseteq \mu)]\},$$

we conclude $(s, q'') \Vdash_{\mathbb{A}(\mathbb{M})} \tau = i$.

But then, by the construction of q we had $(s,q)||_{\mathbb{A}(\mathbb{M})}\ \tau$, a contradiction.

Hence for some finite $F \subseteq \mathrm{Succ}_\nu(q')$, for every $\rho \in \mathrm{Succ}_\nu(q') \setminus F$ we have $i_\rho = 2$. Define

$$q_0 = \bigcup\{q^\rho : \rho \in \mathrm{Succ}_\nu(q') \setminus F\}$$
$$\cup \{\sigma \in q : \nu \not\subseteq \sigma \vee [\exists \mu \in \mathrm{ran}(s)(\nu \subset \mu \subseteq \sigma \vee \sigma \subseteq \mu)]\}.$$

Pick $\sigma_0 \in \mathrm{Split}(q^\rho)$ for some $\rho \in \mathrm{Succ}_\nu(q') \setminus F$ arbitrarily, and let $s_0 = s\,\hat{}\,\langle\sigma_0\rangle$. Then we have $(s_0, q_0) \in \mathbb{A}(\mathbb{M})$ and $(s_0, q_0) \leq (s, q)$.

We mention here that for no $\sigma \in \mathrm{Split}(q_0)$ with $(s\,\hat{}\,\langle\sigma\rangle, q_0) \in \mathbb{A}(\mathbb{M})$ can we have $(s\,\hat{}\,\langle\sigma\rangle, q_0)||_{\mathbb{A}(\mathbb{M})}\ \tau$.

Suppose now that $(s_i, q_i) = (s\,\hat{}\,\langle\sigma_0, \dots, \sigma_i\rangle, q_i) \in \mathbb{A}(\mathbb{M})$ for $i \leq n$ have been constructed such that the following hold:

(1) $(s_n, q_n) \leq (s_{n-1}, q_{n-1}) \leq \cdots \leq (s_0, q_0) \leq (s, q)$;

(2) $\neg((s_n, q)||_{\mathbb{A}(\mathbb{M})}\ \tau)$;

(3) for every $t \in {}^{<\omega}(\mathrm{ran}(s_n))$ with $t(\ell h(t) - 1) \neq \sigma_n$, $(t, q_n) \in \mathbb{A}(\mathbb{M})$ and $(t, q_n) \leq (s, q_n)$ it is true that there is no $\sigma \in \mathrm{Split}(q_n)$ with $(t\,\hat{}\,\langle\sigma\rangle, q_n) \in \mathbb{A}(\mathbb{M})$ for which $(t\,\hat{}\,\langle\sigma\rangle, q)||_{\mathbb{A}(\mathbb{M})}\ \tau$ holds.

So (1), (2), (3) hold for $n = 0$.

Claim 3$'$. *There exist* $q_{n+1} \in \mathbb{M}$ *and* $\sigma_{n+1} \in q_{n+1}$ *such that if* $s_{n+1} = s_n\,\hat{}\,\langle\sigma_{n+1}\rangle$, *then* $(s_{n+1}, q_{n+1}) \in \mathbb{A}(\mathbb{M})$ *and* (1), (2), (3) *above hold for* $n+1$ *instead of* n.

Proof. Let $\langle t_i : i < N\rangle$ be a list of all $t \in {}^{<\omega}(\mathrm{ran}(s_n))$ with $(t, q_n) \in \mathbb{A}(\mathbb{M})$, $(t, q_n) \leq (s, q_n)$ and $t(\ell h(t) - 1) = \sigma_n$. We claim that we can construct $\langle q_n^i : i < N\rangle$ in $\mathbb{M}$ with $q_n \geq q_n^0 \geq \cdots \geq q_n^{N-1}$ such that $(s_n, q_n^{N-1}) \in \mathbb{A}(\mathbb{M})$ and, for every $i < N$, there is no $\sigma \in q_n^i$ with $(t_i\,\hat{}\,\langle\sigma\rangle, q_n^i) \in \mathbb{A}(\mathbb{M})$ for which it is true that $(t_i\,\hat{}\,\langle\sigma\rangle, q)$ decides τ. If this construction is possible, we set $q_{n+1} = q_n^{N-1}$, choose $\sigma_{n+1} \in q_{n+1}$ with $(s_n\,\hat{}\,\langle\sigma_{n+1}\rangle, q_{n+1}) \in \mathbb{A}(\mathbb{M})$, and let $s_{n+1} = s_n\,\hat{}\,\langle\sigma_{n+1}\rangle$. Then it is easy to check that (1), (2), (3) hold for $n+1$ instead of n.

For the construction of $\langle q_n^i : i < N\rangle$, at step $i < N$ just repeat the argument used in constructing q_0 at the beginning of the present induction. By induction hypothesis, (t_i, q) does not decide τ. Hence by the construction of q, (t_i, q_n^{i-1}) (where $q_n^{-1} = q_n$) does not decide τ, so by the argument just quoted we can find q_n^i as desired. □

Finally, suppose that $\langle (s_n, q_n) : n \in \omega\rangle$ has been constructed such that (1), (2), (3) hold for every $n \in \omega$. Then let $\bar{q} \in \mathbb{M}$ be determined by $\mathrm{Split}(\bar{q}) = \bigcup\{\mathrm{ran}(s_n) : n \in \omega\}$. Clearly $(s, \bar{q}) \in \mathbb{A}(\mathbb{M})$, and moreover by construction of q and $\bar{q}$ it follows that no extension of $(s, \bar{q})$ decides τ, a contradiction. □□

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DEFINITION 1.2.2. Suppose that $(t,q),(t',q) \in \mathbb{A}(\mathbb{M})$ with $t' \in {}^{<\omega}(\operatorname{ran}(t))$. Define

$$q(t,t') = \bigcup\{(q)_{\sigma^\frown\langle k\rangle} : \sigma \in \operatorname{ran}(t') \wedge \forall\tau \in \operatorname{ran}(t)\setminus\operatorname{ran}(t')(\sigma^\frown\langle k\rangle \not\subseteq \tau)\}.$$

PROPOSITION 1.2.3. *The forcing* $\mathbb{A}(\mathbb{M})$ *has the Laver property.*

PROOF. Suppose $(s,p) \in \mathbb{A}(\mathbb{M})$, $g \in {}^{\omega}\omega \cap V$ and $\dot f$ is an $\mathbb{A}(\mathbb{M})$-name for a member of ${}^{\omega}\omega$ such that $(s,p) \Vdash_{\mathbb{A}(\mathbb{M})} \forall n(\dot f(n) < g(n))$. We will construct $(s,q) \in \mathbb{A}(\mathbb{M})$ and $\langle H(n) : n<\omega\rangle$ such that $(s,q) \leq (s,p)$, $H(n)$ has size at most 2^{n+2}, and $(s,q) \Vdash_{\mathbb{A}(\mathbb{M})} \forall n(\dot f(n) \in H(n))$. Here (s,q) will be the result of a fusion.

By 1.2.1, there exist K and $(s,q_0) \in \mathbb{A}(\mathbb{M})$ with $(s,q_0) \leq (s,p)$ and $(s,q_0) \Vdash_{\mathbb{A}(\mathbb{M})} \dot f(0) = K$. Let $s_0 = s$ and $H(0) = \{K\}$.

Suppose that, for $i \leq n$, $(s_i,q_i) \in \mathbb{A}(\mathbb{M})$ and $H(i)$ have been constructed such that $(s_n,q_n) \leq \cdots \leq (s_0,q_0)$, $|H(i)| \leq 2^{i+2}$ and $(s,q_i) \Vdash_{\mathbb{A}(\mathbb{M})} \dot f(i) \in H(i)$. Choose $\sigma_n \in q_n$ minimal with respect to the ordering $\prec$ of ${}^{<\omega}\omega$ such that $(s_n{}^\frown\langle\sigma_n\rangle, q_n) \in \mathbb{A}(\mathbb{M})$, hence $(s_n{}^\frown\langle\sigma_n\rangle, q_n) \leq (s_n,q_n)$. Let $s_{n+1} = s_n{}^\frown\langle\sigma_n\rangle$. Then by 1.2.1 we may easily find $(s_{n+1},q') \in \mathbb{A}(\mathbb{M})$, $(s_{n+1},q') \leq (s_{n+1},q_n)$, such that, for every $t \in {}^{<\omega}(\operatorname{ran}(s_{n+1}))$ with $(t,q') \leq (s,q_n)$, for some K, $(t,q'(s_{n+1},t)) \Vdash_{\mathbb{A}(\mathbb{M})} \dot f(n+1) = K$. There are at most 2^{n+1} such t. Hence if H' is the set of all such K, then $|H'| \leq 2^{n+1}$.

Moreover it is easy to see that we may assume that q' has been pruned in such a way that, for every t as above, if $\tau \in q'(s_{n+1},t)$ with $(t^\frown\langle\tau\rangle, q') \in \mathbb{A}(\mathbb{M})$, we have

$$\ell^{t^\frown\langle\tau\rangle}_{\ell h(t)} > \max\{\ell h(s_{n+1}(i)), \ell_i^{s_{n+1}} : 0 < i < \ell h(s_{n+1})\}. \tag{4}$$

Next by a similar fusion argument as in the first part of the proof of 1.2.1 and then by 1.2.1, we may find $(s_{n+1},q'') \in \mathbb{A}(\mathbb{M})$, $(s_{n+1},q'') \leq (s_{n+1},q')$, such that, for every $t \in {}^{<\omega}(\operatorname{Split}(q''))$ with $(t,q'') \in \mathbb{A}(\mathbb{M})$, $(t,q'') \leq (s,q'')$ and $\operatorname{ran}(t) \not\subseteq \operatorname{ran}(s_n)$, (t,q'') decides $\dot f(n+1)$. Certainly we may not drop the last requirement on t here, as otherwise we might not keep $\operatorname{ran}(s_{n+1}) \subseteq q''$.

Now let $\langle(t_i,\tau_i) : i < N\rangle$ list all pairs (t,τ) such that $t \in {}^{<\omega}(\operatorname{ran}(s_{n+1}))$, $\tau \in \operatorname{ran}(s_{n+1})$, $(t^\frown\langle\tau\rangle, q'') \in \mathbb{A}(\mathbb{M})$ and $(t^\frown\langle\tau\rangle, q'') \leq (s,q'')$. By induction on $i < N$ we construct $q^i \in \mathbb{M}$ such that $q^{i+1} \subseteq q^i \subseteq q''$, $(s_{n+1},q^i) \in \mathbb{A}(\mathbb{M})$ and for some K_i, for every $\rho \in \operatorname{Split}(q^i(s_{n+1},t_i{}^\frown\langle\tau_i\rangle))$ which properly extends τ_i, $(t_i{}^\frown\langle\rho\rangle, q^i) \Vdash_{\mathbb{A}(\mathbb{M})} \dot f(n+1) = K_i$.

Suppose we have already constructed q^i. By the choice of q'' and as $(s_{n+1},q^i) \leq (s_{n+1},q'')$, for every $\rho \in \operatorname{Split}(q^i(s_{n+1},t_{i+1}{}^\frown\langle\tau_{i+1}\rangle))$ with $\tau_{i+1} \subset \rho$, $(t_{i+1}{}^\frown\langle\rho\rangle, q^i)$ decides $\dot f(n+1)$. By the same argument as in the proof of Claim 2′ above, for every $\rho \in \operatorname{Succ}_{\tau_{i+1}}(q^i(s_{n+1},t_{i+1}{}^\frown\langle\tau_{i+1}\rangle))$ we may find K_ρ and $q^\rho \in \mathbb{M}$ such that $q^\rho \subseteq (q^i)_\rho$ and, for every $\nu \in \operatorname{Split}(q^\rho)$, $(t_{i+1}{}^\frown\langle\nu\rangle, q^i) \Vdash_{\mathbb{A}(\mathbb{M})} \dot f(n+1) = K_\rho$. As there are infinitely many ρ but only finitely many possible K_ρ, we may find K_{i+1} such that for an infinite set of ρ's, $K_\rho = K_{i+1}$. Now let q^{i+1} be obtained from q^i by pruning off at τ_{i+1} all $(q^i)_\rho$ with $K_\rho \neq K_{i+1}$ and replacing $(q^i)_\rho$ by q^ρ for all ρ with $K_\rho = K_{i+1}$. Then clearly the inductive assumption holds for q^{i+1}, K_{i+1}.

Finally let $q_{n+1} = q^{N-1}$, $H'' = \{K_i : i < N\}$ and $H(n+1) = H' \cup H''$. As clearly $N \leq 2^{n+1}$, we conclude that $|H(n+1)| \leq 2^{n+2}$.

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Claim. $(s, q_{n+1}) \Vdash_{\mathbb{A}(\mathbb{M})} \dot{f}(n+1) \in H(n+1)$.

PROOF. Let $(u, q) \in \mathbb{A}(\mathbb{M})$, $(u, q) \leq (s, q_{n+1})$, and let K be such that $(u, q) \Vdash_{\mathbb{A}(\mathbb{M})} \dot{f}(n+1) = K$. Let $t \subseteq u$ be the maximal initial segment such that $\operatorname{ran}(t) \subseteq \operatorname{ran}(s_{n+1})$. Let $v = u(\ell h(t))$. We distinguish two cases.

In the first case, $v \in q_{n+1}(s_{n+1}, t)$. Then, by (4), $\operatorname{ran}(u) \subseteq q_{n+1}(s_{n+1}, t)$, and hence (u, q) and $q_{n+1}(s_{n+1}, t)$ are compatible. But $(t, q_{n+1}(s_{n+1}, t)) \leq (t, q'(s_{n+1}, t))$ and $(t, q'(s_{n+1}, t)) \Vdash_{\mathbb{A}(\mathbb{M})} \dot{f}(n+1) \in H'$. We conclude that $(u, q) \Vdash_{\mathbb{A}(\mathbb{M})} \dot{f}(n+1) \in H'$.

In the second case, v extends a member of $\operatorname{ran}(s_{n+1})$. Let τ be the longest such one. Then easily $(t^\frown\langle\tau\rangle, q_{n+1}) \in \mathbb{A}(\mathbb{M})$. Hence $(t, \tau) = (t_i, \tau_i)$ for some $i < N$. But then v is a split node of $q^i(s_{n+1}, t_i{}^\frown\langle\tau_i\rangle)$ extending τ_i. Hence $(t_i{}^\frown\langle v\rangle, q^i) \Vdash_{\mathbb{A}(\mathbb{M})} \dot{f}(n+1) = K_i \in H''$. By construction, $(u, q) \leq (t_i{}^\frown\langle v\rangle, q^i)$, and hence $(u, q) \Vdash_{\mathbb{A}(\mathbb{M})} \dot{f}(n+1) = K_i \in H''$. □

In the end let $q \in \mathbb{M}$ be the tree with $\operatorname{Split}(q) = \bigcup\{\operatorname{ran}(s_n) : n < \omega\}$. Then (s, q) and $\langle H(n) : n < \omega\rangle$ are as desired. □

Similarly to Corollary 1.1.6 we obtain the following.

COROLLARY 1.2.4. *$\mathbb{A}(\mathbb{M})$ is proper.*

Suppose that G is $\mathbb{A}(\mathbb{M})$-generic over V. Then clearly there exists a unique $p_G \in \mathbb{M}$ with

$$\operatorname{Split}(p_G) = \bigcup\{\operatorname{ran}(s) : \exists p \in \mathbb{M}((s, p) \in G)\}.$$

For short, we will say that p_G is $\mathbb{A}(\mathbb{M})$-generic over V. The following lemma justifies the name "amoeba forcing" for $\mathbb{A}(\mathbb{M})$.

LEMMA 1.2.5. *Suppose that $D \subseteq \mathbb{M}$ is open and dense $(D \in V)$ and G is $\mathbb{A}(\mathbb{M})$-generic over V. Then, in $V[G]$, $[p_G] \subseteq \bigcup\{[p] : p \in D\}$. Hence every branch through p_G is $\mathbb{M}$-generic over V. Moreover, this is true in every ZFC-model extending $V[G]$.*

PROOF. The proof is similar, and even easier than that of 1.1.8, since we do not have to worry about pure extensions of trees. Instead of [GRShSp, Lemma 2.3.] we use Kechris' theorem (see [K]) which says that every analytic set in the Baire space is either bounded with respect to eventual dominance or contains the branches of a superperfect tree □

REMARK 1.2.6. A similar argument as in Remark 1.1.11 shows that Miller forcing is proper in the stronger sense of 1.1.11.

1.3. The models.

THEOREM 1.3.1. *Suppose that $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2\rangle$ is a countable support iteration of $\mathbb{A}(\mathbb{L})$ (defined in §1.1), i.e., for every $\beta < \omega_2$, $\dot{Q}_\beta$ is a P_β-name for $\mathbb{A}(\mathbb{L})$ defined in V^{P_β}. If G is P_{ω_2}–generic over V, then $V[G] \models \mathfrak{t} = \omega_1 \wedge \operatorname{add}(\ell^0) = \omega_2$.*

PROOF. 1) $V[G] \models \operatorname{add}(\ell^0) = \omega_2$. Suppose in $V[G]$, $\langle X_\alpha : \alpha < \omega_1\rangle$ is a family of members of ℓ^0. Let $X = \bigcup\{X_\alpha : \alpha < \omega_1\}$. Let $p^* \in \mathbb{L}$. We have to find $p' \in \mathbb{L}$ with $p' \subseteq p^*$ and $[p'] \cap X = \emptyset$. For $\alpha < \omega_1$ let $D_\alpha = \{p \in \mathbb{L} : [p] \cap X_\alpha = \emptyset\}$. Clearly D_α is open and dense in $\mathbb{L}$. A standard Löwenheim–Skolem argument combined with the fact that for every $\alpha < \omega_2$ with $\operatorname{cf}(\alpha) = \omega_1$, every $p \in \mathbb{L}^{V[G_\alpha]}$ belongs to $V[G_\beta]$ for some $\beta < \alpha$ shows that the set

$$C_\alpha = \{\beta < \omega_2 : D_\alpha \cap V[G_\beta] \in V[G_\beta] \wedge D_\alpha \cap V[G_\beta] \text{ is open and dense in } \mathbb{L}^{V[G_\beta]}\}$$

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is an ω_1-club in ω_2, i.e. it is unbounded and closed under increasing sequences of length ω_1. Hence $C = \bigcap\{C_\alpha : \alpha \in \omega_1\}$ is also an ω_1-club in ω_2. By the argument just stated and since our forcing conditions have countable support, by genericity there exist $\gamma < \omega_2$ and $s \in {}^{<\omega}(p^*)$ such that $\gamma \in C$ and $(s, p^*) \in G(\gamma)$, where $G(\gamma)$ is the $\dot{Q}_\gamma[G_\gamma]$-generic filter induced by G. By Lemma 1.1.8 we conclude that for every $\alpha < \omega_1$ in $V[G_{\gamma+1}]$ we have

$$[p_{G(\gamma)}] \subseteq \bigcup\{[p] : p \in D_\alpha \cap V[G_\gamma]\}.$$

By Remark 1.1.10. we conclude that this inclusion also holds in $V[G]$. Hence $V[G] \models [p_{G(\gamma)}] \cap X = \emptyset$. But clearly $p_{G(\gamma)} \subseteq p^*$. As p^* was arbitrary, we conclude that $X \in \ell^0$.

2) $V[G] \models \mathfrak{t} = \omega_1$. By Proposition 1.1.2, every iterand of our iteration has the Laver property. By a result of Shelah (see [Sh, pp. 206–207] or [G] for a more accessible proof) the Laver property is preserved under countable support iterations, so P_{ω_2} has it, and hence in $V[G]$ no real is Cohen over V. In terms of cardinal invariants this means $V[G] \models \operatorname{cov}(\mathscr{M}) = \omega_1$, where $\mathscr{M}$ is the ideal of meagre subsets of the real line $\mathbb{R}$. Consequently the following result, which is due to Szymański, concludes the proof of Theorem 1.3.1. For completeness we give the proof. □

LEMMA 1.3.2. $\mathfrak{t} \leq \operatorname{add}(\mathscr{M})$.

PROOF. Suppose $\langle A_\alpha : \alpha < \kappa\rangle$ is a family of nowhere dense subsets of $\mathbb{R}$ and $\kappa < \mathfrak{t}$. We have to show that $\bigcup\{A_\alpha : \alpha < \kappa\}$ is meagre. Inductively we construct $\langle Q_\alpha : \alpha \leq \kappa\rangle$ such that:

(1) Q_α is countable and dense in $\mathbb{R}$;
(2) $Q_\alpha \cap A_\alpha = \emptyset$;
(3) $\alpha < \beta \leq \kappa \Rightarrow Q_\beta \subseteq^* Q_\alpha$.

Let $\langle B_n : n \in \omega\rangle$ be a one-to-one enumeration of all intervals with rational endpoints. For the construction, let $Q_0 = \mathbb{Q} \setminus A_0$. If Q_α has been constructed, let $Q_{\alpha+1} = Q_\alpha \setminus A_{\alpha+1}$. Suppose α is a limit and $\langle Q_\beta : \beta < \alpha\rangle$ has been constructed. For $\beta < \alpha$ and $n < \omega$ let $Q_{\beta,n} = Q_\beta \cap B_n$. By $\kappa < \mathfrak{t}$, for every $n < \omega$ we may choose $Q_{\alpha,n} \in [\mathbb{Q}]^\omega$ with $\forall\beta < \alpha(Q_{\alpha,n} \subseteq^* Q_{\beta,n})$. Let $\langle q_n : n < \omega\rangle$ be a one-to-one enumeration of $Q_{\alpha,n}$. For $\beta < \alpha$ let $f_\beta \in {}^\omega\omega$ be such that $Q_{\alpha,n} \setminus \{q_n(m) : m \leq f_\beta(n)\} \subseteq Q_{\beta,n}$, for every $n \in \omega$. As $\kappa < \mathfrak{t} \leq \mathfrak{b}$, there exists $f \in {}^\omega\omega$ with $\forall\beta < \alpha(f \geq^* f_\beta)$. Define

$$Q_\alpha = \{q_n(m) : n \in \omega \wedge m > f(n)\} \setminus A_\alpha.$$

It is easy to check that Q_α satisfies (1), (2), (3).

Finally let q be a one-to-one enumeration of Q_κ. For $\alpha < \kappa$ define $g_\alpha \in {}^\omega\omega$ by

$$g_\alpha(n) = \begin{cases} 0 & \text{if } q(n) \in A_\alpha, \\ \min\{m : (q(n) - \frac{1}{m}, q(n) + \frac{1}{m}) \cap A_\alpha = \emptyset\} & \text{otherwise.} \end{cases}$$

Let $g \in {}^\omega\omega$ dominate $\langle g_\alpha : \alpha < \kappa\rangle$ and define

$$D = \bigcap_{m\in\omega}\bigcup_{n>m}(q(n) - 1/g(n), q(n) + 1/g(n)).$$

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It is now easy to check that D is comeagre and $\mathbb{R} \setminus D$ covers $\bigcup\{A_\alpha : \alpha < \kappa\}$. $\square\square$

THEOREM 1.3.3. *Suppose that $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2\rangle$ is a countable support iteration of $\mathbb{A}(\mathbb{M})$ (defined in §1.2), i.e., for every $\beta < \omega_2$, $\dot{Q}_\beta$ is a P_β-name for $\mathbb{A}(\mathbb{M})$ defined in V^{P_β}. If G is P_{ω_2}-generic over V, then $V[G] \models \mathfrak{p} = \omega_1 \wedge \operatorname{add}(m^0) = \omega_2$.*

PROOF. 1) $V[G] \models \operatorname{add}(m^0) = \omega_2$. The proof is similar to that of part 1) of Theorem 1.3.1. Instead of Lemma 1.1.8 we use Lemma 1.2.5.

2) $V[G] \models \mathfrak{p} = \omega_1$. By the same reasoning as in the proof of Theorem 1.3.1, part 2), $V[G] \models \mathfrak{t} = \omega_1$. But the inequality $\mathfrak{p} \leq \mathfrak{t}$ holds trivially in ZFC. $\square$

Question 1.3.4. Is either of the inequalities $\operatorname{add}(m^0) < \operatorname{add}(\ell^0)$ or $\operatorname{add}(\ell^0) < \operatorname{add}(m^0)$ consistent with ZFC?

Concerning this question, it would be interesting to know $\operatorname{add}(m^0)$ in the model for Theorem 1.3.1, and $\operatorname{add}(\ell^0)$ in the one for Theorem 1.3.2. One might be tempted to try to construct an amoeba forcing for $\mathbb{M}$ which does not add a dominating real, and then use $\operatorname{cov}(\ell^0) \leq \mathfrak{b}$ from [GRShSp, Theorem 1.1]. However, there is no hope for this, since by Corollary 2.6 below every reasonable forcing increasing $\operatorname{add}(m^0)$ adds a dominating real.

§2. Upper bounds for $\operatorname{add}(m^0)$ and $\operatorname{cov}(\ell^0)$, $\operatorname{cov}(m^0)$.

First we prove a technical lemma on constructing antichains in $\mathbb{M}$, which will have several applications.

LEMMA 2.1. *Let $\langle p_\alpha : \alpha < \kappa\rangle$ be a family in $\mathbb{M}$ with $\kappa < \mathfrak{d}$, and let $p \in \mathbb{M}$ be such that, for every $\alpha < \kappa$, p and p_α are incompatible. Then there exists $q \in \mathbb{M}$ such that $q \subseteq p$ and, for every $\alpha < \kappa$, $[q] \cap [p_\alpha] = \emptyset$.*

PROOF. Choose $\langle f_\alpha : \alpha < \kappa\rangle$ in ${}^\omega\omega$ such that f_α $\leq^*$-bounds $[p] \cap [p_\alpha]$. For this we use Kechris' theorem in [K] which says that every analytic $\leq^*$-unbounded subset of the Baire space contains the branches of a superperfect tree. Let $(N, \in)$ be an elementary substructure of $(H((2^\mathfrak{c})^+), \in)$ such that $|N| = \kappa$, $f_\alpha \in N$ for every $\alpha < \kappa$, and $p \in N$. By $\mathfrak{d} = \mathfrak{c}$ we may choose $g \in {}^\omega\omega$ such that no $f \in {}^\omega\omega \cap N$ $\leq^*$-bounds g.

We prune p, using g, to obtain q with $\operatorname{Split}(p) \cap q \subseteq \operatorname{Split}(q)$, as follows: $\operatorname{Lev}_0(q) = \{\operatorname{stem}(q)\} = \{\operatorname{stem}(p)\}$. At $\operatorname{stem}(p)$ prune p so that $\operatorname{succ}_{\operatorname{stem}(q)}(q) = \operatorname{succ}_{\operatorname{stem}(p)}(p) \cap [g(0), \infty)$. Then $\operatorname{Lev}_1(q) = \operatorname{Succ}_{\operatorname{stem}(q)}(q)$ is determined, by the requirement $\operatorname{Split}(p) \cap q \subseteq \operatorname{Split}(q)$. Suppose that $\operatorname{Lev}_n(q)$ has been determined and $s \in \operatorname{Lev}_n(q)$. Then prune p at s so that $\operatorname{succ}_s(q) = \operatorname{succ}_s(p) \cap [g(n), \infty)$. Then $\operatorname{Lev}_{n+1}(q)$ is determined.

We claim that for no $x \in [q]$ and $y \in {}^\omega\omega \cap N$ can we have $y \geq x$. Then certainly for no $x \in [q]$ and $\alpha < \kappa$ can we have $x \leq^* f_\alpha$, and since $[q] \subseteq [p]$ we are done. For the proof, suppose there were $x \in [q]$ and $y \in {}^\omega\omega \cap N$ with $y \geq x$. Define $h \in N \cap {}^\omega\omega$ as follows: For any $n \in \omega$, $i \leq n$ and $s \in \operatorname{Lev}_n(p)$, let k_i^s be the length of that initial segment of s which belongs to $\operatorname{Lev}_i(p)$. (So $k_0^s = \ell h(\operatorname{stem}(p))$ always.) Now set

$$h(n) = \max\{s(k_n^s) : s \in \operatorname{Lev}_{n+1}(p) \wedge \forall i < n + 1(s(k_i^s) \leq y(k_i^s))\}.$$

It is not difficult to see that here the maximum is taken over finitely many values, and that hence h is well-defined. Moreover, since h is definable from p and y, we have $h \in N$. But it is easy to see that $g \leq h$, a contradiction. $\square$

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A first consequence of Lemma 2.1 is the following:

COROLLARY 2.2. *Suppose $\mathfrak{d} = \mathfrak{c}$. For every dense $D \subseteq \mathbb{M}$ there exists a maximal antichain $A \subseteq D$ such that for distinct $p, q \in A$ we have $[p] \cap [q] = \emptyset$.*

PROOF. Let $\langle q_\alpha : \alpha < \mathfrak{c}\rangle$ enumerate $\mathbb{M}$. Construct $A = \langle p_\alpha : \alpha < \mathfrak{c}\rangle$ inductively as follows. At step $\kappa < \mathfrak{c}$, if q_κ is compatible with some p_α, $\alpha < \kappa$, then p_κ is undefined. Otherwise, since $\kappa < \mathfrak{d}$, we may apply Lemma 2.1 to obtain $q \leq q_\kappa$ with $[q] \cap [p_\alpha] = \emptyset$ for all $\alpha < \kappa$. Choose $p_\kappa \in D$ below q. Then easily A is as desired. □

We do not know whether Corollary 2.2 can be proved in ZFC.

Using Lemma 2.1, exactly as in [GRShSp, Lemma 2.5] we obtain the following:

COROLLARY 2.3. *Suppose $\mathfrak{d} = \mathfrak{c}$. If $D \subseteq \mathbb{M}$ is open and dense, there exists a maximal antichain $A \subseteq D$ such that for every $p \in \mathbb{M}$, if $[p] \subseteq \bigcup\{[q] : q \in A\}$, then $\{q \in A : q$ is compatible with $p\}$ has size $< \mathfrak{c}$.*

DEFINITION 2.4. Suppose P is a forcing notion. Let

(1) "$\kappa(P)$"= the least cardinal κ such that forcing with P adds a bijection between κ and $\mathfrak{c}$;

(2) "$\lambda(P)$"= the least cardinal κ such that forcing with P adds a function with domain κ which is cofinal in $\mathfrak{c}$.

So clearly $\lambda(P) \leq \kappa(P)$. In [GRShSp, Theorem 2.7] it is proved that $\kappa(\mathbb{L}) \leq \mathfrak{h}$ and $\kappa(\mathbb{M}) \leq \mathfrak{h}$ hold. The following lemma improves [GRShSp, Lemma 2.6]:

LEMMA 2.5. *Suppose $\mathfrak{d} = \mathfrak{c}$. Then* $\mathrm{add}(m^0) \leq \lambda(\mathbb{M})$.

PROOF. Since $\mathrm{non}(m^0) = \mathfrak{c}$, clearly $\mathrm{cov}(m^0) \leq \mathrm{cf}(\mathfrak{c})$. Hence we may assume $\lambda(\mathbb{M}) < \mathrm{cf}(\mathfrak{c})$. Let $\dot{f}$ be an $\mathbb{M}$-name such that $\Vdash_{\mathbb{M}}$ "$\dot{f} : \lambda(\mathbb{M}) \to \mathfrak{c}$ is cofinal". For $\alpha < \lambda(\mathbb{M})$ let $D_\alpha \subseteq \mathbb{M}$ be the open dense set of conditions deciding $\dot{f}(\alpha)$, and let $A_\alpha \subseteq D_\alpha$ be as in Corollary 2.3. Then $X_\alpha = {}^\omega\omega \setminus \bigcup\{[p] : p \in A_\alpha\} \in m^0$. Moreover $X = \bigcup\{X_\alpha : \alpha < \lambda(\mathbb{M})\} \notin m^0$, as otherwise $[p] \cap X = \emptyset$ for some $p \in \mathbb{M}$. But that means $[p] \subseteq \bigcup\{[q] : q \in A_\alpha\}$, for every $\alpha < \lambda(\mathbb{M})$. Then, using the property of the A_α, we could construct a cofinal $f : \lambda(\mathbb{M}) \to \mathfrak{c}$ in V, a contradiction. □

COROLLARY 2.6. $\mathfrak{c} = \omega_2$ *implies* $\mathrm{add}(m^0) \leq \lambda(\mathbb{M})$, *and hence also* $\mathrm{add}(m^0) \leq \mathfrak{h}$.

PROOF. If $\mathfrak{d} = \omega_2$, apply Lemma 2.5 and [GRShSp, 2.7]. If $\mathfrak{d} = \omega_1$, then even $\mathrm{cov}(m^0) = \omega_1$ by [GRShSp, Theorem 1.1]. □

Question 2.7. Is $\mathrm{add}(m^0) \leq \mathfrak{b}$ or even $\mathrm{add}(m^0) \leq \mathfrak{h}$ or even $\mathrm{add}(m^0) \leq \lambda(\mathbb{M})$ provable in ZFC?

By Corollary 2.6, every reasonable forcing which increases $\mathrm{add}(m^0)$ increases $\lambda(\mathbb{M})$, hence $\mathfrak{h}$, and hence adds a dominating function. So it seems technically impossible to answer this question negatively.

COROLLARY 2.8. *Suppose that $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \omega_2, \beta < \omega_2\rangle$ is a countable support iteration of $\mathbb{M}$, i.e., for every $\beta < \omega_2$, $\dot{Q}_\beta$ is a P_β-name for $\mathbb{M}$ defined in V^{P_β}. If G is P_{ω_2}-generic over V, then $V[G] \models add(m^0) = \omega_1 \wedge cov(m^0) = \omega_2$.*

PROOF. 1) $V[G] \models \mathrm{add}(m^0) = \omega_1$. Clearly $V[G] \models \mathfrak{c} = \omega_2$. Hence, by Corollary 2.6, $V[G] \models \mathrm{add}(m^0) \leq \mathfrak{h}$. The inequalities $\mathfrak{h} \leq \mathfrak{s} \leq \mathrm{non}(\mathscr{L})$, where $\mathrm{non}(\mathscr{L})$ is the least size of a nonmeasurable set of reals, are well known. Moreover in [BJSh] it is proved that $V[G] \models \mathrm{non}(\mathscr{L}) = \omega_1$.

2) $V[G] \models \mathrm{cov}(m^0) = \omega_2$. This is a standard Löwenheim–Skolem argument and is entirely similar to [GRShSp, Theorem 2.9]. □

COROLLARY 2.9. $\mathrm{Con}(ZFC) \Rightarrow \mathrm{Con}(ZFC + \mathrm{cov}(\ell^0) < \mathrm{cov}(m^0))$.

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PROOF. Use the model $V[G]$ from 2.8: Miller [Mi, Remark on p. 158] proved that $V[G] \models \mathfrak{b} = \omega_1$. But by [GRShSp, Theorem 1.1], $\mathrm{cov}(\ell^0) \leq \mathfrak{b}$ holds in ZFC. □

It is natural to expect that in Laver's model $\mathrm{cov}(m^0) < \mathrm{cov}(\ell^0)$, since $\mathbb{L}$ does not add an $\mathbb{M}$-generic real.

In [Bl] and [E], Blass and Eisworth have shown that in a model obtained by adding $\aleph_1$ random reals to a model for $\omega_1 < \mathfrak{b} = \mathfrak{d}$ the inequalities $\mathrm{cov}(\ell^0) < \mathfrak{b}$ and $\mathrm{cov}(m^0) < \mathfrak{b}$ hold. We will present another way to obtain these consistencies. First we prove the following theorem, which may be of independent interest. The main idea for its proof stems from the proof of the Kulpa-Szymański theorem in [BPS, p. 15]. Moreover, we again apply Lemma 2.1.

THEOREM 2.10. (1) *If* $\mathfrak{b} = \mathfrak{c}$, *then* $\mathrm{cov}(\ell^0) \leq \mathfrak{h}^+$.

(2) *If* $\mathfrak{d} = \mathfrak{c}$, *then* $\mathrm{cov}(m^0) \leq \mathfrak{h}^+$.

PROOF. We may assume $\mathfrak{h} < \mathfrak{c}$. First we prove (1). In [GRShSp, Theorem 2.7] it has been shown that there exists a family $\langle \mathscr{A}_\alpha : \alpha < \mathfrak{h} \rangle$ of antichains in $\mathbb{L}$ with the property that $\bigcup\{\mathscr{A}_\alpha : \alpha < \mathfrak{h}\}$ is dense in $\mathbb{L}$. By the proof of [GRShSp, Lemma 2.5], it follows that under $\mathfrak{b} = \mathfrak{c}$ every dense open set in $\mathbb{L}$ contains a maximal antichain so that no two members of it contain a common branch. Hence without loss of generality we may assume that, for any $\alpha < \mathfrak{h}$, no two members of $\mathscr{A}_\alpha$ have a common branch.

Next, to each $p \in \bigcup\{\mathscr{A}_\alpha : \alpha < \mathfrak{h}\}$ we can assign a family $\langle p(\nu) : \nu < \mathfrak{h}^+ \rangle$ such that for any $\nu \neq \mu$ we have $p(\nu) \leq p$ and $[p(\nu)] \cap [p(\mu)] = \emptyset$. For this we use the fact that every Laver tree has $\mathfrak{c}$ extensions such that no two of them have a common branch. Define

$$X_\eta = {}^\omega\omega \setminus \bigcup\{[p(\nu)] : \eta < \nu < \mathfrak{h}^+,\ p \in \bigcup\{\mathscr{A}_\alpha : \alpha < \mathfrak{h}\}\}.$$

Then $X_\eta \in \ell^0$, since for given $q \in \mathbb{L}$ there exist $\alpha < \mathfrak{h}$ and $p \in \mathscr{A}_\alpha$ such that $p \leq q$; so clearly $p(\nu) \leq q$ and $[p(\nu)] \cap X_\eta = \emptyset$ for every $\eta < \nu < \mathfrak{h}^+$.

On the other hand, we have $\bigcup\{X_\eta : \eta < \mathfrak{h}^+\} = {}^\omega\omega$. In order to see this, for $x \in {}^\omega\omega$ define

$$\eta_x = \sup\{\nu < \mathfrak{h}^+ : x \in [p(\nu)] \text{ for some } p \in \bigcup\{\mathscr{A}_\alpha : \alpha < \mathfrak{h}\}\}.$$

Since, by construction, for any $\alpha < \mathfrak{h}$ there is at most one pair $\langle p, \nu \rangle \in \mathscr{A}_\alpha \times \mathfrak{h}^+$ such that $x \in [p(\nu)]$, we conclude that $\eta_x < \mathfrak{h}^+$. Then $x \in X_{\eta_x}$ by definition.

The proof of (2) is completely similar, but uses Lemma 2.1 instead of [GRShSp, Lemma 2.5]. □

There exists a natural *ccc* forcing adding a dominating function, called Hechler forcing (see [BD] for its definition). Combining this forcing with Theorem 2.10 we obtain the following corollary.

COROLLARY 2.11. *Suppose* $V \models CH$. *Let* P *be a finite-support iteration of Hechler forcing of length* $\aleph_3$, *and let* G *be* P*-generic over* V. *Then* $\mathrm{cov}(\ell^0) < \mathfrak{b}$ *and* $\mathrm{cov}(m^0) < \mathfrak{b}$ *hold in* $V[G]$.

PROOF. The model $V[G]$ satisfies $\mathfrak{b} = \omega_3$. In [BD] it has been shown that Hechler forcing and also a finite-support iteration of it does not increase $\mathfrak{h}$. Hence

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$V[G] \models \mathfrak{h} = \omega_1$ (here we use $V \models CH$). Hence, by Theorem 2.10, $V[G] \models \mathrm{cov}(\ell^0)$, $\mathrm{cov}(m^0) \leq \omega_2 < \mathfrak{b}$. □

REMARK 2.12. In [Bl] and [E] it has been shown that $\mathrm{cov}(\ell^0)$, $\mathrm{cov}(m^0) \leq \mathfrak{g}$ hold, where $\mathfrak{g}$ is the groupwise density number (see [Bl] for its definition). Using the methods of [BD], it can be shown that the Hechler model satisfies $\mathfrak{g} = \omega_1$; hence it even satisfies $\mathrm{cov}(\ell^0) = \mathrm{cov}(m^0) = \omega_1$.

§3. Cohen reals and superperfect trees.

Let $\mathbb{C} = ({}^{<\omega}\omega, \supseteq)$ be Cohen forcing.

LEMMA 3.1. *Suppose that $\mathscr{T}$ is a $\mathbb{C}$-name for a superperfect tree and π is a $\mathbb{C}$-name for a member of ${}^{\omega}\omega$. Then in V there exists $S \in \mathbb{M}$ such that*

$$(*) \qquad \Vdash_{\mathbb{C}} \text{“}S \text{ and } \mathscr{T} \text{ are compatible and } \pi \notin [S]\text{”}.$$

PROOF. For simplicity we assume $\Vdash_{\mathbb{C}} \mathrm{stem}(\mathscr{T}) = \emptyset$. The general case is similar. Let $\langle p_n : n \in \omega \rangle$ be an enumeration of $\mathbb{C}$ such that for every $s \in {}^{<\omega}\omega$, if

$$A_s = \{n \in \omega : s \subset T(n) \wedge \ell h(T(n)) = \ell h(s) + 1\}$$

(where $T : \omega \to ({}^{<\omega}\omega, \prec)$ is the order-preserving enumeration from the notation section), then $\{p_n : n \in A_s\} = \mathbb{C}$.

Inductively we construct families $\langle s_n : n \in \omega \rangle$, $\langle t_n : n \in \omega \rangle$ in ${}^{<\omega}\omega$ and $\langle q_n : n \in \omega \rangle$, $\langle r_n : n \in \omega \rangle$ in $\mathbb{C}$ such that for every $n \in \omega$ the following hold:

(1) $s_0 = \emptyset$, the map sending s_i to $T(i)$ is an isomorphism between $(\{s_i : i \in \omega\}, \subseteq)$ and $(\mathrm{ran}(T), \subseteq)$, and there is $S \in \mathbb{M}$ with $\mathrm{Split}(S) = \{s_i : i \in \omega\}$.

(2) $\ell h(t_n) > \max\{\ell h(s_i) : i \leq n\}$, $r_n \leq p_n$, and $r_n \Vdash_{\mathbb{C}} t_n \subset \pi$.

(3) Suppose $i \leq n$ is such that $T(i) \subset T(n+1)$ and $\ell h(T(n+1)) = \ell h(T(i)) + 1$; then

 (a) if $p_{n+1} \Vdash_{\mathbb{C}}$ “$s_i \in \mathrm{Split}(\mathscr{T})$”, then $q_{n+1} \leq p_{n+1}$, $q_{n+1} \Vdash_{\mathbb{C}}$ “$s_{n+1} \in \mathrm{Succ}_{s_i}(\mathscr{T})$” and $s_{n+1}(\ell h(s_i)) \notin \{t_j(\ell h(s_i)) : j \leq n \wedge \ell h(t_j) > \ell h(s_i)\} \cup \{s_j(\ell h(s_i)) : j \leq n \wedge \ell h(s_j) > \ell h(s_i)\}$;

 (b) if $\neg(p_{n+1} \Vdash_{\mathbb{C}}$ “$s_i \in \mathrm{Split}(\mathscr{T})$”), then $q_{n+1} = p_{n+1}$ and $s_{n+1} \in {}^{<\omega}\omega$ is arbitrary with $s_i \subset s_{n+1}$ and $s_{n+1}(\ell h(s_i)) \notin \{t_j(\ell h(s_i)) : j \leq n \wedge \ell h(t_j) > \ell h(s_i)\} \cup \{s_j(\ell h(s_i)) : j \leq n \wedge \ell h(s_j) > \ell h(s_i)\}$.

The construction is straightforward: Suppose that $\langle s_i : i \leq n \rangle, \langle t_i : i < n \rangle$, $\langle r_i : i < n \rangle$ and $\langle q_i : i < n \rangle$ have been constructed. Then first choose t_n, r_n according to (2), then choose s_{n+1} and q_n with (3).

Claim 1. $\Vdash_{\mathbb{C}}$ “*S and $\mathscr{T}$ are compatible*”.

PROOF. Note that by construction $\emptyset \in \mathrm{Split}(S)$, and whenever $s \in \mathrm{Split}(S)$ then for every $p \in \mathbb{C}$ with $p \Vdash_{\mathbb{C}}$ “$s \in \mathrm{Split}(\mathscr{T})$” there exists $A \in [\mathrm{Succ}_s(S)]^{\omega}$ such that for every $t \in A$ there exists $p' \in \mathbb{C}$ with $p' \leq p$ and $p' \Vdash_{\mathbb{C}} t \in \mathrm{Split}(\mathscr{T})$.

Now suppose that c is Cohen over V. Work in $V[c]$. By pruning S at each split node we construct $S' \in \mathbb{M}$ with $S' \subseteq S \cap \mathscr{T}[c]$, as follows. First note that $\mathrm{Succ}_{\emptyset}(S) \cap \mathrm{Split}(\mathscr{T}[c])$ is infinite, as otherwise some $p \in \mathbb{C}$ with $p \subset c$ forces “$\mathrm{Succ}_{\emptyset}(S) \cap \mathrm{Split}(\mathscr{T})$ is finite”, and hence without loss of generality p forces “$\mathrm{Succ}_{\emptyset}(S) \cap \mathrm{Split}(\mathscr{T}) = F$” for some finite $F \in V$. But this contradicts the observation above. Next, for every $s \in \mathrm{Succ}_{\emptyset}(S) \cap \mathrm{Split}(\mathscr{T}[c])$ we can repeat this

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argument to show that $\mathrm{Succ}_s(S) \cap \mathrm{Split}(\mathscr{T}[c])$ is infinite. Proceeding similarly, we construct S' as desired. □

Claim 2. $\Vdash_{\mathbb{C}} \pi \notin [S]$.

PROOF. Suppose $p \Vdash_{\mathbb{C}} \pi \in [S]$. Let $p \doteq p_n$. Then $r_n \leq p$ and $r_n \Vdash_{\mathbb{C}} t_n \subset \pi$. Let $i \leq n$ be maximal with $s_i \subseteq t_n$. Then $s_i \subset t_n$ by (2). Moreover by (3), for every $s \in \mathrm{Succ}_{s_i}(S)$, $s(\ell h(s_i)) \neq t_n(\ell h(s_i))$, and hence $t_n \notin S$, a contradiction. □□

REMARK 3.2. By taking a bit more care in the proof of Lemma 3.1 we can even construct a maximal antichain in $\mathbb{M}^V$ (not only a predense set) which is still maximal in $(\mathbb{M})^{V^{\mathbb{C}}}$. Hence we obtain the consistency of the existence of a maximal antichain in $\mathbb{M}$ of size $\aleph_1$ with $\mathrm{cov}(\mathscr{M}) > \omega_1$. This contrasts with the situation for perfect trees in the Cantor space (Sacks forcing) for which Repický (in [R]) has proved that a (nontrivial) maximal antichain has size at least $\mathrm{cov}(\mathscr{M})$.

COROLLARY 3.3. *Suppose G is $\mathbb{C}(\kappa)$-generic over V, where κ is an infinite cardinal, $\mathbb{C}(\kappa)$ is the Cohen algebra for adding κ Cohen reals, and $V \models CH$. Then* $V[G] \models \mathrm{cov}(m^0) = \omega_1$.

PROOF. For a given $\mathbb{C}$-name π for a member of ${}^\omega\omega$ let $D_\pi = \{p \in \mathbb{M} : \Vdash_{\mathbb{C}} \pi \notin [p]\}$. By Lemma 3.1, D_π is predense in $\mathbb{M}$ and in $(\mathbb{M})^{V^{\mathbb{C}}}$. Let $X_\pi = {}^\omega\omega \setminus \bigcup\{[p] : p \in D_\pi\}$. We conclude that

$$\Vdash_{\mathbb{C}} X_\pi \in m^0 \wedge \pi \in X_\pi .$$

Note that here the meaning of X_π is ambiguous. It is a definition of a set and its evaluation. By the CH there are only $\aleph_1$ many $\mathbb{C}$-names for members of ${}^\omega\omega$ in V. Hence in V we can build $\langle X_\alpha : \alpha < \omega_1 \rangle$ such that

$$\Vdash_{\mathbb{C}} \forall \alpha < \omega_1 (X_\alpha \in m^0) \wedge \bigcup\{X_\alpha : \alpha < \omega_1\} = \mathbb{R}.$$

By standard arguments on the Cohen algebra, this suffices. □

REMARK 3.4. Some time ago Jörg Brendle wrote me that he had proved Corollary 3.3. He did not tell me the proof, so I thought about it and proved Lemma 3.1. Later when we discussed our proofs we found that they were essentially different. Brendle did not know that everything can be done in the ground model. He constructs his $\aleph_1$ dense sets step by step, extending them each time a new Cohen real is added.

We add another result which has a similar spirit to Lemma 3.1. It is well known that one Cohen real adds a perfect tree such that every branch of it is a Cohen real. The reason is that Cohen forcing is equivalent to forcing with trees, say in ${}^{<\omega}2$, of finite height, ordered by: $p \leq q$ if and only if $p \supseteq q$ and every new split node of p extends a terminal node of q. It is also well known (apply the Ulam–Kuratowski theorem) that after adding one Cohen real, the set of all Cohen reals is not meagre. Hence if we think of Cohen forcing as ${}^{<\omega}\omega$, then it is clear that one Cohen real adds a $\leq^*$–unbounded set of Cohen reals. So one might wonder whether it adds even a superperfect tree of Cohen reals. The next result gives a negative answer.

THEOREM 3.5. *Cohen forcing does not add a superperfect tree of Cohen reals.*

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PROOF. Suppose that $\mathscr{T}$ is a $\mathbb{C}$-name for a superperfect tree. We shall construct a predense set $\{a_n : n \in \omega\} \subseteq \mathbb{C}$ and a $\mathbb{C}$-name π for a member of ${}^{\omega}\omega$ such that

$$\Vdash_{\mathbb{C}} \pi \in [\mathscr{T}] \wedge \pi \notin \bigcup\{[a_n] : n \in \omega\}.$$

This will certainly suffice.

For simplicity we assume $\Vdash_{\mathbb{C}}$ "$\mathscr{T} \in \mathbb{M} \wedge \operatorname{stem}(\mathscr{T}) = \emptyset$". Let $\langle p_n : n \in \omega\rangle$ enumerate $\mathbb{C}$. Inductively we shall construct sequences $\langle a_n : n \in \omega\rangle$, $\langle s_n : n \in \omega\rangle$ and $\langle q_n : n \in \omega\rangle$ in ${}^{<\omega}\omega = \mathbb{C}$ such that for every $m, n \in \omega$ the following five clauses hold:

(1) $a_n < p_n$ and $q_n \leq p_n$;
(2) $q_n < q_m \Rightarrow s_m \subset s_n$, and $q_n \Vdash_{\mathbb{C}} s_n \in \operatorname{Split}(\mathscr{T})$;
(3) $n < m \Rightarrow s_n, a_m$ are incompatible, or $s_n \subset a_m$;
(4) $n \leq m \wedge \forall i < m(s_m \neq s_i) \Rightarrow a_n, s_m$ are incompatible;
(5) $n < m \Rightarrow q_m \leq q_n$ or q_n, q_m are incompatible;

If this construction succeeds, then it is not difficult to see that letting

$$\pi = \bigcup\{s_n : q_n \subseteq \dot{c}\},$$

where $\dot{c}$ is a name for the Cohen real, π is a $\mathbb{C}$-name for a member of ${}^{\omega}\omega$ such that $\Vdash_{\mathbb{C}}$ "$\pi \in [\mathscr{T}]$", and that $\{a_n : n \in \omega\}$ is predense. Moreover, suppose that for some $p \in \mathbb{C}$ we had $p \Vdash_{\mathbb{C}}$ "$\pi \in \bigcup\{[a_n] : n \in \omega\}$". Hence there exist $n, m \in \omega$ such that $p = p_n$ and $p \Vdash_{\mathbb{C}} a_m \subset \pi$. By (1) and (2) there exists $n' \geq \max\{n, m\}$ such that $q_{n'} \leq p$ and $s_{n'} \neq s_i$ for every $i < n'$. But then $q_{n'} \Vdash$ "$s_{n'} \subset \pi$" by (2), and $s_{n'}, a_m$ are incompatible by (4). This contradicts $p \Vdash_{\mathbb{C}} a_m \subset \pi$.

For the construction, suppose that $\langle a_i : i \in n\rangle$, $\langle q_i : i \in n\rangle$, $\langle s_i : i \in n\rangle$ have been constructed such that (1)–(5) hold. First choose $a_n < p_n$ so long that for no $i < n$ do we have $a_n \subseteq s_i$. In order to construct q_n and s_n we distinguish the following three cases:

Case 1. For some $i < n$, $q_i \leq p_n$. Then let i_0 be the maximal such i, and set $q_n = q_{i_0}$ and $s_n = s_{i_0}$. Clearly (1)–(5) hold up to n.

Case 2. Not Case 1, and for some $i < n$, $q_i > p_n$. Then let i_0 be the maximal such i. By the induction hypothesis we know that $p_n \Vdash_{\mathbb{C}} s_{i_0} \in \operatorname{Split}(\mathscr{T})$. So certainly we may find $q_n \leq p_n$ and $s_n \supset s_{i_0}$ such that $q_n \Vdash_{\mathbb{C}} s_n \in \operatorname{Split}(\mathscr{T})$ and, for every $i \leq n$, if $\ell h(a_i) > \ell h(s_{i_0})$, then $s_n(\ell h(s_{i_0})) \neq a_i(\ell h(s_{i_0}))$. By the induction hypothesis for (3) and (4) we conclude that whenever a_i, where $i \leq n$, does not extend s_{i_0}, then s_{i_0}, a_i are incompatible. Consequently (1)–(5) hold up to n.

Case 3. For every $i < n$, p_n, and q_i are incompatible. Then since $\Vdash_{\mathbb{C}} \emptyset \in \operatorname{Split}(\mathscr{T})$, we may find $q_n \leq p_n$ and s_n such that $q_n \Vdash_{\mathbb{C}} s_n \in \operatorname{Succ}_{\emptyset}(\mathscr{T})$, and for every $i \leq n$ we have $s_n(0) \neq a_i(0)$. Then clearly (1)–(5) hold up to n. □

REMARK 3.6. In an earlier version of this paper I gave it as a problem to decide whether a superperfect tree of Cohen reals implies a dominating real. In [Br], Brendle proves that the answer is yes.

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