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# Genus 0 characteristic numbers of the tropical projective plane

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Abstract

Finding the so-called characteristic numbers of the complex projective plane $\mathbb{C}P^2$ is a classical problem of enumerative geometry posed by Zeuthen more than a century ago. For a given $d$ and $g$ one has to find the number of degree $d$ genus $g$ curves that pass through a certain generic configuration of points and at the same time are tangent to a certain generic configuration of lines. The total number of points and lines in these two configurations is $3d-1+g$ so that the answer is a finite integer number. In this paper we translate this classical problem to the corresponding enumerative problem of tropical geometry in the case when $g=0$. Namely, we show that the tropical problem is well posed and establish a special case of the correspondence theorem that ensures that the corresponding tropical and classical numbers coincide. Then we use the floor diagram calculus to reduce the problem to pure combinatorics. As a consequence, we express genus 0 characteristic numbers of $\mathbb{C}P^2$ in terms of open Hurwitz numbers.

**Contents**



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## 1. Introduction

Characteristic numbers were considered by nineteenth-century geometers among which are Maillard [Mai71], who computed them in degree 3, Zeuthen [Zeu73], who did third-degree and fourth-degree cases, and Schubert [Sch79]. Modern mathematicians confirmed and extended their predecessors' results, thanks, in particular, to intersection theory. Aluffi computed, for instance, all characteristic numbers for plane cubics and some of them for plane quartics (see [Alu88, Alu90, Alu91] and [Alu92]), and Vakil completed to confirm Zeuthen's computation of all characteristic numbers of plane quartics in [Vak99]. Pandharipande computed characteristic numbers in the rational case in [Pan99], Vakil achieved the genus 1 case in [Vak01], and Graber, Kock and Pandharipande computed genus 2 characteristic numbers of plane curves in [GKP02]. In this article, we provide a new insight to the genus 0 case and give new formulas via its tropical counterpart and by reducing the problem to combinatorics of floor diagrams.

Before going into the details of the novelties of this paper, let us recall the classical problem of computing characteristic numbers of $\mathbb{C}P^2$.

Let $d$, $g$ and $k$ be non-negative integer numbers such that $g \leqslant (d-1)(d-2)/2$ and $k \leqslant 3d+g-1$, and let $d_1, \dots, d_{3d+g-1-k}$ be positive integer numbers. For any configurations $\mathcal{P} = \{p_1, \dots, p_k\}$ of $k$ points in $\mathbb{C}P^2$, and $\mathcal{L} = \{L_1, \dots, L_{3d+g-1-k}\}$ of $3d+g-1-k$ complex non-singular algebraic curves in $\mathbb{C}P^2$ such that $L_i$ has degree $d_i$, we consider the set $\mathcal{S}(d, g, \mathcal{P}, \mathcal{L})$ of holomorphic maps $f : C \to \mathbb{C}P^2$ from an irreducible non-singular complex algebraic curve of genus $g$, passing through all points $p_i \in \mathcal{P}$, tangent to all curves $L_i \in \mathcal{L}$, and such that $f(C)$ has degree $d$ in $\mathbb{C}P^2$.

If the constraints $\mathcal{P}$ and $\mathcal{L}$ are chosen generically, then the set $\mathcal{S}(d, g, \mathcal{P}, \mathcal{L})$ is finite, and the characteristic number $N_{d,g}(k; d_1, \dots, d_{3d+g-1-k})$ is defined as

$$N_{d,g}(k; d_1, \dots, d_{3d+g-1-k}) = \sum_{f \in \mathcal{S}(d,g,\mathcal{P},\mathcal{L})} \frac{1}{|\mathrm{Aut}(f)|}$$

where $\mathrm{Aut}(f)$ is the group of automorphisms of the map $f : C \to \mathbb{C}P^2$, i.e. isomorphisms $\Phi : C \to C$ such that $f \circ \Phi = f$. It depends only on $d$, $g$, $k$ and $d_1, \dots, d_{3d+g-1-k}$ (see for example [Vak01]). In this text, we will use the shorter notation $N_{d,g}(k; d_1^{i_1}, \dots, d_l^{i_l})$ which indicates that the integer $d_j$ is chosen $i_j$ times. Let us describe the characteristic numbers in some special instances.

*Example* 1.1. The number $N_{d,g}(3d-1+g)$ is the usual Gromov–Witten invariant of degree $d$ and genus $g$ of $\mathbb{C}P^2$.

*Example* 1.2. The numbers $N_{2,0}(5)$, $N_{2,0}(4;1)$, and $N_{2,0}(3;1^2)$ are easy to compute by hand, and thanks to projective duality we have

$$N_{2,0}(k;1^{5-k}) = N_{2,0}(5-k;1^k) = 2^k \quad \text{for } 0 \leqslant k \leqslant 2.$$

*Example* 1.3. All characteristic numbers $N_{3,0}(k;1^{8-k})$ for rational cubic curves have been computed by Zeuthen [Zeu72] and confirmed by Aluffi [Alu91]. We sum up part of their results in the following table.

| $k$ | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|---|---|---|---|
| $N_{3,0}(k;1^{8-k})$ | 12 | 36 | 100 | 240 | 480 | 712 | 756 | 600 | 400 |

*Example* 1.4. The number $N_{2,0}(0;2^5)$ has been computed independently by Chasles [Cha64] and De Jonquiere. More than one century later, Ronga, Tognoli, and Vust showed in [RTV97] that it is possible to choose five real conics in such a way that all conics tangent to these five conics are real. See also [Ghy08] and [Sot] for a historical account and digression on this subject. See also Example 7.2 for a tropical version of the arguments from [RTV97]. We list below the numbers $N_{2,0}(k;2^{5-k})$:

$$N_{2,0}(4;2) = 6, \quad N_{2,0}(3;2^2) = 36, \quad N_{2,0}(2;2^3) = 184,$$
$$N_{2,0}(1;2^4) = 816, \quad N_{2,0}(0;2^5) = 3264.$$

More generally, the characteristic numbers $N_{d,g}(k;1^{3d-1+g-k})$ of $\mathbb{C}P^2$ determine all the numbers $N_{d,g}(k;d_1,\ldots,d_{3d-1+g-k})$. Indeed, by degenerating the non-singular curve $L_{d_{3d-1+g-k}}$ to the union of two non-singular curves of lower degrees intersecting transversely, we obtain the following formula (see for example [RTV97, Theorem 8]):

$$\begin{aligned} N_{d,g}(k;d_1,\ldots,d_{3d-1+g-k}) &= 2d'_{3d-1+g-k}d''_{3d-1+g-k}N_{d,g}(k+1;d_1,\ldots,d_{3d-2+g-k}) \\ &\quad + N_{d,g}(k;d_1,\ldots,d'_{3d-1+g-k}) + N_{d,g}(k;d_1,\ldots,d''_{3d-1+g-k}), \end{aligned} \tag{1}$$

where $d_{3d-1+g-k} = d'_{3d-1+g-k} + d''_{3d-1+g-k}$.

This paper contains three main contributions. First we identify tropical tangencies between two tropical morphisms. Then we deduce from the location of tropical tangencies a correspondence theorem which allows one to compute characteristic numbers of $\mathbb{C}P^2$ in genus 0. Finally, using the floor decomposition technique, we provide a new insight on these characteristic numbers and their relation to Hurwitz numbers.

Thanks to a series of correspondence theorems initiated in [Mik05] (see also [Nis10, NS06, Mik, Shu05, Tyo12]), tropical geometry turned out to be a powerful tool to solve enumerative problems in complex and real algebraic geometry. However, until now, correspondence theorems dealt with problems only involving simple incidence conditions, i.e. with enumeration of curves intersecting transversally a given set of constraints. Correspondence theorems, Theorems 3.8 and 3.12, in this paper are the first ones concerning plane curves satisfying tangency conditions to a given set of curves. In the case of simple incidences, the (finitely many) tropical curves arising as the limit of amoebas of the enumerated complex curves could be identified considering the tropical limit of embedded complex curves. This is no longer enough to identify the tropical limit of tangent curves, and we first refine previous studies by considering the tropical limit of a family of holomorphic maps to a given projective space. This gives rise to the notion of *phase-tropical curves* and morphisms. For a treatment of phase-tropical geometry more general than the one in this paper, we refer to [Mik]. Using tropical morphisms and their approximation by holomorphic maps, we identify tropical tangencies between tropical morphisms, and prove Theorems 3.8 and 3.12. Note that Dickenstein and Tabera also studied in [DT12] tropical tangencies but in a slightly different context, i.e. tangencies between tropical cycles instead of tropical morphisms.

As in the works cited above, Theorems 3.8 and 3.12 allow us to solve our enumerative problem by exhibiting some special configurations of constraints for which we can actually find all complex curves satisfying our conditions. In particular we obtain more information than using only the intersection theoretical approach from complex geometry. As a consequence, when all constraints are real we are also able to identify all real curves matching our constraints. This is the starting observation in applications of tropical geometry in real algebraic geometry, which already turned out to be fruitful (see for example [Ber, Ber08, BP, IKS03, Mik05]). In this paper we just provide a few examples of such applications to real enumerative geometry in § 7.2. However, there is no doubt that Theorems 3.8 and 3.12 should lead to further results in real enumerative geometry.

The next step after proving our correspondence theorems is to use generalized *floor diagrams* to reduce the computations to pure combinatorics. In the particular case where only incidence conditions are considered they are equivalent to those defined in [BM07] (see also [BM09]), and used later in several contexts (see for example [AB13, ABL11, Ber, BGM12, BP, FM10]). Note that the floor decomposition technique has strong connections with the Caporaso and Harris method (see [CH98]) extended later by Vakil (see [Vak00]), and with the neck-stretching method in symplectic field theory (see [EGH00, IP04]). This method allows one to solve an enumerative problem by induction on the dimension of the ambient space, i.e. to reduce enumerative problems in $\mathbb{C}P^n$ to enumerative problems in $\mathbb{C}P^{n-1}$. In the present paper, the enumerative problem we are concerned with is to count curves which interpolate a given configuration of points and are tangent to a given set of curves. On the level of maps, tangency conditions are naturally interpreted as ramification conditions. In particular, the one-dimensional analogues of characteristic numbers are *Hurwitz numbers*, which are the number of maps from a (non-fixed) genus $g$ curve to a (fixed) genus $g_0$ curves with a fixed ramification profile at some fixed points. Hence, using floor diagrams, we express characteristic numbers of $\mathbb{C}P^2$ in terms of Hurwitz numbers. Surprisingly, other one-dimensional enumerative invariants also appear in this expression. These are the so-called *open Hurwitz numbers*, a slight generalization of Hurwitz numbers defined and computed in [BBM11]. Computations of characteristic numbers of $\mathbb{C}P^2$ performed in [Pan99, GKP02], and [Vak01], were done by induction on the degree of the enumerated curves. To our knowledge, this is the first time that characteristic numbers are expressed in terms of their analogue in dimension 1, i.e. in terms of (open) Hurwitz numbers.

Here is the plan of the paper. In § 2 we review standard definitions we need from tropical geometry. In § 3 we define tropical tangencies and state our correspondence theorems. Even though we can reduce all degrees $d_i$ of the constraints to one by (1), we still leave $d_1, \ldots, d_{3d-1-k}$ in the statement of correspondence Theorem 3.8, in view of possible application to real geometry (see e.g. Example 7.2). Section 4 is devoted to the proof of technical lemmas on generic configurations of constraints. The multiplicity of a tropical curve in Theorems 3.8 and 3.12 are defined by a determinant, and we give in § 5 a practical way of computing this determinant which will be used in § 8. We introduce in § 6 the notion of phase-tropical curves and morphisms, and the tropical limit of a family of holomorphic maps, and we prove Theorems 3.8 and 3.12 in § 7. In § 8, floor diagrams are introduced, formulas for characteristic numbers in which they appear are proved, and examples given.

We end this introduction elaborating on possible natural generalizations of the techniques presented in this paper. All definitions, statements, and proofs should generalize with no difficulty to the case of rational curves in $\mathbb{C}P^n$ intersecting cycles and tangent to non-singular hypersurfaces. The resulting floor diagrams would then be a generalization of those defined in [BMa] and [BM07]. The enumeration of plane curves with higher-order tangency conditions to other curves should also be doable in principle using our methods. This would first necessitate identifying tropicalizations of higher-order tangencies between curves, generalizing the simple tangency case treated in § 3.1. However, this identification might be intricate, and will certainly lead to much more different cases than for simple tangencies (third-order tangencies to a line are dealt with in [BL12]). In turn, the use of tropical techniques in the computation of higher-genus characteristic numbers requires some substantial additional work. The main difficulty is that superabundancy appears for positive genus: some combinatorial types appearing as solution of the enumerative problem might be of actual dimension strictly bigger than the expected one (see Remark 2.5). Hence, before enumerating tropical curves, in addition to the balancing condition one has first to understand extra necessary conditions for a tropical morphism to be the tropical limit of a family of algebraic maps (see § 6). Using the techniques developed in [BMb], we succeeded to compute genus 1 characteristic numbers of $\mathbb{C}P^2$. These results will appear in a separate paper. Also for a small number of tangency constraints, it is possible to find a configuration of constraints for which no superabundant curve shows up. In this case Theorem 3.8 applies, the proof only requiring minor adjustments.

## 2. Tropical curves and morphisms

In this section, we define abstract tropical curves, their morphisms to $\mathbb{R}^n$, and tropical cycles in $\mathbb{R}^2$.

### 2.1 Tropical curves

Given a finite graph $C$ (i.e. $C$ has a finite number of edges and vertices) we denote by $\mathrm{Vert}(C)$ the set of its vertices, by $\mathrm{Vert}^0(C)$ the set of its vertices which are not 1-valent, and by $\mathrm{Edge}(C)$ the set of its edges. By definition, the valency of a vertex $v \in \mathrm{Vert}(C)$, denoted by $\mathrm{val}(v)$, is the number of edges in $\mathrm{Edge}(C)$ adjacent to $v$. Throughout the text we will identify a graph and any of its topological realization. The next definition is taken from [BBM11]. Tropical curves with boundary will be needed in § 8.

Definition 2.1. An *irreducible tropical curve* $C$ with boundary is a finite compact connected graph with $\mathrm{Edge}(C) \neq \emptyset$, together with a set of 1-valent vertices $\mathrm{Vert}^\infty(C)$ of $C$, such that:

- $C \backslash \mathrm{Vert}^\infty(C)$ is equipped with a complete inner metric;
- the vertices of $\mathrm{Vert}^0(C)$ have non-negative integer weights, i.e. $C$ is equipped with a map

$$\begin{aligned} \mathrm{Vert}^0(C) &\longrightarrow \mathbb{Z}_{\geqslant 0} \\ v &\longmapsto g_v; \end{aligned}$$

- any 2-valent vertex $v$ of $C$ satisfies $g_v \geqslant 1$.

If $v$ is an element of $\mathrm{Vert}^0(C)$, the integer $g_v$ is called the genus of $v$. The genus of $C$ is defined as

$$g(C) = b_1(C) + \sum_{v \in \mathrm{Vert}^0(C)} g_v$$

where $b_1(C)$ denotes the first Betti number of $C$.

An element of $\mathrm{Vert}^\infty(C)$ is called a *leaf* of $C$, and its adjacent edge is called an *end* of $C$. A 1-valent vertex of $C$ not in $\mathrm{Vert}^\infty(C)$ is called a *boundary component* of $C$.

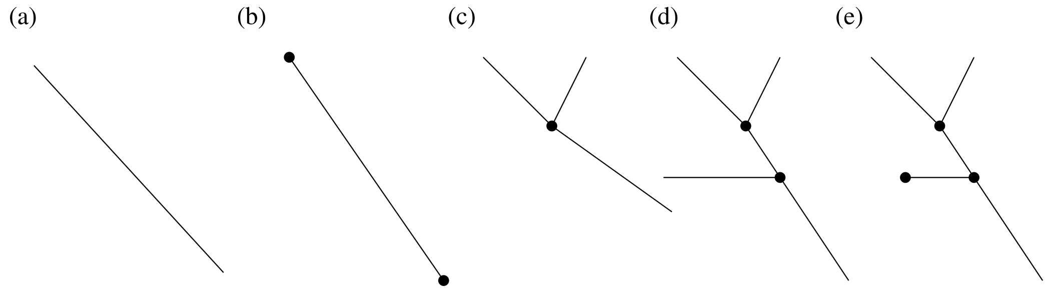



Figure 1. A few rational tropical curves.

Definition 2.1 might appear very general, whereas we essentially deal with *rational* tropical curves, i.e. of genus 0, in the rest of the paper. The reason for us to give such a general definition is that the general framework for correspondence theorems between complex algebraic and tropical curves we set up in § 6.1 is valid for tropical curves as in Definition 2.1, and may be used in future correspondences between complex algebraic and tropical curves of any genus.

It follows immediately from Definition 2.1 that the leaves of $C$ are at the infinite distance from all the other points of $C$. We denote by $\partial C$ the set of the boundary components of $C$, by $\mathrm{Edge}^\infty(C)$ the set of ends of $C$, and by $\mathrm{Edge}^0(C)$ the set of its edges which are not adjacent to a 1-valent vertex.

A *punctured tropical curve* $C'$ is given by $C\backslash\mathcal{P}$ where $C$ is a tropical curve, and $\mathcal{P}$ is a subset of $\mathrm{Vert}^\infty(C)$. Note that $C'$ has a tropical structure inherited from $C$. Elements of $\mathcal{P}$ are called *punctures*. An end of $C'$ is said to be *open* if it is adjacent to a puncture, and *closed* otherwise.

*Example* 2.2. In Figure 1 we depict some examples of the simplest rational tropical curves. Boundary components and vertices of $\mathrm{Vert}^0(C)$ are depicted as black dots and vertices of $\mathrm{Vert}^\infty(C)$ are omitted so that an edge not ending at a black vertex is of infinite length and that no difference is made on the picture between punctured and non-punctured tropical curves.

Given $C$ a tropical curve, and $p$ a point on $C$ which is not a 1-valent vertex, we can construct a new tropical curve $\widetilde{C}_p$ by attaching to $C$ a closed end $e_p$ (i.e. an end with a leaf point at infinity) at $p$, and by setting $g_p = 0$ if $p \notin \mathrm{Vert}^0(C)$. The natural map $\pi : \widetilde{C}_p \to C$ which contracts the edge $e_p$ to $p$ is called a *tropical modification*. If we denote by $v$ the 1-valent vertex of $\widetilde{C}_p$ adjacent to $e_p$, the restriction of the map $\pi$ to the punctured tropical curve $\widetilde{C}_p\backslash\{v\}$ is called an *open tropical modification*.

*Example* 2.3. The closed curve depicted in Figure 1(d) can be obtained by a tropical modification from the curve of Figure 1(c) which in turn can be obtained modifying the infinite closed segment of Figure 1(a). More generally, every rational tropical curve without boundary can be obtained from the infinite closed segment (Figure 1(a)) by a finite sequence of tropical modifications.

### 2.2 Tropical morphisms

Given $e$ an edge of a tropical curve $C$, we choose a point $p$ in the interior of $e$ and a unit vector $u_e$ of the tangent line to $C$ at $p$ (recall that $C$ is equipped with a metric). Of course, the vector $u_e$ depends on the choice of $p$ and is well defined only up to multiplication by $-1$, but this will not matter in the following. We will sometimes need $u_e$ to have a prescribed direction, and we will then specify this direction. The standard inclusion of $\mathbb{Z}^n$ in $\mathbb{R}^n$ induces a standard inclusion of $\mathbb{Z}^n$ in the tangent space of $\mathbb{R}^n$ at any point of $\mathbb{R}^n$. A vector in $\mathbb{Z}^n$ is said to be *primitive* if the greatest common divisor of its coordinates equals 1.

Definition 2.4. Let $C$ be a punctured tropical curve. A continuous map $f : C \to \mathbb{R}^n$ is a *tropical morphism* if the following hold.

- For any edge $e$ of $C$, the restriction $f_{|e}$ is a smooth map with $df(u_e) = w_{f,e} u_{f,e}$ where $u_{f,e} \in \mathbb{Z}^n$ is a primitive vector, and $w_{f,e}$ is a non-negative integer.
- For any vertex $v$ in $\mathrm{Vert}^0(C)$ whose adjacent edges are $e_1, \ldots, e_k$, one has the balancing condition
$$\sum_{i=1}^{k} w_{f,e_i} u_{f,e_i} = 0$$
where $u_{e_i}$ is chosen so that it points away from $v$.

The integer $w_{f,e}$ is called the *weight of the edge e with respect to f*. When no confusion is possible, we simply speak about the weight of an edge, without referring to the morphism $f$. If $w_{f,e} = 0$, we say that the morphism $f$ *contracts* the edge $e$. The morphism $f$ is called *minimal* if $f$ does not contract any edge. If the morphism $f$ is proper then an open end of $C$ has to have a non-compact image, while a closed end has to be contracted. Hence if a proper tropical morphism $f : C \to \mathbb{R}^n$ is minimal, then $\mathrm{Vert}^\infty(C) = \emptyset$. The morphism $f$ is called an *immersion* if it is a topological immersion, i.e. if $f$ is a local homeomorphism on its image.

*Remark* 2.5. Definition 2.4 is a rather coarse definition of a tropical morphism when $C$ has positive genus. Indeed, in contrast to the case of rational curves, one can easily construct tropical morphisms from a positive-genus tropical curve which are superabundant, i.e. whose space of deformation has a strictly bigger dimension that the expected one (see [Mik05, § 2]). In particular, when the corresponding situation in classical geometry is regular (i.e. with no superabundancy phenomenon) as in the case of projective plane curves, such a superabundant tropical morphism is unlikely to be presented as the tropical limit of a family of holomorphic maps (see § 6.1 for the definition of the tropical limit).

One may refine Definition 2.4, still using pure combinatoric, to get rid of many of these superabundant tropical morphisms. One of these possible refinements, explored in [BMb], is to require in addition that the map $f : C \to \mathbb{R}^n$ should be *modifiable*: the map $f$ has to be liftable to any sequence of tropical modifications of $\mathbb{R}^n$ with smooth center (see [Mik06]). This definition of modifiable tropical morphisms relies on the more general definition of a tropical morphism $f : C \to X$ where $X$ is any non-singular tropical variety. In addition to the balancing condition, a tropical morphism $f : C \to X$ has to satisfy some combinatorial conditions coming from complex algebraic geometry, such as the Riemann–Hurwitz condition at points of $C$ mapped to the skeleton of codimension one of $X$ (see [BBM11] for the case when $X$ is a tropical curve).

However, since this paper is about enumeration of rational curves, Definition 2.4, even if coarse, is sufficient for our purposes here.

*Example* 2.6. We represent morphisms from a curve to $\mathbb{R}$ as in Figure 2. The weights of the edges with respect to the morphism label the edge if they exceed 1. Figure 2(a) represents a double cover of the real line by a tropical curve without boundary, and Figure 2(b) represents an open cover of $\mathbb{R}$ (see § Appendix) by a tropical curve with one boundary component.

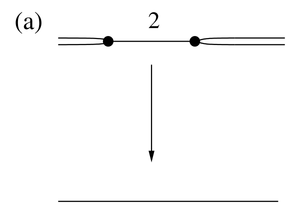



Figure 2. Example of tropical morphisms $f : C \to \mathbb{R}$.

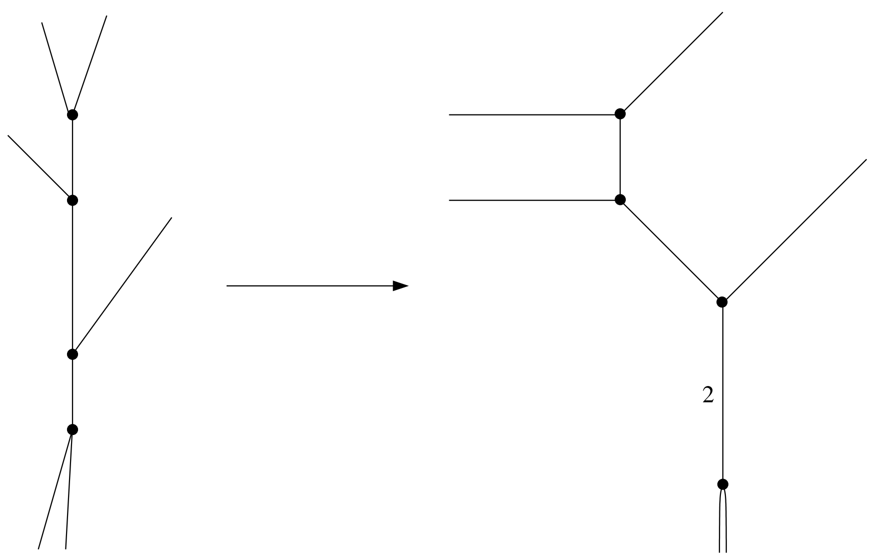



Figure 3. Example of a morphism: a plane conic.

*Example* 2.7. In contrast to the one-dimensional case, for morphisms from a curve to $\mathbb{R}^2$, we label the **image** of an edge with the corresponding weight as it often allows us to omit the source of the morphism which is then implicit. In Figure 3 we depicted a plane conic $C$, which is the image in $\mathbb{R}^2$ of a morphism from a trivalent punctured curve with four vertices in $\mathrm{Vert}^0(C)$.

Two tropical morphisms $f_1 : C_1 \to \mathbb{R}^n$ and $f_2 : C_2 \to \mathbb{R}^n$ are said to be *isomorphic* if there exists an isometry $\phi : C_1 \to C_2$ such that $f_1 = f_2 \circ \phi$ and $g_{\phi(v)} = g_v$ for any $v \in \mathrm{Vert}^0(C_1)$. In this text, we consider tropical curves and tropical morphisms up to isomorphism.

A less restrictive equivalence relation is the one associated to *combinatorial types*. Two tropical morphisms $f_1 : C_1 \to \mathbb{R}^n$ and $f_2 : C_2 \to \mathbb{R}^n$ are said to have the same combinatorial type if there exists a homeomorphism of graphs $\phi : C_1 \to C_2$ inducing two bijections $\mathrm{Vert}^\infty(C_1) \to \mathrm{Vert}^\infty(C_2)$ and $\partial C_1 \to \partial C_2$, and such that for all edges $e$ of $C_1$ and all vertices $v \in \mathrm{Vert}^0(C)$ one has

$$w_{f_1,e} = w_{f_2,\phi(e)}, \quad u_{f_1,e} = u_{f_2,\phi(e)} \quad \text{and} \quad g_{\phi(v)} = g_v.$$

Given a combinatorial type $\alpha$ of tropical morphisms, we denote by $\mathcal{M}_\alpha$ the space of all such tropical morphisms having this combinatorial type. Given $f \in \mathcal{M}_\alpha$, we say that $\mathcal{M}_\alpha$ is the *rigid deformation space* of $f$.

Lemma 2.8 (Mikhalkin, [Mik05, Proposition 2.14]). *Let $\alpha$ be a combinatorial type of tropical morphisms $f : C \to \mathbb{R}^n$ where $C$ is a rational tropical curve with $\partial C = \emptyset$. Then the space $\mathcal{M}_\alpha$ is an open convex polyhedron in the vector space $\mathbb{R}^{n+|\mathrm{Edge}^0(\alpha)|}$, and*

$$\dim \mathcal{M}_\alpha = |\mathrm{Edge}^\infty(\alpha)| + n - 3 - \sum_{v \in \mathrm{Vert}^0(\alpha)} (\mathrm{val}(v) - 3).$$

*Proof.* We recall the proof in order to fix notations we will need later in § 5. If $\mathrm{Vert}^0(\alpha) \neq \emptyset$, choose a root vertex $v_1$ of $\alpha$, and an ordering $e_1, \dots, e_{|\mathrm{Edge}^0(\alpha)|}$ of the edges in $\mathrm{Edge}^0(\alpha)$. Given $f : C \to \mathbb{R}^n$ in $\mathcal{M}_\alpha$, we write $f(v_1) = (x_1, \dots, x_n) \in \mathbb{R}^n$, and we denote by $l_i \in \mathbb{R}^*$ the length of the edge $e_i \in \mathrm{Edge}^0(C)$. Then

$$\mathcal{M}_\alpha = \{(x_1, \dots, x_n, l_1, \dots, l_{|\mathrm{Edge}^0(\alpha)|}) \mid l_1, \dots, \dots, l_{|\mathrm{Edge}^0(\alpha)|} > 0\} = \mathbb{R}^n \times \mathbb{R}_{>0}^{|\mathrm{Edge}^0(\alpha)|}.$$

If $\mathrm{Vert}^0(C) = \emptyset$, then $\mathcal{M}_\alpha = \mathbb{R}^n / \mathbb{R}u_{f,e}$, where $e$ is the only edge of $\alpha$. □

Other choices of $v_1$ and of the ordering of elements of $\mathrm{Edge}^0(\alpha)$ provide other coordinates on $\mathcal{M}_\alpha$, and the change of coordinates is given by an element of $\mathrm{GL}_{n+|\mathrm{Edge}^0(\alpha)|}(\mathbb{Z})$. □

*Example* 2.9. In the simplest case when $\alpha$ is the combinatorial type of tropical morphisms $f : C \to \mathbb{R}^2$ with $\mathrm{Vert}^0(C) = \{v\}$, the space $\mathcal{M}_\alpha$ is $\mathbb{R}^2$ and the coordinates are given by $f(v)$.

If $\alpha$ is the morphism depicted in Figure 3, $\mathcal{M}_\alpha$ is the unbounded polyhedron $\mathbb{R}^2 \times \mathbb{R}^3_{>0}$ with coordinates $\{x_1, x_2, l_1, l_2, l_3\}$ where $x_1$ and $x_2$ are the coordinates of the image of the lowest vertex and $l_1, l_2$ and $l_3$ are the lengths of the bounded edges at the source ordered from bottom to top.

### 2.3 Tropical cycles

Here we fix notations concerning standard facts in tropical geometry. We refer for example to [IMS07, Mik06, RST05], or [BPS08] for more details.

An *effective tropical 1-cycle* in $\mathbb{R}^2$ is the tropical divisor defined by some tropical polynomial $P(x, y)$, whose Newton polygon is denoted by $\Delta(P)$. Given a positive integer $d$, we denote by $T_d$ the triangle in $\mathbb{R}^2$ with vertices $(0, 0)$, $(0, d)$ and $(d, 0)$.

The number 1 in 1-cycle stands for dimension 1 as such a cycle is necessarily a graph. Since any smaller-dimensional cycles in $\mathbb{R}^2$ can only be 0-cycles which are just linear combinations of points in $\mathbb{R}^2$, in this paper we will be just saying 'cycles in $\mathbb{R}^2$' for 1-cycles.

Definition 2.10. An effective tropical cycle in $\mathbb{R}^2$ defined by a tropical polynomial $P(x, y)$ is said to have *degree* $d \geqslant 1$ if $\Delta(P) \subset T_d$ and $\Delta(P) \nsubseteq T_{d-1}$.

Recall that a tropical polynomial induces a subdivision of its Newton polygon.

Definition 2.11. An effective tropical cycle in $\mathbb{R}^2$ of degree $d$ defined by a tropical polynomial $P(x, y)$ is said to be *non-singular* if $\Delta(P) = T_d$ and the subdivision of $\Delta(P)$ induced by $P(x, y)$ is primitive, i.e. contains only triangles of Euclidean area $\frac{1}{2}$.

The tropical cycle is said to be *simple* if this subdivision of $\Delta(P)$ contains only triangles and parallelograms.

Two effective tropical cycles are said to be of the same *combinatorial type* if they have the same Newton polygon, and the same dual subdivision. As in the case of tropical morphisms, we denote by $\mathcal{M}_\alpha$ the set of all effective tropical cycles of a given combinatorial type $\alpha$.

Lemma 2.12. *If $\alpha$ is a combinatorial type of non-singular effective tropical cycles, then $\mathcal{M}_\alpha$ is an open convex polyhedron in $\mathbb{R}^{|\Delta(\alpha) \cap \mathbb{Z}^2| - 1}$.*

If $f : C \to \mathbb{R}^2$ is a non-constant tropical morphism, then the image $f(C)$ is a balanced polyhedral graph in $\mathbb{R}^2$, and hence is defined by some tropical polynomial $P(x, y)$ (see [Mik04, Proposition 2.4]). The Newton polygon of the morphism $f : C \to \mathbb{R}^2$, denoted by $\Delta(f)$, is defined as the Newton polygon of $P(x, y)$. The polygon $\Delta(f)$ is well defined up to translation by a vector in $\mathbb{Z}^2$, and in particular the *degree of* $f$ is well defined.

The *genus* of a tropical cycle $A$ is the smallest genus of a tropical immersion $f : C \to \mathbb{R}^2$ such that $f(C) = A$. Note that if $A$ is simple and $f : C \to \mathbb{R}^2$ is such a tropical immersion of minimal genus then the tropical curve $C$ contains only 3-valent vertices, which are called *the vertices* of $A$.

We can refine the notion of Newton polygon of a tropical cycle $A$ (or of an algebraic curve in $(\mathbb{C}^*)^2$) by encoding how $A$ intersect the toric divisors at infinity. Namely, we define the *Newton fan* of $A$ to be the multiset of vectors $w_e u_e$ where $e$ runs over all unbounded edges of $A$, $w_e$ is the weight of $e$ (i.e. the integer length of its dual edge), and $u_e$ is the primitive integer vector of $e$ pointing to the unbounded direction of $e$. The definition in the case of algebraic curves in $(\mathbb{C}^*)^2$ works similarly.

*Remark* 2.13. Note that our simple effective cycles are in 1–1 correspondence with 3-valent immersed tropical curves such that their self-intersections are isolated points that come as intersection of different edges at their interior points. We call them tropical cycles (instead of calling them tropical curves) to emphasize their role as constraints.

## 3. Tangencies

### 3.1 Tropical pretangencies in $\mathbb{R}^2$

Our definition of tropical pretangencies between tropical morphisms is motivated by the study of tropical tangencies in [BMb], to which we refer for more details. We also refer to [DT12] for the notion of tropical tangencies between two tropical cycles in $\mathbb{R}^2$. Let $f : C \to \mathbb{R}^2$ be a tropical morphism, and $L$ be a simple tropical cycle in $\mathbb{R}^2$.

Definition 3.1. The tropical morphism $f$ is said to be *pretangent* to $L$ if there exists a connected component $E$ of the set theoretic intersection of $f(C)$ and $L$ which contains either a vertex of $L$ or the image of a vertex of $C$.

The set $E \subset \mathbb{R}^2$ is called a *pretangency set* of $f$ and $L$. A connected component of $f^{-1}(E) \subset C$ is called a *pretangency component* of $f$ with $L$ if $E$ contains either a vertex of $C$ or a point $p$ such that $f(p)$ is a vertex of $L$.

*Example* 3.2. In Figure 4 we depicted several examples of the image of a morphism pretangent to a cycle which is represented by dotted lines.

It is clear that not any pretangency set corresponds to some classical tangency point. For example, the two tropical lines in Figure 4(b) are pretangent, but this pretangency set does not correspond to any tangency point between two complex algebraic lines in $\mathbb{C}P^2$. However, given any approximation of $f$ (if one exists) and any approximation of $L$ by algebraic curves, the accumulation set of tangency points of these approximations must lie inside the pretangency sets of $f$ and $L$ (see § 7, or [BMb]).

### 3.2 Correspondence

As in § 1 let $d \geqslant 1$, $k \geqslant 0$, and $d_1, \ldots, d_{3d-1-k} > 0$ be some integer numbers, and choose $\mathcal{P} = \{p_1, \ldots, p_k\}$ a set of $k$ points in $\mathbb{R}^2$, and $\mathcal{L} = \{L_1, \ldots, L_{3d-1-k}\}$ a set of $3d - 1 - k$ non-

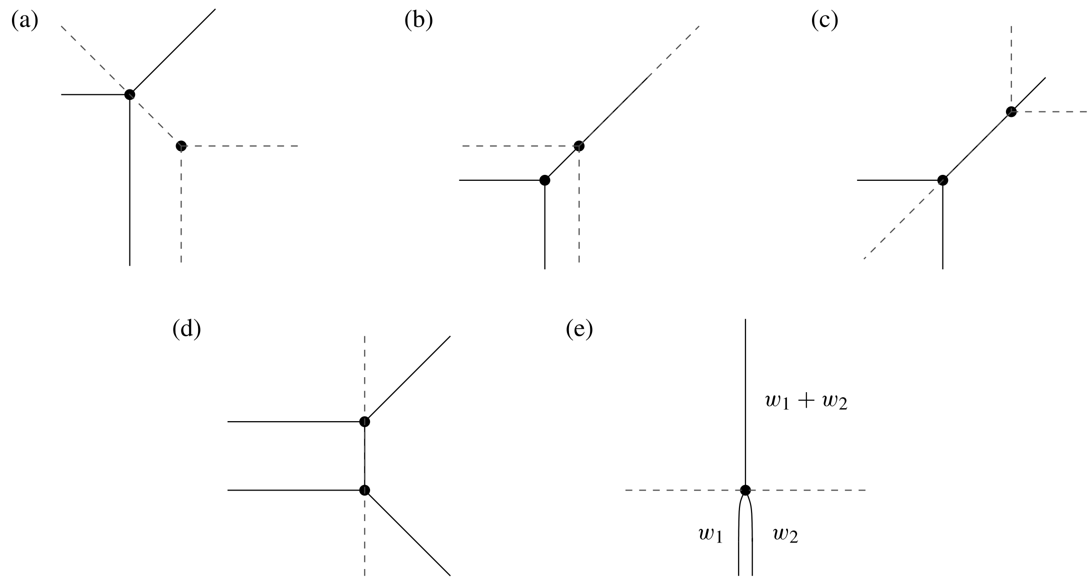



Figure 4. Pretangent morphisms.

singular effective tropical cycles in $\mathbb{R}^2$ such that $L_i$ has degree $d_i$. We denote by $\mathcal{S}^{\mathbb{T}}(d,\mathcal{P},\mathcal{L})$ the set of minimal tropical morphisms $f: C \to \mathbb{R}^2$ of degree $d$, where $C$ is a rational tropical curve with $\partial C = \emptyset$, passing through all points $p_i$ and pretangent to all curves $L_i$.

We suppose now that the configuration $(\mathcal{P},\mathcal{L})$ is *generic*. The precise definition of this word will be given in § 4, Definition 4.7. For the moment, it is sufficient to have in mind that the set of generic configurations is a dense open subset of the set of all configurations with a given number of points and tropical cycles. The proof of the next three statements will be given in § 4. Proposition 3.3 and Lemma 3.5 are direct consequences of Corollary 4.6.

Proposition 3.3. *The set $\mathcal{S}^{\mathbb{T}}(d,\mathcal{P},\mathcal{L})$ is finite, and any of its elements $f: C \to \mathbb{R}^2$ satisfy:*

- *$C$ is a 3-valent curve with exactly $3d$ leaves, all of them of weight 1;*
- *$f(\mathrm{Vert}^0(C)) \bigcap (\bigcup_{L\in\mathcal{L}} \mathrm{Vert}^0(L) \cup \mathcal{P}) = \emptyset$, i.e. no vertex of $C$ is mapped to a vertex of a curve in $\mathcal{L}$ nor to a point in $\mathcal{P}$;*
- *given $p \in \mathcal{P}$, if $x$ and $x'$ are in $f^{-1}(p)$, then the (unique) path $\gamma$ in $C$ from $x$ to $x'$ is mapped to a segment in $\mathbb{R}^2$ by $f$;*
- *given $L \in \mathcal{L}$, there exists a connected subgraph $\Gamma \subset C$ which contains all pretangency components of $f$ with $L$, and such that $f(\Gamma)$ is a segment in $\mathbb{R}^2$.*

Let $f: C \to \mathbb{R}^2$ be an element of $\mathcal{S}^{\mathbb{T}}(d,\mathcal{P},\mathcal{L})$, and let us denote by $\alpha$ its combinatorial type. Given $p \in \mathcal{P}$ (respectively $L \in \mathcal{L}$), we denote by $\lambda_p$ (respectively $\lambda_L$) the set of elements of $\mathcal{M}_\alpha$ in a small neighborhood $U_f$ of $f$ which pass through $p$ (respectively are pretangent to $L$).

Lemma 3.4. *If $U_f$ is small enough, then $\lambda_q$ spans a classical affine hyperplane $\Lambda_q$ defined over $\mathbb{Z}$ in $\mathcal{M}_\alpha$ for any $q$ in $\mathcal{P} \cup \mathcal{L}$.*

*Proof.* This is an immediate consequence of the $\mathbb{Z}$-linearity of the evaluation and forgetful maps $ev$ and $ft$ (see § 4). □

Lemma 3.5. *We have*

$$\bigcap_{q\in\mathcal{P}\cup\mathcal{L}} \Lambda_q = \{f\}.$$

Let us associate a weight to the (classical) linear spaces $\Lambda_q$. Given $p\in\mathcal{P}$, we denote by $\mathcal{E}(p)$ the set of edges of $C$ which contain a point of $f^{-1}(p)$ and we define

$$w_p = \sum_{e\in\mathcal{E}(p)} w_{f,e}.$$

Given $L\in\mathcal{L}$, we denote by $E_L$ the union of all pretangency components of $f$ with $L$, by $\mu$ the cardinal of $E_L\cap \mathrm{Vert}^0(C)$, and by $\lambda$ the number of ends of $C$ contained in $E_L$. If $v\in\mathrm{Vert}^0(L)$, we denote by $\mathcal{E}(v)$ the set of edges of $C$ which contain a point of $f^{-1}(v)$, and we define

$$w_L = \left(\sum_{v\in\mathrm{Vert}^0(L)} \sum_{e\in\mathcal{E}(v)} w_{f,e}\right) + \mu - \lambda. \tag{2}$$

Equivalently, we can define $w_L$ as follows:

$$w_L = \sum_{e\in\mathcal{E}(v)} w_{f,e} \quad \text{if } f(E_L) = v\in\mathrm{Vert}^0(L),$$

$$w_L = \left(\sum_{v\in\mathrm{Vert}^0(L)} \sum_{e\in\mathcal{E}(v)} (w_{f,e}+1)\right) + \kappa - 2b_0(E_L) \quad \text{otherwise},$$

where $\kappa$ is the number of edges of $C\backslash E_L$ adjacent to a vertex of $C$ in $E_L$.

Definition 3.6. The $(\mathcal{P},\mathcal{L})$-*multiplicity of* $f: C\to\mathbb{R}^2$, denoted by $\mu_{(\mathcal{P},\mathcal{L})}(f)$, is defined as the tropical intersection number in $\mathcal{M}_\alpha$ of all the tropical hypersurfaces $\mu_q\Lambda_q$ divided by the number of automorphisms of $f$. That is to say,

$$\mu_{(\mathcal{P},\mathcal{L})}(f) = \frac{|\det(\Lambda_{p_1},\dots,\Lambda_{p_k},\Lambda_{L_1},\dots,\Lambda_{L_{3d-1-k}})|}{|\mathrm{Aut}(f)|}\prod_{q\in\mathcal{P}\cup\mathcal{L}} w_q.$$

The morphism $f$ is said to be *tangent* to $\mathcal{L}$ if $\mu_{\mathcal{P},\mathcal{L}}(f)\neq 0$.

*Example* 3.7. For the morphism of Figure 5, in the coordinate system described in Example 2.9, let $y=\alpha_1$ and $x=\alpha_2$ be the equations respectively of the horizontal and vertical lines and $(a_1,b_1)$, $(a_2,b_2)$, and $(a_3,b_3)$ be the coordinates of the three points from bottom to top. Then the hyperplanes $\Lambda_1,\dots,\Lambda_5$ have equations

$$\begin{cases} y=\alpha_1,\\ x=a_1,\\ y+2l_1+a_2-a_1=b_2,\\ y+l_2=\alpha_2,\\ y+2l_1+l_2+l_3+a_3-\alpha_2=b_3, \end{cases}$$

and

$$|\det(\Lambda_1\dots\Lambda_5)| = \begin{vmatrix} 0&1&0&1&0\\ 1&0&1&0&1\\ 0&0&2&0&2\\ 0&0&0&1&1\\ 0&0&0&0&1 \end{vmatrix} = 2.$$

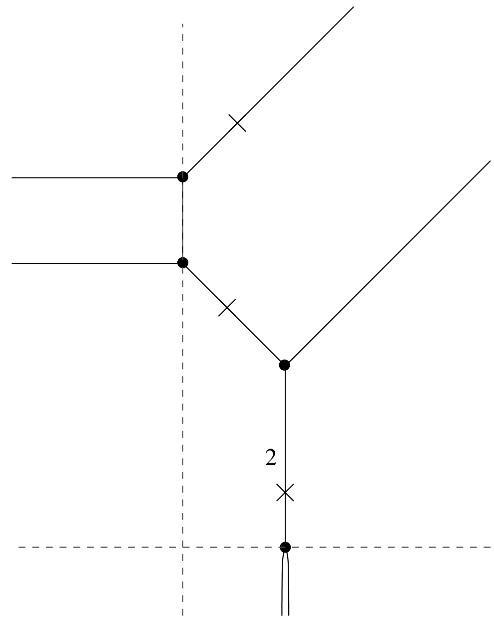



Figure 5. A conic tangent to two lines and passing through three points.

All the weights are 1 except the ones associated to the bottom-most point and the one associated to the tangency to the vertical line, which are 2, and thus the formula of Definition 3.6 becomes $\mu(f) = \frac{2}{2} \times 2^2 = 4$.

Theorem 3.8 (Correspondence theorem). *With the hypothesis above, we have*

$$N_{d,0}(k; d_1, \ldots, d_{3d-1-k}) = \sum_{f \in \mathcal{S}^{\mathbb{T}}(d,\mathcal{P},\mathcal{L})} \mu_{(\mathcal{P},\mathcal{L})}(f).$$

*Example* 3.9. For the configuration of points and lines considered in Example 3.7, there is only one morphism in $S^{\mathbb{T}}(2, \mathcal{P}, \mathcal{L})$, and $N_{2,0}(3; 1, 1)$ is indeed 4. Other examples are given in § 5.2.

*Remark* 3.10. In the proof of Theorem 3.8 (see § 7), we establish not only equality of both numbers, but we also give a correspondence between *phase-tropical curves* (see § 6 for the definition) and complex curves close to the tropical limit in the sense of § 6.1. In particular, if we choose real phases, in the sense of Definition 6.1 and Remark 6.3, for all constraints in $(\mathcal{P}, \mathcal{L})$, it is possible to recover all real algebraic curves passing through a configuration of real points and tangent to a configuration of real lines when these points and lines are close to the tropical limit. See § 7.2 for a few examples.

The definition of $\mu_{(\mathcal{P},\mathcal{L})}(f)$ we have given so far is not very convenient for actual computations. We will give in § 5 a practical way to compute this tropical multiplicity. We first clarify the notion of generic configurations in § 4.

### 3.3 Generalization to immersed constraints

In this section we generalize Theorem 3.8 to the case when the constraints are not necessarily non-singular tropical or complex curves but any immersed curves. Instead of considering curves of degree $d$ in the projective plane we now consider curves with a given Newton fan $\mathcal{N}$ in the two-dimensional torus $(\mathbb{C}^*)^2$.

We first pose the problem in complex geometry. Let $s \geqslant 2$, $k \leqslant s-1$, $g_1, \ldots, g_{s-1-k}$ be some non-negative integer numbers, and $\mathcal{N}, \mathcal{N}_1, \ldots, \mathcal{N}_{s-1-k}$ some Newton fans such that the number of elements of $\mathcal{N}$ is $s$ (recall that a Newton fan is a multiset). Choose a set $\mathcal{P} = \{p_1, \ldots, p_k\}$ of $k$ points in $(\mathbb{C}^*)^2$, and a set $\mathcal{L} = \{L_1, \ldots, L_{s-1-k}\}$ of $s-1-k$ immersed complex curves such that $L_i$ is of genus $g_i$ and has Newton fan $\mathcal{N}_i$.

We denote by $\mathcal{S}(\mathcal{N}, \mathcal{P}, \mathcal{L})$ the set of all rational complex algebraic maps $f : \mathbb{C}P^1 \backslash \{s \text{ points}\} \to (\mathbb{C}^*)^2$ with Newton fan $\mathcal{N}$, passing through all points $p_i \in \mathcal{P}$ and tangent to all curves $L_i \in \mathcal{L}$. The cardinal of the set $\mathcal{S}(\mathcal{N}, \mathcal{P}, \mathcal{L})$ is finite as long as $\mathcal{P}$ and $\mathcal{L}$ are chosen generically, and we define

$$N_{\mathcal{N},\mathcal{P},\mathcal{L}}(k; \mathcal{N}_1, g_1, \ldots, \mathcal{N}_{3d-1-k}, g_{3d-1-k}) = \sum_{f \in \mathcal{S}(\mathcal{N},\mathcal{P},\mathcal{L})} \frac{1}{|\mathrm{Aut}(f)|}.$$

The problem in the tropical set-up is similar. Choose now $\mathcal{P}^{\mathbb{T}} = \{p_1^{\mathbb{T}}, \ldots, p_k^{\mathbb{T}}\}$ a set of $k$ points in $\mathbb{R}^2$, and $\mathcal{L}^{\mathbb{T}} = \{L_1^{\mathbb{T}}, \ldots, L_{s-1-k}^{\mathbb{T}}\}$ a set of $s-1-k$ simple effective tropical cycles in $\mathbb{R}^2$ such that $L_i^{\mathbb{T}}$ has Newton fan $\mathcal{N}_i$ and genus $g_i$. We denote by $\mathcal{S}^{\mathbb{T}}(\mathcal{N}, \mathcal{P}^{\mathbb{T}}, \mathcal{L}^{\mathbb{T}})$ the set of minimal tropical morphisms $f : C \to \mathbb{R}^2$ with Newton fan $\mathcal{N}$, where $C$ is a rational tropical curve with $\partial C = \emptyset$, passing through all points $p_i^{\mathbb{T}}$ and pretangent to all curves $L_j^{\mathbb{T}}$. Next proposition is a direct consequence of Corollary 4.6.

Proposition 3.11. *The set $\mathcal{S}^{\mathbb{T}}(\mathcal{N}, \mathcal{P}^{\mathbb{T}}, \mathcal{L}^{\mathbb{T}})$ is finite and its elements are extreme in the sense of Definition 4.4.*

Moreover, Lemmas 3.4 and 3.5 still hold in this situation. If $p^{\mathbb{T}} \in \mathcal{P}^{\mathbb{T}}$, we define $w_p$ in the same way as in § 3.2. If $L^{\mathbb{T}} \in \mathcal{L}^{\mathbb{T}}$, we define $w_L$ as follows:

$$w_{L^{\mathbb{T}}} = \left( \sum_{v \in \mathrm{Vert}^0(L^{\mathbb{T}})} \sum_{e \in \mathcal{E}(v)} w_{f,e} \right) + \delta(\mu - \lambda)$$

where $\delta$ is the weight of the edge of $L$ containing the tangency set of $f$ and $L^{\mathbb{T}}$ (if any, otherwise $\mu = \lambda = 0$). The $(\mathcal{P}^{\mathbb{T}}, \mathcal{L}^{\mathbb{T}})$-multiplicity of a tropical morphism $f$ in $\mathcal{S}^{\mathbb{T}}(d, \mathcal{P}^{\mathbb{T}}, \mathcal{L}^{\mathbb{T}})$ is given by Definition 3.6.

Theorem 3.12 (Correspondence theorem in $(\mathbb{C}^*)^2$). *Let $\mathcal{N}$, $s$, $k$, $g_1, \ldots, g_{s-1-k}$, $\mathcal{N}_1, \ldots, \mathcal{N}_{s-1-k}$, $\mathcal{L}^{\mathbb{T}}$ and $\mathcal{P}^{\mathbb{T}}$ be as above.*

*There exists a generic configuration of points $\mathcal{P} = \{p_1, \ldots, p_k\}$ and immersed complex curves $\mathcal{L} = \{L_1, \ldots, L_{s-1-k}\}$ in $(\mathbb{C}^*)^2$ having respective Newton fans $\mathcal{N}_1, \ldots, \mathcal{N}_{s-1-k}$ and genera $g_1, \ldots, g_{s-1-k}$ such that*

$$N_{\mathcal{N},\mathcal{P},\mathcal{L}}(k; \mathcal{N}_1, g_1, \ldots, \mathcal{N}_{3d-1-k}, g_{s-1-k}) = \sum_{f \in \mathcal{S}^{\mathbb{T}}(\mathcal{N},\mathcal{P}^{\mathbb{T}},\mathcal{L}^{\mathbb{T}})} \mu_{(\mathcal{P}^{\mathbb{T}},\mathcal{L}^{\mathbb{T}})}(f).$$

Theorem 3.12 we will be proved in § 7.

## 4. Generic configurations of constraints

### 4.1 Marked tropical curves

In order to prove Proposition 3.3, it is convenient to consider marked tropical curves as a technical tool.

Definition 4.1. A *tropical curve with $n$ marked points* is a $(n+1)$-tuple $(C, x_1, \ldots, x_n)$ where $C$ is a tropical curve and the $x_i$ are $n$ points on $C$.

A *marked tropical morphism with* $n$ *marked points* is a $(n+2)$-tuple $(C, x_1, \dots, x_n, f)$ where $(C, x_1, \dots, x_n)$ is a tropical curve with $n$ marked points, and $f : C \to \mathbb{R}^2$ is a tropical morphism.

Note that we do not require the marked points to be distinct. As in the case of unmarked tropical curves, we have the notion of *isomorphic marked tropical morphisms* and *combinatorial types of marked tropical morphisms*. The definition is the same as in § 2.1, where we require in addition that the map $\phi : C_1 \to C_2$ sends the $i$th marked point of $C_1$ to the $ith$ marked point of $C_2$.

Lemma 2.8 has a straightforward extension to the case of marked tropical morphisms.

Lemma 4.2. *Let $\alpha$ be a combinatorial type of marked tropical morphisms $f : C \to \mathbb{R}^2$ with $n_\alpha$ marked points, $m_\alpha$ of which lie on edges of $C$, and where $C$ is a rational curve with $\partial C = \emptyset$. Then the space $\mathcal{M}_\alpha$ is an open convex polyhedron in the vector space $\mathbb{R}^{2+|\mathrm{Edge}^0(\alpha)|+m_\alpha}$, and*

$$\dim \mathcal{M}_\alpha = |\mathrm{Edge}^\infty(\alpha)| - 1 - \sum_{v \in \mathrm{Vert}^0(\alpha)} (\mathrm{val}(v) - 3) + m_\alpha.$$

Let $\alpha$ be a combinatorial type of marked minimal tropical morphisms $f : C \to \mathbb{R}^2$ where $C$ is a rational tropical curve with $\partial C = \emptyset$, with $s$ (open) ends, and with $n_\alpha$ marked points, $m_\alpha$ of which lie on edges of $C$. We denote by $\overline{\alpha}$ the combinatorial type of unmarked tropical morphisms underlying $\alpha$. We define the evaluation map $ev$ and the forgetful map $ft$ by

$$ev : \begin{array}{ccc} \mathcal{M}_\alpha & \longrightarrow & (\mathbb{R}^2)^{n_\alpha} \\ (C, x_1, \dots, x_{n_\alpha}, f) & \longmapsto & (f(x_1), \dots, f(x_{n_\alpha})) \end{array}, \qquad ft : \begin{array}{ccc} \mathcal{M}_\alpha & \longrightarrow & \mathcal{M}_{\overline{\alpha}} \\ (C, x_1, \dots, x_{n_\alpha}, f) & \longmapsto & (C, f). \end{array}$$

Lemma 4.3. *The maps $ev$ and $ft$ are $\mathbb{Z}$-affine linear on $\mathcal{M}_\alpha$, and*

$$\dim ev(\mathcal{M}_\alpha) \leqslant s - 1 + m_\alpha.$$

*Moreover, if equality holds, then $\alpha$ is 3-valent, and the evaluation map is injective on $\mathcal{M}_\alpha$.*

*Proof.* The proof of the $\mathbb{Z}$-linearity is the same as [GM07, Proposition 4.2], and the dimension is given by Lemma 4.2. □

## 4.2 Generic configurations

A *parameter space* is a space of the form

$$\mathrm{Par}(k, \beta_1, \dots, \beta_{s-1-k}) = (\mathbb{R}^2)^k \times \mathcal{M}_{\beta_1} \times \cdots \times \mathcal{M}_{\beta_{s-1-k}}$$

where $s \geqslant 2$ and $0 \leqslant k \leqslant s-1$ are two integer numbers, and $\beta_1, \dots, \beta_{s-1-k}$ are $s-1-k$ combinatorial types of simple effective tropical cycles in $\mathbb{R}^2$. According to Lemma 2.12, $\mathrm{Par}(k, \beta_1, \dots, \beta_{s-1-k})$ is an open convex polyhedron in some Euclidean vector space. In particular, it has a natural topology induced by this Euclidean space. Note that if $\mathrm{Vert}^0(\beta_i) = \emptyset$ (i.e. $\beta_i$ is just a ray), then $\mathcal{M}_{\beta_i} = \mathbb{R}$.

Given an element $(\mathcal{P}, \mathcal{L})$ of $\mathrm{Par}(k, \beta_1, \dots, \beta_{s-1-k})$, where $\mathcal{P} = \{p_1, \dots, p_k\}$ and $\mathcal{L} = \{L_1, \dots, L_{s-1-k}\}$, we denote by $\mathcal{C}(s, \mathcal{P}, \mathcal{L})$ the set of all minimal tropical morphisms $f : C \to \mathbb{R}^2$ where $C$ is a rational tropical curve with $s$ ends and $\partial C = \emptyset$, and $f$ passes through all points $p_i$, and is pretangent to all curves $L_j$.

Definition 4.4. A tropical morphism $f : C \to \mathbb{R}^2$ in $\mathcal{C}(s, \mathcal{P}, \mathcal{L})$ is said to be *extreme* if it satisfies the four following properties:

(i) $C$ is a 3-valent curve;

(ii) $f(\mathrm{Vert}^0(C)) \bigcap (\bigcup_{L \in \mathcal{L}} \mathrm{Vert}^0(L) \cup \mathcal{P}) = \emptyset$, i.e. no vertex of $C$ is mapped to a vertex of a curve in $\mathcal{L}$ nor to a point in $\mathcal{P}$;

(iii) given $p \in \mathcal{P}$, if $x$ and $x'$ are in $f^{-1}(p)$, then the path $\gamma$ in $C$ from $x$ to $x'$ is mapped to a segment by $f$;

(iv) given $L \in \mathcal{L}$, there exists a connected subgraph $\Gamma \subset C$ which contains all pretangency components of $f$ with $L$, and such that $f(\Gamma)$ is a segment in $\mathbb{R}^2$.

Note that in the case where $\mathrm{Vert}^0(L) = \emptyset$, property (iv) is equivalent to the fact that $E_L$ is connected.

Proposition 4.5. *Suppose that* $\mathrm{Vert}^0(\beta_i) = \emptyset$ *for all* $i$. *Then there exists a dense open subset* $\widetilde{\mathrm{Par}}(k, \beta_1, \ldots, \beta_{s-1-k})$ *in the parameter space* $\mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k}) = \mathbb{R}^{s-1+k}$ *such that the set* $\mathcal{C}(s, \mathcal{P}, \mathcal{L})$ *is finite and contains only extreme tropical morphisms.*

*Proof.* Denote by $\mathcal{C}'(s, \mathcal{P}, \mathcal{L})$ the set of all marked minimal tropical morphisms $f : (C, x_1, \ldots, x_{s-1}) \to \mathbb{R}^2$ where $(C, x_1, \ldots, x_{s-1})$ is a rational tropical curve with $s$ leaves, with $\partial C = \emptyset$, and with $s-1$ marked points, $x_{k+1}, \ldots, x_{s-1}$ being on vertices of $C$, such that $f(x_i) = p_i$ for $1 \leqslant i \leqslant k$ and $f(x_i) \in L_{i-k}$ for $k+1 \leqslant i \leqslant s-1$. By Definition 3.1, the set $\mathcal{C}(s, \mathcal{P}, \mathcal{L})$ is contained in the set $ft(\mathcal{C}'(s, \mathcal{P}, \mathcal{L}))$. Hence, it is sufficient to prove that there exists a dense open subset of the parameter space such that the set $\mathcal{C}'(s, \mathcal{P}, \mathcal{L})$ is finite and contains only extreme tropical morphisms.

Let $\alpha$ be a combinatorial type of marked tropical morphisms $f : (C, x_1, \ldots, x_{s-1}) \to \mathbb{R}^2$ where $(C, x_1, \ldots, x_{s-1})$ is a rational tropical curve with $s$ leaves, with $\partial C = \emptyset$, and with $s-1$ marked points, $x_{k+1}, \ldots, x_{s-1}$ being on vertices of $C$. Let us define the incidence variety $\mathfrak{I}_\alpha \subset \mathcal{M}_\alpha \times \mathbb{R}^{s-1+k}$ containing elements $(f, p_1, \ldots, p_k, L_1, \ldots, L_{s-1-k})$ where $f$ is a tropical morphisms with combinatorial type $\alpha$ such that:

(a) for $j \leqslant k$, $f(x_i) = p_i$;

(b) for $k+1 \leqslant j \leqslant s-1$, $f(x_j) \in L_{j-k}$.

Any of the above conditions (a) (respectively (b)) demands 2 (respectively 1) affine conditions on elements of $\mathfrak{I}_\alpha$. Moreover, all these conditions are independent since any variable $p_i$ or $L_i$ is contained in exactly one of these equations. Hence, the set $\mathfrak{I}_\alpha$ is an open polyhedral complex of dimension at most $\dim \mathcal{M}_\alpha$. Let us consider the two natural projections $\pi_1 : \mathfrak{I}_\alpha \to \mathcal{M}_\alpha$ and $\pi_2 : \mathfrak{I}_\alpha \to \mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k})$. For each $1 \leqslant i \leqslant s-1-k$, there is a natural linear isomorphism between $\mathcal{M}_{\beta_i}$ and the quotient of $\mathbb{R}^2$ by the linear direction of elements of $\beta_i$. This provides a natural map $\psi : (\mathbb{R}^2)^{s-1} \to \mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k})$ by taking the quotient of the $s-1-k$ lasts copies of $\mathbb{R}^2$ by the direction of the corresponding $\beta_i$. We have $\pi_2 = \psi \circ ev \circ \pi_1$. Hence according to Lemma 4.3, if $\dim \pi_2(\mathfrak{I}_\alpha) = s+k-1$, then $\alpha$ is necessarily trivalent, the evaluation map is injective on $\mathcal{M}_\alpha$, and $x_i \notin \mathrm{Vert}^0(C)$ for $i \leqslant k$.

We define $\widetilde{\mathrm{Par}}(k, \beta_1, \ldots, \beta_{s-1-k})$ as the complement in $\mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k})$ of the union of the sets $\pi_2(\mathfrak{I}_\alpha)$ where $\alpha$ ranges over all combinatorial types such that $\dim \pi_2(\mathcal{M}_\alpha) < s-1+k$. This is an open and dense subset of $\mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k})$. Moreover, by injectivity of the evaluation map, if $(\mathcal{P}, \mathcal{L})$ is in $\widetilde{\mathrm{Par}}(s, k, \beta_1, \ldots, \beta_{s-1-k})$, then the fiber $\pi_2^{-1}(\mathcal{P}, \mathcal{L})$ consists of at most one point for any combinatorial type. The number of possible combinatorial types $\alpha$ is finite, and so is the set $\mathcal{C}'(s, \mathcal{P}, \mathcal{L})$. Moreover, all its elements satisfy properties (i) and (ii) of Definition 4.4.

Let us now find an open dense subset of $\widetilde{\mathrm{Par}}(k, \beta_1, \ldots, \beta_{s-1-k})$ which will ensure properties (iii) and (iv). Let $\alpha$ be a combinatorial type considered above of marked tropical morphisms, and let $\alpha'$ be a combinatorial type of marked tropical morphisms consisting of $\alpha$ with an additional marked point $x_s$. We also fix $1 \leqslant i \leqslant s-1$, and we suppose that the path $\gamma$ in $C$ joining $x_s$ to $x_i$ is not mapped to a segment by $f$.

If $i \leqslant k$, we define $\mathfrak{I}_{\alpha',i} \subset \mathcal{M}_{\alpha'} \times \mathbb{R}^{s-1+k} \times \mathbb{R}^2$ by the conditions (a) and (b) above and the condition $f(x_s) = p_s$ ($p_s$ is the coordinate corresponding to the copy of $\mathbb{R}^2$ we added to $\mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k})$). We still have the two projections $\pi_1 : \mathfrak{I}_{\alpha',i} \to \mathcal{M}_{\alpha'}$ and $\pi_2 : \mathfrak{I}_{\alpha',i} \to \mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k}) \times \mathbb{R}^2$, in addition to the projection $\pi : \mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k}) \times \mathbb{R}^2 \to \mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k})$. According to the previous study, the set $\pi_2(\mathfrak{I}_{\alpha',i})$ has codimension at least 1 in $\mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k}) \times \mathbb{R}^2$, and none of the two sets $\pi_2(\mathfrak{I}_{\alpha',i})$ and $\{p_i = p_s\}$ contains the other. Since the latter has codimension 2 in $\mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k}) \times \mathbb{R}^2$, the intersection $X_{\alpha',i}$ of $\pi_2(\mathfrak{I}_{\alpha',i})$ and $\{p_i = p_s\}$ has codimension at least 3, and $\pi(X_{\alpha',i})$ has codimension at least 1 in $\mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k})$.

If $k+1 \leqslant i \leqslant s-1$, we define $\mathfrak{I}_{\alpha',i} \subset \mathcal{M}_{\alpha'} \times \mathbb{R}^{s-1+k} \times \mathcal{M}_{\beta_{i-k}}$ by the conditions (a) and (b) above and the condition $f(x_s) \in L_s$ ($L_s$ is the coordinate corresponding to the copy of $\mathcal{M}_{\beta_{i-k}}$ we added to $\mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k})$). We still have the two projections $\pi_1 : \mathfrak{I}_{\alpha',i} \to \mathcal{M}_{\alpha'}$ and $\pi_2 : \mathfrak{I}_{\alpha',i} \to \mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k}) \times \mathcal{M}_{\beta_{i-k}}$, in addition to the projection $\pi : \mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k}) \times \mathcal{M}_{\beta_{i-k}} \to \mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k})$. According to the previous study, the set $\pi_2(\mathfrak{I}_{\alpha',i})$ has codimension at least 1 in $\mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k}) \times \mathcal{M}_{\beta_{i-k}}$. Moreover, since the path $\gamma$ in $C$ joining $x_s$ to $x_i$ is not mapped to a segment by $f$, none of the two sets $\pi_2(\mathfrak{I}_{\alpha',i})$ and $\{L_{i-k} = L_s\}$ contains the other. Since the latter has codimension 1 in $\mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k}) \times \mathcal{M}_{\beta_{i-k}}$, the intersection $X_{\alpha',i}$ of $\pi_2(\mathfrak{I}_{\alpha',i})$ and $\{L_{i-k} = L_s\}$ has codimension at least 2, and $\pi(X_{\alpha',i})$ has codimension at least 1 in $\mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k})$.

We define

$$\widehat{\mathrm{Par}}(k, \beta_1, \ldots, \beta_{s-1-k}) = \widetilde{\mathrm{Par}}(k, \beta_1, \ldots, \beta_{s-1-k}) \setminus \left( \bigcup_{\alpha',i} \pi(X_{\alpha',i}) \right)$$

where $\alpha'$ and $i$ range over all possible choices in the preceding construction. Since the number of choices is finite, $\widehat{\mathrm{Par}}(k, \beta_1, \ldots, \beta_{s-1-k})$ is a dense open subset of $\mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k})$, and for $(\mathcal{P}, \mathcal{L})$ in this set, the set $\mathcal{C}'(s, \mathcal{P}, \mathcal{L})$ is finite and all its elements are extreme. □

Corollary 4.6. *There exists a dense open subset* $\widehat{\mathrm{Par}}(k, \beta_1, \ldots, \beta_{s-1-k})$ *in the parameter space* $\mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k})$ *such that the set* $\mathcal{C}(s, \mathcal{P}, \mathcal{L})$ *is finite and only contains extreme tropical morphisms.*

*Proof.* Choose $0 \leqslant k' \leqslant s-1-k$, and $1 \leqslant i_1 < \cdots < i_{k'} \leqslant k$. We denote by $\{j_1, \ldots, j_{s-1-k-k'}\}$ the complement of $\{i_1, \ldots, i_{k'}\}$ in $\{1, \ldots, s-1-k\}$. Choose a vertex $v_{i_n}$ on each $\beta_{i_n}$, and an edge $e_{j_n}$ on each $\beta_{j_n}$. Given an element $L_{j_n}$ in $\mathcal{M}_{\beta_{j_n}}$, we denote by $L'_{j_n}$ the effective tropical cycle in $\mathbb{R}^2$ spanned by $e_{j_n}$ (this is just the classical line of $\mathbb{R}^2$ spanned by $e_{j_n}$), and denote by $\delta_{j,n}$ its combinatorial type. Then we have a natural surjective linear map

$$\begin{array}{rrcl} \chi : & \mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k}) & \longrightarrow & \mathrm{Par}(k+k', \delta_{j_1}, \ldots, \delta_{j_{s-1-k-k'}}) \\ & (p_1, \ldots, p_k, L_1 \ldots, L_{s-1-k}) & \longmapsto & (p_1, \ldots, p_k, v_{i_1}, \ldots, v_{i_{k'}}, L'_{j_1} \ldots, L'_{j_{s-1-k-k'}}). \end{array}$$

We define

$$\widetilde{\mathrm{Par}}(k, \beta_1, \ldots, \beta_{s-1-k}) = \bigcap \chi^{-1}(\widehat{\mathrm{Par}}(k+k', \delta_{j_1}, \ldots, \delta_{j_{s-1-k-k'}}))$$

where $k', i_1, \ldots, i_{k'}, v_{i_1}, \ldots, v_{i_{k'}}, e_{j_1}, \ldots, e_{j_{s-1-k-k'}}$ run over all possible choices. Note that this number of choices is finite, and that $\widetilde{\mathrm{Par}}(k, \beta_1, \ldots, \beta_{s-1-k})$ is open and dense in $\mathrm{Par}(k, \beta_1, \ldots, \beta_{s-1-k})$.

Let $(\mathcal{P}, \mathcal{L}) \in \widetilde{\operatorname{Par}}(k, \beta_1, \ldots, \beta_{s-1-k})$, and $f : C \to \mathbb{R}^2$ an element of $\mathcal{C}(s, \mathcal{P}, \mathcal{L})$. By Definition 3.1, the tropical morphism $f : C \to \mathbb{R}^2$ is also an element of some set

$$\mathcal{C}(s, \{p_1, \ldots, p_k, v_{i_1}, \ldots, v_{i_{k'}}\}, \{L'_{j_1} \ldots, L'_{j_{s-1-k-k'}}\})$$

constructed as above. Hence, according to Proposition 4.5, the set $\mathcal{C}(s, \mathcal{P}, \mathcal{L})$ is finite and all its elements satisfy properties (i)–(iii) of Definition 4.4.

Moreover, for any tangency component $E$ of $f$ with an element $L$ of $\mathcal{L}$, the set $f(E)$ is contained in a classical line of $\mathbb{R}^2$. Indeed, otherwise there would exist $v \in \operatorname{Vert}^0(C)$ such that $f(v) \in \operatorname{Vert}^0(L)$, in contradiction with the fact that $f$ satisfies property (ii) of Definition 4.4. Hence, we can use the same technique we used at the end of the proof of Proposition 4.5 to construct an open dense subset $\widehat{\operatorname{Par}}(k, \beta_1, \ldots, \beta_{s-1-k})$ of $\widetilde{\operatorname{Par}}(k, \beta_1, \ldots, \beta_{s-1-k})$ such that if $(\mathcal{P}, \mathcal{L}) \in \widehat{\operatorname{Par}}(k, \beta_1, \ldots, \beta_{s-1-k})$, then all elements of $\mathcal{C}(s, \mathcal{P}, \mathcal{L})$ satisfy property (iv) of Definition 4.4. That is to say, all elements of $\mathcal{C}(s, \mathcal{P}, \mathcal{L})$ are extreme. □

Definition 4.7. Let $p_1, \ldots, p_k$ be $k$ points in $\mathbb{R}^2$ and $L_1, \ldots, L_l$ be $l$ simple effective tropical cycles in $\mathbb{R}^2$ of combinatorial types $\beta_1, \ldots, \beta_l$. The $(k+l)$-tuple $(p_1, \ldots, p_k, L_1, \ldots, L_l)$ is called *weakly generic* if it is an element of $\widehat{\operatorname{Par}}(k, \beta_1, \ldots, \beta_l)$.

The $(k+l)$-tuple $(p_1, \ldots, p_k, L_1, \ldots, L_l)$ is called *generic* if any of its sub-tuple is weakly generic.

Automorphisms of elements of $\mathcal{C}(s, \mathcal{P}, \mathcal{L})$ are pretty simple when dealing with generic configurations.

Lemma 4.8. *Let $(\mathcal{P}, \mathcal{L})$ be a generic configuration of $s$ points and simple effective tropical cycles, and let $f : C \to \mathbb{R}^2$ be an element of $\mathcal{C}(s, \mathcal{P}, \mathcal{L})$. Denote by $(e_1^1, e_2^1), \ldots, (e_1^l, e_2^l)$ all pairs of open ends of $C$ with $u_{f,e_1^i} = u_{f,e_2^i}$ and adjacent to a common vertex in $\operatorname{Vert}^0(C)$. Denote also by $\phi_{e_1^i, e_2^i}$ the automorphism of $C$ such that $\phi_{e_1^i, e_2^i}(e_1^i) = e_2^i$ and $\phi_{e_1^i, e_2^i | C \setminus \{e_1^i, e_2^i\}} = \mathrm{Id}$. Then $\operatorname{Aut}(f)$ is the abelian group generated by the automorphisms $\phi_{e_1^i, e_2^i}$, i.e.*

$$\operatorname{Aut}(f) = \langle \phi_{e_1^i, e_2^i}, i = 1, \ldots, l \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^l.$$

*Proof.* It is clear that the automorphisms $\phi_{e_1^i, e_2^i}$ commute and are of order 2. We just have to prove that they generate $\operatorname{Aut}(f)$. Suppose that this group is non-trivial and let $\phi \neq \mathrm{Id}$ be an element of $\operatorname{Aut}(f)$. Since $\phi$ is non-trivial, there exist two distinct edges $e_1$ and $e_2$ in $\operatorname{Edge}(C)$ such that $\phi(e_1) = e_2$. Since two vertices of $C$ cannot have the same image by $f$, the edges $e_1$ and $e_2$ must be adjacent to the same vertices. The tropical curve $C$ is rational, so $e_1$ and $e_2$ must be adjacent to exactly one vertex, which means that they are open ends of $C$. □

## 5. Practical computation of tropical multiplicities

### 5.1 Combinatorial multiplicity

Here we give a practical way to compute the determinant in Definition 3.6 by a standard cutting procedure (see for example [GM08]). Let us fix a generic configuration $(\mathcal{P}, \mathcal{L}) \in \operatorname{Par}(k, \beta_1, \ldots, \beta_{3d-1-k})$ and an element $f : C \to \mathbb{R}^2$ of $\mathcal{S}^{\mathbb{T}}(d, \mathcal{P}, \mathcal{L})$ tangent to $\mathcal{L}$. We choose a marking of $C$ such that $f : (C, x_1, \ldots, x_{3d-1}) \to \mathbb{R}^2$ is an element of one of the sets $\mathcal{C}'(3d, \mathcal{P}', \mathcal{L}')$ defined in the proof of Proposition 4.5, and we define $\mathring{C} = C \setminus \{x_1, \ldots, x_{3d-1}\}$.

First, we define an orientation on $\mathring{C}$. Let $x$ be a point on an edge of $\mathring{C}$. Since $C$ is rational, $C \setminus \{x\}$ has two connected components $C_1$ and $C_2$ containing respectively $s_1$ and $s_2$ ends, and $s_1 + s_2 = 3d + 2$. Moreover, since $(\mathcal{P}, \mathcal{L})$ is generic, $C_1$ (respectively $C_2$) contains $k_1 \leqslant s_1 - 1$

(respectively $k_2 \leqslant s_2 - 1$) marked points. Since $k_1 + k_2 = 3d - 1 = s_1 + s_2 - 3$, up to exchanging $C_1$ and $C_2$, we have $k_1 = s_1 - 1$ and $k_1 = s_2 - 2$. We orient $\overset{\circ}{C}$ at $x$ from $C_1$ to $C_2$. Note that $\overset{\circ}{C}$ and its orientation depends on the choice of the marking of $C$ we have chosen, but this will not play a role in what follows.

Now we define a multiplicity $\mu_{(\mathcal{P},\mathcal{L})}(v)$ for each vertex $v$ in $\mathrm{Vert}^0(C)$. If $f(v) \notin \bigcup_{L\in\mathcal{L}} L$, then the genericity of $(\mathcal{P},\mathcal{L})$ implies that there exist two edges $e_1, e_2 \in \mathrm{Edge}(C)$ adjacent to $v$ and oriented toward $v$. We define

$$\mu_{(\mathcal{P},\mathcal{L})}(v) = |\det(u_{f,e_1}, u_{f,e_2})|.$$

If $f(v) \in L_i$, we denote by $u_{L_i}$ the primitive integer direction of the edge of $L_i$ containing $f(v)$. If $f(v) \in L_i \setminus \bigcup_{L\neq L_i} L$, then the genericity of $(\mathcal{P},\mathcal{L})$ implies that there exists exactly one edge $e \in \mathrm{Edge}(C)$ oriented toward $v$ with $u_{f,e} \neq u_{L_i}$, and we define

$$\mu_{(\mathcal{P},\mathcal{L})}(v) = |\det(u_{f,e_1}, u_{L_i})|.$$

If $f(v) \in L_i \cap L_j$, we define

$$\mu_{(\mathcal{P},\mathcal{L})}(v) = |\det(u_{L_i}, u_{L_j})|.$$

Proposition 5.1. *For any tropical morphism $f : C \to \mathbb{R}^2$ in $\mathcal{S}^{\mathbb{T}}(d,\mathcal{P},\mathcal{L})$, we have*

$$\mu_{(\mathcal{P},\mathcal{L})}(f) = \frac{1}{|\mathrm{Aut}(f)|} \prod_{q\in\mathcal{P}\cup\mathcal{L}} w_q \prod_{e\in\mathrm{Edge}^0(C)} w_{f,e} \prod_{v\in\mathrm{Vert}^0(C)} \mu_{(\mathcal{P},\mathcal{L})}(v).$$

We prove Proposition 5.1 by writing down explicitly the linear part of the equations of the hyperplanes $\Lambda_q$. Before going deeper into the details, we remark that the definition of the multiplicity of a tropical morphism in $\mathcal{S}^{\mathbb{T}}(d,\mathcal{P},\mathcal{L})$ and of the orientation of $\overset{\circ}{C}$ are based only on the fact that all elements of $\mathcal{C}(3d,\mathcal{P},\mathcal{L})$ are extreme. Hence, we can extend Definition 3.6 and the orientation of $\overset{\circ}{C}$ to any tropical morphism in $\mathcal{C}(s,\mathcal{P},\mathcal{L})$ for any $s$, as long as $(\mathcal{P},\mathcal{L})$ is generic.

For the rest of this section, we fix some positive integer $s$, a generic configuration $(\mathcal{P},\mathcal{L})$ of constraints containing $s$ elements, and an element $f : C \to \mathbb{R}^2$ of $\mathcal{C}(s,\mathcal{P},\mathcal{L})$ tangent to $\mathcal{L}$. In particular, $\mu_{(\mathcal{P},\mathcal{L})}(f) \neq 0$. Let us choose a vertex $v_1 \in \mathrm{Vert}^0(C)$, and some ordering of edges in $\mathrm{Edge}^0(C)$. If $v \in \mathrm{Vert}^0(C)$, we denote by $(v_1 v)$ the path joining $v_1$ to $v$ in $C$. In particular we have

$$f(v) = f(v_1) + \sum_{e\in(v_1 v)} l_e w_{f,e} u_{f,e} \tag{3}$$

where the vectors $u_{f,e}$ are oriented toward $v_1$ (recall that $w_{f,e}$ and $l_e$ are respectively the weight and the length of $e$).

Given $p \in \mathcal{P}$, we choose $v \in \mathrm{Vert}^0(C)$ adjacent to an edge $e$ of $C$ such that $e \cap f^{-1}(p_i) \neq \emptyset$. Then, in the coordinate used in the proof of Lemma 2.8, the linear part of the equation of $\Lambda_p$ is given by

$$|\det(f(v), u_{f,e})| = 0. \tag{4}$$

Let $L$ be an element of $\mathcal{L}$. If there exist $v_0 \in \mathrm{Vert}^0(L)$, $e \in \mathrm{Edge}(C)$ adjacent to $v \in \mathrm{Vert}^0(C)$, such that $e \cap f^{-1}(v_0) \neq \emptyset$, then the linear part of the equation of $\Lambda_L$ is given by

$$|\det(f(v), u_{f,e})| = 0. \tag{5}$$

If there exists a vertex $v \in \mathrm{Vert}^0(C)$ such that $f(v) \in L$, then the linear part of the equation of $\Lambda_L$ is given by

$$|\det(f(v), u_L)| = 0 \tag{6}$$

where $u_L$ is the primitive integer direction of the edges of $L$ containing $f(v)$.

Equations (4)–(6) do not depend on the choice of $e$ and $v$ thanks to properties (iii) and (iv) in Definition 4.4.

To sum up, if $q \in \mathcal{P} \cup \mathcal{L}$, the linear part of the equation of $\Lambda_q$ is always of the form $|\det(f(v_q), u_q)| = 0$, for some $v_q \in \mathrm{Vert}^0(C)$ and $u_q = (u_{q,1}, u_{q,2}) \in \mathbb{R}^2$. Thus, the coefficients of the matrix $M(f)$ of intersection of all the $\Lambda_q$ are given by (3), as follows.

- The coefficient corresponding to the hyperplane $\Lambda_q$ and $x_{v_1}$ is $u_{q,2}$.
- The coefficient corresponding to the hyperplane $\Lambda_q$ and $y_{v_1}$ is $-u_{q,1}$.
- The coefficient corresponding to the hyperplane $\Lambda_q$ and $e \in \mathrm{Edge}^0(C)$ is

  $w_{f,e} \det(u_{f,e}, u_q)$, if $e \in (v_1 v_q)$,
  0 otherwise.

Clearly, the matrix $M(f)$ depends on the choice of the coordinates we choose on the deformation space of $f$. However, $M(f)$ is well defined up to a multiplication by a matrix in $\mathrm{GL}_{s-1}(\mathbb{Z})$, hence the absolute value of its determinant does not depend on this choice.

Lemma 5.2. *Suppose that there exists $L \in \mathcal{L}$ such that we are in one of the two following situations.*

- *There exists a vertex $v \in \mathrm{Vert}^0(C)$ adjacent to two edges $e_1, e_2 \in \mathrm{Edge}^0(C)$ with $u_{f,e_1} = u_L$ and $u_{f,e_2} \neq u_L$, and such that $f(v) \in L$; in this case, choose $p$ in $f(e_1) \cap L$.*
- *There exists a point $x \in C$ such that $f(x) \in \mathrm{Vert}^0(L)$; in this case, put $p = f(x)$.*

*Define $\mathcal{P}' = \mathcal{P} \cup \{p\}$ and $\mathcal{L}' = \mathcal{L} \backslash \{L\}$. Then*

$$\mu_{(\mathcal{P},\mathcal{L})}(f) = \frac{w_L}{w_p} \mu_{(\mathcal{P}',\mathcal{L}')}(f).$$

*In particular, Proposition 5.1 for $\mu_{(\mathcal{P},\mathcal{L})}(f)$ follows from Proposition 5.1 applied to $\mu_{(\mathcal{P}',\mathcal{L}')}(f)$.*

*Proof.* The linear part of the equations of $\Lambda_L$ and $\Lambda_p$ are the same. □

Let us now explain the cutting procedure to compute $\mu_{(\mathcal{P},\mathcal{L})}(f)$ in general. Suppose that there exist $e \in \mathrm{Edge}^0(C)$ and $x \in e$ such that $f(x) \notin \mathcal{P} \bigcup_{L \in \mathcal{L}} L$. Recall that we have defined an orientation on $C$ at $x$. The space $C \backslash \{x\}$ has two connected components, $C_1$ and $C_2$, containing respectively $s_1$ and $s_2$ ends. We choose $C_1$ and $C_2$ so that $C$ is oriented from $C_1$ to $C_2$ at $x$. It is clear that $s_1 + s_2 = s + 2$. The graph $C_i$ inherits a tropical structure from the tropical curve $C$, and there is a unique way to extend $C_i$ to a rational tropical curve $\overline{C}_i$ without boundary components. Note that the tropical morphism $f : C \to \mathbb{R}^2$ induces tropical morphisms $f_i : \overline{C}_i \to \mathbb{R}^2$.

Let $\mathcal{P}_i \subset \mathcal{P}$ be the points of $\mathcal{P}$ through which $f_i$ passes, and $\mathcal{L}_i \subset \mathcal{L}$ be the curves of $\mathcal{L}$ which are pretangent to $f_i$. The configuration $(\mathcal{P}, \mathcal{L})$ is generic, so we have $|\mathcal{P}_i| + |\mathcal{L}_i| \leqslant s_i - 1$. Since we also have $s_1 + s_2 = s + 2$ and $|\mathcal{P}| + |\mathcal{L}| = s - 1$, we deduce that $|(\mathcal{P}_1 \cup \mathcal{L}_1) \cap (\mathcal{P}_2 \cup \mathcal{L}_2)| \leqslant 1$.

If $(\mathcal{P}_1 \cup \mathcal{L}_1) \cap (\mathcal{P}_2 \cup \mathcal{L}_2) = \emptyset$, then $|\mathcal{P}_2| + |\mathcal{L}_2| = s_2 - 2$ because of the orientation of $C$ at $x$. In this case we define $\mathcal{P}_2' = \mathcal{P}_2 \cup \{f(x)\}$ and $\nu = 1$.

If $|(\mathcal{P}_1 \cup \mathcal{L}_1) \cap (\mathcal{P}_2 \cup \mathcal{L}_2)| = \{q_0\}$, then we define $\mathcal{P}_2' = \mathcal{P}_2$ and

$$\nu = \frac{w_{q_0}(f) w_{f,e}}{w_{q_0}(f_2) w_{q_0}(f_1)}.$$

Since the three distinct tropical morphisms $f$, $f_1$, and $f_2$ pass through or are tangent to $q_0$, we specify to which morphism the quantity $w_{q_0}$ refers in the previous formula.

Lemma 5.3. *We have*

$$\mu_{(\mathcal{P},\mathcal{L})}(f) = \nu\mu_{(\mathcal{P}_1,\mathcal{L}_1)}(f_1)\mu_{(\mathcal{P}'_2,\mathcal{L}_2)}(f_2).$$

*In particular, Proposition 5.1 for $\mu_{(\mathcal{P},\mathcal{L})}(f)$ follows from Proposition 5.1 applied to $\mu_{(\mathcal{P}_1,\mathcal{L}_1)}(f_1)$ and $\mu_{(\mathcal{P}'_2,\mathcal{L}_2)}(f_2)$.*

*Proof.* For the coordinates of the deformation space of $f_1$, we choose the root vertex to be the vertex of $C_1$ adjacent to $e$. For the deformation space of $f_2$, we choose the root vertex to be the vertex of $C_2$ adjacent to $e$. We also suppose that the first line of $M(f_2)$ is given by $\Lambda_{f(x)}$ or $\Lambda_{q_0}$. Choose an order $q_2,\dots,q_{s_2-1}$ on the other elements of $\mathcal{P}_2 \cup \mathcal{L}_1$. Then we have

$$M(f_2) = \begin{pmatrix} u_{e,2} & -u_{e,1} & 0 \\ U_1 & U_2 & A \end{pmatrix}$$

where $A$ is a $(s_2-2)\times(s_2-3)$ matrix and

$$U_1 = \begin{pmatrix} u_{q_2,2} \\ \vdots \\ u_{q_{s_2-1},2} \end{pmatrix} \quad \text{and} \quad U_2 = \begin{pmatrix} -u_{q_2,1} \\ \vdots \\ -u_{q_{s_2-1},1} \end{pmatrix}.$$

Hence, eliminating the coefficient $-u_{e,1}$ by elementary operations on the column of $M(f_2)$ we get

$$\det(M(f_2)) = \det\left(\begin{pmatrix} U_3 & A \end{pmatrix}\right)$$

where

$$U_3 = \begin{pmatrix} \det(u_{f,e}, u_{q_2}) \\ \vdots \\ \det(u_{f,e}, u_{q_{s_2-1}}) \end{pmatrix}.$$

We choose coordinates on the deformation space of $f$ correspondingly to the one we chose for $f_1$ and $f_2$: the root vertex is the root vertex we chose for $f_1$, and the order on the edges in $\mathrm{Edge}^0(C)$ is given by first the edges in $\mathrm{Edge}^0(C_1)$, then $e$, and then the edges in $\mathrm{Edge}^0(C_2)$. Then, we have

$$M(f) = \begin{pmatrix} M(f_1) & 0 & 0 \\ * & w_{f,e}U_3 & A \end{pmatrix}.$$

Hence $\det(M(f)) = w_{f,e}\det(M(f_1))\det(M(f_2))$. To conclude, we remark that according to Lemma 4.8, we have $\mathrm{Aut}(f) = \mathrm{Aut}(f_1)\times\mathrm{Aut}(f_2)$. □

*Proof of Proposition 5.1* Applying recursively Lemmas 5.3 and 5.2, we reduce to cases when $\mathrm{Vert}^0(C)$ has at most one element, for which we can easily check by hand that Proposition 5.1 is true. □

### 5.2 Examples of computations

Let us first compute again the multiplicity of the conic of Example 3.7 pictured in Figure 5 using the combinatorial procedure described above.

The automorphism group of the morphism is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, the multiplicities $\mu_q$ of the constraints is 1 except for the point sitting on an edge of weight 2 and for the vertical line in which case it is 2. There is an edge of weight 2 contributing for 2 in the multiplicity and

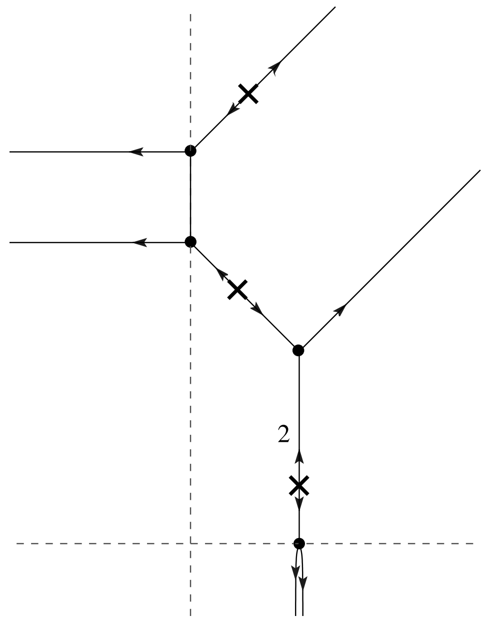



FIGURE 6. Conic (with orientations) tangent to two lines.

the vertices all have multiplicity 1 as can be seen using the formula given at the beginning of previous section and the orientations described in Figure 6. The multiplicity of this morphism is thus $\mu_{(\mathcal{P},\mathcal{L})}(f) = \frac{1}{2} \times 2^2 \times 2 \times 1 = 4$ which is indeed the number of conics tangent to two lines and passing through three points provided that the configuration is generic.

One can check using the same techniques and Figure 7 the number of conics tangent to $5-k$ lines when $k$ varies from 0 to 5. In each case there is only one tropical curve satisfying the constraints on which all the classical conics degenerate. Their multiplicities are $\min(2^k, 2^{5-k})$.

Finally, we draw on Figure 8 one of the rational cubics tangent to seven lines and passing through one point.

In this case the automorphism group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, all multiplicities of constraints are 1 except that of the point and of the line having its vertex on a vertical edge of weight 2 which are 3. The product of the weights of interior edges is $2^4 \times 3$ and all vertices have multiplicity 1. This morphism thus contributes $\mu_{(\mathcal{P},\mathcal{L})}(f) = 54$ to the seven hundred classical rational cubics tangent to seven lines.

## 6. Phases and tropical limits

Our strategy is to approximate our tropical ambient space with constraints $(\mathbb{R}^2, \mathcal{P}, \mathcal{L})$ with a family of corresponding classical ambient spaces. In order to prove correspondence theorems with tangency conditions as some of the constraints, we introduce the notion of *phase-tropical structure* on a tropical variety and that of *tropical limit*. For simplicity, in this paper we define both concepts only in the special case we need, namely for points and curves in toric surfaces. See [Mik] for a more general case. Such an approach allows us to generalize correspondence statements started in [Mik05] and followed in [Nis10, NS06, Tyo12] to curves tangent to given curves.

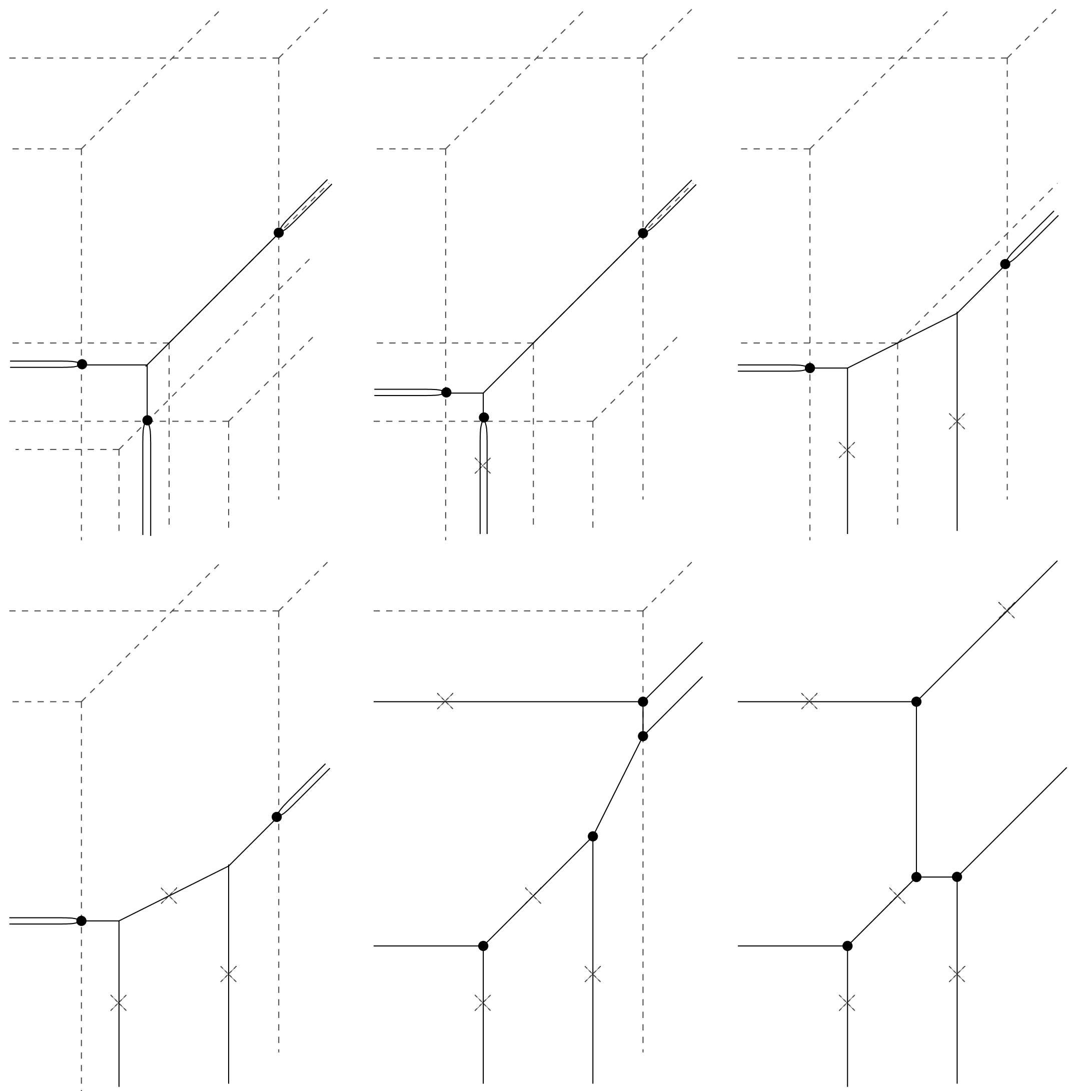


Figure 7. Conics tangent to $5-k$ lines, $k=0,\dots,5$.

All tropical curves $C$ considered in this section have no boundary component, i.e. $\partial C=\emptyset$ (see § 2.1). We do not make any assumption on the genus of the curves in § 6.1: the hypothesis of being rational will be necessary only starting from § 6.2.

## 6.1 Phase-tropical structures and tropical limits

Let $p\in\mathbb{R}^n$ be a point.

Definition 6.1. The *phase of* $p$ is a choice of a point $\phi(p)\in(S^1)^n$. Alternatively, we may think of it as a choice of point in $(\mathbb{C}^*)^n$ as long as we identify two phases $\phi(p),\phi(p')\in(\mathbb{C}^*)^n$ whenever they have the same argument

$$\operatorname{Arg}(\phi(p))=\operatorname{Arg}(\phi'(p))\in(S^1)^n.$$

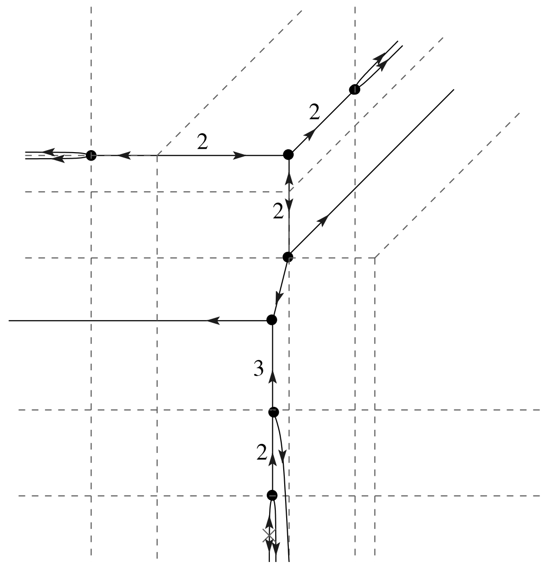



Figure 8. Cubic (with orientations) tangent to seven lines.

The phase of $p$ is *real* if $\phi(p) \in \mathrm{Arg}((\mathbb{R}^*)^n) = \{0, \pi\}^n$. The phase of the collection $\mathcal{P}$ of points is a choice of phase for each point of $\mathcal{P}$.

Clearly, in the case of points the phase is nothing else but prescription of arguments to the coordinates. In the case of curves we need to prescribe a phase for each vertex of the curve so that this prescription is compatible at each edge.

Before defining the phase-tropical structure for curves we consider some motivations for it. We refer to the upcoming paper [Mik] for a more thorough treatment, but as this paper has not appeared yet we give below a preview relevant to our purposes here. A tropical curve can be thought of as a certain degeneration of a sequence (or a family, which can be thought of as a generalized sequence) of complex curves, i.e. Riemann surfaces $S_{t_j}$ whose genus $g$ and number of punctures $k$ do not depend on the parameter $t_j$.

From the hyperbolic geometry viewpoint, each $S_{t_j}$ is a hyperbolic surface and is completely determined by the length of any collection of $3g - 3 + k$ disjoint closed embedded geodesics (see [Thu97]). Such a collection defines a decomposition of $S_{t_j}$ into pairs-of-pants. This means that every connected component of the complement of this collection is homeomorphic to a sphere punctured three times. Some of these punctures correspond to the punctures of $S$ while others correspond to a geodesic from our collection. Note that each geodesic corresponds to two punctures of pairs-of-pants, cf. Figure 9. Conversely, each decomposition into pairs-of-pants gives a collection of $3g - 3 + k$ disjoint closed embedded geodesics once we represent each cutting circle by a geodesic in the hyperbolic metric of $S_{t_j}$. Once all surfaces $S_{t_j}$ are marked (i.e. a homotopy equivalence with a 'standard surface' $S$ of genus $g$ with $k$ punctures is fixed) the pairs-of-pants decomposition in $S_{t_j}$ with different $t_j$ can be chosen in a compatible way.

There may not be more than $3g - 3 + k$ disjoint closed embedded geodesics, but we may also consider collection consisting less than that number. The result can be thought of as a generalized (or partial) pairs-of-pants decomposition. Some component of the complement of such a collection are spheres punctures three times while the others have a greater number of punctured or even a genus. Thus the conformal structure of some component is no longer

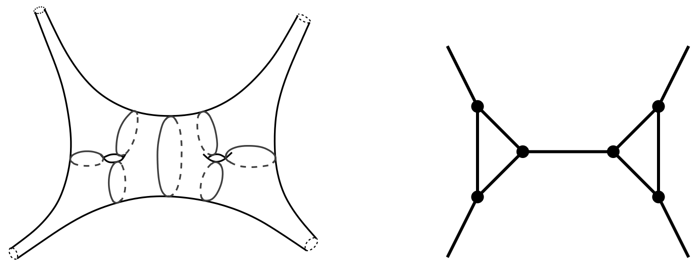


Figure 9. Pairs-of-pants decomposition of a punctured Riemann surface and a tropical curve realizing the dual graph.

determined by the lengths of the boundary geodesic and we need to specify it separately. As we are preparing the framework for the tropical limit where the hyperbolic lengths of all these boundary geodesics vanish we need only to consider conformal structures of finite type (i.e. such that all the end components correspond to punctures conformally).

Any non-compact Riemann surface $S$ of finite type is obtained from a closed Riemann surface $\tilde{S}$ by puncturing it in finitely many points. It is convenient to define a compact surface $\bar{S}$ obtained by an oriented real blowup of $\tilde{S}$ at the points of puncture as in [MO07, § 6]. Then each puncture $\epsilon$ of $S$ gets transformed to a boundary component $b_\epsilon \subset \partial\bar{S}$ that is naturally oriented as the boundary of $\bar{S}$. We refer to the resulting $b_\epsilon \approx S^1$ as the *boundary circles* of $\epsilon$.

Similar considerations work in the case of so-called *nodal* surfaces $S$. Topologically, such a surface $S$ is obtained from a (possibly disconnected) punctured Riemann surface by choosing some number of distinct pairs of points and identifying the points in these pairs. The points resulting in this procedure are called the *nodes*. The surface $S^\circ$ is defined from $S$ by removing all nodes. Thus, in addition to the punctures, $S^\circ$ gets new punctures. A nodal Riemann surface consists of $S$ and the choice of a conformal structure of finite type on $S^\circ$.

We set $\bar{S} = \bar{S}^\circ$ by oriented blowup of the closed (possibly disconnected) Riemann surface $\tilde{S}$ obtained by attaching back a point to each puncture. Each node $\delta$ contributes to two boundary components $b'_\delta, b''_\delta \subset \partial\bar{S}$ which we call the *vanishing boundary circles* of $\delta$.

We call a map $\Phi : S^\circ \to (S^1)^n$ *pluriharmonic* if it can be obtained from a holomorphic map $\tilde{\Phi} : S^\circ \to (\mathbb{C}^*)^n$ by composing it with the argument map $\mathrm{Arg} : (\mathbb{C}^*)^n \to (S^1)^n$. It is easy to see (cf., for example, [Mik, § 2] and [MO07, § 6.2]) that any pluriharmonic map $\Phi : S^\circ \to (S^1)^n$ which is proper in a neighborhood of a boundary circle $b$ induces a map

$$\Phi^b : b \to (S^1)^n. \tag{7}$$

Its image is a geodesic on the flat torus $(S^1)^n = (\mathbb{R}/2\pi i)^n$. The map $\Phi^b$ comes as a limit of the map $\Phi$ restricted to a small simple loop around the corresponding puncture or node when this loop tends to $b$ (in the Hausdorff topology on closed subsets of $\bar{S}$). This map allows us to define a natural translation-invariant metric of circumference $2\pi$ on each such $b$ (cf. [Mik]).

If a pluriharmonic map $\Phi : S^\circ \to (S^1)^n$ has a removable singularity at some puncture with corresponding boundary circle $b$, then $\Phi^b(b)$ is a point of $(S^1)^n$. We can treat such point as a degenerate geodesic of slope 0. Otherwise we say that $b$ is an *essential* boundary circle for $\Phi$.

Recall that a tropical morphism $f : C \to \mathbb{R}^n$ is called minimal if no edge of $C$ is contracted to a point by $f$. From now on we assume that $f$ is minimal to simplify our definitions (we will only use minimal morphisms in the applications of this paper, and refer to [Mik] for the general case).

Definition 6.2. The *phase $\phi$ of $f$* is the following data:

- a choice of nodal Riemann surface $\Gamma_v$ of genus $g_v$ with $k$ punctures for each inner vertex $v \in \mathrm{Vert}^0(C)$ of valence $k$;
- a one-to-one correspondence between the punctures of $\Gamma_v$ and the edges of $C$ adjacent to $v$;
- an orientation-reversing isometry
$$\rho_e : b^v_\epsilon \approx b^{v'}_\epsilon \tag{8}$$
between the boundary circles of the punctures corresponding to $e$ for any edge $e$ connecting vertices $v, v' \in \mathrm{Vert}^0$;
- a pluriharmonic map
$$\Phi_v : \Gamma^\circ_v \to (S^1)^n$$
for each $v \in \mathrm{Vert}^0(C)$, where $\Gamma^\circ_v$ is obtained from $\Gamma_v$ by removing all nodes;
- an orientation-reversing isomorphism
$$\rho_\delta : b'_\delta \approx b''_\delta \tag{9}$$
between the vanishing boundary circles for each node $\delta$ of $\Gamma_v$,

subject to the following properties.

(i) For any edge $e$ adjacent to $v \in \mathrm{Vert}^0(C)$ and the puncture $\epsilon$ corresponding to $e$ we have the following identity for the homology class $[\Phi^{b_\epsilon}_v(b_\epsilon)] \in H_1((S^1)^n) = \mathbb{Z}^n$
$$[\Phi^{b_\epsilon}_v(b_\epsilon)] = w_{f,e} u_{f,e} \in \mathbb{Z}^n.$$
Here $u_{f,e} \in \mathbb{Z}^n$ is the outgoing (from $v$) primitive integer vector parallel to $f(e) \subset \mathbb{R}^n$.

(ii) For any edge $e$ connecting $v, v' \in \mathrm{Vert}^0(C)$ we have
$$\Phi^{b_\epsilon}_v = \Phi^{b_\epsilon}_{v'} \circ \rho_e : b^v_\epsilon \to (S^1)^n,$$
where $\epsilon$ is the puncture corresponding to $e$.

(iii) For any node $\delta$ of $\Gamma_v$ we have
$$\Phi^{b'_\delta}_v = \Phi^{b''_\delta}_v \circ \rho_\delta : b'_\delta \to (S^1)^n,$$
where the node $\delta$ is considered as puncture for the components of $\Gamma^\circ_v$ whose closures intersect at $\delta$.

(iv) Any node $\delta$ of $\Gamma_v$ with essential boundary circle is adjacent to two distinct connected components of $\Gamma^\circ_v$.

A tropical morphism $f$ equipped with a phase $\phi$ is called the *phase-tropical morphism* $(f, \phi)$.

Note that by the last property the dual graph of $\Gamma_v$ does not have edges adjacent to the same vertex (loop-edges) corresponding to a node with essential boundary circle. Such a case may be equivalently treated via perturbing $v$ into several vertices via inserting a corresponding length 0 edge for each node of $\Gamma_v$. The slopes of the new edges are determined by the homology classes of the geodesics $\Phi^{b'_\delta}_v(b_\delta)$ and are allowed to be zero. In the simplification considered in this paper (restricting to minimal tropical morphisms and coarse phase-tropical limits) we always treat such edges as having zero length, though in a refined version a length of such an edge may be positive if it has a zero slope.

*Remark* 6.3. As in the case of points, one has the notion of *real phase* of a tropical morphism: a phase is real if there exists a continuous involution $\sigma : C \to C$ such that $\Phi_v$ has a real algebraic lift if $\sigma(v) = v$, and $\Phi_v = -\Phi_{\sigma(v)}$ if $v \neq \sigma(v)$.

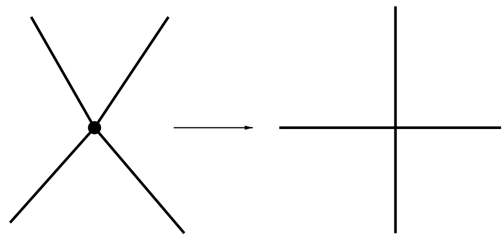


Figure 10. Parameterization of a four-valent node.

*Example* 6.4. Consider a tropical curve $C$ that consists of four lines emanating from the same point and a tropical morphism $f:C\to\mathbb{R}^2$ that maps this curve onto the union of the $x$- and $y$-coordinate axes, see Figure 10. To specify the phase of $f$ we need to choose a conformal structure on $\Gamma_v$, a sphere punctured four times corresponding to the only vertex $v\in C$ and a pluriharmonic map $\Phi_v:\Gamma_v^\circ\to S^1\times S^1$. If $\Gamma_v$ is irreducible then this defines the phase $\phi$ completely.

The choice of a conformal structure on $\Gamma_v$ is the choice of an element in $\mathcal{M}_{0,4}\simeq\mathbb{C}P^1$, the compactified moduli space of rational curves with four marked points. Exactly three of these structures correspond to a reducible surface $\Gamma_v=\Gamma'\cup\Gamma''$ made of two components intersecting at a node $\delta$. If $\Gamma'$ contains two punctures corresponding to the horizontal rays of $C$ then the boundary circles $b'_\delta$ and $b''_\delta$ are contracted by $\Phi_v^{b'_\delta}$ and $\Phi_v^{b''_\delta}$ and define a point in $S^1\times S^1$. The orientation-reversing isometry $\rho_\delta$ can be chosen arbitrarily. For all other reducible cases these boundary circles define closed geodesics on $S^1\times S^1$ which should coincide and thus induce an orientation-reversing isomorphism $\rho_\delta$.

Now we are ready to define tropical limits. As usual, it is especially easy to do for the case of points. For $t>0$ we define the renormalization isomorphism $H_t:(\mathbb{C}^*)^n\approx(\mathbb{C}^*)^n$ (cf. [Mik04, Mik05]) by

$$H_t(z_1,\dots,z_n)=\left(|z_1|^{1/\log t}\frac{z_1}{|z_1|},\dots,|z_n|^{1/\log t}\frac{z_n}{|z_n|}\right). \tag{10}$$

Let $p_{t_j}\in(\mathbb{C}^*)^n$, $t_j>0$, $j\in\mathbb{N}$ be a sequence of points indexed by a sequence of real numbers $t_j\to+\infty$.

Definition 6.5. We say that a sequence $p_{t_j}$ *converges tropically* if the limit $\lim_{j\to+\infty}H_{t_j}(p_{t_j})$ exists as a point of $(\mathbb{C}^*)^n$. The *phase-tropical limit* of this sequence is the point $p$ enhanced with the phase $\phi(p)$, where

$$p=\operatorname{Log}\left(\lim_{j\to+\infty}H_{t_j}(p_{t_j})\right),\quad \phi(p)=\lim_{j\to+\infty}\operatorname{Arg}(H_{t_j}(p_{t_j}))=\lim_{j\to+\infty}\operatorname{Arg}(p_{t_j}).$$

Also we say that the point $p$ itself is the tropical limit of $p_{t_j}$.

It is easy to see that in this case $p=\lim_{j\to+\infty}\operatorname{Log}_{t_j}p_{t_j}$, where $\operatorname{Log}_t(z_1,\dots,z_n)=(\log_t|z_1|,\dots,\log_t|z_n|)$, since

$$\operatorname{Log}_t=\operatorname{Log}\circ H_t.$$

Note that the limit depends not only on the sequence of points $p_{t_j}\in(\mathbb{C}^*)^n$, but also on the parameterizing sequence $t_j\in\mathbb{R}$.

Let $f:C\to\mathbb{R}^n$ be a minimal tropical morphism enhanced with a phase $\phi$. Let $U\subset\mathbb{R}^n$ be a convex bounded open set with connected intersection $f(C)\cap U$, containing not more than a single vertex of $f(C)$ (this vertex may be an image of more than one vertex of $C$) and such that its boundary does not contain any such vertex of $f(C)$. Each connected component $W$ of $f^{-1}(U)$ contains not more than a single vertex $v$. For any such vertex the phase $\phi$ of $f$ associates to $v$ a Riemann surface $\Gamma_v$ and a pluriharmonic map $\Phi_v : \Gamma_v^\circ \to (S^1)^n$. In the case $\Gamma_v$ is smooth, we let $\Gamma_W = \Gamma_v$ and $\Phi_W = \Phi_v$.

If $\Gamma_v$ is not smooth but has nodal points, then we prepare a new smooth surface $\hat{\Gamma}_v$ by resolving each node, i.e. replacing each node $\delta$ with the boundary circle $b_\delta$ corresponding to this node, i.e. either side of the isometry (9). In other words $\hat{\Gamma}_v$ is obtained from $\bar{\Gamma}_v$ by identifying all pairs of vanishing boundary circles with (9). Naturally we have an inclusion $\Gamma_v^\circ \subset \hat{\Gamma}_v$ and a surjective map

$$\hat{\Gamma}_v \to \Gamma_v$$

that collapses each boundary circle to a point. Also we have a map

$$\hat{\Phi}_v : \hat{\Gamma}_v \to (S^1)^n$$

that extends the map $\Phi_v$ from $\Gamma_v^\circ$ to $\hat{\Gamma}_v$ with the help of the boundary circle isomorphism (9). In this case we let $\Gamma_W = \hat{\Gamma}_v$ and $\Phi_W = \hat{\Phi}_v$. Note that, in this case, $\Gamma_W$ is not a Riemann surface in the conventional sense, but $\Gamma_W^\circ = \Gamma_v^\circ$ is.

If the connected component $W$ does not contain a vertex then it is an open arc of an edge $e \subset C$. Let $v$ be an endpoint vertex of $e$. Consider the boundary circle $b_\epsilon \subset \bar{\Gamma}_v$ corresponding to the edge $e$. The map $\Phi_\epsilon : b_\epsilon \to (S^1)^n$ induces a holomorphic map $\mathbb{C}^* \approx b_\epsilon \times \mathbb{R} \to (\mathbb{C}^*)^n$ by identifying $\mathbb{C}^*$ with the tangent space of $b_\epsilon$ and $(\mathbb{C}^*)^n$ with the tangent space of $(S^1)^n$. Composing this map with Arg, we obtain a pluriharmonic map $\Phi_e : \mathbb{C}^* \to (S^1)^n$. Note that $\Phi_e(\mathbb{C}^*) = \Phi_\epsilon(b_\epsilon)$. We denote $b_\epsilon \times \mathbb{R}$ enhanced with this complex structure (of $\mathbb{C}^*$) by $\Gamma_e$. Note that also we have an embedding of $b_\epsilon$ to $\Gamma_e$ as well as to $\Gamma_v$ for any endpoint $v$ of $e$. In this case we let $\Gamma_W = \Gamma_W^\circ = \Gamma_e$ and $\Phi_W = \Phi_e$.

Let $f_{t_j} : C_{t_j} \to (\mathbb{C}^*)^n$ be a sequence of holomorphic maps from Riemann surfaces $C_{t_j}$ of finite type. Note that the inverse image $f_{t_j}^{-1}(\mathrm{Log}_{t_j}^{-1}(U))$ is a Riemann surface with finitely many ends (as we can enlarge $U$ and extend the map $f|_{f_{t_j}^{-1}(\mathrm{Log}_{t_j}^{-1}(U))}$).

A map $\tau : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ defined by $(z_1, \dots, z_n) \mapsto (a_1 z_1, \dots, a_n z_n)$ for some $a_1, \dots, a_n > 0$ is called a *positive multiplicative translation* in $(\mathbb{C}^*)^n$.

Definition 6.6. We say that $f : C \to \mathbb{R}^n$ enhanced with the phase $\phi$ is the *coarse phase-tropical limit* of $f_{t_j}$ if for any choice of open set $U \subset \mathbb{R}^n$ as above and all sufficiently large $t_j$ there is a one-to-one correspondence between connected components $W_{t_j}$ of $f_{t_j}^{-1}(\mathrm{Log}_{t_j}^{-1}(U))$ and connected components $W$ of $f^{-1}(U)$ with the following properties of the corresponding components.

- There exists an open embedding $\Xi_{t_j}^W : W_{t_j} \to \Gamma_W$ and, for each connected component $\Gamma^\circ$ of $\Gamma_W^\circ$, a holomorphic map

$$\tilde{\Phi}_{\Gamma^\circ} : \Gamma^\circ \to (\mathbb{C}^*)^n$$

  such that $\mathrm{Arg} \circ \tilde{\Phi}_{\Gamma^\circ} = \Phi_W|_{\Gamma^\circ}$ and a sequence of positive multiplicative translations $\tau_{t_j} : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ such that for any $z \in \Gamma^\circ$

$$\lim_{t_j \to +\infty} \tau_{t_j} \circ f_{t_j} \circ (\Xi_{t_j}^W)^{-1}(z) = \tilde{\Phi}_{\Gamma^\circ}(z).$$

  In particular, we require that $z \in \Xi_{t_j}^W(W_{t_j})$ for large $t_j$.
- For any pair of boundary circles $b$ and $b'$ identified by (8) (in the case when these pair correspond to an edge of $C$, i.e. $b$ and $b'$ are boundary circles for $\Gamma = \Gamma_v$ and $\Gamma' = \Gamma_{v'}$ for

distinct vertices $v, v'$) or (9) (in the case when they are vanishing boundary circles for components $\Gamma$ and $\Gamma'$ of $\Gamma_v^\circ$ for the same vertex $v=v'$), any point $z\in b$, any $\eta>0$ and a sufficiently large $t_j$ there exist:

- a point $z_\eta\in\Gamma$ and a point $z'_\eta\in\Gamma'$;
- a path $\gamma_\eta\subset\bar{\Gamma}$ connecting $z_\eta$ and $z$ and a path $\gamma'_\eta\subset\bar{\Gamma}'$ connecting $z'_\eta$ and $z'=\rho_e(z)$ (see (8)) such that the diameter of $\hat{\Phi}_v(\gamma_\eta)\subset(S^1)^n$ and that of $\hat{\Phi}_{v'}(\gamma'_\eta)\subset(S^1)^n$ are less than $\eta$ (in the standard product metric on $(S^1)^n$);
- a path $\gamma_{t_j}\subset C_{t_j}$ connecting $(\Xi^v_{t_j})^{-1}(z_\eta)$ and $(\Xi^{v'}_{t_j})^{-1}(z'_\eta)$ such that the diameter of $\mathrm{Arg}(f_{t_j}(\gamma_{t_j}))\subset(S^1)^n$ is less than $\eta$ and the path $\mathrm{Log}_{t_j}\circ f_{t_j}\circ\gamma_{t_j}$ is contained in a small neighborhood of the interval connecting $\mathrm{Log}_{t_j}(f_{t_j}(v))$ and $\mathrm{Log}_{t_j}(f_{t_j}(v'))$ in $\mathbb{R}^n$.

In plain words, if our open set $U\subset\mathbb{R}^n$ is disjoint from $f(C)$ then it should not intersect $\mathrm{Log}_{t_j}(f_{t_j}(C_{t_j}))$ for large $t_j$. If it intersects $f(C)$ along an interval that is simply covered by an edge $e\subset C$ then $\mathrm{Log}_{t_j}^{-1}(U)\cap f_{t_j}(C_{t_j})$ should be an annulus whose image by Arg is close to $\Phi_e(\Gamma_e)$. Similarly, if this interval is covered by several edges then we should see an annulus for each such edge. Finally, if $U$ contains a neighborhood of a vertex $v\in C$ then there should exist holomorphic maps $\tilde{\Phi}_{\Gamma^\circ}:\Gamma^\circ\to(\mathbb{C}^*)^n$ whose argument agrees with the argument of the map to which $\mathrm{Log}_{t_j}^{-1}(U)\cap f_{t_j}(C_{t_j})$ accumulates. All these components should glue in accordance with (8) and (9).

Altogether we may think of $C_{t_j}$ as decomposed into generalized pairs-of-pants (cf. the discussion preceding Definition 6.2) made from pairs-of-pants (or more general Riemann surfaces) close to $\Gamma_v$ that are glued along embedded circles dual to the edges of $C$. Existence of paths $\gamma_\delta$ ensures that the gluing of these pairs-of-pants agrees with the orientation reversing isometries specified by the phase $\phi$ in the case when these geodesics represent a divisible class in $H_1((S^1)^n)$, i.e. when the weight of the corresponding edge is greater than 1. If this class is primitive then this condition holds automatically.

*Remark* 6.7. In Definition 6.6 we use the term *coarse* tropical limit as for simplicity we ignore the issue of non-minimal morphisms by implicit identification of the limit with the corresponding minimal morphism. This procedure can actually be refined (although we do not need it for this paper) so that the lengths and the phases of the contracted edges will also be well defined in the limit, see [Mik].

*Example* 6.8. Let us consider the tropical morphism $f:C\to\mathbb{R}^2$ from Example 6.4, equipped with a phase $\Phi_v:\Gamma_v^\circ\to S^1\times S^1$ at the only vertex $v$ of $C$.

We may present this phase-tropical morphism as a phase-tropical limit of a generalized sequence of holomorphic maps $f_t:S_t\to(\mathbb{C}^*)^2$ with $S_t$ a Riemann surface of genus 0 with four punctures. For simplicity we rather present $(f,\phi)$ as the limit of a family of embedded curves in $(\mathbb{C}^*)^2$ defined by an implicit equation, the passage from one point of view to the other being straightforward. Note that by definition of $\Phi_v$, we can see $\Gamma_v^\circ$ as embedded in $(\mathbb{C}^*)^2$ and $\Phi_v=\mathrm{Arg}_{|\Gamma_v^\circ}$.

If $\Gamma_v$ is irreducible, then $\Gamma_v=\Gamma_v^\circ$ and we choose $S_t=\Gamma_v$.

Suppose that now $\Gamma_v$ is reducible and made of two components $\Gamma'$ and $\Gamma''$ intersecting at a node $\delta$. Both $\Gamma'$ and $\Gamma''$ are spheres with two punctures, and we may assume without loss of generality that $\Gamma'$ (respectively $\Gamma''$) has one puncture corresponding to the left-horizontal (respectively a vertical) ray of $C$.

If the boundary circle of $\delta$ has slope 0, then the pluriharmonic map $\Phi_v:\Gamma_v^\circ\to S^1\times S^1$ extends to the whole curve $\Gamma_v=\Gamma'\cup\Gamma''$. In this case we can still assume that $\Gamma_v\subset(\mathbb{C}^*)^2$, and that it is given by the equation $(x-x_0)(y-y_0)=0$ with $(x_0,y_0)\in(\mathbb{C}^*)^2$. We then choose $S_t$ to be the algebraic curve given by the equation $(x-x_0)(y-y_0)+t^{-1}=0$.

If the boundary circle of $\delta$ has a non-zero slope, then one cannot extend $\Phi_v$ at the node $\delta$. If the other puncture of $\Gamma'$ corresponds to the down-vertical ray of $C$, then $\Gamma'\backslash\delta$ (respectively $\Gamma''\backslash\delta$) has an equation of the form $1+c_1x+c_2y=0$ (respectively $c_1x+c_2y+c_3xy=0$) with $c_1,c_2,c_3\in\mathbb{C}^*$. In this case we choose $S_t$ to be the algebraic curve given by the equation $1+c_1x+c_2y+e^{-t}c_3xy=0$.

In the other case $\Gamma'\backslash\delta$ (respectively $\Gamma''\backslash\delta$) has an equation of the form $1+c_1y+c_2xy=0$ (respectively $1+c_2xy+c_3x=0$) with $c_1,c_2,c_3\in\mathbb{C}^*$, and we choose $S_t$ to be the algebraic curve given by the equation $1+c_1y+c_2xy+e^{-t}c_3x=0$.

In all cases, the phase-tropical morphism $(f,\phi)$ is the phase-tropical limit of the curves $S_t\subset(\mathbb{C}^*)^2$.

Also it is worth noting that the tropical limit we define here is defined for convergence in compact sets in $(\mathbb{C}^*)^n$. Thus the limit may have strictly smaller degree than the terms in the sequence. It is possible to compactify $(\mathbb{C}^*)^n$ to a toric variety and define convergence there so that the total degree will be preserved under the limit (some components of the limit will be contained in the toric divisors). Nevertheless we have the following compactness property once we allow the limit to have smaller degree.

Proposition 6.9. *Let $f_{t_j}:C_t\to(\mathbb{C}^*)^n$ be a (generalized) sequence of holomorphic curves with $t_j\to+\infty$, $j\to+\infty$. Then there exists a tropical morphism $f:C\to\mathbb{R}^n$ and a subsequence of $t_j$ such that $f$ is the coarse phase-tropical limit of the subsequence.*

*Proof.* The proposition is essentially contained in [Mik05, Proposition 8.7]. Indeed, by [Mik05, Proposition 3.9 and Corollary 8.6] we can extract from the family $f_t$ a sequence $f_{t_j}$ such that $\operatorname{Log}_{t_j}(f_{t_j}(C_{t_j}))$ converges in the Hausdorff metric on compact sets in $\mathbb{R}^n$ to a set $A$ that is an image of a tropical curve.

However, we need convergence not only to a set in $\mathbb{R}^n$ but also to a tropical morphism $f:C\to\mathbb{R}^n$ whose image is $A$. Thus we need to construct such a morphism. For that we need to know the vertices and the edges of $C$.

Let $p=(p_1,\ldots,p_n)\in A\subset\mathbb{R}^n$ be a point and $U\ni p$ be a convex open set such that the closure $\bar{U}$ is compact, does not contain vertices of $A$ other than $p$ and its intersection with $A$ is connected. Let $p^{(t_j)}\in\mathbb{R}^n_{>0}$ be a sequence whose tropical limit is $p$, i.e. such that $\lim_{j\to\infty}\operatorname{Log}_{t_j}p^{(t_j)}=p$.

For each $j$ we have a positive multiplicative translation

$$\tau_{t_j}:(z_1,\ldots,z_n)\to\left(\frac{z_1}{p_1^{(t_j)}},\ldots,\frac{z_n}{p_n^{(t_j)}}\right)$$

that is a biholomorphic automorphism of the torus $(\mathbb{C}^*)^n$. The projective compactifications of the curves

$$\tau_{t_j}\circ f_{t_j}:C_{t_j}\to\mathbb{CP}^n$$

must contain a converging subsequence. Its limit may be reducible and some of the components may be contained in the boundary divisors $\mathbb{CP}^n\backslash(\mathbb{C}^*)^n$. The restriction of the limit to $(\mathbb{C}^*)^n$ produces a holomorphic map $\Gamma_{\{p^{(t_j)}\}}\to(\mathbb{C}^*)^n$ which we can use for $\tilde{\Phi}_{\Gamma^\circ}$ for every component $\Gamma^\circ$ of $\Gamma^\circ_{\{p^{(t_j)}\}}$. The boundary circles of $\Phi_{\Gamma^\circ}$ must be geodesics in $(S^1)^n$ whose homology classes are proportional to the slope vectors of the edges of $A$ adjacent to $p$ with positive proportionality coefficients as any non-proportional class would contradict to the maximum principle. Indeed, otherwise we choose a codimension 1 subtorus of $(S^1)^n$ parallel to the boundary circle, but transverse to the corresponding edge adjacent to $p$ in $\mathbb{R}^n$. Then some multiplicative translate of the corresponding codimension 1 complex subgroup of $(\mathbb{C}^*)^n$ would have to have a negative intersection point with our curve.

Thus we just need to find sequences $\{p^{(t_j)}\}$ that produce meaningful (in particular, non-empty) curves $\mathrm{Log}_{t_j}^{-1}(U)$ Note that for all but finitely many $p$ the Riemann surface $\Gamma_{\{p^{(t_j)}\}}$ is a collection of annuli if $U$ is sufficiently small, no matter what is the approximation $\{p^{(t_j)}\}$ of $p$.

Indeed, if $\Gamma_{\{p^{(t_j)}\}}$ has a non-annulus component then by the maximum principle its Euler characteristic is negative and its image under a character $\chi_{a_1,\dots,a_n} : (\mathbb{C}^*)^n \to \mathbb{C}^*$, $(z_1, \dots, z_n) \mapsto z_1^{a_1} \dots z_n^{a_n}$, must have a critical point whenever $(a_1, \dots, a_n)$ is transverse to the edges of $A$. However, the algebraic map $\chi_{a_1,\dots,a_n} \circ f_{t_j}$ may only have finitely many critical points, so after passing to a (diagonal) subsequence of $t_j$ so that we get a convergence to infinitely many non-annulus surfaces $\Gamma_{\{p^{(t_j)}\}}$ we get a contradiction. Since non-annulus irreducible components of surfaces in $(\mathbb{C}^*)^n$ have negative Euler characteristic we call them *hyperbolic components*.

By passing to even smaller open neighborhoods $U \ni p$ if needed we may ensure that each component of $\mathrm{Log}_{t_j}^{-1}(U)$ is an annulus except for finitely many $p$ for sufficiently large $t_j$. Furthermore, once $t_j$ is sufficiently large and $U \ni p$ is sufficiently small we have a natural one-to-one correspondence between the annuli components of $\mathrm{Log}_{t_j}^{-1}(U)$ and the annuli components of $\Gamma_{\{p^{(t_j)}\}}$. In the same time a hyperbolic component of $\mathrm{Log}_{t_j}^{-1}(U)$ may correspond to more than one hyperbolic component of $\Gamma_{\{p^{(t_j)}\}}$ as it may converge to a reducible curve.

To reconstruct the limiting curve $C$ and the tropical map $f : C \to A \subset \mathbb{R}^n$ we take a vertex $v$ for each hyperbolic component of $\mathrm{Log}_{t_j}^{-1}(U)$, and we define $f(v) = p$ and $g_v$ as the genus of this hyperbolic component. The edges of $C$ are obtained by gluing the corresponding annulus components of $\mathrm{Log}_{t_j}^{-1}(U)$ along paths in $A$. The tropical length on the edges of $C$ comes from the length of the corresponding edges of $A$ divided by the proportionality coefficient between the homology class of the boundary circle and the slope vector of the edge of $A$ (a positive integer number).

This procedure gives the limiting tropical morphism $f : C \to \mathbb{R}^n$ for a subsequence $f_{t_j} : C_{t_j} \to (\mathbb{C}^*)^n$. The limits of hyperbolic components of $\mathrm{Log}_{t_j}^{-1}(U)$ in the Deligne–Mumford compactifications of the corresponding moduli spaces give us (possibly nodal and reducible) curves $\Gamma_v$. Consider a component $\Gamma^\circ$ of $\Gamma_v^\circ$. Choosing a point $q \in \Gamma^\circ$ and a family of approximating points $q^{(t_j)}$ in the corresponding hyperbolic component of $\mathrm{Log}_{t_j}^{-1}(U)$ we ensure that the curve $\Gamma^\circ_{\{f_{t_j}(q^{(t_j)})\}}$ contains $\Gamma^\circ$ as a component. This defines the map $\tilde{\Phi}_{\Gamma^\circ}$ for the limiting phase-tropical curve. Note that punctures of $\Gamma^\circ$ with boundary circle of slope 0 are precisely the removable singularities of the map $\tilde{\Phi}_{\Gamma^\circ}$. In other words the boundary circles of a node of $\Gamma$ has slope 0, and condition (4) of Definition 6.2 is satisfied. $\square$

### 6.2 Rational tropical morphisms as tropical limits

Proposition 6.9 can be reversed in the rational case; any phase-tropical rational morphism can be presented as a tropical limit of a family of holomorphic curves parameterized by a real positive parameter. Furthermore, we can do this procedure consistently for all phase-tropical curve in a neighborhood of $(f, \phi)$. For the purpose of this paper it suffices to consider tropical curves supported on graphs with no vertices of valence higher than 3. With a slight abuse of terminology (ignoring the ever-present 1-valent leaves) we call such curves 3-valent. For the sake of shortness in definitions we restrict to this case.

Let $f : C \to \mathbb{R}^n$ be a 3-valent rational tropical curve and $\phi$ be its phase. Choose a reference vertex $v \in C$. Recall that a neighborhood of $f$ is obtained by varying the image $f(v) \in \mathbb{R}^n$ as well as the lengths of the bounded edges of $C$ while keeping the slope vectors of all edges of $f(C)$ unchanged. Since $C$ is a 3-valent tree, the image $f(v)$ and the length of all edges define $f$ and so the space of deformation of $f$ is locally $\mathbb{R}^{b+n}$, where $b$ is the number of the bounded edges.

To define a *neighborhood of* $(f, \phi)$ *in the space of phase-tropical curves* we take a neighborhood of $f$ in the space of tropical curves and add to it all phases obtained by sufficiently small multiplicative translations of $\Phi_v : \Gamma_v \to (S^1)^n$ by $(a_1, \ldots, a_n) \in (\mathbb{C}^*)^n$, $|a_1| = \cdots = |a_n| = 1$ as well as small perturbations of the isometry (8) for all bounded edges $e$. Clearly we get another $b + n$ real parameters.

Let us note that the length of each bounded edge $e$ connecting vertices $v$ and $v'$ has a preferred direction for deformation, say increasing of its length. Similarly, the orientation-reversing isometry $\rho_\epsilon : b^v_\epsilon \approx b^{v'}_\epsilon$ given by the phase structure $\phi$ also has a preferred direction for deformation. Namely, we may compose $\rho_\epsilon$ with a small translation of the circle $b^{v'}_\epsilon$ in the direction coherent with the orientation of $b^{v'}_\epsilon$ (induced from the complex orientation of $\Gamma_{v'}$). Note that this direction of deformation of $\rho_\epsilon$ stays the same if we exchange the roles of $v$ and $v'$. We call it *the positive twist*.

We can couple translation of $f(v)$ in $\mathbb{R}^n$ with $(\arg a_1, \ldots, \arg a_n)$ and also couple varying the lengths of $e$ with varying the isometry (8) so that the positive twist corresponds to increase of the length. Thus a neighborhood of $(f, \phi)$ can be locally identified with $\mathbb{R}^{b+n} \oplus i\mathbb{R}^{b+n} = \mathbb{C}^{b+n}$. Recall that $b$ is the number of bounded edges. For a 3-valent tree $C$ it is equal to the number of ends minus two.

Let us revisit the notion of the degree for a curve and generalize Definition 2.10 to arbitrary dimension $n$. Recall (cf. [FM10, Mik, Mik07]) that the degree of $f$ can also be defined by the following formula:

$$d = \sum_{e \in \mathrm{Vert}^\infty(C)} w_{f,e} \max_{j=1}^{n} \{0, s_j(e)\}, \tag{11}$$

where $s(e) = (s_j(e))_{s=1}^n$ is the slope vector of the end $e$ in the outgoing direction.

As the degree is invariant with respect to permutations of the basic directions $-E_1, \ldots, -E_n, \sum_{j=1}^n E_j$, where $(E_j)_{j=1}^n$ is the standard basis for $\mathbb{R}^n$ that are used for the compactification $\mathbb{TP}^n$, we have an alternative formula for computing the degree,

$$d = \sum_{e \in \mathrm{Vert}^\infty(C)} w_{f,e} \max_{j=1}^{n} \{s_j(e) - s_k(e), -s_k(e)\}, \tag{12}$$

that holds for any $k = 1, \ldots, n$. It is easy to see that (11) and (12) are consistent for any curve in $\mathbb{R}^n$ that satisfies the balancing condition and that for the case $n = 2$ these formulas give the same number as Definition 2.10.

Note that each end of $C$ has to contribute to at least one of these $n + 1$ formulas for degree. Therefore the maximal number of ends for a minimal curve of degree $d$ is $(n + 1)d$ and that in the case when we have $(n + 1)d$ ends the slope vectors of the ends are exactly $-E_1, \ldots, -E_n, \sum_{j=1}^n E_j$. Such curves are called *generic at* $\infty$ in $\mathbb{R}^n$.

Let $U$ be a neighborhood of $(f : C \to \mathbb{R}^n, \phi)$ in the space of phase-tropical curves. Denote by $\mathcal{M}_C$ the space of all rational curves in $(\mathbb{C}^*)^n$ whose collection of boundary circles realizes the same classes in $H_1((S^1)^n)$ as the ends of the curve $f$. This means that the homology class of each boundary circle agrees with the slope vector and the weight of the corresponding end, cf. the first property imposed by Definition 6.2.

Theorem 6.10. *For all sufficiently large values $t \gg 1$ there exists an open embedding $\Lambda_t : U \to \mathcal{M}_C$ such that for any sequence $f_{t_j} \in \mathcal{M}_C$, $t_j \to +\infty$, the following conditions are equivalent.*

(i) *The sequence $f_{t_j}$ converges to a phase-tropical curve $(f', \phi') \in U$ in the sense of Definition 6.6.*

(ii) *For all sufficiently large $t_j$ we have $f_{t_j} \in \Lambda_{t_j}(U)$ and*

$$\lim_{t_j \to +\infty} \Lambda_{t_j}^{-1}(f_{t_j}) = (f', \phi') \in U.$$

*(Recall that $U$ may be topologically viewed as an open set in $\mathbb{C}^N$ for some $N$.)*

In particular this proposition allows us to present any holomorphic curve sufficiently close to $(f, \phi)$ in the $t_j$-framework as $\Lambda_{t_j}(f', \phi')$ for a phase-tropical curve $(f', \phi')$ close to $(f, \phi)$. Proposition 6.10 is true also for curves in higher genus mapping to realizable tropical varieties under the condition of regularity (which means that the dimension of the deformation space of the curve is of expected dimension) and is proved in a more general case in [Mik] with some intermediate generalizations proved in [Mik05, Nis10, NS06, Tyo12]. However, the assumptions we make in this paper imply that the curve is rational and the target variety is a projective space (even more specifically in dimension 2 but it does not make much difference). This is an especially easy case and to prove it under these assumptions it suffices to consider the lines in $\mathbb{P}^n$.

We say that $\pi : (\mathbb{C}^*)^{s-1} \to (\mathbb{C}^*)^n$ is a *multiplicatively affine map* if it is obtained by composition of a multiplicatively linear map $(\mathbb{C}^*)^{s-1} \to (\mathbb{C}^*)^n$ (given by a $(s-1) \times n$ matrix with integer entries) and an arbitrary multiplicative translation in $(\mathbb{C}^*)^n$. Note that $\pi$ induces a map $\pi_{\mathbb{R}} : \mathbb{R}^{s-1} \to \mathbb{R}^n$ such that $\mathrm{Log} \circ \pi = \pi_{\mathbb{R}} \circ \mathrm{Log}$, and a map $\pi_{\mathrm{Arg}} : (S^1)^{s-1} \to (S^1)^n$ such that $\mathrm{Arg} \circ \pi = \pi_{\mathrm{Arg}} \circ \mathrm{Arg}$. Given a phase-tropical curve $(\tilde{f} : C \to \mathbb{R}^{s-1}, \tilde{\phi})$ we may compose it with $\pi$ to obtain a phase-tropical curve $(f : C \to \mathbb{R}^n, \phi)$ with $f = \pi_{\mathbb{R}} \circ \tilde{f}$ by setting the phase maps $\Phi_v$ at each vertex $v$ to be the composition maps $\pi_{\mathrm{Arg}} \circ \tilde{\Phi}_v$. Note that we may assume that $n < s$, otherwise all the ends of $f(C)$ must be parallel to a $(s-1)$-dimensional affine space in $\mathbb{R}^n$, so the whole curve $f(C)$ is contained in such space and we may replace the target with this smaller-dimensional space $\mathbb{R}^{s-1}$.

We say that a tropical curve $f : C \to \mathbb{R}^n$ is a *line* if it has degree 1 in the natural compactification $\mathbb{TP}^n \supset \mathbb{R}^n$. A *phase-tropical line* is a phase-tropical curve of degree 1 (i.e. a tropical line enhanced with any phase structure).

Lemma 6.11. *Let $(f : C \to \mathbb{R}^n, \phi)$ be a phase-tropical curve where $C$ is a rational tropical curve with $s$ leaves. Then there exists a phase-tropical line $(\tilde{f} : C \to \mathbb{R}^{s-1}, \tilde{\phi})$ generic at $\infty$ and a multiplicatively affine map $\pi : (\mathbb{C}^*)^{s-1} \to (\mathbb{C}^*)^n$ such that $\pi \circ (\tilde{f}, \tilde{\phi}) = (f, \phi)$.*

*If $\tilde{f}_{t_j} : C_{t_j} \to (\mathbb{C}^*)^{s-1}$ is a family of holomorphic curves coarsely converging to $(\tilde{f}, \tilde{\phi})$ then $f_{t_j} = \pi \circ \tilde{f}_{t_j}$ coarsely converges to $(f, \phi)$.*

*Furthermore, if $U$ is a small neighborhood of $(f, \phi)$ in the space of phase-tropical curves to $\mathbb{R}^n$ then we can find a locus $\tilde{U} \ni (\tilde{f}, \tilde{\phi})$ inside the space of deformations of $(\tilde{f}, \tilde{\phi})$ and an isomorphism $U \approx \tilde{U}$ so that for any $(f', \phi') \in U$ the corresponding point of $\tilde{U}$ is its lift in the above sense.*

*Proof.* We start by lifting the tropical curve $f : C \to \mathbb{R}^n$ to $\mathbb{R}^{s-1}$. To do this we choose a vertex $v$ in $\mathrm{Vert}^0(C)$ and we arbitrarily associate the outgoing unit tangent vectors to the $s$ leaves of $C$ with the $s$ preferred vectors in $\mathbb{R}^{s-1}$: $E_1 = (-1, 0, \ldots, 0), \ldots, E_{s-1} = (0, \ldots, 0, -1), E_s = (1, \ldots, 1)$. This identification defines slope vectors for the remaining (bounded) edges. Namely, the tangent vector to a point inside an edge of $C$ can be associated to the sum of the vectors associated to the leaves of $C$ in the direction of the edge (recall that $C$ is a tree).

Let us choose a multiplicatively affine map $\pi : (\mathbb{C}^*)^{s-1} \to (\mathbb{C}^*)^n$ such that $\pi_{\mathbb{R}}$ sends the vector $E_j$ to the corresponding outgoing unit tangent vector multiplied by its weight. Then we choose $\tilde{f}(v)$ to be an arbitrary point of $\pi^{-1}(f(v))$. The slope vectors of the edges along with the tropical structure on $C$ define $\tilde{f} : C \to \mathbb{R}^{s-1}$. To lift the phase we choose an arbitrary pluriharmonic map $\tilde{\Phi}_v : \Gamma_v \to (S^1)^{s-1}$ that lifts $\Phi_v : \Gamma_v \to (S^1)^n$. The isometry (8) defines the lifts of the phases on the vertices connected to $v$ with a single edge. Inductively we get the lift of the phase.

Note that the only ambiguities in the choice of $(\tilde{f}, \tilde{\phi})$ is the choice of a point in $\pi^{-1}(f(v)) \approx \mathbb{R}^{s-1-n}$ and the translation of the phase $\tilde{\Phi}_v$ by the corresponding $(s-1-n)$-dimensional subgroup of the torus $(S^1)^{s-1}$. Taken together these ambiguities form $(\mathbb{C}^*)^{s-1-n}$. □

Lemma 6.12. *Theorem 6.10 holds if $U$ is a small neighborhood of $(f : C \to \mathbb{R}^n, \phi)$, where $f : C \to \mathbb{R}^n$ is a 3-valent line generic at $\infty$.*

*Proof.* We prove the lemma by induction on $n$. Note that for $n \leqslant 1$ the lemma holds trivially. Consider the multiplicative affine map $\pi : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^{n-1}$ $(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-1})$. By the induction hypothesis $\pi(f, \phi)$ is realizable. The map $\pi_{\mathbb{R}}|_{f(C)}$ contracts an end $E \subset f(C)$ adjacent to a leaf (1-valent vertex) $u \in C$. By the 3-valency assumption the point $\pi(f(u)) \in \pi(f(C)) \subset \mathbb{R}^{n-1}$ is inside an edge of $\pi(f(C))$. Thus it can be obtained as a transverse intersection point of $\pi(f(C))$ and a hyperplane $\{x_j = c\} \subset \mathbb{R}^{n-1}$ for some $j = 1, \ldots, n-1$. Since $f(C) \subset \mathbb{R}^n$ has degree 1 so has $\pi(f(C)) \in \mathbb{R}^{n-1}$. Therefore

$$\{\pi(f(u))\} = \pi(f(C)) \cap \{x_j = c\}$$

and the intersection in the right-hand side has tropical intersection number 1.

Let $f_t^\pi : C_t^\pi \to (\mathbb{C}^*)^{n-1}$ be a sequence of holomorphic curves that converges to $\pi(f, \phi)$. Let $f_t^{\pi,j} : C_t^\pi \to \mathbb{C}^*$ be the $j$th coordinate of $f_t^\pi$. We are looking for the family $f_t$ converging to $(f, \phi)$ in the form

$$(f_t^\pi, \alpha t^a (f_t^{\pi,j} - \beta t^c)) : C_t^\pi \backslash \{f_t^{\pi,j} = \beta t^c\} \to (\mathbb{C}^*)^n,$$

$\alpha, \beta \in \mathbb{C}^*$, $|\alpha| = |\beta| = 1$, $a \in \mathbb{R}$.

We set $a$ to be the maximum of the $n$th coordinate of the contracted edge $E \subset f(C)$. This maximum is attached at the vertex $v \in C$. This ensures the convergence of $\mathrm{Log}_t \circ f_t(C_t^\pi \backslash \{f_t^{\pi,j} = \beta t^c\})$ to $f$ as we have

$$f_t(C_t^\pi \backslash \{f_t^{\pi,j} = \beta t^c\}) \subset P_t = \{z_n = \alpha t^a (z_j - \beta t^c)\}$$

and clearly $\mathrm{Log}_t(P_t)$ converges to the tropical hyperplane $P$ given by the tropical polynomial "$z_n + a(z_j + c)$" and we have $P \supset f(C)$.

It remains to choose the arguments $\alpha$ and $\beta$ to guarantee convergence at the phase level. These unit complex numbers are determined by the phase structure $\phi$, namely by its value $\Phi_v : \Gamma_v \to (S^1)^n$. As $v$ is 3-valent the Riemann surface $\Gamma_v$ is a standard pair-of-pants. The argument $\beta$ is determined by the boundary circle of $\Phi_v$ of slope vector $-e_n$ while the argument $\alpha$ is determined by the boundary circle of $\Phi_v$ of slope vector $-e_j$. (Recall that we already have $\pi \circ \Phi_v$ compatible with the limit of $f_t^\pi$.) □

Theorem 6.10 now follows by combining Lemmas 6.11 and 6.12 together.

## 7. Tangency conditions in the phase-tropical world

### 7.1 Proof of Theorems 3.8 and 3.12

Note that the *classical* number $N_{d,0}(k; d_1, \ldots, d_{3d-1-k})$ does not depend on the choice of configuration of the $k$ points and $3d-1-k$ curves (as long as these constraints are generic and all the curves we count are regular so that the corresponding enumerative problem is well defined). Thus we can take a family of constraints $\mathcal{P}_t, \mathcal{L}_t$ depending on the real parameter $t > 1$ to compute the (independent of $t$) number $N_{d,0}(k; d_1, \ldots, d_{3d-1-k})$.

Recall that in the hypothesis of Theorem 3.8 the tropical constraints $\mathcal{P}$, $\mathcal{L}$ are already fixed and they are in tropical general position.

We choose any family $\mathcal{P}_t$ so that its tropical limit in the sense of Definition 6.5 is $\mathcal{P}$. It is easy to see that we can always make such choice. Indeed a point $p_t \in (\mathbb{C}^*)^2$ is determined by $\mathrm{Log}_t(p_t) \in \mathbb{R}^n$ and $\mathrm{Arg}(p_t) \in S^1 \times S^1$. Once we choose an arbitrary phase $\phi(p)$ the point $p_t$ is determined by $\mathrm{Log}_t(p_t) = p$ and $\mathrm{Arg}(p_t) = \phi(p)$. We do this for every point $p \in \mathcal{P}$.

Consider a tropical curve $L$ from $\mathcal{L}$. Even though this curve is not rational, since it is immersed to $\mathbb{R}^2$, it can still be presented as the tropical limit of a 1-parametric real family of complex curves in $\mathbb{CP}^2$, see [Mik05]. Namely, there exists a phase for $L$ (viewed as an embedding $L \to \mathbb{R}^2$) in the sense of Definition 6.2 and a family $L_t \subset (\mathbb{C}^*)^2$ such that $L$ is the tropical limit of $L_t$ in the sense of Definition 6.6.

For construction of $L_t$ in the case of general immersion we refer to [Mik05, Proposition 8.12]. Note that in the case when $L \to \mathbb{R}^2$ is an embedding it is especially easy to construct this approximating family and this can be done directly by *patchworking* [Vir01] (see also [GKZ94, Chapters 7 and 11]).

Namely, any embedded tropical curve $L \subset \mathbb{R}^2$ is given by a tropical polynomial

$$F(x, y) = \max_{j,k \in \mathbb{Z}} \{a_{jk} + jx + ky\} = \text{“}\sum_{j,k \in \mathbb{Z}} a_{jk} x^j y^k\text{”}$$

in two real variables $x, y$, where $a_{jk} \in [-\infty, +\infty)$ and $a_{jk} = -\infty$ except for some finitely many values of $(j, k)$ with $j \geqslant 0$ and $k \geqslant 0$, see [Mik04, Proposition 2.4]. The quotation marks here signify tropical arithmetic operations, and the formula above can be viewed as the definition of these operations (addition and multiplication). We set

$$F_t(z, w) = \sum_{j,k \in \mathbb{Z}} \alpha_{jk} t^{a_{jk}} z^j w^k,$$

$t > 1$, where we choose $\alpha_{jk} \in \mathbb{C}$ with $|\alpha_{jk}| = 1$ arbitrarily. The function $F_t$ is a polynomial in two complex variables $z, w$. We define the curve $L_t \subset (\mathbb{C}^*)^2$ as the zero set of $F_t$.

Recall (see [GKZ94, ch. 7]) that the polynomial $F$ defines a subdivision of the Newton polygon

$$\Delta_F = \text{Convex Hull}\{(j, k) \in \mathbb{Z}^2 \mid a_{jk} \neq 0\} \subset \mathbb{R}^2.$$

This subdivision is defined by projections of the faces of the extended Newton polygon $\tilde{\Delta}_F$ of which is the undergraph of $(j, k) \mapsto a_{jk}$, i.e. the set

$$\tilde{\Delta}_F = \text{Convex Hull} \bigcup_{j,k \mid a_{jk} \neq 0} \{(j, k, t) \mid t \leqslant a_{jk}\}.$$

Recall that the tropical polynomial $F$ is smooth if the projection of each finite face of $\tilde{\Delta}_F$ is a triangle of area $\frac{1}{2}$ (it is easy to see that this area is the minimal possible for a lattice polygon).

Let $(x_0, y_0) \in \mathbb{R}^2$ be a point. Each open set $U \subset \mathbb{R}^2$ defines a real 1-parametric family of open sets in $(\mathbb{C}^*)^2$

$$U_t^{(x_0,y_0)} = (\mathrm{Log}_t \circ \tau_t^{(x_0,y_0)})^{-1}(U),$$

where $\tau_t^{(x_0,y_0)} : (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$ is a coordinatewise multiplication (i.e. multiplicative translation) by $(t^{-x_0}, t^{-y_0})$. We set $L_t^{(x_0,y_0)} = \tau_t^{(x_0,y_0)}(L_t)$.

By definition of tropical hypersurfaces we have $(x_0, y_0) \in L$ if and only if there are at least two tropical monomials in $F$ that assume the same value at $(x_0, y_0)$ and have value greater than the other monomials of $F$. Note that $L_t^{(x_0,y_0)}$ is defined by a polynomial $F_t^{(x_0,y_0)}$ which is obtained by multiplying the coefficients of the $x^j y^k$ monomials of $F_t$ by $t^{x_0 j + y_0 k}$. Thus, if $(x_0, y_0) \notin L$, then for sufficiently large $t$ the absolute value of one monomial in $F_t$ is larger than the sum of the absolute values of all the other monomials and $U_t^{x_0,y_0} \cap L_t = \emptyset$ for any bounded open set $U \subset \mathbb{R}^2$.

Similarly, if $U$ is bounded, $t \gg 1$ is large and $(x_0, y_0) \in L$, then we have more than one dominating monomial for $F_t^{(x_0,y_0)}$ (after division by an appropriate power of $t$). Furthermore, if $L$ is smooth, all indices $(j, k) \in \Delta_F$ of the dominating monomials for $F_t^{(x,y)}$ must be contained in a lattice triangle of area $\frac{1}{2}$. Thus the convex hull $\Delta_{(x_0,y_0)}$ of such indices is either such a triangle itself or one of its sides which is an interval of integer length 1.

Consider the *truncation*

$$F^{(x_0,y_0)}_{\Delta_{(x_0,y_0)},t}(z, w) = \sum_{(j,k)\in\Delta_{(x_0,y_0)}} t^{jx_0+ky_0} \alpha_{jk} t^{a_{jk}} z^j w^k,$$

cf. [Vir01]. Clearly in $U_t^{(x_0,y_0)}$ the polynomial $F_t^{(x_0,y_0)}$ is a small perturbation of $F^{(x_0,y_0)}_{\Delta_{(x_0,y_0)},t}$ for large $t$. However, if $L$ is smooth, then so is the hypersurface $L^{(x_0,y_0)}_{\Delta_{(x_0,y_0)},t}$ defined by $F^{(x_0,y_0)}_{\Delta_{(x_0,y_0)},t}$. Thus $L_t^{(x_0,y_0)} \cap U_t^{(x_0,y_0)}$ is a small perturbation of $L^{(x_0,y_0)}_{\Delta_{(x_0,y_0)},t} \cap U_t^{(x_0,y_0)}$.

*Proof of Theorem 3.8.* Consider the constraints $(\mathcal{P}_t, \mathcal{L}_t)$. As in the very beginning of this paper, we set $\mathcal{S}_t = \mathcal{S}_t(d, 0, \mathcal{P}_t, \mathcal{L}_t)$ to be the set of all degree $d$ genus 0 curves that are passing through $\mathcal{P}_t$ and tangent to $\mathcal{L}_t$ (see § 1). By Proposition 6.9 there exists a tropical limit for a (generalized) subsequence of $\mathcal{S}_t$. We note that it must be a curve from $\mathcal{S}^{\mathbb{T}}(d, \mathcal{P}, \mathcal{L})$. To see this we assume $f : C \to \mathbb{R}^2$ is such a limit. Since $\mathcal{P}_t \subset f_t(C_t)$ for some $f_t : C_t \to (\mathbb{C}^*)^2$ from $\mathcal{S}_t$ we see that $\mathcal{P} \subset f(C)$. Let us assume that there exists a line $L \in \mathcal{L}$ such that $f : C \to \mathbb{R}^2$ is not pretangent to it. In such a case every intersection of $L$ and $f(C)$ is disjoint from the vertices and thus contained inside the edges of both $L$ and $C$. Therefore, all intersection points $p \in f(C) \cap L$ must be transverse intersections of the edges of $C$ and $L$. However, this means that for sufficiently large $t$ the intersections of $L_t$ and $f_t : C_t \to (\mathbb{C}^*)^2$ in $\mathrm{Log}_t^{-1}(U)$ are also transverse for any bounded $U \ni p$; but therefore all intersection points of $L_t$ and $f_t(C_t)$ are transverse, and we get a contradiction.

Thus any accumulation point of $\mathcal{S}_t$ when $t \to +\infty$ must be contained in $\mathcal{S}^{\mathbb{T}}(d, \mathcal{P}, \mathcal{L})$ which is a finite set of 3-valent curves by Proposition 3.3. In turn, Proposition 6.10 describes all curves that have a chance to converge to an element $f : C \to \mathbb{R}^2$ from $\mathcal{S}^{\mathbb{T}}(d, \mathcal{P}, \mathcal{L})$ through its neighborhood $U$ in the space of all deformations of $f$ in the class of phase-tropical morphisms. Thus it suffices to describe those curves from $\Lambda_t(U)$ that pass through $\mathcal{P}_t$ and are tangent to $\mathcal{L}_t$. We do it below.

Consider a small neighborhood $U$ of $f$ in the space of tropical curves. Such a neighborhood itself consists of 3-valent curves $f' \in U$, $f' : C' \to \mathbb{R}^2$. As we have already seen, the curve $f' \in U$ is parameterized by $\mathbb{R}^2 \times \mathbb{R}^b$, where $b$ is the number of bounded edges of $C$, once we fix a reference vertex $v \in C$. In these coordinates we define the curve $f' : C' \to \mathbb{R}^2$ to be such a curve that $C'$ is isomorphic to the curve $C$ as a graph, the first $\mathbb{R}^2$ coordinate corresponds to $f'(v) - f(v) \in \mathbb{R}^2$, and the remaining coordinates correspond to the difference in lengths of the corresponding edges of $C'$ and $C$. Note that, in these coordinates, $f$ corresponds to the origin of $\mathbb{R}^2 \times \mathbb{R}^b$.

These coordinates are naturally coupled with the coordinates responsible for the phase structure. For this we choose an arbitrary reference phase-structure $\phi$ for $f$. Namely, the first $\mathbb{R}^2$-coordinates are coupled with the $(S^1 \times S^1)$-coordinates of the arguments of the same coordinates in $(\mathbb{C}^*)^2$. Each bounded edge defines a parameter for identifying the corresponding boundary circles. This coordinate couples with the $\mathbb{R}$-coordinate in $\mathbb{R}^2 \times \mathbb{R}^b$ corresponding to the same edge so that the increment in length corresponds to the positive direction of the twist. Together the coupled coordinates form $\mathbb{C}^*$. Thus the space of all phase-tropical curves corresponding to tropical curves from $U$ is

$$\mathcal{U} = U \times (S^1 \times S^1) \times (S^1)^b$$

that can be considered as a subspace of $(\mathbb{C}^*)^2 \times (\mathbb{C}^*)^b$ obtained as $\mathrm{Log}_t^{-1}(U)$ for arbitrary $t > 1$.

By Lemma 3.4 each constraint $q \in \mathcal{P} \cup \mathcal{L}$ imposes a condition defining a hyperplane (in the classical sense) $\Lambda_p$ with a rational slope in $U \subset \mathbb{R}^2 \times \mathbb{R}^b$. Similarly, the space of all phase-tropical curves that can be approximated by classical curves passing through $p_t$ (in the case when $q = p \in \mathcal{P}$) or tangent to $L_t$ (in the case when $q = L \in \mathcal{L}$) is defined by a subtorus $N_q \subset (S^1 \times S^1) \times (S^1)^b$ of the same slope as shown below.

Consider a point $p \in \mathcal{P}$. The condition that $f'$ passes through $p$ is a linear condition on $f'$ in $\mathbb{R}^2 \times \mathbb{R}^b$. It defines a hyperplane (in the classical sense) with a rational slope. The phase $\phi(p)$ must be contained in $\phi(e) = \Phi_v^\epsilon(b_\epsilon^v) \subset S^1 \times S^1$, where $e$ is an edge of $C$ passing through $p$. This imposes a linear condition in $(S^1 \times S^1) \times (S^1)^b$. Note that there might be several edges of $C$ passing through $p$ but Proposition 3.3 ensures that all such edges have the same slope vector (as they are connected by a chain in the same line). As $\phi(e)$ is an annulus covered $w_{f,e}$ times by $b_\epsilon^v$, the point $\phi(p)$ is covered $w_{f,e}$ times.

Let us choose a straight line $G_p$ in $U$ that is transversal to $\Lambda_p$ in the sense that the primitive integer vector parallel to $G_p$ forms an (integer) basis of $\mathbb{Z}^2 \times \mathbb{Z}^b$ when taken with some integer vectors parallel to $\Lambda_p$. Such a line can be equipped with a phase that is a geodesic of the same slope vector in $(S^1)^2 \times (S^1)^b$. Together with the phase, the line $G_p$ defines a proper annulus $\mathcal{G}_p \subset \mathcal{U}$. The image $\Lambda_t(\mathcal{G}_p)$ contains some holomorphic curves passing through $p_t$. Their number is determined by the edges of $C$ passing through $p \in \mathbb{R}^2$.

We claim that this number is equal to the sum of the weights of all edges of $C$ containing $p$. Indeed, this number does not depend on the choice of $\mathcal{G}_p$ by topological reasons. Thus we may choose for $\mathcal{G}_p$ the phased straight line obtained by multiplicative translation of $(f, \phi)$ by a given $\mathbb{C}^*$-subgroup of $(\mathbb{C}^*)^2$. The points of $\Lambda_t(\mathcal{G}_p)$ passing through $p_t$ will correspond to the points of $b_\epsilon^v$ covering $\phi(p)$.

Consider $L \in \mathcal{L}$. First we suppose that the pretangency set consists of a single point $v \in \mathbb{R}^2$. If $v$ is a vertex of $L$ we look at the *coamoeba* of $L$ at $v$, i.e. the closure $\phi(v) \subset S^1 \times S^1$ of $\mathrm{Arg}(L^v_{\Delta_v})$. As the curve $L$ is smooth, its coamoeba is an image of the coamoeba of a line in $(\mathbb{C}^*)^2$ under a linear automorphism of the torus $S^1 \times S^1$. This coamoeba consists of two triangles of equal area cut by three geodesics in the torus whose slope vectors coincide with the slope vectors of the edges of $L$ adjacent to $v$. The two triangles share their three vertices; see Figure 11. We call these three vertices the *coamoeba vertices*.

The interior of the triangles of the coamoeba correspond to the interior of amoeba of $L^v_{\Delta_v}$; see e.g. the Theorem of [Pas08]. In turn, the logarithmic Gauss map (i.e. the map taking each point of $L$ to the slope of its tangent space after applying a branch of the holomorphic logarithm

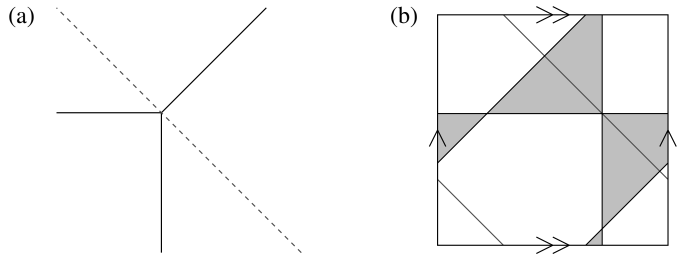



Figure 11. Two tangent tropical curves and their phases.

map) on this interior takes imaginary values, see [Mik00, Lemma 3], while the real values are assumed on the boundary of the amoeba of the line. However, the three arcs of the boundary of the amoeba are contracted to the coamoeba vertices while three points where the logarithmic Gauss map takes values equal to the slope vector of the edges adjacent to $v$ are blown up to the three geodesics on the coamoeba.

However, if the pretangency set is a point, this means that any edge $e$ of $C$ that contains $v$ cannot be parallel to one of the edges adjacent to $v$. Therefore $\phi(e)$ and $\phi(v)$ must intersect in a coamoeba vertex $u$, otherwise $C_t$ and $L_t$ cannot be tangent for large $t \gg 1$ as there are no nearby points with the same value of the logarithmic Gauss map. Furthermore, locally near $u \in S^1 \times S^1$, $\phi(e)$ must be contained in the coamoeba $\operatorname{Arg}(L^v_{\Delta_v})$ as only this arc contains the real value of the logarithmic Gauss map corresponding to the slope vector of $e$. This completely determines $\phi(e)$ as well as making a linear condition on the space of phases for $\Lambda_q \subset U$.

As before we choose a straight line $G_L$ in $U$ transversal to $\Lambda_L$ and a phase for $G_L$ that forms $\mathcal{G}_L \subset \mathcal{U}$. The image $\Lambda_t(\mathcal{G}_L)$ contains some holomorphic curves tangent to $L_t$. By the same argument as before, their number is the sum of the weights of all edges of $C$ passing through $v$.

The situation is similar if $v$ is an image of a vertex $\tilde{v}$ of $C$ and $f$ is an immersion near $\tilde{v}$. In this case the coamoeba of the corresponding edge of $L$ must pass through one of the three coamoeba vertices of the phase of $\tilde{v}$, the one determined by the slope vector of the edge $e$ of $L$ containing $v$. As we assume that $L$ is smooth we have the weight of $e$ as well as the weight of $\Lambda_L$ equal to 1.

This reasoning can be easily modified to include the case when $f$ is not necessarily an immersion near $\tilde{v}$. Consider the linear projection $\mathbb{R}^2 \to \mathbb{R}$ such that its kernel is parallel to $e$. The exponentiation of this map gives us a multiplicatively linear map $\pi : (\mathbb{C}^*)^2 \to \mathbb{C}^*$. Let $\Psi_{\tilde{v}} : \Gamma^\circ_{\tilde{v}} \to (\mathbb{C}^*)^2$ be a holomorphic map such that $\Phi_{\tilde{v}} = \operatorname{Arg} \circ \Psi_{\tilde{v}}$ (see Definition 6.2). By the Riemann–Hurwitz formula, the holomorphic map $\pi \circ \Psi_{\tilde{v}} : \Gamma^\circ_{\tilde{v}} \to \mathbb{C}^*$ has a unique ramification point $r \in \Gamma^\circ_{\tilde{v}}$ (recall that as the vertex $\tilde{v}$ is 3-valent the surface $\Gamma^\circ_{\tilde{v}}$ is a pair of pants). Note that $\Phi_{\tilde{v}}(r)$ must be contained in the phase-boundary circle corresponding to $e$ in order for the corresponding approximation curves to be tangent.

Let $e \subset C$ be any edge adjacent to $\tilde{v}$. Varying the phase structure of $C$ by slightly changing the orientation-reversing isometry $\rho_e$ (from Definition 6.2) and applying $\Lambda_t$ (from Proposition 6.10) for large but finite values of $t$ we have a unique curve tangent to $L_t$, so that the weight of $\Lambda_L$ is again 1.

Applying Proposition 5.1 inductively we see that the determinant $\det(\Lambda_{p_1}, \dots, \Lambda_{p_k}, \Lambda_{L_1}, \dots, \Lambda_{L_{3d-1-k}})$ computes the number of different phase structures satisfying to the phase tangency conditions at $v$. However, the same phase structure is counted several times, once for every automorphism of our tropical curve $f : C \to \mathbb{R}^2$, as inducing a phase structure by an automorphism of $f$ gives an isomorphic phase structure. Thus we have to divide the result by $|\mathrm{Aut}(f)|$.

Finally we consider the general case when the pretangency components are not necessarily points. Let $E_L$ be a pretangency component. This component is a tree (as it is contained in the tree $C$). Furthermore, the coamoeba of this component (by which we mean the union of the argument of phases of all vertices and edges contained in $E_L$) is a circle which we denote $\phi(E_L) \subset S^1 \times S^1$. Indeed all edges adjacent to a vertex of $E_L$ are parallel to the same line, and therefore the only possible change in coamoeba of an edge in $E_L$ is a number of times it runs through the same circle $\phi(E_L)$ while the number of times coincides in turn with the weight of the edge. The same can be said about the coamoeba of the part of $L$ corresponding to the same pretangency set. To obtain tangencies for $L_t$ and $C_t$ these two coamoebas must coincide.

To compute the weight of $\Lambda_L$ we consider again the properly embedded annulus $\mathcal{G}_L \subset \mathcal{U}$ obtained as a phased straight line in an integer direction transversal to $\Lambda_L$ as well as its image under $\Lambda_t$ for large $t \gg 1$.

To compute the number of curves in $\Lambda_t(\mathcal{G}_L)$ tangent to $\mathcal{L}_t$ we prepare an auxiliary phase tropical curve from $(f, \phi)$ and $E_L$. Namely we take the subtree $C_L \subset C$ formed by the closed edges of $C$ intersecting $E_L$ and continue the resulting new 1-valent edges to infinity. This defines $f_L : C_L \to \mathbb{R}^2$. The phase $\phi$ induces a phase $\phi_L$ for $f_L$.

Proposition 6.10 produces a real 1-parametric family of complex curves $C_t^{(E_L)}$ whose tropical limit is $f_L : C_L \to \mathbb{R}^2$. As $\phi_L$ is induced by $\phi$ we can compute the weight of $\Lambda_L$ with the help of $C_t^{(E_L)}$ and a 1-parametric family obtained by multiplicative translation in a direction transversal to $\Lambda_L$.

Furthermore, in this computation we can replace the family $L_t$ with the family $L_t^{(E_L)}$ obtained in a similar way as $C_t^{(E_L)}$. Namely we take the subtree $E_L \subset L$ formed by the closed edges of $L$ intersecting $E_L$ and continue the new 1-valent edges to infinity. Denote the resulting (rational immersed) tropical curve by $L_L$. Then we take the approximating family $L_t^{(E_L)}$ provided by Proposition 6.10 for the phase structure induced by that of $L$.

Then the number of tangencies of $C_t^{(E_L)}$ and $\tau_\lambda(L_t^{(E_L)})$, the multiplicative translation of $L_t^{(E_L)}$ by $\lambda \in \mathcal{G}_L$, can be computed by Euler characteristic calculus as follows. Projection of $C_t^{(E_L)}$ and $L_t^{(E_L)}$ along $G_L$ to $\mathbb{C}^*$ allows one to define the fiber product of $C_t^{(E_L)}$ and $L_t^{(E_L)}$, which we denote by $A_t$. As there exists infinitely many directions $\mathbb{Z}$-transversal to $E_L$, we may assume that the direction of $G_L$ is not parallel to any edge of $L$ or $C$.

Note that minus the Euler characteristic of $A_t$ is equal to the number of tangencies between $C_t^{(E_L)}$ and $\tau_\lambda(L_t^{(E_L)})$, $\lambda \in \mathcal{G}_L$, plus a correction term $\delta$ at infinity, by the Fubini theorem for the calculus based on the Euler characteristic (see [Vir88, Theorem 3.A]). The correction at infinity is computed for each end of $L$ contained in the (classical) line $D \subset \mathbb{R}^2$ extending the pretangency set $E_L$. Whenever $L$ has an end contained in $D$, we add to $\delta$ the number of the ends of $C_L$ contained in $D$ and going in the same direction. Since there are two possible infinite directions in $D$, $\delta$ is the sum of two possible corrections.

Indeed, unless the point of the target $\mathbb{C}^*$ is an image of a tangency point for some $\lambda \in \mathcal{G}_L$, the number of inverse images in the projection $A_t \to \mathbb{C}^*$ is equal to the product of the degrees of the projections $C_t^{(E_L)} \to \mathbb{C}^*$ and $L_t^{(E_L)} \to \mathbb{C}^*$. Thus the Euler characteristic of $A_t$ is equal to a multiple of $\chi(\mathbb{C}^*) = 0$ minus the number of the tangency points in the family parameterized by $\lambda$.

The Euler characteristic of the fiber product $A_t$ for large $t \gg 1$ can be computed from $C$ and $L$. Indeed, each vertex $v$ of $L$ or $C$ (respectively) that belongs to $D$ gives a contribution to this Euler characteristic of the fiber product $A_t$. The contribution is equal to the degree of projection of $C_t^{(E_L)} \to \mathbb{C}^*$ or $L_t^{(E_L)} \to \mathbb{C}^*$ (respectively). In turn this degree can be computed from $f : C \to \mathbb{R}^2$ and $L$ (respectively) and it is equal to the number of intersections of the edges $E$ of $C_L$ and $L_L$ (respectively) and the line $G_L^v$ parallel to $G_L$ and passing through $v$. Each point of $E \cap G_L^v$ contributes the corresponding tropical intersection number, i.e. the absolute value of the determinant of the matrix formed by a primitive integer vector parallel to $E$ and a primitive integer vector parallel to $G_L$ multiplied by the weight of $E$.

Note that, if $E$ is not contained in $D$, the corresponding contribution can be excluded from the Euler characteristic by passing to a smaller surface $C_t^{(E_L)}$ or $L_t^{(E_L)}$ (respectively) by taking an intersection with $\mathrm{Log}_t^{-1}(V)$ for a small neighborhood $V \supset D$. Indeed this contribution corresponds to tangencies with $\tau_\lambda(L_t^{(E_L)})$ for large $\lambda$ for large $t \gg 1$ and $\lambda$ cannot tend to zero when $t \to +\infty$. Summarizing all contributions together, we get $w_L$ from § 3 as the sum of the weights of the vertices $v$ of $L$ such that $v \in C$, where each weight is equal to the sum of the weights of $C_L$ passing through it plus the number of vertices of $C$ that are mapped on $L$ (as $L$ is embedded and all its weights are 1) minus the contribution $\delta$ at infinity.

To conclude the proof we note that the intersection number of proper submanifolds

$$\Lambda_q \times N_q \subset (\mathbb{C}^*)^2 \times (\mathbb{C}^*)^b = (\mathbb{R}^2 \times \mathbb{R}^b) \times ((S^1 \times S^1) \times (S^1)^b)$$

for all $q \in \mathcal{P} \cup \mathcal{L}$ is determined by their homology classes with closed support and thus coincides with the corresponding tropical intersection number as both coincide with the same intersection numbers in $H_*((S^1 \times S^1) \times (S^1)^b)$. Thus each $f \in \mathcal{S}^{\mathbb{T}}(d, \mathcal{P}, \mathcal{L})$ contributes $\mu_{(\mathcal{P},\mathcal{L})}(f)$ to the Zeuthen number. □

*Proof of Theorem 3.12.* To prove Theorem 3.12 we need to construct a suitable configuration of immersed complex curves $L_j$ of genus $g_j$ starting from our configuration $L_j^{\mathbb{T}}$. Such construction can be provided by Proposition 6.10 (with large value of $t$) once we equip each tropical curve $L_j^{\mathbb{T}}$ with the phase structure in the case when the tropical curves $L_j^T$ are rational, i.e. all $g_j = 0$. In the general case a family of immersed complex curves $L_j^t$ for large $t$ converging to $L_j^{\mathbb{T}}$ in the sense of Definition 6.6 is provided by [Mik05, proof of Theorem 1], more precisely by Proposition 8.23.

The rest of the proof is similar to the proof of Theorem 3.8. The only difference is that the constraints $L_j^{\mathbb{T}}$ no longer have to be smooth near their 3-valent vertices. However, since the curve is immersed there is always a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ such that near that vertex $L_j^{\mathbb{T}}$ is the image of a tropical line in $\mathbb{R}^2$. Here the determinant $m$ of this linear map is the *multiplicity* of our vertex, cf. [Mik05, Definition 2.16]. The exponent of this map is a multiplicatively linear map $M : (\mathbb{C}^*)^2 \to (\mathbb{C}^*)^2$ of degree $m$. Note that the genus of the part of $L_j^t$ approximating $L_j^{\mathbb{T}}$ is zero by [Mik05, Proposition 8.14]. Taking the pull-back by $M$ we reduce the problem to the case of smooth curves which is already considered in the proof of Theorem 3.8. □

### 7.2 Enumeration of real curves

As already mentioned, the proof of Theorem 3.8 establishes a correspondence between phase-tropical curves and complex curves close to the tropical limit. In particular, if we choose real phases for all constraints in $(\mathcal{P}, \mathcal{L})$, it is possible to recover all real algebraic curves passing through a configuration of real points and tangent to a configuration of real lines when these points and lines are close to the tropical limit.

*Example* 7.1. Let us revisit Example 3.7 from a real point of view. For example, if all the three points in Example 3.7 have phase $(1,1)$, then the tropical curve in Figure 5 ensures that there exists a configuration of three points and two lines in $\mathbb{R}P^2$ such that all four conics passing through these points and tangent to these lines are real. On the other hand, if the middle point has phase $(-1,1)$ and the two other points have phase $(1,1)$, then there exists a corresponding configuration of three points and two lines in $\mathbb{R}P^2$ such that none of the four conics passing through these points and tangent to these lines are real.

*Example* 7.2. One can interpret in tropical terms the method used in [RTV97] to construct a configuration of five real conics such that all 3264 conics tangent to these five conics are real. The main step in this construction is to find five real lines $L_1,\dots,L_5$ in $\mathbb{R}P^2$ and five points $p_1\in L_1,\dots,p_5\in L_5$ such that for any set $I\subset\{1,\dots,5\}$, all the conics passing through the points $p_i$, $i\in I$, and tangent to the lines $L_j$, $j\in\{1,\dots,5\}\backslash I$ are real. As in [RTV97], let us start with the configuration depicted in Figure 12(a), whose tropical analog is depicted in Figure 12(b) (without phase) and Figure 12c (equipped with the appropriate real phases). Next, we perturb the double lines $L_i^2$ as depicted in Figure 12(d) (without phase, the cycle defined by the image is a twice a line) and Figure 12e (equipped with the appropriate real phases). Then there exist five families of real conics converging to our five phase conics and producing 3264 real conics as in [RTV97].

It would be interesting to explore the possible numbers of real conics tangent to five real conics, in connection to [Ber08] and [Wel06]. In particular, does there exist a configuration of five real conics, any one of which lying outside the others, such that exactly 32 real conics are tangent to them?

Note that once the lines and points $L_i$ and $p_i$ are chosen as above, arguments used in Example 7.2 also prove the next proposition.

Proposition 7.3. *For any $0\leqslant k\leqslant 5$, any $d_1,\dots,d_{5-k}\geqslant 1$, and any $g_1,\dots,g_{5-k}\geqslant 0$, there exists a generic configuration of $k$ points $p_1,\dots,p_k$ in $\mathbb{R}P^2$ and $5-k$ immersed real algebraic curves $C_1,\dots,C_{5-k}$ with $C_i$ of degree $d_i$ and genus $g_i$ such that all conics passing through $p_1,\dots,p_k$ and tangent to $C_1,\dots,C_{5-k}$ are real.*

For other examples of totally real enumerative problems concerning conics in $\mathbb{R}P^n$, see [BP].

## 8. Floor decompositions

### 8.1 Motivation

In this section we give a purely combinatorial solution to the computation of characteristic numbers. To obtain totally combinatorial objects we stretch our configuration of constraints in the vertical direction, i.e. we only consider configurations $(\mathcal{P},\mathcal{L})$ for which the difference of the $y$-coordinates of any two elements of the set $\mathcal{P}\cup_{L\in\mathcal{L}}\mathrm{Vert}(L)$ is very big compared to the difference of their $x$-coordinates. For a sufficiently stretched configuration $(\mathcal{P},\mathcal{L})$, tropical morphisms $f: C\to\mathbb{R}^2$ in $\mathcal{S}^{\mathbb{T}}(d,\mathcal{P},\mathcal{L})$ will have a very simple decomposition into *floors* linked together by *shafts*. *Marked floor diagrams* and their multiplicities will encode the combinatoric of these decompositions together with the distribution of $f^{-1}(\mathcal{P})$ and the tangency components of $f$ with elements of $\mathcal{L}$. In the case where no tangency condition is imposed, these new floor diagrams get simplified to an equivalent of those introduced in [BM07] and [BM09]. The floor diagrams we define here allow us to compute characteristic numbers of the plane in terms of a slight generalization of Hurwitz number. Indeed the multiplicity of a marked floor diagram will be

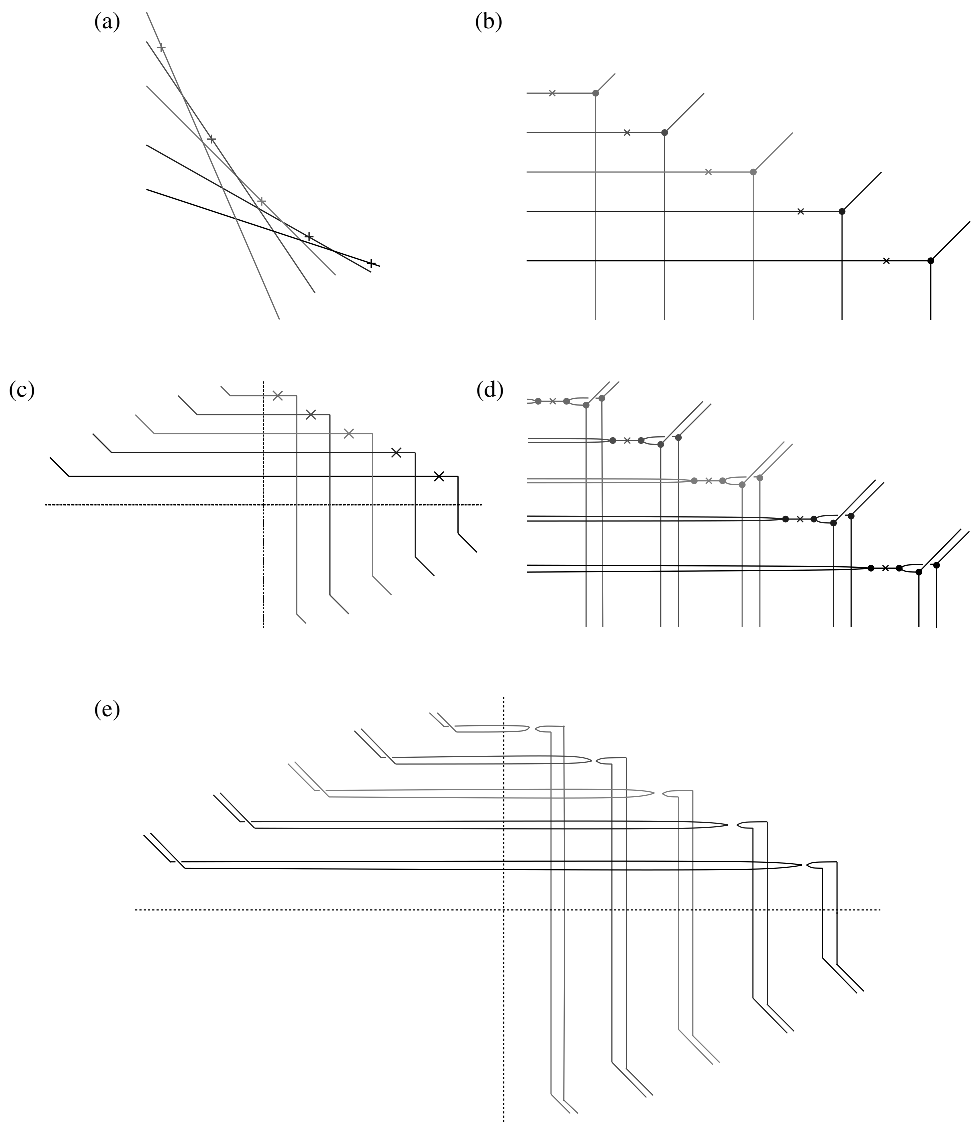



FIGURE 12. 3264 real conics tangent to five real conics.

expressed in terms of *open Hurwitz numbers* which appear in two distinct ways in the count of $N_{d,0}(k; d_1, \ldots, d_{3d-1-k})$. These numbers were introduced in [BBM11]. We give in Appendix the definitions and result from [BBM11] we need in this paper.

DEFINITION 8.1. An *elevator* of a tropical morphism $f: C \to \mathbb{R}^2$ is an edge $e$ of $C$ with $u_{f,e} = \pm(0, 1)$. A *shaft* of $f$ is a connected component of the topological closure of the union of all elevators of $f$. The set of shafts of $f: C \to \mathbb{R}^2$ is denoted by $\mathrm{Sh}(f)$.

A *floor* of a tropical morphism $f: C \to \mathbb{R}^2$ is a connected component of the topological closure of $C \setminus \mathrm{Sh}(f)$. The *degree* of a floor $\mathcal{F}$ of $f$, denoted by $\deg(\mathcal{F})$, is the tropical intersection number of $f(\mathcal{F})$ with a generic vertical line of $\mathbb{R}^2$.

Let us illustrate our approach on a simple case. Let us consider $\mathcal{L}$ the set composed of the five tropical lines depicted in Figure 13. The set $\mathcal{S}^{\mathbb{T}}(2, \emptyset, \mathcal{L})$ is then reduced to the tropical

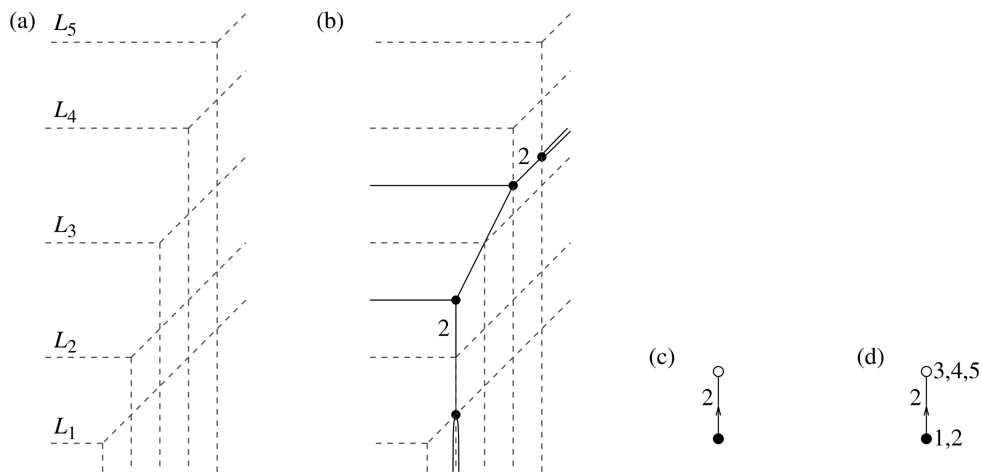



Figure 13. A tropical conic tangent to five lines, and its associated marked floor diagram.

morphism $f : C \to \mathbb{R}^2$ depicted in Figure 13(b), which is of multiplicity 1. This morphism has one floor of degree 2, and one shaft made of three elevators. Let us represent the morphism $f$ by the graph depicted in Figure 13(c), where the black vertex represents the shaft of $f$, the white vertex represents the floor of $f$, and the edge represents the weight 2 elevator of $f$ which join the shaft and the floor of $f$. By remembering on this graph how are distributed the tangency components of $f$ with the lines $L_i$, we obtain the labeled graph depicted in Figure 13(d).

Let us define the two projections $\pi_x$ and $\pi_y$ as follows:

$$\begin{array}{rrcl} \pi_x : & \mathbb{R}^2 & \to & \mathbb{R} \\ & (x,y) & \mapsto & x \end{array} \quad \text{and} \quad \begin{array}{rrcl} \pi_y : & \mathbb{R}^2 & \to & \mathbb{R} \\ & (x,y) & \mapsto & y. \end{array}$$

Our main observation is the following pair of properties.

- The map $\pi_x \circ f$ restricted to the floor of $f$ is a tropical ramified covering of $\mathbb{R}$ of degree 2; its critical values correspond to the vertical edge of the lines $L_4$ and $L_5$ (see Figure 14).
- The map $\pi_y \circ f$ restricted to the shaft of $f$ is a tropical morphism with source a tropical curve with one boundary component; its critical value correspond approximatively to the horizontal edge of $L_1$; the image of its boundary component corresponds approximatively to the horizontal edge of $L_3$.

Vice versa, the morphism $f : C \to \mathbb{R}^2$ can be reconstructed out of the labeled diagram of Figure 13(d) in the following way: we first find the tropical solutions of two tropical open Hurwitz problems, one for the floor of $f$ and one for its shaft; next we glue them according to the elevator joining this floor and this shaft, and the lines $L_2$ and $L_3$.

The floor of $f$ leads to the Hurwitz number $H(2) = \frac{1}{2}$; we have two possibilities to attach the weight 2 elevator; making the floor tangent to $L_3$ gives a factor 1; the Hurwitz number we have to compute to reconstruct the shaft of $f$ is $H(\delta, n) = \frac{1}{2}$ where $\delta(0) = 2$, $\delta(1) = n(1) = 0$, and $n(0) = 1$ (see Appendix for the definition of Hurwitz numbers); making the shaft tangent to $L_2$ gives us a factor 1; gluing the floor and the shaft along the weight 2 elevator gives an extra factor 2. Hence, the total multiplicity of $f$ is

$$\tfrac{1}{2} \times 2 \times 1 \times \tfrac{1}{2} \times 1 \times 2 = 1$$

as expected.

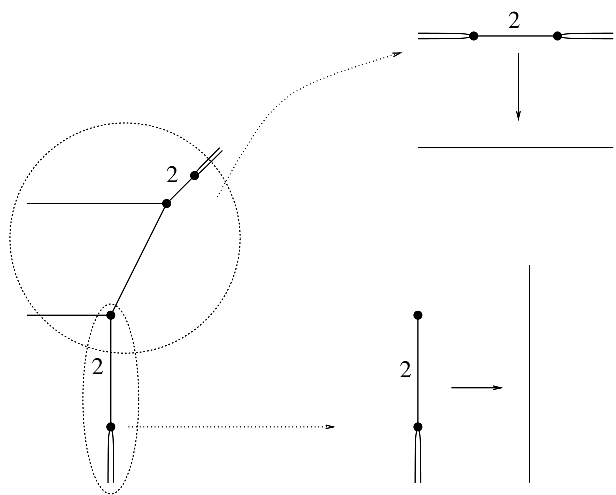



Figure 14. From characteristic numbers to open Hurwitz numbers.

The next section is devoted to the generalization of the previous computation to arbitrary degree and set of constrains.

### 8.2 Floor diagrams

Here we define floor diagrams and their markings. Our definitions are similar in the spirit of those given in [BM07, BM09], and [FM10], but are somewhat different since our enumerative problems also involve tangency conditions. For simplicity, we only explain in detail how to turn the problem of computing the numbers $N_{d,0}(k; 1^{3d-1-k})$ into the enumeration of marked floor diagrams. The general computation of the numbers $N_{d,0}(k; d_1, \dots, d_{3d-1-k})$ in terms of floor diagrams require no more substantial efforts, but makes the exposition heavier. Hence we restrict ourselves to the case of tangency with lines, which by (1) is enough to recover all genus 0 characteristic numbers of $\mathbb{C}P^2$.

The floor diagrams we deal with in this paper underlie bipartite trees, whose vertices are divided between *white* and *black* vertices. As usual, the divergence of a vertex $v$ of $\mathcal{D}$, denoted by $\mathrm{div}(v)$, is the sum of the weight of all its incoming adjacent edges minus the sum of the weight of all its outgoing adjacent edges.

Definition 8.2. A *floor diagram* (of genus 0) is an oriented bipartite tree $\mathcal{D}$ equipped with a weight function $w : \mathrm{Edge}(\mathcal{D}) \to \mathbb{Z}_{>0}$ such that white vertices have positive divergence, and black vertices have non-positive divergence.

The sum of the divergence of all white vertices is called the *degree* of $\mathcal{D}$. We denote by $\mathrm{Vert}^\circ(\mathcal{D})$ the set of white vertices of $\mathcal{D}$, and by $\mathrm{Vert}^\bullet(\mathcal{D})$ the set of its black vertices.

As explained in § 8.1, a white vertex represents a *floor* of a tropical morphism, whereas a black vertex represents one of its shafts.

*Example* 8.3. All floor diagrams of degree 2 are depicted in Figure 15. We specify the weight of an edge of $\mathcal{D}$ only if this latter is not 1.

Given a vertex $v$ of $\mathcal{D}$, we denote by $\mathrm{Vert}(v)$ the set of vertices of $\mathcal{D}$ adjacent to $v$.

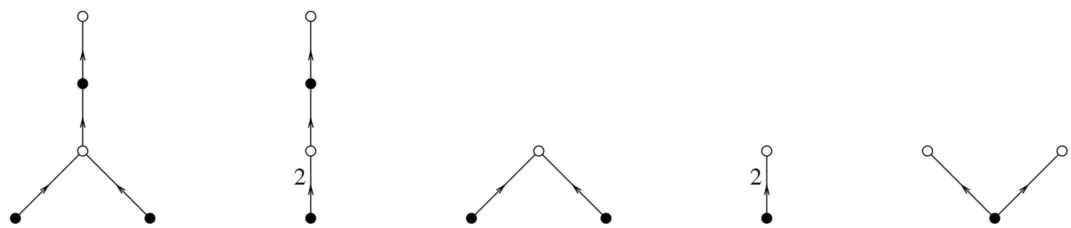


Figure 15. Floor diagrams of degree 2.

Definition 8.4. Let $\mathcal{L}^{\mathrm{comb}} \sqcup \mathcal{P}^{\mathrm{comb}}$ be a partition of the set $\{1, \ldots, 3d-1\}$. A $\mathcal{L}^{\mathrm{comb}}$*-marking* of a floor diagram $\mathcal{D}$ of degree $d$ is a surjective map $m : \{1, \ldots, 3d-1\} \to \mathrm{Vert}(\mathcal{D})$ such that.

- For any $v \in \mathrm{Vert}(\mathcal{D})$, the set $m^{-1}(v)$ contains at most one point in $\mathcal{P}^{\mathrm{comb}}$; moreover, if $v \in \mathrm{Vert}^\circ(\mathcal{D})$ and $m^{-1}(v) \cap \mathcal{P}^{\mathrm{comb}} = \{i\}$, then $i = \min(m^{-1}(v))$.
- For any $v \in \mathrm{Vert}^\circ(\mathcal{D})$, $|m^{-1}(v)| = 2\,\mathrm{div}(v) - 1$.
- For any $v \in \mathrm{Vert}^\bullet(\mathcal{D})$, $|m^{-1}(v)| = \mathrm{val}(v) - \mathrm{div}(v) - 1$; moreover there exists at most one element $i$ in $m^{-1}(v)$ such that $i > \max_{v' \in \mathrm{Vert}(v)} \min(m^{-1}(v'))$, and if such an element exists then we have $m^{-1}(v) \cap \mathcal{P}^{\mathrm{comb}} = \emptyset$.

Two $\mathcal{L}^{\mathrm{comb}}$-markings $m : \{1, \ldots, 3d-1\} \to \mathrm{Vert}(\mathcal{D})$ and $m' : \{1, \ldots, 3d-1\} \to \mathrm{Vert}(\mathcal{D}')$ are *isomorphic* if there exists an isomorphism of bipartite graphs $\phi : \mathcal{D} \to \mathcal{D}'$ such that $m = m' \circ \phi$. In this text, $\mathcal{L}^{\mathrm{comb}}$-marked floor diagrams are considered up to isomorphism.

The set $\{1, \ldots, 3d-1\}$ represents the configuration of constraints in the increasing height order (see § 8.1), the set $\mathcal{L}^{\mathrm{comb}}$ represents the lines in the configuration, and the set $\mathcal{P}^{\mathrm{comb}}$ represents the points. Note that, unlike [BM07, BM09, FM10], we do not consider the partial order on $\mathcal{D}$ defined by its orientation. In particular, it makes no sense here to require the marking $m$ to be an increasing map.

In order to define the multiplicity of an $\mathcal{L}^{\mathrm{comb}}$-marked floor diagram, we first define the multiplicity of a vertex of $\mathcal{D}$. Recall that the definitions of open Hurwitz numbers $H(\delta, n)$ and Hurwitz numbers $H(d) = H(d, 0)$ are given in Definition A.1.

Definition 8.5. The *multiplicity of a vertex* $v$ *in* $\mathrm{Vert}^\circ(\mathcal{D})$ is defined as follows.

- If $\min(m^{-1}(v)) \in \mathcal{P}^{\mathrm{comb}}$, then
$$\mu_{\mathcal{L}^{\mathrm{comb}}}(v) = \mathrm{div}(v)^{\mathrm{val}(v)+1} H(\mathrm{div}(v)).$$
- Otherwise,
$$\mu_{\mathcal{L}^{\mathrm{comb}}}(v) = (\mathrm{div}(v) - 2 + \mathrm{val}(v))\,\mathrm{div}(v)^{\mathrm{val}(v)} H(\mathrm{div}(v)).$$

*Example* 8.6. We give in Figure 16 some examples of multiplicities of white vertices of a marked floor diagram. The corresponding Hurwitz numbers are given in Proposition A.3. We write the elements of $m^{-1}(v)$ close to the vertex $v$.

The definition of the multiplicity of a black vertex $v$ of $\mathcal{D}$ requires a preliminary construction. The order on $\{1, \ldots, 3d-1\}$ induces an order on $\mathrm{Vert}(v)$ via the map $v' \mapsto \min(m^{-1}(v'))$. Note that this order doesn't have to be compatible with the orientation of $\mathcal{D}$. Let us denote by $v'_1 < \cdots < v'_s$ the elements of $\mathrm{Vert}(v)$ according to this order. We denote by $e_i$ the edge of $\mathcal{D}$ joining the vertices $v$ and $v'_i$, and define $\varepsilon_i = 1$ if $e_i$ is oriented toward $v$, and $\varepsilon_i = -1$ otherwise.


$a \in \mathcal{P}^{\text{comb}}$, $\mu = 1$ $\quad$ $a \in \mathcal{P}^{\text{comb}}$, $\mu = 324$ $\quad$ $a \in \mathcal{P}^{\text{comb}}$, $\mu = 1$ $\quad$ $a \in \mathcal{P}^{\text{comb}}$, $\mu = \frac{d^{d-1}(2d-2)!}{d!}$

$a \in \mathcal{L}^{\text{comb}}$, $\mu = 2$ $\quad$ $a \in \mathcal{L}^{\text{comb}}$, $\mu = 432$ $\quad$ $a \in \mathcal{L}^{\text{comb}}$, $\mu = 0$ $\quad$ $a \in \mathcal{L}^{\text{comb}}$, $\mu = \frac{d^{d-3}(2d-2)!}{(d-2)!}$

Figure 16. Example of multiplicities of white vertices of $\mathcal{D}$.

Given $j \in m^{-1}(v)$, we define the integer $i_j$ by:
$i_j = 0$ if $j < \min(m^{-1}(v'_1))$;
$i_j = i$ if $\min(m^{-1}(v'_i)) < j < \min(m^{-1}(v'_{i+1}))$;
$i_j = s$ if $j > \min(m^{-1}(v'_s))$.
We define two functions $\delta, \tilde{n} : \{0, \dots, s\} \to \mathbb{Z}$ by:

- $\delta(0) = -\operatorname{div}(v)$,
  $\delta(i+1) = \delta(i) + \varepsilon_{i+1} w(e_{i+1})$;
- $\tilde{n}(i) = |\{j \in m^{-1}(v) \mid i_j = i\}|$.

Given $i_0 \in \{0, \dots, s\}$, we define the function $n_{i_0} : \{0, \dots, s\} \to \mathbb{Z}_{\geqslant 0}$ by $n_{i_0}(i_0) = \tilde{n}(i_0) - 1$ and $n_{i_0}(i) = \tilde{n}(i)$ if $i \neq i_0$. Finally, we define $\tilde{N}(i) = \sum_{l=0}^{i} \tilde{n}(l)$ and $\tilde{N}(-1) = 0$.

Definition 8.7. The *multiplicity of a vertex $v$ in* $\mathrm{Vert}^\bullet(\mathcal{D})$ is defined by the following rules.

- If $m^{-1}(v) \cap \mathcal{P}^{\text{comb}} = \{j\}$, then
$$\mu_{\mathcal{L}^{\text{comb}}}(v) = \delta(i_j) H(\delta, n_{i_j}).$$
- If $m^{-1}(v) \cap \mathcal{P}^{\text{comb}} = \emptyset$ and $m^{-1}(v)$ contains an element $j$ such that $j > \max_{v' \in \mathrm{Vert}(v)} \min(m^{-1}(v'))$, then
$$\mu_{\mathcal{L}^{\text{comb}}}(v) = (2\,\mathrm{val}(v) - 2) H(\delta, n_s).$$
- Otherwise,
$$\mu_{\mathcal{L}^{\text{comb}}}(v) = \frac{1}{2} \sum_{i=0}^{s} (\tilde{n}(i)(2\delta(i) + 2i + \tilde{N}(i) + \tilde{N}(i-1) - 1 + 2\operatorname{div}(v)) H(\delta, n_i)).$$

*Example* 8.8. We give in Figure 17 some examples of multiplicities of black vertices of a marked floor diagram. The corresponding open Hurwitz numbers are given in Proposition A.3 and Example A.6.

Definition 8.9. The *multiplicity of an* $\mathcal{L}^{\text{comb}}$*-marked floor diagram* is defined as
$$\mu_{\mathcal{L}^{\text{comb}}}(\mathcal{D}, m) = \prod_{e \in \mathrm{Edge}(\mathcal{D})} w(e) \prod_{v \in \mathrm{Vert}(\mathcal{D})} \mu_{\mathcal{L}^{\text{comb}}}(v).$$

Note that $\mu_{\mathcal{L}^{\text{comb}}}(\mathcal{D}, m)$ can be equal to 0.

*Example* 8.10. We give in Figure 18 a few examples of multiplicities of $\mathcal{L}^{\text{comb}}$-marked floor diagrams.

The next theorem is a direct consequence of Proposition 8.21 and Theorem 3.8.

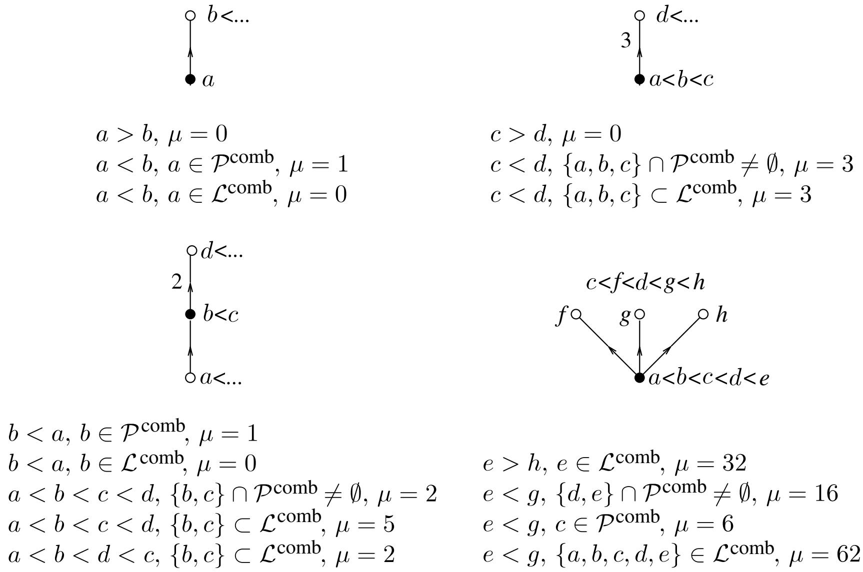



Figure 17. Example of multiplicities of black vertices of $\mathcal{D}$.

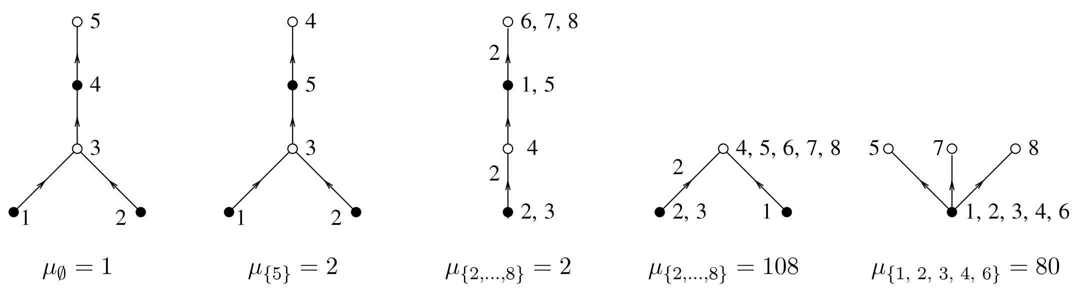



Figure 18. Example of multiplicities of $\mathcal{L}^{\mathrm{comb}}$-marked floor diagrams.

Theorem 8.11. *For any $d \geqslant 1$, $k \geqslant 0$, and $\mathcal{L}^{\mathrm{comb}} \subset \{1, \dots, 3d-1\}$ of cardinal $3d-1-k$, we have*

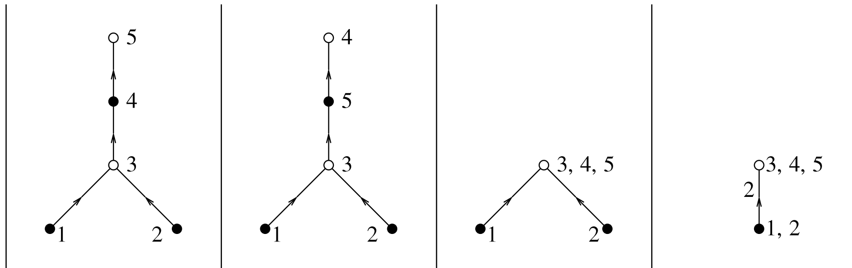
$$N_{d,0}(k; 1^{3d-1-k}) = \sum \mu_{\mathcal{L}^{\mathrm{comb}}}(\mathcal{D}, m)$$

*where the sum ranges over all $\mathcal{L}^{\mathrm{comb}}$-marked floor diagrams of degree $d$.*

Note that in the case $k = 3d-1$, Theorem 8.11 agrees with [BM07, Theorem 1] and [BM09, Theorem 3.6]. Indeed, a $\emptyset$-marked floor diagram has non-null multiplicity if and only if the marking is increasing with respect to the partial order on $\mathcal{D}$ defined by its orientation; in this case, according to Example A.2, the different definitions of multiplicity of a marked floor diagram coincide.

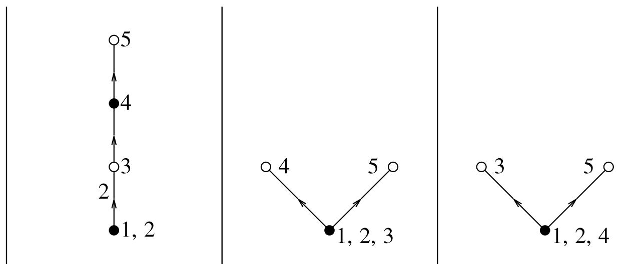
*Example* 8.12. We first compute the numbers $N_{2,0}(k; 1^{5-k})$, with $\mathcal{L}^{\mathrm{comb}} = \{k+1, \dots, 5\}$. In each case, there is exactly one marked floor diagram of positive multiplicity, depicted in Figure 19.

*Example* 8.13. In order to get used with floor diagrams, we compute again the numbers $N_{2,0}(k; 1^{5-k})$, using now $\mathcal{L}^{\mathrm{comb}} = \{1, \dots, 5-k\}$. In this case we have one (respectively 3, and 2)




| | | | | |
|---|---|---|---|---|
| $\mu_\emptyset$ | 1 | 0 | 0 | 0 |
| $\mu_{\{5\}}$ | 0 | 2 | 0 | 0 |
| $\mu_{\{4,\,5\}}$ | 0 | 0 | 4 | 0 |
| $\mu_{\{3,\,4,\,5\}}$ | 0 | 0 | 4 | 0 |
| $\mu_{\{2,\,3,\,4,\,5\}}$ | 0 | 0 | 0 | 2 |
| $\mu_{\{1,\,2,\,3,\,4,\,5\}}$ | 0 | 0 | 0 | 1 |

FIGURE 19. Computation of $N_{2,0}(k; 1^{5-k})$ with $\mathcal{L}^{\text{comb}} = \{k+1, \dots, 5\}$.




| | | | |
|---|---|---|---|
| $\mu_{\{1\}}$ | 2 | 0 | 0 |
| $\mu_{\{1,\,2\}}$ | 1 | 2 | 1 |
| $\mu_{\{1,\,2,\,3\}}$ | 1 | 3 | 0 |
| $\mu_{\{1,\,2,\,3,\,4\}}$ | 2 | 0 | 0 |

FIGURE 20. Computation of $N_{2,0}(k; 1^{5-k})$ with $\mathcal{L}^{\text{comb}} = \{1, \dots, 5-k\}$.

marked floor diagrams of positive multiplicity for $k = 5, 4, 1,$ and $0$ (respectively 3, and 2). These marked floor diagrams are depicted in Figure 20.

*Example* 8.14. Figure 21 represents all marked floor diagrams of degree 3 with positive multiplicity when $\mathcal{L}^{\text{comb}} = \{2, \dots, 8\}$. Hence there are exactly 600 rational cubics passing through one point and tangent to seven lines.

*Example* 8.15. We give in Figure 22 the sum of multiplicities of floor diagrams of degree 3 with $\mathcal{L}^{\text{comb}} = \{1, 2, 3, 4, 6\}$, when this sum is positive. In particular, we find that there are exactly 712 rational cubics passing through three points and tangent to five lines.

### 8.3 Proof of Theorem 8.11

The strategy of the proof of Theorem 8.11 is the same as in the proof of [BM09, Theorem 3.6]: we first prove that if points in $\mathcal{P}$, and vertices of elements of $\mathcal{L}$, lie in some strip $I \times \mathbb{R}$, then all the vertices of a curve in $\mathcal{S}^{\mathbb{T}}(d, \mathcal{P}, \mathcal{L})$ tangent to $\mathcal{L}$ also lie in the strip $I \times \mathbb{R}$. As a consequence, if the configuration $(\mathcal{P}, \mathcal{L})$ is sufficiently stretched in the vertical direction, then all floors of a

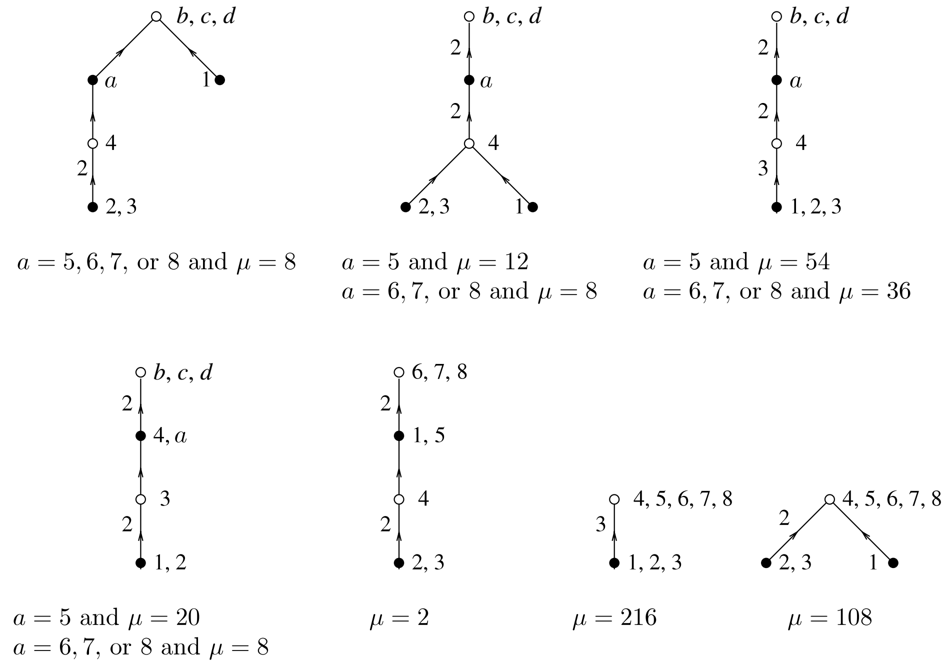



Figure 21. Computation of $N_{3,0}(1; 1^7) = 600$ with $\mathcal{L}^{\mathrm{comb}} = \{2, \ldots, 8\}$.

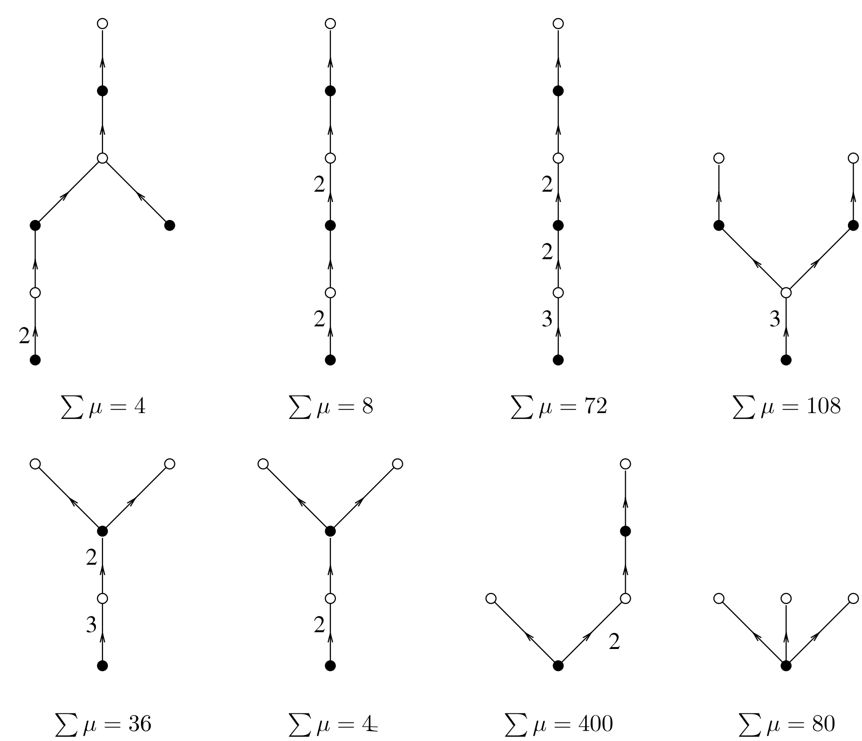



Figure 22. Computation of $N_{3,0}(3; 1^5) = 712$ with $\mathcal{L}^{\mathrm{comb}} = \{1, 2, 3, 4, 6\}$.

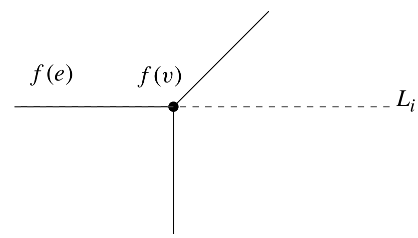



FIGURE 23. No vertex $v \in \mathrm{Vert}^0(C)$ with $f(v) < a$ if $f$ is tangent to $\mathcal{L}$.

curve in $\mathcal{S}^{\mathbb{T}}(d, \mathcal{P}, \mathcal{L})$ tangent to $\mathcal{L}$ contain exactly one horizontal constraint. It will then remain to study how the constraints can be distributed among the floors and the shafts of the tropical curves we are counting.

The rest of this section is devoted to make precise the latter explanations.

From now on, we fix $I = [a; b] \subset \mathbb{R}$ a bounded interval, and a generic configuration $(\mathcal{P}, \mathcal{L})$ where $\mathcal{P} = \{p_1, \ldots, p_k\}$ and $\mathcal{L} = \{L_1, \ldots, L_{3d-1-k}\}$ is a set of $3d - 1 + k$ tropical lines in $\mathbb{R}^2$. We denote by $\eta_i$ the vertex of the line $L_i$. Recall that an element of $\mathcal{S}^{\mathbb{T}}(d, \mathcal{P}, \mathcal{L})$ with positive $(\mathcal{P}, \mathcal{L})$-multiplicity is called tangent to $\mathcal{L}$.

LEMMA 8.16. *Suppose that* $\mathcal{P} \bigcup \{\eta_1, \ldots, \eta_{3d-1-k}\} \subset I \times \mathbb{R}$. *Then for any tropical morphism* $f : C \to \mathbb{R}^2$ *in* $\mathcal{S}^{\mathbb{T}}(d, \mathcal{P}, \mathcal{L})$ *tangent to* $\mathcal{L}$, *and for any vertex* $v \in \mathrm{Vert}^0(C)$, *we have* $f(v) \in I \times \mathbb{R}$.

*Proof.* The proof follows the same lines as the proof of [BM09, Proposition 5.3]. Suppose that there exists a tropical morphism $f : C \to \mathbb{R}^2$ in $\mathcal{S}^{\mathbb{T}}(d, \mathcal{P}, \mathcal{L})$ and a vertex $v$ in $\mathrm{Vert}^0(C)$ such that $f(v) = (x_v, y_v)$ with $x_v < a$. We can choose $v$ such that no vertex of $C$ is mapped by $f$ to the half-plane $\{(x, y) \mid x < x_v\}$. Since the configuration $(\mathcal{P}, \mathcal{L})$ is generic, the set $\mathcal{S}^{\mathbb{T}}(d, \mathcal{P}, \mathcal{L})$ is finite and $v$ is a trivalent vertex of $C$. Hence, the proof of [BM09, Proposition 5.3] shows that $v$ is adjacent to an end $e$ of direction $(-1, 0)$, and two other edges of direction $(0, \pm 1)$ and $(1, \alpha)$, and that one line $L_i$ has its horizontal edge passing through $f(v)$ (see Figure 23). Thus, $e \cup v$ is the pretangency component of $f$ with $L_i$, and $\mu_{L_i}(f) = 0$.

The case where there exist an element $f : C \to \mathbb{R}^2$ in $\mathcal{S}^{\mathbb{T}}(d, \mathcal{P}, \mathcal{L})$ and a vertex $v$ in $\mathrm{Vert}^0(C)$ such that $f(v) = (x_v, y_v)$ with $x_v > b$ works analogously. □

Recall that pretangency sets are defined in § 3.1. A line $L \in \mathcal{L}$ is called a *vertical constraint* (respectively a *horizontal constraint*) of a morphism $f \in \mathcal{S}^{\mathbb{T}}(d, \mathcal{P}, \mathcal{L})$ if the pretangency set of $f$ and $L$ is (respectively is not) contained in the vertical edge of $L$. A point in $\mathcal{P}$ is at the same time a horizontal and a vertical constraint of any morphism $f \in \mathcal{S}^{\mathbb{T}}(d, \mathcal{P}, \mathcal{L})$. We say that a floor $\mathcal{F}$ (respectively a shaft $S$) of $f$ *matches* the constraint $q \in \mathcal{P} \cup \mathcal{L}$ if $\mathcal{F}$ (respectively $S$) contains $f^{-1}(q)$ or contains the pretangency component of $f$ with $q$.

COROLLARY 8.17. *If the points in* $\mathcal{P} \bigcup \{\eta_1, \ldots, \eta_{3d-1-k}\}$ *are in* $I \times \mathbb{R}$ *and far enough apart, then any floor of any tropical morphism* $f : C \to \mathbb{R}^2$ *in* $\mathcal{S}^{\mathbb{T}}(d, \mathcal{P}, \mathcal{L})$ *tangent to* $\mathcal{L}$ *matches at most one horizontal constraint.*

*Proof.* If $e$ is an edge of $C$ with $u_{f,e} \neq (0, \pm 1)$, then the slope of $f(e)$ is uniformly bounded in terms of the degree of $f$. Hence, the result is an immediate consequence of Lemma 8.16. □

DEFINITION 8.18. We say that a generic configuration $(\mathcal{P}, \mathcal{L})$ is *vertically stretched* if it satisfies the hypothesis of Corollary 8.17.

To any morphism $f: C \to \mathbb{R}^2$ in $\mathcal{S}^{\mathbb{T}}(d, \mathcal{P}, \mathcal{L})$, we can naturally associate a floor diagram $\mathcal{D}_f$ of degree $d$ in the following way: white vertices of $\mathcal{D}_f$ correspond to floors of $C$, black vertices correspond to shafts of $C$, and a white vertex and a black vertex of $\mathcal{D}_f$ are joined by an edge of weight $w$ if and only if the corresponding floor and shaft are joined by an elevator of weight $w$; an elevator of $C$ has an orientation inherited from the standard orientation of the line $\{x = 0\}$ in $\mathbb{R}^2$, and edges of $\mathcal{D}_f$ inherits this orientation as well. Given a shaft $S$ of $f$, we define $l(S)$ to be the sum of the number of floors of $C$ adjacent to $S$ and the number of ends of $C$ contained in $S$. In other words, if $S$ corresponds to the black vertex $v_S$ of $\mathcal{D}_f$, then $l(S) = \mathrm{val}(v_S) - \mathrm{div}(v_S)$.

Corollary 8.19. *Let $(\mathcal{P}, \mathcal{L})$ be a vertically stretched configuration of constraints and let $f$ be an element of $\mathcal{S}^{\mathbb{T}}(d, \mathcal{P}, \mathcal{L})$ tangent to $\mathcal{L}$. Then we have the following.*

- *Any floor $\mathcal{F}$ of $f$ matches exactly $2\deg(\mathcal{F}) - 1$ constraints, and exactly one of them is horizontal.*
- *Any shaft $S$ matches exactly $l(S) - 1$ constraints, and exactly one of them is vertical.*

*Proof.* A floor $\mathcal{F}$ and a shaft $S$ of $f$ have respectively exactly $2\deg(\mathcal{F}) - 2$ and $l(S) - 2$ vertices, since $\pi_x \circ f_{|\mathcal{F}} : \mathcal{F} \to \mathbb{R}$ and $\pi_y \circ f_{|S} : S \to \mathbb{R}$ are tropical (open) ramified coverings with respectively $2\deg(\mathcal{F})$ and $l(S)$ ends. According to Corollary 8.17, a floor $\mathcal{F}$ matches at most one horizontal constraint so $\mathcal{F}$ matches at most $2\deg(\mathcal{F}) - 1$ constraints. Since the configuration $(\mathcal{P}, \mathcal{L})$ is generic, a shaft $S$ matches at most one vertical constraint, and hence at most $l(S) - 1$ constraints. All together, we get the inequality

$$\sum_{\mathcal{F}} (2\deg(\mathcal{F}) - 1) + \sum_S (l(S) - 1) \leqslant 3d - 1.$$

We have

$$\sum_{\mathcal{F}} (2\deg(\mathcal{F}) - 1) + \sum_S (l(S) - 1) = 2d - |\mathrm{Vert}^\bullet(\mathcal{D}_f)| - |\mathrm{Vert}^\circ(\mathcal{D}_f)| + \sum_{v \in \mathrm{Vert}^\bullet(\mathcal{D}_f)} \mathrm{val}(v) + d$$

and the fact that $\mathcal{D}_f$ is a bipartite tree gives us

$$|\mathrm{Vert}^\bullet(\mathcal{D}_f)| + |\mathrm{Vert}^\circ(\mathcal{D}_f)| - \sum_{v \in \mathrm{Vert}^\bullet(\mathcal{D}_f)} \mathrm{val}(v) = |\mathrm{Vert}^\bullet(\mathcal{D}_f)| + |\mathrm{Vert}^\circ(\mathcal{D}_f)| - |\mathrm{Edge}(\mathcal{D}_f)| = 1$$

so we obtain

$$\sum_{\mathcal{F}} (2\deg(\mathcal{F}) - 1) + \sum_S (l(S) - 1) = 3d - 1.$$

Hence all the inequalities above are, in fact, equalities. □

The set $\mathcal{P} \cup \{\eta_1, \ldots, \eta_{3d-k-1}\}$ inherits a total order from the map $\pi_y$, and we relabel elements of this set by $\overline{q}_1 < \cdots < \overline{q}_{3d-1}$. We define the set $\mathcal{L}^{\mathrm{comb}} \subset \{1, \ldots, 3d-1\}$ by

$$i \in \mathcal{L}^{\mathrm{comb}} \Longleftrightarrow \overline{q}_i \in \mathcal{L}.$$

Given a morphism $f: C \to \mathbb{R}^2$ in $\mathcal{S}^{\mathbb{T}}(d, \mathcal{P}, \mathcal{L})$, we define a map $m_f : \{1, \ldots, 3d-1\} \to \mathcal{D}_f$ by

$$m_f(i) = v \Longleftrightarrow \text{the floor or shaft of } C \text{ corresponding to } v \text{ matches } \overline{q}_i.$$

Lemma 8.20. *Let $(\mathcal{P}, \mathcal{L})$ be a vertically stretched configuration of constraints. If $f$ is tangent to $\mathcal{L}$, then $(\mathcal{D}_f, m_f)$ is an $\mathcal{L}^{\mathrm{comb}}$-marked floor diagram of degree $d$.*

*Proof.* This is an immediate consequence of Corollary 8.19 and the genericity of the configuration $(\mathcal{P}, \mathcal{L})$. □

Hence we have constructed a map $\phi_{(\mathcal{P},\mathcal{L})}$ which maps an element of $f \in \mathcal{S}^{\mathbb{T}}(d,\mathcal{P},\mathcal{L})$ tangent to $\mathcal{L}$ to the $\mathcal{L}^{\text{comb}}$-marked floor diagram $(\mathcal{D}_f, m_f)$. Theorem 8.11 is a direct consequence of Theorem 3.8 and of the following proposition.

PROPOSITION 8.21. *Let $(\mathcal{P},\mathcal{L})$ be a vertically stretched configuration of constraints. For any $\mathcal{L}^{\text{comb}}$ marked floor diagram $(\mathcal{D}, m)$ of degree $d$, we have*

$$\sum_{f\in\phi^{-1}_{(\mathcal{P},\mathcal{L})}(\mathcal{D},m)} \mu_{(\mathcal{P},\mathcal{L})}(f) = \mu_{\mathcal{L}^{\text{comb}}}(\mathcal{D}, m).$$

*Proof.* Let $(\mathcal{D}, m)$ be a $\mathcal{L}^{\text{comb}}$-marked floor diagram. We start with two easy observations. Given a tropical morphism $f \in \phi^{-1}_{(\mathcal{P},\mathcal{L})}(\mathcal{D}, m)$ and $v \in \text{Vert}^{\circ}(\mathcal{D})$ corresponding to the floor $\mathcal{F}_v$ of $f$, then the horizontal constraint matched by $\mathcal{F}_v$ is $\overline{q}_{\min(m^{-1}(v))}$. If $v \in \text{Vert}^{\bullet}(\mathcal{D})$ corresponds to the shaft $S_v$ of $f$, then the vertical constraint matched by $S_v$ is:

(i) the point $\overline{q}_{i_v}$ in $\mathcal{P}$ where $i_v$ is the unique element of $m^{-1}(v) \cap \mathcal{P}^{\text{comb}}$ if this set is non-empty;

(ii) the line $\overline{q}_{i_v}$ in $\mathcal{L}$ where $i_v = \max(m^{-1}(v))$ if $i_v > \max_{v' \in \text{Vert}(v)} \min((m^{-1}(v')))$;

(iii) a line $\overline{q}_{i_v}$ where $i_v \in m^{-1}(v)$, otherwise.

Hence, we see that the horizontal constraint matched by $S_v$ is not determined by $\mathcal{D}$ only in case (iii) above. We denote by $\text{Vert}^{\bullet,3}(\mathcal{D})$ the subset of $\text{Vert}^{\bullet}(\mathcal{D})$ composed of vertices in this case.

Given $\varepsilon > 0$ and $p \in \mathcal{P}$ (respectively $L \in \mathcal{L}$ with vertex $\eta$), we denote by $I_{\varepsilon,p}$ (respectively $I_{\varepsilon,L}$) the interval of $\mathbb{R}$ centered in $\pi_y(p)$ (respectively $\pi_y(\eta)$) and of length $\varepsilon$. Let us denote by $\overline{q}'_1 < \cdots < \overline{q}'_{\text{val}(v)}$ the elements of $\mathcal{P} \cup \mathcal{L}$ which are horizontal constraints for a floor of $f$ adjacent to the shaft $S_v$, and by $x_i$ the center of the interval $I_{\varepsilon,\overline{q}'_i}$. We define the set $\mathcal{Q}^{\mathbb{T}}$ to be the set of centers of all intervals $I_{\varepsilon,L}$ with $L$ a non-vertical constraint of $S_v$ matched by $S_v$. Now we can consider the set $\mathcal{H}^{\mathbb{T}}(\delta, n_{i_v})$ corresponding to the points $x_i$ and the set $\mathcal{Q}^{\mathbb{T}}$ (see Appendix for the definition of $\mathcal{H}^{\mathbb{T}}$, and Definition 8.7 for the definition of $\delta$ and $n_{i_v}$). Since the points of $\mathcal{P} \cup \{\eta_1, \ldots, \eta_{3d-1-k}\}$ are contained in the strip $I \times \mathbb{R}$ and very far apart, Lemma 8.16 tells us that there exists $\varepsilon > 0$ depending only on $a$,$b$, and $d$ (in particular independent on $f$, $\mathcal{P}$, and $\mathcal{L}$), such that the following hold.

- If $v'$ is a vertex of $S_v$ such that $f(v') \in L \in \mathcal{L}$ with $L \neq \overline{q}_{i_v}$, then $\pi_y \circ f(v') \in I_{\varepsilon,L}$.
- If $v'$ is a vertex of $S_v$ which is also a vertex of a floor matching the horizontal constraint $q$, then $\pi_y \circ f(v') \in I_{\varepsilon,q}$.
- If $q \neq q'$, then $I_{\varepsilon,q} \cap I_{\varepsilon,q'} = \emptyset$.

Hence, there exists a unique element $g$ of $\mathcal{H}^{\mathbb{T}}(\delta, n_{i_v})$ which can be obtained from a deformation $(g_t : C'_t \to \mathbb{R}^2)_{t\in[0;1]}$ of the restriction $f_{|S_v}$ in its deformation space such that $g_t(\text{Vert}(C'_t)) \subset \bigcup_{q\in\mathcal{P}\cup\mathcal{L}} I_{\varepsilon,q}$ for all $t$. Moreover, when $f$ ranges over all elements of $\phi^{-1}_{(\mathcal{P},\mathcal{L})}(\mathcal{D}, m)$ with fixed $i_v$, there is a natural bijection between all possible restrictions $f_{|S_v}$ and the set $\mathcal{H}^{\mathbb{T}}(\delta, n_{i_v})$.

Once again, since the points of $\mathcal{P} \cup \{\eta_1, \ldots, \eta_{3d-1-k}\}$ are contained in the strip $I \times \mathbb{R}$ and very far apart, to construct a morphism $f$ in $\phi^{-1}_{(\mathcal{P},\mathcal{L})}(\mathcal{D}, m)$, it is enough to construct independently the restriction of $f$ on the floors and on the shafts of $C$, and to glue all these pieces together along elevators. It follows from Definition 8.7, Proposition 5.1, and Theorem A.5, that the contribution of a vertex $v \in \text{Vert}^{\bullet}(\mathcal{D})\backslash\text{Vert}^{\bullet,3}(\mathcal{D})$ is equal to $\mu_{\mathcal{L}^{\text{comb}}}(v)$. In the case of a vertex $v \in \text{Vert}^{\bullet,3}(\mathcal{D})$, an easy Euler characteristic computation and (2) for the weight associated to a tangency show that the contribution of $v$ with a fixed $i_v$ is, in the notation of Definition 8.7,

$$r_{i,v} H(\delta, n_{i_v})$$

where $r_{i_v} = (\delta(i_v) + i + |\{i \in m^{-1}(v) \ | \ i < i_v\}| + \operatorname{div}(v))$. Hence, according to Lemma 8.22, the sum of the $(\mathcal{P},\mathcal{L})$-multiplicity of all morphisms $f \in \phi^{-1}_{(\mathcal{P},\mathcal{L})}(\mathcal{D}, m)$ with fixed $i_v$ for all $v \in \operatorname{Vert}^{\bullet,3}(\mathcal{D})$ is exactly

$$\prod_{e\in\operatorname{Edge}(\mathcal{D})} w(e) \prod_{v\in\operatorname{Vert}(\mathcal{D})\setminus\operatorname{Vert}^{\bullet,3}(\mathcal{D})} \mu_{\mathcal{L}^{\mathrm{comb}}}(v) \prod_{v\in\operatorname{Vert}^{\bullet,3}(\mathcal{D})} (r_{i_v} H(\delta, n_{i_v})).$$

Given $v \in \operatorname{Vert}^{\bullet,3}(\mathcal{D})$, we have

$$\begin{aligned}
&\sum_{i_v} (r_{i_v} H(\delta, n_{i_v}))\\
&\quad= \sum_{i=0}^{s} \left(\left(\tilde{n}(i)(\delta(i) + i + \operatorname{div}(v)) + \sum_{j=\tilde{N}(i-1)}^{\tilde{N}(i)-1} j\right) H(\delta, n_i)\right)\\
&\quad= \sum_{i=0}^{s} \left(\left(\tilde{n}(i)(\delta(i) + i + \operatorname{div}(v)) + \frac{\tilde{N}(i)^2 - \tilde{N}(i-1)^2 - (\tilde{N}(i) - \tilde{N}(i-1))}{2}\right) H(\delta, n_i)\right)\\
&\quad= \mu_{\mathcal{L}^{\mathrm{comb}}}(v).
\end{aligned}$$

Hence we have

$$\sum_{f\in\phi^{-1}_{(\mathcal{P},\mathcal{L})}(\mathcal{D},m)} \mu_{(\mathcal{P},\mathcal{L})}(f) = \prod_{e\in\operatorname{Edge}(\mathcal{D})} w(e) \prod_{v\in\operatorname{Vert}(\mathcal{D})} \mu_{\mathcal{L}^{\mathrm{comb}}}(v) = \mu_{\mathcal{L}^{\mathrm{comb}}}(\mathcal{D}, m),$$

as desired. □

Lemma 8.22. *Let $d_v \geqslant 1$ and $w_1, \ldots, w_l > 0$ be integer numbers. Choose a generic configuration $(\mathcal{P},\mathcal{L})$ of constraints such that $\mathcal{P} = \{p_1, \ldots, p_l\} \subset I \times \mathbb{R}$, and $\mathcal{L} = \{L_1, \ldots, L_{2d_v-2}\} \subset I \times \mathbb{R}$ is a set of vertical lines. Choose also a point $p_0 \in I \times \mathbb{R}$ and a tropical line $L_0$ whose vertex $\eta_0$ is in $I \times \mathbb{R}$ such that the configuration $(\mathcal{P} \cup \{p_0\}, \mathcal{L} \cup \{L_0\})$ is generic. Suppose that the points $p_i$ and $\eta_0$ are very far apart, and denote by $\mathcal{C}_{p_0}(2d_v + l, \mathcal{P}, \mathcal{L})$ (respectively $\mathcal{C}_{L_0}(2d_v + l, \mathcal{P}, \mathcal{L})$) the set of all minimal tropical morphisms $f : C \to \mathbb{R}^2$ such that the following hold:*

(i) *$C$ has $l$ ends of direction $(0, \pm 1)$; given such an end $e$ of $C$, $f(e)$ passes through one point $p_i$ of $\mathcal{P}$, and $w_{f,e} = w_i$;*
(ii) *$C$ has $d_v$ ends of direction $(-1, 0)$ and weight 1, and $d_v$ ends of direction $(1, 1)$ and weight 1;*
(iii) *$f$ is pretangent to all lines in $\mathcal{L}$;*
(iv) *$f$ passes through $p_0$ (respectively is pretangent to $L_0$).*

*Then*

$$\sum_{f\in\mathcal{C}_{p_0}(2d_v+l,\mathcal{P},\mathcal{L})} \mu_{(\mathcal{P}\cup p_0,\mathcal{L})}(f) = d_v^{l+1} H(d_v) \prod_{e\in\operatorname{Edge}^\infty(C), u_{f,e}=(0,\pm1)} w(e)$$

*and*

$$\sum_{f\in\mathcal{C}_{L_0}(2d_v+l,\mathcal{P},\mathcal{L})} \mu_{(\mathcal{P},\mathcal{L}\cup L_0)}(f) = (l - 2 + d_v) d_v^{l} H(d_v) \prod_{e\in\operatorname{Edge}^\infty(C), u_{f,e}=(0,\pm1)} w(e).$$

*Moreover, any tropical morphism $f \in \mathcal{C}_{p_0}(2d_v + l, \mathcal{P}, \mathcal{L})$ (respectively $f \in \mathcal{C}_{L_0}(2d_v + l, \mathcal{P}, \mathcal{L})$) has exactly one floor, and this floor matches $p_0$ (respectively $L_0$).*

*Proof.* The fact that $f$ has only one floor is straightforward: $C$ has $2d_v + l - 2$ vertices, exactly $l$ of which are adjacent to a vertical end of $C$, so $C$ has exactly one vertex mapped to each line $L_i$, $i>0$; in particular, it has no other vertical edge than its vertical ends. Since the floor of $f$ has to match a horizontal constraint, it matches necessarily $p_0$ or $L_0$.

In the following, we use the notations of Appendix. Let us denote by $\mathcal{Q}^{\mathbb{T}}$ the set of intersection points of all tropical lines $L_i$ with $\{y=0\}$ when $i>0$, and let us consider the set $\mathcal{H}^{\mathbb{T}}(\delta, n)$ with $s=0$, $\delta(0)=d_v$, and $n(0)=2d_v-2$. Let us fix an element $f_0: C_0 \to \{y=0\}=\mathbb{R}$ in $\mathcal{H}^{\mathbb{T}}(\delta, n)$. Consider a sequence of $l$ open tropical modifications $\pi: C_1 \to C_0$, and a minimal tropical morphism $f_1: C_1 \to \mathbb{R}^2$ satisfying conditions (i)–(iii) such that $\pi_x \circ f_1 = f_0 \circ \pi$. Composing $f_1$ with a translation in the $(0,1)$ direction, we construct a finite number of elements of $\mathcal{C}_{p_0}(2d_v+l, \mathcal{P}, \mathcal{L})$ and $\mathcal{C}_{L_0}(2d_v+l, \mathcal{P}, \mathcal{L})$.

- To construct an element of $\mathcal{C}_{p_0}(2d_v+l, \mathcal{P}, \mathcal{L})$, we have to make one of the edges of $C_1$ pass through $p_0$. Once this is done, the orientation of the curve $C_1$ defined in § 5 is as follows: the rays emanating from $p_0$ are oriented away from $p_0$, and exactly one edge $e$ with $u_{f_1,e} \neq (0,\pm 1)$ is oriented toward $v_1$ at a vertex $v_1$ of $C_1$. Hence the multiplicity of such a vertex $v_1$ is $\mu(v_1)=|\alpha|$ if $u_{f_1,e}=(\alpha,\beta)$. The multiplicity can then be computed via Proposition 5.1. Notice that for a given choice of morphism $f_0$, and every thing being fixed but for one vertical edge (respectively the edge containing the marked point), summing the corresponding $\mu(v_1)$ (respectively weights of the supporting edges) over all the possible choices, one gets a factor $d_v$. Thus adding the $(\mathcal{P}\cup p_0, \mathcal{L})$-multiplicity of all possible morphisms $f_1$ constructed in this way starting with a fixed $f_0$, we obtain $d_v^{l+1}\mu_H(f_0)$. Considering all possible tropical morphisms $f_0 \in \mathcal{H}^{\mathbb{T}}(\delta, n)$, Theorem A.5 tells us that we obtain

$$\sum_{f\in\mathcal{C}_{p_0}(2d_v+l,\mathcal{P},\mathcal{L})} \mu_{(\mathcal{P}\cup p_0,\mathcal{L})}(f) = d_v^{l+1} H(d_v) \prod_{e\in \mathrm{Edge}^\infty(C), u_{f,e}=(0,\pm 1)} w(e).$$

- To construct an element of $\mathcal{C}_{L_0}(2d_v+l, \mathcal{P}, \mathcal{L})$, we have two possibilities: either $f_1(C_1)$ passes through the vertex $\eta_0$ of $L_0$, or a vertex of $C_1$ is mapped to $L_0$. Once this is done, the orientation of the curve $C_1$ defined in § 5 is as follows ($E_0$ denotes the union of all pretangency components of $f_1$ with $L_0$): an edge $e$ with $u_{f_1,e} \neq (0,\pm 1)$ intersecting $E_0$ but not included in $E_0$ is oriented away from $E_0$, and exactly one edge $e$ with $u_{f_1,e} \neq (0,\pm 1)$ is oriented toward $v_1$ at a vertex $v_1$ of $C_1 \backslash E_0$. Note that if there exists an end of $C_1$ with direction $(-1,0)$ or $(1,1)$ and intersecting $E_0$ in infinitely many points, then $\mu_{(\mathcal{P},\mathcal{L}\cup L_0)}(f_1)=0$. Hence, adding the $(\mathcal{P}, \mathcal{L}\cup L_0)$-multiplicity of all possible morphisms $f_1$ constructed in this way starting with a fixed $f_0$, we obtain $d_v^l K \mu_H(f_0)$, where $K = d_v + |\mathrm{Vert}^0(C_1)| - 2d_v = l-2+d_v$. Considering all possible tropical morphisms $f_0 \in \mathcal{H}^{\mathbb{T}}(\delta, n)$, Theorem A.5 tells us that we obtain

$$\sum_{f\in\mathcal{C}_{L_0}(2d_v+l,\mathcal{P},\mathcal{L})} \mu_{(\mathcal{P},\mathcal{L}\cup L_0)}(f) = (l-2+d_v) d_v^{l} H(d_v) \prod_{e\in \mathrm{Edge}^\infty(C), u_{f,e}=(0,\pm 1)} w(e).$$

Hence the lemma is proved. □

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## Appendix. Open Hurwitz numbers

We recall here the definition of open Hurwitz numbers. These numbers were introduced in [BBM11] and are a slight generalization of well-known Hurwitz numbers. For simplicity, we restrict ourselves to the special cases we need in this paper. We refer to [BBM11] for more details and examples about open Hurwitz numbers and their tropical counterpart (see also [CJM10]).

Let $s \geqslant 0$ be an integer number, and $\delta,\ n : \{0, \dots, s\} \to \mathbb{Z}_{\geqslant 0}$ be two functions. Choose a collection of $s$ embedded circles $c_1, \dots, c_s$ in the sphere $S^2$ such that $c_1$ (respectively $c_s$) bounds a disk $D_0$ (respectively $D_s$), and $c_i$ and $c_{i+1}$ bound an annulus $D_i$ for $1 \leqslant i \leqslant s-1$. Choose also a collection $\mathcal{Q}$ of points in $S^2 \setminus \bigcup_{i=1}^{s} c_i$, such that each $D_i$ contains exactly $n(i)$ points of $\mathcal{Q}$. Let us consider the set $\mathcal{H}(\delta, n)$ of all equivalence class of ramified coverings $f : \Sigma \to S^2$ where the following hold:

- $\Sigma$ is a connected compact oriented surface of genus 0 with $s$ boundary components;
- $f(\partial\Sigma) \subset \bigcup_{i=1}^{s} c_i$;
- $f$ is unramified over $S^2 \setminus \mathcal{Q}$;
- $f_{|f^{-1}(D_i)}$ has degree $\delta(i)$ for each $i$;
- each point in $\mathcal{Q}$ is a simple critical value of $f$;
- for each circle $c_i$, the set $f^{-1}(c_i)$ contains exactly one connected component $c$ of $\partial\Sigma$, and $f_{|c} : c \to c_i$ is an unramified covering of degree $|\delta(i) - \delta(i-1)|$.

Two continuous maps $f : \Sigma \to S^2$ and $f' : \Sigma' \to S^2$ are considered equivalent if there exists a homeomorphism $\Phi : \Sigma \to \Sigma'$ such that $f' \circ \Phi = f$.

Note that the cardinal of the set $\mathcal{Q}$ is prescribed by the Riemann–Hurwitz formula

$$|\mathcal{Q}| = \delta(0) + \delta(s) + s - 2.$$

Definition A.1. The *open Hurwitz number* $H(\delta, n)$ is defined as:

$$H(\delta, n) = \sum_{f \in \mathcal{H}(\delta, n)} \frac{1}{|\mathrm{Aut}(f)|}.$$

Note that we can naturally extend the definition of the numbers $H(\delta, n)$ to the case where $\delta : \{0, \dots, s\} \to \mathbb{Z}$ is any function by setting

$$H(\delta, n) = 0 \quad \text{if } \mathrm{Im}\ \delta \not\subseteq \mathbb{Z}_{\geqslant 0}.$$

*Example* A.2. If $s = 2$, $\delta(0) = \delta(2) = n(0) = n(1) = n(2) = 0$, and $\delta(1) = d$, one computes easily

$$H(\delta, n) = \frac{1}{d}.$$

In the special case where $s = 0$, we recover the usual Hurwitz numbers. In particular $\delta$ is just a positive integer number, the degree of the maps we are counting and that we just denote by $d$. We simply denote this Hurwitz number by $H(d)$.

Proposition A.3 (Hurwitz). *For any* $d \geqslant 1$, *then*

$$H(d) = \frac{d^{d-3}(2d-2)!}{d!}.$$

Let us now define tropical open Hurwitz numbers, and let us relate them to the open Hurwitz numbers we have just defined. Recall that we have chosen $s \geqslant 0$ an integer number, and $\delta,\ n : \{0, \dots, s\} \to \mathbb{Z}_{\geqslant 0}$ two functions.

Choose a collection of $s$ points $x_1 < \cdots < x_s$ in $\mathbb{R}$, and define $D_0^{\mathbb{T}} = (-\infty; x_1)$, $D_i^{\mathbb{T}} = (x_i, x_{i+1})$, and $D_s^{\mathbb{T}} = (x_s; -\infty)$. Choose another collection of points $\mathcal{Q}^{\mathbb{T}}$ in $\mathbb{R} \setminus \{x_1, \dots, x_s\}$, such that each $D_i^{\mathbb{T}}$ contains exactly $n(i)$ points of $\mathcal{Q}^{\mathbb{T}}$. Let us consider the set $\mathcal{H}^{\mathbb{T}}(\delta, n)$ of all equivalence class of minimal tropical morphisms $f : C \to \mathbb{R}$ where the following hold:

- $C$ is a rational tropical curve with $s$ boundary components;
- $f(\partial C) \subset \{x_1, \ldots, x_s\}$;
- $f(\mathrm{Vert}^0(C)) \subset \mathcal{Q}^{\mathbb{T}}$;
- given $0 \leqslant i \leqslant s$ and $p \in \mathcal{D}_i^{\mathbb{T}}$, if we denote by $e_1, \ldots, e_r$ the edges of $C$ which contain a point of $f^{-1}(p)$, then we have $\sum_{j=1}^{r} w_{f,e_j} = \delta(i)$;
- for each $p \in \mathcal{Q}^{\mathbb{T}}$, $f^{-1}(p)$ contains exactly one element of $\mathrm{Vert}^0(C)$, which is a 3-valent vertex of $C$;
- for each $1 \leqslant i \leqslant s$, the set $f^{-1}(x_i)$ contains exactly one boundary component of $C$, adjacent to an edge of weight $|\delta(i) - \delta(i+1)|$;
- each end of $C$ is of weight 1.

As previously, two tropical morphisms $f : C \to \mathbb{R}$ and $f' : C' :\to \mathbb{R}$ are considered equivalent if there exists a tropical isomorphism $\Phi : C \to C'$ such that $f' \circ \Phi = \phi \circ f$.

Once again, we have

$$|\mathcal{Q}^{\mathbb{T}}| = \delta(0) + \delta(s) + s - 2.$$

Given $v$ a puncture of $C$ adjacent to the end $e$, we set $w_{f,v} = w_{f,e}$, and we define $w_{f,\infty}$ as the product of the weights $w_{f,v}$ when $v$ ranges over all punctures of $C$. The multiplicity of an element of $\mathcal{H}^{\mathbb{T}}(\delta, n)$ is then defined as

$$\mu_H(f) = \frac{\prod_{e \in \mathrm{Edge}(C)} w_{f,e}}{w_{f,\infty}}.$$

DEFINITION A.4. The *tropical open Hurwitz number* $H^{\mathbb{T}}(\delta, n)$ is defined as

$$H^{\mathbb{T}}(\delta, n) = \sum_{f \in \mathcal{H}^{\mathbb{T}}(\delta,n)} \frac{1}{|\mathrm{Aut}(f)|} \mu_H(f).$$

As previously we extend the definition of the numbers $H^{\mathbb{T}}(\delta, n)$ by setting

$$H^{\mathbb{T}}(\delta, n) = 0 \quad \text{if } \mathrm{Im}\, \delta \not\subseteq \mathbb{Z}_{\geqslant 0}.$$

THEOREM A.5 [BBM11, Theorem 2.11]. *For any two functions* $\delta : \{0, \ldots, s\} \to \mathbb{Z}$ *and* $n : \{0, \ldots, s\} \to \mathbb{Z}_{\geqslant 0}$, *we have*

$$H(\delta, n) = H^{\mathbb{T}}(\delta, n).$$

*Example* A.6. Using Theorem A.5 we compute easily the following open Hurwitz numbers:
if $s = 1$, $\delta(0) = 2$, $\delta(1) = 0$, $n(0) = 1$, $n(1) = 0$, then $H(\delta, n) = \frac{1}{2}$;
if $s = 2$, $\delta(0) = 1$, $\delta(1) = 2$, $\delta(2) = 0$, $n(0) = n(2) = 0$, $n(1) = 1$, then $H(\delta, n) = 1$;
if $s = 1$, $\delta(0) = 3$, $\delta(1) = 0$, $n(0) = 2$, $n(1) = 0$, then $H(\delta, n) = 1$.

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