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Torsion classes in the cohomology of congruence subgroups

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Introduction

For any prime number p, let $\Gamma_{n,p}$ denote the *congruence subgroup* of $SL_n(\mathbb{Z})$ of level p, i.e. the kernel of the surjective homomorphism $f_p: SL_n(\mathbb{Z}) \to SL_n(\mathbb{F}_p)$ induced by the reduction mod p ($\mathbb{F}_p$ is the field with p elements). We define

$$\Gamma_p := \varinjlim_n \Gamma_{n,p}$$

using upper left inclusions $\Gamma_{n,p} \hookrightarrow \Gamma_{n+1,p}$. Recall that the groups $\Gamma_{n,p}$ are homology stable with M-coefficients, for instance if $M = \mathbb{Q}$, $\mathbb{Z}[1/p]$, or $\mathbb{Z}/q$ with q prime and $q \neq p$: $H_i(\Gamma_{n,p}; M) \cong H_i(\Gamma_p; M)$ for $n \geqslant 2i+5$ from [7] (but the homology stability fails if $M = \mathbb{Z}$ or $\mathbb{Z}/p$).

If p is an odd prime, then the group Γ_p is torsion-free. The main objective of this paper is to detect torsion classes in the integral cohomology of Γ_p. It is actually of general interest to provide examples of torsion-free groups having torsion in their integral cohomology; this is called *strange torsion* in [**16**]. Let us mention that $H_1(\Gamma_{n,p}; \mathbb{Z})$ has been computed for $n \geqslant 3$ in [**10**]: it is a p-group (but does not stabilize).

In order to examine the cohomology of the congruence subgroups, our method is based on the study of the corresponding problem on the space level. As usual we denote by $BSL(\mathbb{Z})^+$ and $BSL(\mathbb{F}_p)^+$ the spaces obtained by performing the plus construction on the classifying spaces of

$$SL(\mathbb{Z}) = \varinjlim_n SL_n(\mathbb{Z}) \quad \text{and} \quad SL(\mathbb{F}_p)$$

respectively; notice that these spaces have the same (co)homology as the corresponding groups. The reduction mod p induces a map $h_p: BSL(\mathbb{Z})^+ \to BSL(\mathbb{F}_p)^+$, and we call $F(p)$ its fibre and i_p the inclusion map $F(p) \hookrightarrow BSL(\mathbb{Z})^+$. Consider the commutative diagram

$$\begin{array}{ccccc}
B\Gamma_p & \longrightarrow & BSL(\mathbb{Z}) & \xrightarrow{Bf_p} & BSL(\mathbb{F}_p) \\
\downarrow & & \downarrow + & & \downarrow + \\
F(p) & \xrightarrow{i_p} & BSL(\mathbb{Z})^+ & \xrightarrow{h_p} & BSL(\mathbb{F}_p)^+.
\end{array}$$

We proved in lemma 1·2 and corollary 1·3 of [**2**] that the map $B\Gamma_p \to F(p)$ induces an isomorphism on (co)homology with M-coefficients if M is again $\mathbb{Q}$, $\mathbb{Z}[1/p]$, or $\mathbb{Z}/q$

with $q \neq p$ (this is of course also true for the plus construction). In particular, the cohomological restriction $H^*(SL(\mathbb{Z}); M) \to H^*(\Gamma_p; M)$ may be identified with the homomorphism $i_p^*: H^*(BSL(\mathbb{Z})^+; M) \to H^*(F(p); M)$.

Our main result is given in the first section of the paper, which is devoted to mod 2 cohomology. If p is a prime and $p \equiv 3 \bmod 4$, we are able to compute the algebra $H^*(\Gamma_p; \mathbb{Z}/2)$. We prove for instance that the even-dimensional Stiefel–Whitney classes in $H^*(SL(\mathbb{Z}); \mathbb{Z}/2)$ do not become trivial when restricted to Γ_p, for all primes $p \equiv 3 \bmod 4$; see [**11**] for a more general discussion of elements in the cohomology of an arithmetic group which are not killed by passage to subgroups of finite index. Our calculation produces many 2-torsion classes in the integral cohomology of Γ_p.

In the second section we introduce a general argument which shows that for infinite loop spaces the Hurewicz homomorphism is injective on q-torsion elements if q is a sufficiently large prime (in comparison with the dimension we are looking at). The q-torsion classes discovered in the algebraic K-theory of $\mathbb{Z}$ in [**13**] (where q is related to the numerator of the Bernoulli numbers) survive in the homotopy groups of $F(p)$, and consequently also in the integral (co)homology of Γ_p via the Hurewicz homomorphism, assuming $p \neq q$.

1. *Mod 2 cohomology*

In this section we determine the restriction homomorphism $H^*(SL(\mathbb{Z}); \mathbb{Z}/2) \to H^*(\Gamma_p; \mathbb{Z}/2)$ for all prime numbers $p \equiv 3 \bmod 4$. Let $GL(\mathbb{Z})$ be the infinite general linear group of $\mathbb{Z}$ and $w_i \in H^i(GL(\mathbb{Z}); \mathbb{Z}/2)$ the ith Stiefel–Whitney class of the inclusion $GL(\mathbb{Z}) \hookrightarrow GL(\mathbb{R})$ for $i \geqslant 1$; we shall also denote by w_i the image of w_i under the restriction $H^i(GL(\mathbb{Z}); \mathbb{Z}/2) \to H^i(SL(\mathbb{Z}); \mathbb{Z}/2)$ for $i > 1$ (the first Stiefel–Whitney class of $SL(\mathbb{Z})$ is zero). Let Q_n denote the subgroup of diagonal matrices in $GL_n(\mathbb{Z})$; notice that $Q_n \cong (\mathbb{Z}/2)^n$ and $H^*(Q_n; \mathbb{Z}/2) = \mathbb{Z}/2[v_1, v_2, \ldots, v_n]$, where $\deg v_j = 1$ for $1 \leqslant j \leqslant n$. As usual, write σ_i for the ith elementary symmetric function of $v_1, v_2, \ldots, v_n$. Now let us consider the inclusion $\phi_n: Q_n \hookrightarrow GL_n(\mathbb{Z}) \hookrightarrow GL(\mathbb{Z})$ and the induced homomorphism $\phi_n^*: H^*(GL(\mathbb{Z}); \mathbb{Z}/2) \to H^*(Q_n; \mathbb{Z}/2)$.

LEMMA 1·1. (*a*) *For any element* $\alpha \in H^*(GL(\mathbb{Z}); \mathbb{Z}/2)$ *and for each* $n \geqslant 1$, $\phi_n^*(\alpha)$ *is a polynomial in* $\sigma_1, \sigma_2, \ldots, \sigma_n$; (*b*) $\phi_n^*(w_i) = \sigma_i$ *for* $i \geqslant 1$, $n \geqslant i$.

Proof. The action of the symmetric group S_n on $H^*(Q_n; \mathbb{Z}/2)$ is induced by conjugation by an element of $GL_n(\mathbb{Z})$, therefore it is trivial on the image of ϕ_n^*. This shows (*a*) and assertion (*b*) is proved in theorem 22·7 of [**5**].

PROPOSITION 1·2. *There is a commutative graded algebra A such that*

(*a*) $H^*(GL(\mathbb{Z}); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_1, w_2, w_3, \ldots] \otimes A$,

(*b*) $H^*(SL(\mathbb{Z}); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3, w_4, \ldots] \otimes A$.

Proof. It is obvious that the classes w_i, for $i \geqslant 1$, are non-trivial and algebraically independent in $H^*(GL(\mathbb{Z}); \mathbb{Z}/2)$ by assertion (*b*) of the previous lemma. On the other hand, $H^*(GL(\mathbb{Z}); \mathbb{Z}/2)$ is the cohomology of the H-space $BGL(\mathbb{Z})^+$ and thus may be written in the following form:

$$H^*(GL(\mathbb{Z}); \mathbb{Z}/2) = \bigotimes_{j=1}^{\infty} B_j,$$

where for each j, B_j is either $\mathbb{Z}/2[x_j]$ or $\mathbb{Z}/2[x_j]/(x_j^{e_j}=0)$, where e_j is a power of 2. It is now sufficient to prove that the w_i's are not decomposable in $H^*(GL(\mathbb{Z});\mathbb{Z}/2)$ (we then may replace some of the x_j's by the w_i's and deduce (*a*)). But the assumption that w_i is decomposable would imply, after applying the homomorphism ϕ_n^* for some $n \geqslant i$, that σ_i is a polynomial in the σ_j's with $j < i$; this is not the case.

The second part is immediate because $BSL(\mathbb{Z})^+$ is the fundamental cover of $BGL(\mathbb{Z})^+$ and $BGL(\mathbb{Z})^+ \simeq BSL(\mathbb{Z})^+ \times B\mathbb{Z}/2$.

Remark 1·3. It has been conjectured, in relation to the Quillen–Lichtenbaum conjecture, that A is an exterior algebra $\Lambda(u_3, u_5, \ldots, u_{2k+1}, \ldots)$ where $\deg u_{2k+1} = 2k+1$ (cf. [**8**], corollary 4·3). Observe that, if Z denotes the fibre of the obvious infinite loop map $\psi: BSL(\mathbb{Z})^+ \to BSO$, the Serre spectral sequence of the fibration $Z \to BSL(\mathbb{Z})^+ \xrightarrow{\psi} BSO$ collapses because $\psi^*: H^*(BSO;\mathbb{Z}/2) \to H^*(BSL(\mathbb{Z})^+;\mathbb{Z}/2)$ is injective (cf. [**5**], theorem 15·2) and one can conclude that $A \cong H^*(Z;\mathbb{Z}/2)$. But the study of the Eilenberg–Moore spectral sequence (with rational coefficients) of this fibration shows that the rational cohomology of Z is an exterior algebra with one generator in each odd dimension $\geqslant 3$. This exhibits non-trivial elements u_{2k+1} in A for all $k \geqslant 1$.

The mod 2 cohomology algebra of the infinite general linear group $GL(\mathbb{F}_p)$ (p odd) is known (cf. [**12**], or § IV·8 of [**9**]): it is generated by classes c_i and e_i for $i \geqslant 1$, where $\deg c_i = 2i$ and $\deg e_i = 2i-1$. If p is a prime $\equiv 3 \bmod 4$, one has the relations

$$e_i^2 = \sum_{0 \leqslant j < i} c_j c_{2i-1-j} \qquad (*)$$

and the mod 2 cohomology algebra is polynomial:

$$H^*(GL(\mathbb{F}_p);\mathbb{Z}/2) = \mathbb{Z}/2[e_1, e_2, e_3, \ldots, c_2, c_4, c_6, \ldots].$$

We write also c_i (respectively e_i) for the restriction of c_i (resp. e_i) to $SL(\mathbb{F}_p)$:

$$H^*(SL(\mathbb{F}_p);\mathbb{Z}/2) = \mathbb{Z}/2[e_2, e_3, e_4, \ldots, c_2, c_4, c_6, \ldots].$$

LEMMA 1·4. *Let p be an odd prime, $g_p: GL(\mathbb{Z}) \to GL(\mathbb{F}_p)$ the homomorphism induced by reduction* mod p *and $g_p^*: H^*(GL(\mathbb{F}_p);\mathbb{Z}/2) \to H^*(GL(\mathbb{Z});\mathbb{Z}/2)$ the induced homomorphism. Then $g_p^*(c_i) = w_i^2$ for $i \geqslant 1$.*

Proof. Recall that the cohomology class c_i comes from the reduction mod 2 of the ith universal Chern class in the cohomology of BU, via the Brauer lift. According to [**4**], $g_p^*(c_i)$ is then the reduction mod 2 of the ith Chern class of the inclusion $GL(\mathbb{Z}) \hookrightarrow GL(\mathbb{C})$; but this is w_i^2 by theorem 5·3 of [**15**] or proposition 25·6 of [**5**].

LEMMA 1·5. *If $p \equiv 3 \bmod 4$ and $i \geqslant 1$, then*

$$g_p^*(e_i) = \sum_{0 \leqslant j < i} w_j w_{2i-1-j} + \gamma_i,$$

where $\phi_n^*(\gamma_i) = 0$ for $n \geqslant 2i-1$.

Proof. We deduce from the relation (*) and the previous lemma that

$$(g_p^*(e_i))^2 = \sum_{0 \leqslant j < i} w_j^2 w_{2i-1-j}^2,$$

and consequently that

$$(\phi_n^*(g_p^*(e_i)))^2 = (\sum_{0\leqslant j<i} \sigma_j \sigma_{2i-1-j})^2.$$

Since $\phi_n^*(g_p^*(e_i))$ is a polynomial in $\sigma_1, \sigma_2, \ldots, \sigma_n$ by Lemma 1·1, $\phi_n^*(g_p^*(e_i))$ must be $\Sigma_{0\leqslant j<i}\sigma_j\sigma_{2i-1-j}$. The proof is then complete because

$$\phi_n^*(g_p^*(e_i)) = \phi_n^*(\sum_{0\leqslant j<i} w_j w_{2i-1-j}).$$

Remark 1·6. The fact that $\phi_n^*(\gamma_i) = 0$ implies that $\gamma_i \notin \mathbb{Z}/2[w_1, w_2, w_3, \ldots]$. It follows in particular from Lemma 1·5 that $g_p^*(e_i) \neq 0$.

PROPOSITION 1·7. *Let p be a prime congruent to* 3 mod 4. *Then the homomorphisms* $g_p^*: H^*(GL(\mathbb{F}_p); \mathbb{Z}/2) \rightarrow H^*(GL(\mathbb{Z}); \mathbb{Z}/2)$ *and* $f_p^*: H^*(SL(\mathbb{F}_p); \mathbb{Z}/2) \rightarrow H^*(SL(\mathbb{Z}); \mathbb{Z}/2)$, *induced by reduction* mod p, *are injective.*

Proof. Recall that $H^*(GL(\mathbb{F}_p); \mathbb{Z}/2) = \mathbb{Z}/2[e_1, e_2, e_3, \ldots, c_2, c_4, c_6, \ldots]$. We have seen that $g_p^*(c_i)$ and $g_p^*(e_i)$ are non-trivial in $H^*(GL(\mathbb{Z}); \mathbb{Z}/2)$ for $i \geqslant 1$. In order to establish the injectivity of g_p^*, we must verify that there are no polynomial relations between the elements $g_p^*(c_i)$ (i even, $i \geqslant 2$), $g_p^*(e_i)$ ($i \geqslant 1$). But this follows after applying the homomorphism ϕ_n^* (with n large enough), because we know that

$$\phi_n^*(g_p^*(c_i)) = \sigma_i^2 \quad \text{and} \quad \phi_n^*(g_p^*(e_i)) = \sum_{0\leqslant j<i} \sigma_j \sigma_{2i-1-j}$$

(and that there are no polynomial relations among the σ_j's). A similar proof provides the injectivity of f_p^*.

Remark 1·8. This argument is not valid if $p \equiv 1$ mod 4; the assertion of Proposition 1·7 is actually wrong for primes $p \equiv 1$ mod 8 (see [**1**], proposition 1).

Remember that $H^*(SL(\mathbb{Z}); \mathbb{Z}/2) \cong \mathbb{Z}/2[w_2, w_3, w_4, \ldots] \otimes A$ and write $w_i(\Gamma_p)$ for the image of w_i under the restriction homomorphism $H^i(SL(\mathbb{Z}); \mathbb{Z}/2) \rightarrow H^i(\Gamma_p; \mathbb{Z}/2)$, $i \geqslant 2$.

THEOREM 1·9. *If p is a prime congruent to* 3 mod 4, *then the restriction homomorphism* $H^*(SL(\mathbb{Z}); \mathbb{Z}/2) \rightarrow H^*(\Gamma_p; \mathbb{Z}/2)$ *is surjective and its image is*

$$H^*(\Gamma_p; \mathbb{Z}/2) \cong \Lambda(w_2(\Gamma_p), w_4(\Gamma_p), \ldots, w_{2k}(\Gamma_p), \ldots) \otimes A.$$

Proof. We actually work on the space level. In the fibration

$$F(p) \xrightarrow{i_p} BSL(\mathbb{Z})^+ \xrightarrow{h_p} BSL(\mathbb{F}_p)^+,$$

observe that all spaces are infinite loop spaces and all maps are infinite loop maps. Therefore, since

$$h_p^*: H^*(BSL(\mathbb{F}_p)^+; \mathbb{Z}/2) \rightarrow H^*(BSL(\mathbb{Z})^+; \mathbb{Z}/2)$$

is injective by the previous proposition, the homomorphism

$$i_p^*: H^*(BSL(\mathbb{Z})^+; \mathbb{Z}/2) \rightarrow H^*(F(p); \mathbb{Z}/2)$$

is surjective and the Serre spectral sequence collapses:

$$E_\infty = E_2 \cong H^*(BSL(\mathbb{F}_p)^+; \mathbb{Z}/2) \otimes H^*(F(p); \mathbb{Z}/2)$$

(cf. [**5**], theorem 15·2). It remains to examine the kernel of i_p^*.

Let i be an integer with $i \geqslant 2$ and K_i the subgroup of $H^i(BSL(\mathbb{Z})^+; \mathbb{Z}/2)$ generated by elements of the form xy, where x is of positive degree in the image of h_p^* and $y \in H^*(BSL(\mathbb{Z})^+; \mathbb{Z}/2)$. Clearly the kernel of $i_p^*: H^i(BSL(\mathbb{Z})^+; \mathbb{Z}/2) \to H^i(F(p); \mathbb{Z}/2)$ contains K_i; we want to show that it is exactly K_i by checking that the dimension of this kernel (as $\mathbb{Z}/2$-vector space), i.e.

$$\dim \bigoplus_{j=1}^{i} H^j(BSL(\mathbb{F}_p)^+; \mathbb{Z}/2) \otimes H^{i-j}(F(p); \mathbb{Z}/2),$$

is less than or equal to the dimension of K_i. Take a basis of

$$\bigoplus_{j=1}^{i} H^j(BSL(\mathbb{F}_p)^+; \mathbb{Z}/2) \otimes H^{i-j}(F(p); \mathbb{Z}/2)$$

and associate to each element $\zeta \otimes \eta$ of this basis an element xy of K_i as follows: $x := h_p^*(\zeta)$ and y is a cohomology class in $H^*(BSL(\mathbb{Z})^+; \mathbb{Z}/2)$ satisfying $i_p^*(y) = \eta$; we choose y such that it is a sum of elements of the form $w \otimes a$, where w is a square-free polynomial in the even-dimensional Stiefel–Whitney classes and $a \in A$ (since i_p^* vanishes on the image of h_p^*, Lemmas 1·4 and 1·5 prove inductively that this is possible). With this choice it is not hard to verify that these associated elements are linearly independent in K_i.

Consequently

$$\begin{aligned} H^*(F(p); \mathbb{Z}/2) &\cong H^*(BSL(\mathbb{Z})^+; \mathbb{Z}/2)/\mathrm{Ker}\, i_p^* \\ &\cong H^*(BSL(\mathbb{Z})^+; \mathbb{Z}/2)/(\mathrm{Im}\, h_p^*) \\ &\cong \Lambda(w_2, w_4, \ldots, w_{2k}, \ldots) \otimes A. \end{aligned}$$

In particular, i_p^* does not vanish on even-dimensional Stiefel–Whitney classes or on elements of A.

Corollary 1·10. *If $p \equiv 3 \bmod 4$, then $H^i(\Gamma_p; \mathbb{Z}/2) \neq 0$ for all $i \geqslant 2$.*

Proof. The algebra A contains a non-trivial element u_3 of degree 3 (cf. Remark 1·3). For $i \geqslant 2$, define $\omega_i \in H^i(\Gamma_p; \mathbb{Z}/2)$ to be $w_i(\Gamma_p)$ if i is even and $w_{i-3}(\Gamma_p) \otimes u_3$ if i is odd; the previous theorem asserts that $\omega_i \neq 0$. Notice that since $F(p)$ is simply connected $H^1(\Gamma_p; \mathbb{Z}/2) = H^1(F(p); \mathbb{Z}/2) = 0$.

We close this section by mentioning that the non-vanishing of the even-dimensional Stiefel–Whitney classes of Γ_p (for $p \equiv 3 \bmod 4$) produces 2-torsion classes in its integral cohomology groups in arbitrarily large dimensions. (According to the definition given in [**16**], Γ_p has *very strange 2-torsion*.)

Corollary 1·11. *Let p be a prime congruent to $3 \bmod 4$ and i an even integer $\geqslant 2$. Then $w_i(\Gamma_p)$ detects 2-torsion in $H^i(\Gamma_p; \mathbb{Z})$ or in $H^{i+1}(\Gamma_p; \mathbb{Z})$.*

Proof. The Stiefel–Whitney class $w_i(\Gamma_p)$ is not the image of an element of infinite order under the reduction mod 2, because i is even and the rational cohomology of Γ_p is an exterior algebra generated by classes of degree $5, 9, \ldots, 4k+1, \ldots$ (see [**2**], theorem 1·4).

2. *Spherical classes in the homology of infinite loop spaces*

The purpose of this section is to prove the following

THEOREM 2·1. *Let X be an m-connected infinite loop space (where $m \geqslant 0$), i an integer greater than m and q a prime greater than $(i-m)/2+1$. If $\pi_i X$ contains an element α of order q^r (with $r \geqslant 1$), then the image of α under the Hurewicz homomorphism is also of order q^r in $H_i(X;\mathbb{Z})$.*

This will be useful for the study of odd torsion classes in the (co)homology of congruence subgroups. The proof of Theorem 2·1 is based on the next result which follows from the discussion of the k-invariants of iterated loop spaces: see [3].

PROPOSITION 2·2. *There exist positive integers S_j $(j \geqslant 1)$ with the following property: if X is an m-connected infinite loop space (where $m \geqslant 0$ and i an integer greater than m, then there is a map $f\colon X \to K(\pi_i X, i)$ such that the induced homomorphism $f_*\colon \pi_i X \to \pi_i X$ is multiplication by a divisor of S_{i-m}. (These integers S_j are defined in [3]; a prime number q divides S_j if and only if $q \leqslant j/2+1$.)*

Proof of Theorem 2·1. Let us look at the commutative diagram induced by the map f introduced in the previous proposition:

$$\begin{array}{ccc} \pi_i X & \xrightarrow{f_*} & \pi_i K(\pi_i X, i) \\ \Big\downarrow \mathrm{Hu} & & \wr\Big\downarrow \mathrm{Hu} \\ H_i(X;\mathbb{Z}) & \xrightarrow{f_*} & H_i(K(\pi_i X, i);\mathbb{Z}) \end{array}$$

(Hu denotes the Hurewicz homomorphism). If α is an element of order q^r in $\pi_i X$, $f_* \circ \mathrm{Hu}(\alpha)$ is again of order q^r by Proposition 2·2. With this argument we may conclude in fact that if α generates a cyclic direct summand of order q^r in $\pi_i X$, then the same is true for $\mathrm{Hu}(\alpha)$ in $H_i(X;\mathbb{Z})$.

We obtain also information on the Pontryagin ring structure of $H_*(X;\mathbb{Z})$.

COROLLARY 2·3. *Let X be an m-connected infinite loop space (where $m \geqslant 0$), i an even integer greater than m and q a prime greater than $(i-m)/2+1$. Assume that $\pi_i X$ contains a cyclic direct summand of order q^r (with $r \geqslant 1$) generated by α and define $\beta := \mathrm{Hu}(\alpha) \in H_i(X;\mathbb{Z})$. Then β^k is a non-trivial q-torsion element of $H_{ki}(X;\mathbb{Z})$ for each k such that $((k-1)!)_q < q^r$. (Here $(\;)_q$ denotes the q-primary part.)*

Proof. Consider the composition

$$\lambda\colon H_*(X;\mathbb{Z}) \xrightarrow{f_*} H_*(K(\pi_i X, i);\mathbb{Z}) \to H_*(K(\mathbb{Z}/q^r, i);\mathbb{Z})$$

where the second arrow is induced by the projection $\pi_i X \to \mathbb{Z}/q^r$; λ is a ring homomorphism since it is possible to choose a loop map for f. Define $\gamma := \lambda(\beta)$: it is a generator of $H_i(K(\mathbb{Z}/q^r, i);\mathbb{Z})$ by Theorem 2·1. The ring structure of $H_*(K(\mathbb{Z}/q^r, i);\mathbb{Z})$ is known (see [6]) and provides an interesting conclusion if i is even: γ^k is of order $(kq^r)_q/(k!)_q$. Thus $\gamma^k \neq 0$ if $q^r > ((k-1)!)_q$ and the non-vanishing of β^k follows from $\lambda(\beta^k) = \gamma^k$.

Remark 2·4. If X is an m-connected s-fold loop space (where $m \geqslant 0$, $s \geqslant 0$), then the above results remain valid if i satisfies $m < i \leqslant s+2m$ (cf. [3]).

The next assertion on the infinite loop space $BSL(\mathbb{Z})^+$ is given on page 290 in [13]: let i be an even integer, q a properly irregular prime with $q > i$ and such that q divides the numerator of B_i/i (where B_i is the ith Bernoulli number: $B_2 = 1/6$, $B_4 = 1/30$, ...); then there exists q-torsion in $K_{2i-2}\mathbb{Z} = \pi_{2i-2}BSL(\mathbb{Z})^+$ and in $H_{2i-2}(BSL(\mathbb{Z})^+; \mathbb{Z})$. Notice that the localization exact sequence in algebraic K-theory ([14], théorème 1) shows that this q-torsion survives into $K_{2i-2}\mathbb{Q}$ and therefore into $H_{2i-2}(BSL(\mathbb{Q})^+; \mathbb{Z})$ according to Theorem 2·1. Observe also that Corollary 2·3 implies actually the existence of q-torsion in $H_{k(2i-2)}(BSL(\mathbb{Z})^+; \mathbb{Z})$ for $1 \leqslant k \leqslant q$. It is interesting to remark that the corresponding q-torsion classes in $H^{k(2i-2)+1}(BSL(\mathbb{Z})^+; \mathbb{Z})$ do not come from the cohomology of BU via the homomorphism induced by a complex representation of $SL(\mathbb{Z})$, because $H^j(BU; \mathbb{Z}) = 0$ if j is odd.

Now let us look at the homotopy sequence of the fibration $F(p) \to BSL(\mathbb{Z})^+ \to BSL(\mathbb{F}_p)^+$:

$$\ldots \to \pi_{2i-2}F(p) \to K_{2i-2}\mathbb{Z} \to K_{2i-2}\mathbb{F}_p \to \ldots.$$

Since $K_{2i-2}\mathbb{F}_p = 0$ from [12], the group $\pi_{2i-2}F(p)$ (which is finitely generated) also contains q-torsion. We apply Corollary 2·3 to the infinite loop space $F(p)$ and detect q-torsion in $H_{k(2i-2)}(F(p); \mathbb{Z})$ for $1 \leqslant k \leqslant q$. If we suppose $p \neq q$, we obtain q-torsion in the integral (co)homology of Γ_p because $H_*(\Gamma_p; \mathbb{Z}[1/p]) \cong H_*(F(p); \mathbb{Z}[1/p])$. We obtain

THEOREM 2·5. *Let i be an even integer, q a properly irregular prime with $q > i$ and such that q divides the numerator of B_i/i. Then there exists q-torsion in $H_{k(2i-2)}(\Gamma_p; \mathbb{Z})$ for $1 \leqslant k \leqslant q$ and all primes $p \neq q$.*

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