Ergod. Th. & Dynam. Sys. (1983), **3**, 13–46
Printed in Great Britain

Positive Liapunov exponents and absolute continuity for maps of the interval

P. COLLET AND J.-P. ECKMANN

Centre de Physique Théorique, Ecole Polytechnique, 91128 *Palaiseau, France*;
Département de Physique Théorique, Universite de Genève, 1211 *Genève* 4,
Switzerland

(*Received* 16 *September* 1981 *and revised* 22 *September* 1982)

Abstract. We give a sufficient condition for a unimodal map of the interval to have an invariant measure absolutely continuous with respect to the Lebesgue measure. Apart from some weak regularity assumptions, the condition requires positivity of the forward and backward Liapunov exponent of the critical point.

1. *Introduction and statement of results*

Continuous maps of an interval to itself can be viewed as dynamical systems, whose time evolution is given by iterating a given map. Despite their innocent looking simplicity, iterated maps can serve as important models for testing general ideas about dynamical systems.

One such circle of ideas concerns the existence of invariant measures which are absolutely continuous (with respect to Lebesgue measure). If, in addition, the measure is ergodic, then erratic behaviour can be expected for many orbits. One would like to argue that if a system has positive characteristic (Liapunov) exponents, then it behaves erratically. One is still far from a complete understanding of these matters, because of the presence of stable directions, see e.g. [**9**]. The analogue of this question for maps of the interval is easier to handle because, when there is an unstable direction, then there is no space for a stable direction. The Liapunov exponent is clearly positive if the map is everywhere expanding, and this is the easiest case in which existence of an absolutely continuous invariant measure can be shown [**6**], [**12**]. If the map of the interval has a critical point, it is of course not uniformly expanding. It may nevertheless possess an absolutely continuous invariant measure [**11**]. It was then discovered that this result can be generalized to maps with the property that the critical points have orbits which eventually land on unstable fixed points [**8**], [**1**]. One can further generalize this to maps whose critical points have orbits staying away from the critical points [**7**], [**10**]. In the present paper, we give different conditions for the existence of an absolutely continuous invariant measure.

For simplicity, we shall consider maps with a single critical point, x_0. Apart from technical conditions to be given below, we require that the map f satisfies

$$\text{(C1)} \qquad \liminf_{n\to\infty} \frac{1}{n} \log \left| \frac{d}{dx} (f^n)(f(x_0)) \right| > 0,$$

$$\text{(C2)} \qquad \liminf_{n\to\infty} \frac{1}{n} \inf_{y\in f^{-n}(x_0)} \log \left| \frac{d}{dx} f^n(y) \right| > 0,$$

where $f^n = f\circ\cdots\circ f$ (n times) and $f^{-n}(x_0)$ denotes the set of n'th pre-images of x_0.

These conditions can be fulfilled even if the orbit of x_0 does not stay away from x_0, and they are thus weaker than those mentioned above. One can show that they are met for a large set of maps among the one-parameter family $\delta \to f_\delta$ given by

$$f_\delta(x) = \begin{cases} 1-2|x| & \text{if } |x| \geq \delta, \\ 1-\delta-(x^2/\delta) & \text{if } |x| \leq \delta, \end{cases}$$

for $0<\delta<\frac{1}{2}$. We have studied these maps in [**2**], and a slight extension of that work shows that there is a set of positive Lebesgue measure in δ such that f_δ satisfies all conditions to be enumerated below, and hence f_δ has an absolutely continuous invariant measure. In addition, this set of δ has a Lebesgue point (i.e. full relative measure) at $\delta = 0$. Similar results for one-parameter families of maps have been obtained earlier in [**5**].

If we consider our conditions in the general framework of dynamical systems then (C1) corresponds to requiring that the Liapunov exponent is positive, while condition (C2) says that the *inverse* of f (which only exists as a set function in our case) is contracting. It is tempting to conjecture that (C1) might imply (C2), maybe with some additional convexity condition, but our insight into this question is incomplete. Note also that, if the orbit of the critical point stays at a distance from the critical point, then Misiurewicz' conditions are stronger than (C1) and (C2), see the Appendix.

We now state our hypotheses in detail, followed by the statement of the theorem and some remarks.

Hypotheses

(H1) f is defined on $\Omega = [f(1), 1]$ and takes values in Ω. It is strictly increasing on $[f(1), 0]$ and strictly decreasing on $[0, 1]$. The function f is of class $\mathscr{C}^1$. In addition, $f(0)=1$.

(H2) The function f' is Lipschitz continuous, and $|f'|^{-\frac{1}{2}}$ is convex on $[f(1), 0]$ and on $[0, 1]$.

(H3)

$$\limsup_{x\to 0} |f'(x)/x| < \infty, \qquad \inf_{x\in[f(1),\,1]} |f'(x)/x| > 0.$$

(H4) There is a $C_1 > 0$ and a $\theta > 0$ such that

(1) $$\left|\left(\frac{d}{dx} f^n\right)(1)\right| \geq C_1 \exp(n\theta) \quad \text{for all } n \geq 0.$$

(2) If $f^m(z) = 0$ for some $m > 0$, then

$$\left|\left(\frac{d}{dx} f^m\right)(z)\right| \geq C_1 \exp(m\theta).$$

Our main result is the following:

THEOREM. *If f satisfies* (H1)–(H4) *then f has an invariant measure which is absolutely continuous with respect to Lebesgue measure.*

The conditions (H1)–(H4) can be formulated more concisely under the assumption that f is of class $\mathscr{C}^3$. Namely, we can replace (H2), (H3) by the more readable
(H2′) $Sf(x) \leq 0$ for $x \in \Omega$, where

$$Sf = f'''/f' - \tfrac{3}{2}(f''/f')^2$$

is the Schwarzian derivative.
(H3′) $f''(0) < 0$ and $f'(f(1)) \neq 0$ and $f'(1) \neq 0$.
Although the Schwarzian derivative is not defined in the general case, the results of the corresponding theory do apply, see e.g. [**3**]. In particular, (H1)–(H3) and (H4.1) imply that the map f has no stable periodic orbit.

We next outline the main steps of the proof of the main theorem. Define the operator $\mathscr{L}$ by

$$(\mathscr{L}g)(y) = \sum_{x \in f^{-1}(y)} \frac{g(x)}{|f'(x)|},$$

where $f^{-1}(E) = \{x \in \Omega, f(x) \in E\}$.

The density h of the invariant measure, if it exists, satisfies the equation

$$\mathscr{L}h = h.$$

We shall consider the sequence of functions

$$h_n = \mathscr{L}^n 1, \qquad n = 0, 1, 2, \ldots.$$

It is easy to see that

$$\int h_n(y)\, dy = 1 - f(1). \tag{1}$$

We shall show:

THEOREM 1.1. *Define $h_n(y) = (\mathscr{L}^n 1)(y)$ if $y \in \Omega$, $h_n(y) = 0$ if $y \notin \Omega$. Then, for all n, one has*

$$\int_{\mathbb{R}} |h_n(y) - h_n(y + \varepsilon)|\, dy \leq \exp(-|\log \varepsilon|^{\frac{1}{5}}), \tag{2}$$

for all $\varepsilon > 0$, ε sufficiently small.

The conclusions of theorem 1.1 together with (1) are the main input to Kolmogorov's compactness criterion (see e.g. [**13**]). It then follows from [**4**] (Mazur's theorem) that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} h_j = h$$

exists, is in L_1, is not zero and satisfies $\mathscr{L}h = h$. Thus the main theorem follows from theorem 1.1.

To discuss our method of proof, we define

$$Df^n = \frac{d}{dx}(f^n).$$

The difficulty in analysing $h_n(y) - h_n(y+\varepsilon)$ comes from those regions in which $1/|Df^n|$ varies quickly. These regions are located near the pre-images of 0, where $1/|Df^n|$ is infinite. In other words, since

$$h_n(y) = (\mathscr{L}^n 1)(y) = \sum_{x\in f^{-n}(y)} \frac{1}{|Df^n(x)|},$$

h_n varies rapidly around the points of the forward orbit of the critical point, i.e. of zero. This orbit is allowed to come close to the critical point, (it may be dense), and we have to subdivide carefully the space into pieces where $h_n(y) - h_n(y+\varepsilon)$ is regular, and their complement where $\int h_n$ will be small. To be more precise, fix $\varepsilon > 0$ sufficiently small and define the following ε dependent quantities.

Definitions. The following symbols have fixed meaning throughout the paper:

$$L_1 = \sup_{t\in[f(1),\,1]} |f'(t)|.$$

L_2 will be a constant which is fixed in (18) on p. 25. The constant τ is defined by

$$\tau = \frac{\theta}{4(\log L_1 - \theta)}.$$

We shall assume for simplicity that $\tau < \frac{1}{2}$. (This can always be achieved by making θ smaller.) Our main definitions are now as follows:

$$n'_\varepsilon = \frac{(1 - 3\tau/2)|\log \varepsilon| + L_2}{\log L_1 - \theta}; \tag{3}$$

$$n_\varepsilon = 1000|\log \varepsilon|/\theta;$$

$$\sigma_{m,\varepsilon} = \begin{cases} 0 & \text{if } m \le n'_\varepsilon, \\ \varepsilon^{1+\tau} & \text{if } n'_\varepsilon < m \le n_\varepsilon, \\ \exp(-m\theta/20) & \text{if } n_\varepsilon < m. \end{cases} \tag{4}$$

Sometimes we shall use conditions of the form $l = 1, 2, \ldots, n_\varepsilon$. Then we tacitly assume n_ε is rounded to that integer which allows for a larger set of l; similarly for n'_ε.

We next define *m, ε-regular points*. Let J be a maximal connected component of $f^{-m}([-\varepsilon, \varepsilon])$. The set J is called *m, ε-regular* if

(1) $f(1) \notin J$, $1 \notin J$, $0 \notin f^k(J)$, $\quad k = 0, 1, \ldots, m-1$,

(these conditions imply that $f^m|J$ is strictly monotone);

(2) let $\{z\} = f^{-m}(\{0\}) \cap J$. Then, if $m \geq n'_\varepsilon$,

$$|f^j(z)| \geq \sigma_{m-j,\varepsilon}$$

for $j = 0, 1, \ldots, m - n'_\varepsilon$.

Every point in a m, ε-regular set is called an m, ε-regular point. We define

$$h^{\text{reg}}_{m,\varepsilon}(y) = \sum_{\substack{x \in f^{-m}(y) \\ x \text{ is } m,\varepsilon\text{-regular}}} \frac{1}{|Df^m(x)|}. \tag{5}$$

Our first main estimate is:

THEOREM 1.2. *For all sufficiently small $\rho > 0$, all small $\varepsilon > 0$, and all $n \geq 0$,*

$$\int_{|y| \leq \rho} h^{\text{reg}}_{n,\varepsilon}(y)\, dy \leq 4(\rho^{\tau/4} + \rho/\alpha_0), \tag{6}$$

for some universal constant $\alpha_0 > 0$.

We shall then show that there are so many m, ε-regular points that h_n can be bounded in terms of $h^{\text{reg}}_{n,\varepsilon}$. This will lead to:

THEOREM 1.3. *For all sufficiently small $\varepsilon > 0$,*

$$\int_{|y| < \varepsilon} h_n(y)\, dy \leq \varepsilon^{\tau/40}. \tag{7}$$

This shows that h_n is relatively well-behaved near $y = 0$. To analyse the global situation, we need other subsets of Ω.

We define new cut-off functions which are similar to the $\sigma_{m,\varepsilon}$. Namely, let

$$\rho_{m,\varepsilon} = \begin{cases} 1 & \text{if } m \leq 0, \\ \exp(-|\log \varepsilon|^{\frac{1}{3}}) & \text{if } 0 < m \leq n''_\varepsilon, \\ \exp(-m\theta/20) & \text{if } n''_\varepsilon < m, \end{cases}$$

where $n''_\varepsilon = 7|\log \varepsilon|^{\frac{1}{2}}$. Note that ρ is decreasing in m and increasing as $\varepsilon \downarrow 0$.

Definition. We define, for given $\varepsilon > 0$,

$$E^\varepsilon_{m,\gamma} = \{t \in \Omega \mid |t| \leq \gamma \rho^{\frac{1}{2}}_{m,\varepsilon}\}; \tag{8}$$

$$F^\varepsilon_{m,\gamma} = \{t \in \Omega \mid |1 - t| \leq \gamma \rho_{m,\varepsilon}\}; \tag{9}$$

$$G^\varepsilon_{m,\gamma} = \{t \in \Omega \mid |f(1) - t| \leq \gamma \rho_{m,\varepsilon}\}. \tag{10}$$

These are small sets around the critical point of f and near the endpoints of $\Omega = [f(1), 1]$. We define χ_E as the characteristic function of the set E, and set

$$\Omega^R_{n,\varepsilon}(x) = \prod_{p=0}^{n} (1 - \chi_{E^\varepsilon_{n-p,1} \cup F^\varepsilon_{n-p,1} \cup G^\varepsilon_{n-p,1}})(f^p(x)), \tag{11}$$

and, for some ε-independent constant $\gamma_0 > 0$,

$$\Omega^T_{n,\varepsilon}(x) = 1 - \left\{ \prod_{p=0}^{n} (1 - \chi_{G^\varepsilon_{n-p,\gamma_0+1}})(f^p(x)) \right\} \cdot (1 - \chi_{E^\varepsilon_{0,\gamma_0} \cup F^\varepsilon_{0,\gamma_0}})(f^n(x)) \cdot (1 - \chi_{E^\varepsilon_{1,\gamma_0}})(f^{n-1}(x)). \tag{12}$$

Then we have three parts of h_n:

$$h^R_{n,\varepsilon}(y) = \sum_{x\in f^{-n}(y)} \frac{\Omega^R_{n,\varepsilon}(x)}{|Df^n(x)|}, \tag{13}$$

$$h^S_{n,\varepsilon}(y) = h_n(y) - h^R_{n,\varepsilon}(y), \tag{14}$$

$$h^T_{n,\varepsilon}(y) = \sum_{x\in f^{-n}(y)} \frac{\Omega^T_{n,\varepsilon}(x)}{|Df^n(x)|}. \tag{15}$$

The relations between these quantities are given by the following:

THEOREM 1.4. *For sufficiently small $\varepsilon > 0$ and all $n \geq 0$ one has*

$$|h^R_{n,\varepsilon}(y) - h^R_{n,\varepsilon}(y+\varepsilon)| \leq \mathcal{O}(\varepsilon^{\frac{1}{10}})(h^R_{n,\varepsilon}(y) + h^R_{n,\varepsilon}(y+\varepsilon)) + h^T_{n,\varepsilon}(y) + h^T_{n,\varepsilon}(y+\varepsilon), \tag{16}$$

and

$$\int_{\mathbb{R}} h^S_{n,\varepsilon}(y)\,dy \leq \int_{\mathbb{R}} h^T_{n,\varepsilon}(y)\,dy \leq \exp(-|\log \varepsilon|^{\frac{1}{4}}). \tag{17}$$

This theorem is based on a careful control of the sets $E^\varepsilon_{m,\gamma_0}$, $F^\varepsilon_{m,\gamma_0}$, $G^\varepsilon_{m,\gamma_0}$, occurring in the definition of $\Omega^R_{n,\varepsilon}$ and $\Omega^T_{n,\varepsilon}$, and on theorem 1.3.

It is now straightforward to see that (1), (14), (16), (17), and the fact that

$$0 \leq h^R_{n,\varepsilon}(y) \leq h_n(y),$$

imply theorem 1.1. Hence the main theorem will follow if we prove theorems 1.2, 1.3, 1.4. This will be done in the subsequent sections.

2. *Notation and terminology, preliminary estimates*

We adopt all definitions of the preceding section. To make the statements of the theorems, propositions and lemmas more readable, we use the following terminology:

'If $\varepsilon < \varepsilon_0 \ldots$' is the short version of 'There is an $\varepsilon_0 > 0$ such that if $\varepsilon \in [0, \varepsilon_0) \ldots$', and similarly for other letters of the alphabets.

'ε_0 small' stands for 'choosing $\varepsilon_0 > 0$ sufficiently small'.

$C_1, C_2, \ldots, L_1, L_2, \ldots, L$, and U denote finite positive constants whose value and meaning do not change throughout the paper. The corresponding clauses should typically be 'There is a finite positive constant C_1 such that'.

$K_1, K_2, \ldots,$ denote finite positive constants whose value and meaning is only unchanged in each single proof. It changes from one proof to the next.

Equations are numbered (1), (2), ... in each proof.

$[x; y]$ denotes the closed interval whose endpoints are x and y, where the order of x and y is arbitrary, i.e. $[x; y] = [y; x]$. We extend this notation by analogy to open and half-open intervals.

All maps under consideration are assumed to satisfy (H1) to (H4).

A maximal interval of monotonicity of f^n is called a *homterval*. The endpoints of homtervals are either critical points of f^n, or 1 or $f(1)$. (This definition does not agree with [7] who requires monotonicity for all n.)

The letter z always denotes a pre-image of the critical point 0.

LEMMA 2.1. *The function $|Df^n|^{-\frac{1}{2}}$ is convex on every homterval of f^n, for $n = 1, 2, \ldots$.*

Proof. If f is $\mathscr{C}^3$ then the concavity of $|f'|$ (i.e. (H2)) implies $f'''f' \le 0$. This implies that the Schwarzian derivative of f, defined as

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2,$$

satisfies $Sf \le 0$. It then follows that $S(f^n) \le 0$ for all $n \ge 1$ (cf. e.g. § 4 in [**3**]). From the remarks on page 104 in [**3**] it is easy to see that the same line of reasoning applies if $S'f \le 0$ is replaced by the weaker convexity $|f'|^{-\frac{1}{2}}$ if $f \in \mathscr{C}^1$. □

LEMMA 2.2. *There are constants $L > 0$ and $U < \infty$ such that*

$$Lt^2 < |f(t) - f(0)| < Ut^2,$$

and

$$2L|t| < |f'(t)| < 2U|t|.$$

Proof. This follows at once from (H3). □

LEMMA 2.3. *If $xy > 0$ and $|x - y| < \frac{1}{2}\max(|x|, |y|)$ then*

$$\left|1 - \frac{f'(x)}{f'(y)}\right| \le \frac{L_3|x - y|}{\max(|x|, |y|)}.$$

This is the typical kind of basic estimate needed in proofs of existence of absolutely continuous invariant measures.

Proof. Recall that 0 is the critical point of f. For definiteness, we only consider the case $0 < x, y$. If $x < y$ we have, since f' is Lipschitz,

$$\frac{|f'(x) - f'(y)|}{|f'(y)|} \le K_1 \frac{|x - y|}{|f'(y)|} \le K_2 \frac{|x - y|}{|y|}.$$

Now

$$\frac{|f'(x) - f'(y)|}{|f'(x)|} \le K_2 \frac{|x - y|}{|x|}.$$

But since $|y - x| < \frac{1}{2}|y|$, we find

$$\frac{|x - y|}{|x|} \le \frac{|x - y|}{\frac{1}{2}|y|},$$

and hence

$$\frac{|f'(x) - f'(y)|}{|f'(x)|} \le 2K_2 \frac{|x - y|}{|y|}.$$ □

COROLLARY 2.4. *Assume f^n is monotone on (x, y), and*

$$\frac{|f^j(x) - f^j(y)|}{|f^j(x)|} < \delta_j \qquad \text{for } j = 0, \ldots, n-1.$$

If $\Delta = \sum_{j=0}^{n-1} \delta_j < L_5$ for some universal constant L_5, then

$$\exp(-L_4\Delta) \le \frac{Df^n(x)}{Df^n(y)} \le \exp(L_4\Delta).$$

Proof. By the chain rule of differentiation

$$Df^n(x) = \prod_{j=0}^{n-1} f'(f^j(x))$$
$$= Df^n(y) \cdot \prod_{j=0}^{n-1} \left\{1 - \left[1 - \frac{f'(f^j(x))}{f'(f^j(y))}\right]\right\}.$$

By lemma 2.3,

$$\frac{Df^n(x)}{Df^n(y)} = \prod_{j=0}^{n-1} (1+\Delta_j),$$

with

$$|\Delta_j| < L_3\delta_j.$$

The assertion follows from

$$\exp(-2|x|) < 1 + x < \exp(|x|) \qquad \text{for } |x| < \tfrac{1}{2}.$$ □

LEMMA 2.5. *If* $xy > 0$ *then*

$$|f(x) - f(y)| \geq L_6(|f'(x)| + |f'(y)|)|x - y|.$$

Proof. For definiteness assume $0 < x < y$. Then

$$|f(x) - f(y)| = \int_x^y |f'(t)|\, dt,$$

since $f'(t) \neq 0$ if $t \neq 0$, and $0 \notin (x, y)$. Therefore, using lemma 2.2 twice, we find

$$|f(x) - f(y)| \geq 2L \int_x^y |t|\, dt$$
$$= L(y^2 - x^2) = L(|x| + |y|)|y - x|$$
$$\geq \frac{L}{2U}(|f'(y)| + |f'(x)|)|y - x|.$$ □

LEMMA 2.6. *Assume that* f^n *is monotone on* (x, y). *Then*

$$|f^n(x) - f^n(y)| \geq |Df^n(x) Df^n(y)|^{\frac{1}{2}} |x - y|.$$

Remark. The assumption of the lemma can be stated as follows. Assume x, y are such that $0 \notin f^l((x, y))$ for $l = 0, 1, \ldots, n-1$.

This lemma is crucial and we shall use it quite often.

Proof. If either x or y is zero, the assertion is trivial. So we assume $x, y \neq 0$. Define

$$u(t) = |Df^n(t)|^{-\frac{1}{2}}.$$

By lemma 2.1, u is convex on (x, y). Hence there are two constants α, β such that

$$u(t) \leq \alpha t + \beta, \qquad \textit{when } t \in (x, y),$$
$$u(x) = \alpha x + \beta, \qquad u(y) = \alpha y + \beta.$$

Therefore, assuming $0 < x < y$ for definiteness,

$$|f^n(x) - f^n(y)| = \int_x^y |Df^n(t)|\, dt \geq \int_x^y (\alpha t + \beta)^{-2}\, dt$$
$$= \frac{|x - y|}{(\alpha x + \beta)(\alpha y + \beta)}.$$

The result follows. □

LEMMA 2.7. *Assume* $|f^n(1)|<\varepsilon$. Then

$$n>\frac{C_8+|\log\varepsilon|}{\log L_1-\theta}.$$

Proof. By the definition of L_1 and the chain rule of differentiation, we have

$$|Df^n(1)|\le L_1^n.$$

By (H4), we have

$$|Df^{n+1}(1)|>C_1\exp[(n+1)\theta].$$

By lemma 2.2, and the assumption, we find

$$|f'(f^n(1))|<2U|f^n(1)|<2U\varepsilon.$$

Hence

$$\begin{aligned}L_1^n\ge|Df^n(1)|&=|Df^{n+1}(1)/f'(f^n(1))|\\&>\frac{C_1\exp[(n+1)\theta]}{2U\varepsilon},\end{aligned}$$

so that

$$n>\frac{\log(C_1e^{\theta}/2U)+|\log\varepsilon|}{\log L_1-\theta}.\qquad\square$$

We approach now the first delicate estimate and we introduce some further notation.

We call z an *n-pre-image of zero* if $f^n(z)=0$. This definition is satisfactory, since (H4.1) and (H1)–(H3) imply [**3**] that f has no stable periodic orbits, in particular 0 cannot be a periodic point.

We say x is *n, ε-close to z* if

(1) $0\notin f^l((x;z))$ for $l=0,1,\ldots,n-1$;

(2) $|f^n(x)|\le\varepsilon$.

Observe that if x is n, ε-close to z then f^n is strictly monotone on $(x;z)$. The next result states that, if z is not too close to 0 and n is not too large, then $|x-z|$ is small if $|f^n(x)|$ is small. More precisely, we have:

PROPOSITION 2.8. *For $\alpha>0$ sufficiently small, the following is true. Assume z is an n-pre-image of zero and x, x' are n, α-close to z.*

If

$$n\le n_\alpha\qquad(\text{i.e. }\alpha\le\exp(-n\theta/1000))\tag{1}$$

and

$$|f^p(z)|\ge\alpha^{1+\tau}\qquad\text{for }p=0,1,\ldots,n-n'_\alpha,$$

then

$$\frac{|x-x'|}{|z|}<|f^n(x)-f^n(x')|^{\tau/2},\tag{2}$$

where τ was defined above (3) *on p.* 16.

Remark. The condition on $f^p(z)$ can also be written as

$$|f^p(z)|\ge\sigma_{n-p,\alpha}\qquad\text{for }p\le n-n'_\alpha.$$

Proof. We shall prove the following stronger statement by recursion:

$$\frac{|x-x'|}{|z|}<|f^n(x)-f^n(x')|^\tau \exp\left(n\frac{(\log|\log\alpha|)^2}{|\log\alpha|}\right), \tag{2'}$$

which implies (2), since x, x' are n, α-close to z and α is small.

The bound on α will be adapted during the proof. We proceed by induction on n, proving, for every n, first the case $x'=z$ and then, by a verbatim repetition of all parts of the proof but one, the general case, $x'\neq z$, $x\neq z$. All steps except the one which is special for $x'=z$ are written for x'. We write $y=f^n(x)$, $y'=f^n(x')$.

Case $n=1$. By lemmas 2.2 and 2.5, we have, if $|x|<|x'|$,

$$\begin{aligned} L|x-x'||x| &\le \tfrac{1}{2}|f'(x)||x-x'|\le L_6^{-1}|f(x)-f(x')|\\ &\le L_6^{-1}|f(x)-f(x')|^{\frac{2}{3}}(2\alpha)^{\frac{1}{3}}. \end{aligned}$$

Using now that 0 has at most two pre-images, which are away from 0, and the continuity of f', we obtain for small α,

$$\frac{|x-x'|}{|z|}\le\frac{L_6|f(x)-f(x')|^{\frac{2}{3}}(2\alpha)^{\frac{1}{3}}}{L\inf_{w:f(w)\in[-\alpha,\alpha]}|w|^2}<|f(x)-f(x')|^\tau, \tag{3}$$

as asserted.

Induction step from $n-1$ to n. Assume z, x, x', n fulfil the assumptions of the proposition. Then for every l, $1\le l<n$, the quadruple

$$z_{n-l}=f^{n-l}(z),\qquad x_{n-l}=f^{n-l}(x),\qquad x'_{n-l}=f^{n-l}(x'),\qquad n'=l$$

also fulfils the assumptions. Therefore the inductive hypothesis implies

$$\frac{|f^{n-l}(x)-f^{n-l}(x')|}{|f^{n-l}(z)|}<|f^n(x)-f^n(x')|^{\tau/2},\qquad l=1,\dots,n-1. \tag{4}$$

On the other hand, lemma 2.6 implies

$$|f(x)-f(x')|\le\frac{|f^{s-1}(f(x))-f^{s-1}(f(x'))|}{|Df^{s-1}(f(x))Df^{s-1}(f(x'))|^{\frac{1}{2}}},\qquad 2\le s\le n. \tag{5}$$

If we use (4) with $x'=z$ in corollary 2.4, we see that

$$|Df^{s-1}(f(x))|\ge|Df^{s-1}(f(z))|\exp(-s\alpha^{\tau/3}),\qquad 2\le s\le n.$$

Repeating the argument with x' and z, we get from (5),

$$|f(x)-f(x')|\le|f^s(x)-f^s(x')|\exp(s\alpha^{\tau/3})/|Df^{s-1}(f(z))|,\qquad 1\le s\le n. \tag{6}$$

For the next argument we first assume nothing has been shown for n, and we show a certain bound in the case $x'=z$ only. Then assuming the proposition has been shown for n, x, and $x'=z$ we show that the same bound holds for arbitrary x'.

Case $x'=z$. By lemma 2.5,

$$|x-z|\le L_6^{-1}|f(x)-f(z)|/|f'(z)|,$$

so that (6) implies

$$|x-z|\le L_7|f^s(x)-f^s(z)|\exp(s\alpha^{\tau/3})/|Df^s(z)|,\qquad s\le n. \tag{7}$$

Case $x' \neq z$. Assume the conclusion of the proposition has been shown for $x' = z$ and n. By this inductive assumption,

$$\frac{|x-z|}{|z|} \leq (2\alpha)^{\tau/2},$$

and hence by lemma 2.3, $|(f'(z)/f'(x)) - 1| \leq 2L_3\alpha^{\tau/2}$. Combining this with lemma 2.5, we get

$$\begin{aligned}|x-x'| &\leq L_6^{-1}|f(x)-f(x')|/|f'(z)| \cdot |f'(z)/f'(x)| \\ &\leq 2L_6^{-1}|f(x)-f(x')|/|f'(z)|.\end{aligned}$$

Thus we get again, from (6),

$$|x-x'| \leq L_7|f^s(x)-f^s(x')| \exp(s\alpha^{\tau/3})/|Df^s(z)|, \qquad s \leq n. \tag{7'}$$

The proof proceeds once again in parallel for $x' = z$ and $x' \neq z$.

Combining (7) with (H4), we get

$$|x-x'| \leq |f^n(x)-f^n(x')| K_1 \exp[n(\alpha^{\tau/3}-\theta)]. \tag{8}$$

Note now that

$$K_1 \exp[n(\alpha^{\tau/3}-\theta)] < \alpha^{-\frac{1}{10}}, \tag{9}$$

provided α is small.

For later use we note the analogous bounds, valid for $n < n_\alpha$

$$\exp(n\alpha^{\tau/3}) < 2, \qquad \exp(n/|\log\alpha|) < K_2. \tag{10}$$

Coming back to (8), and using (9), we get

$$|x-x'| \leq |f^n(x)-f^n(x')|\alpha^{-\frac{1}{10}} \tag{11}$$

We shall now prove (2′) for cases of increasing complexity. The easiest case is $|z| \geq \alpha^{\frac{1}{5}}$. Then, by (11),

$$\frac{|x-x'|}{|z|} \leq \frac{|f^n(x)-f^n(x')|}{|z|}\alpha^{-\frac{1}{10}} < |y-y'|^\tau,$$

which is (2′).

We now consider the case $|z| < \alpha^{\frac{1}{5}}$. Let $p \geq 1$ be the smallest integer for which, with $a = |\log\alpha|$,

$$|f^p(0)-f^p(z)| > |f^p(z)|/a. \tag{12}$$

Since $f^n(z) = 0$, we have $p \leq n$. Note that lemma 2.2 and $|z| < \alpha^{\frac{1}{5}}$ imply

$$|f(0)-f(z)| < U\alpha^{\frac{2}{5}} < |f(z)|/a,$$

since $|z| < \alpha^{\frac{1}{5}}$ implies $|f(z)| > \frac{1}{2}$. Thus $p > 1$. We shall now bound $|f^p(0)-f^p(z)|$ from above. Using (12), we have for $l = 1, 2, \ldots, p-1$,

$$|f^l(0)-f^l(z)| \leq |f^l(z)|/a,$$

and hence by lemma 2.3, since $0 \notin (f^l(0); f^l(z))$,

$$\begin{aligned}|f^{l+1}(0)-f^{l+1}(z)| &\leq \sup_{t\in(f^l(0);f^l(z))} |f'(t)||f^l(0)-f^l(z)| \\ &\leq (1+(L_3/a))|f'(f^l(z))||f^l(0)-f^l(z)|.\end{aligned}$$

Using this inequality successively for all l, we get the desired bound

$$|f^p(0)-f^p(z)|\le K_4^{(p-1)/a}|Df^{p-1}(f(z))||1-f(z)|$$
$$\le K_5K_6^{p/a}|Df^{p-1}(f(z))||z|^2\le K_7|zDf^p(z)|, \tag{13}$$

making use twice of lemma 2.2 and by (10). We now rewrite (7′) as

$$\frac{|x-x'|}{|z|}\le L_7\exp(p\alpha^{\tau/3})\frac{|f^p(x)-f^p(x')|}{|f^p(z)|}\,\frac{|f^p(z)|}{|f^p(0)-f^p(z)|}\,\frac{|f^p(0)-f^p(z)|}{|zDf^p(z)|}. \tag{14}$$

Combining (13), (14) and (10), we get the following bounds, which are better than using (9) only. Namely

$$\frac{|x-x'|}{|z|}\le K_8\frac{|f^p(x)-f^p(x')|}{|f^p(0)-f^p(z)|},\qquad \text{when } p\le n, \tag{15}$$

and (by (12)), when $p<n$,

$$\frac{|x-x'|}{|z|}\le K_8\frac{|f^p(x)-f^p(x')|}{|f^p(z)|}a. \tag{16}$$

Note that just substituting the induction hypothesis in (16) will not suffice to get (2′) because of the factor a. We next bound $|f^p(x)-f^p(x')|$ in terms of $|y-y'|$. Applying (11) with $n'=n-p$ to $f^p(x)$, $f^p(x')$, $f^p(z)$ we get

$$|f^p(x)-f^p(x')|\le|y-y'|\alpha^{-\frac{1}{10}}<|y-y'|^{\frac{9}{10}}. \tag{17}$$

We now combine the above bounds in several ways according to subcases of $|z|<\alpha^{\frac{1}{5}}$.

Case 1. $p<n$, $|f^p(z)|\ge\alpha^{\frac{1}{5}}$. By (16) and (17),

$$\frac{|x-x'|}{|z|}\le K_8|y-y'|^{\frac{9}{10}}a/|f^p(z)|<|y-y'|^\tau$$

Case 2. $p<n$, $|f^p(z)|<\alpha^{\frac{1}{5}}$, $|f^p(0)|\ge 2\alpha^{\frac{1}{5}}$. Then we use (15) and (17) and $|f^p(0)-f^p(z)|>\alpha^{\frac{1}{5}}$ to get

$$\frac{|x-x'|}{|z|}\le K_8|y-y'|\alpha^{-\frac{3}{10}}<|y-y'|^\tau.$$

Case 3. $p<n$, $|f^p(z)|<\alpha^{\frac{1}{5}}$, $|f^p(0)|<2\alpha^{\frac{1}{5}}$. By lemma 2.7,

$$p>(K_9+\tfrac{1}{5}|\log\alpha|)/(\log L_1-\theta).$$

Now we use (16) and (2′) and we get

$$\frac{|x-x'|}{|z|}\le|f^n(x)-f^n(x')|^\tau K_7\exp\left[(n-p)\frac{(\log|\log\alpha|)^2}{|\log\alpha|}\right]\cdot a$$
$$\le|f^n(x)-f^n(x')|^\tau\exp\left[n\frac{(\log|\log\alpha|)^2}{|\log\alpha|}\right],$$

since

$$aK_7<a^2=\exp(2\log a)<\exp\left[\frac{p}{a}(\log a)^2\right].$$

Case 4. $p=n$, $|f^n(0)|>K_8\alpha^{1-(3\tau/2)}$. Then, by (15),

$$\frac{|x-x'|}{|z|}<K_8\frac{|f^n(x)-f^n(x')|}{|f^n(0)|}<\alpha^{\frac{3}{2}\tau-1}2\alpha^{1-\tau}|f^n(x)-f^n(x')|^\tau.$$

Case 5. $p=n$, $|f^n(0)|\le K_8\alpha^{1-(3\tau/2)}$. We show:

(5.1) $$n>n'_\alpha.$$

(5.2) $$|z|<\alpha^{1+\tau}/2.$$

Therefore this case will not occur and the proposition is proved.

Proof of (5.1). By lemma 2.7,

$$n>\frac{C_8-\log K_8+\left(1-\frac{3\tau}{2}\right)|\log\alpha|}{\log L_1-\theta}=n'_\alpha. \tag{18}$$

We define $L_2=C_8-\log K_8$.

Note that $n'_\alpha=\mathcal{O}(|\log\alpha|)$, since we are assuming $\tau<\frac{1}{2}$.

Proof of (5.2). By (18) and lemma 2.2, we get

$$\begin{aligned}|Df^{n-1}(1)|&=|Df^n(1)/f'(f^n(0))|\\&\ge C_1\exp(n\theta)/K_{10}\alpha^{1-\frac{3}{2}\tau}\\&\ge K_{11}\alpha^{-4\tau-1},\end{aligned}$$

by the definition of τ, see page 16. We now bound z. By lemma 2.2, we see that

$$\begin{aligned}|z|^2&<\frac{1}{L}|1-f(z)|\\&\le|f^n(z)-f^n(0)|/|Df^{n-1}(t)|\end{aligned}$$

for some $t\in[f(z),1]$. Applying corollary 2.4 to t and $f(z)$, and using (12),

$$|f^p(t)-f^p(f(z)|<|f^{p+1}(0)-f^{p+1}(z)|<|f^{p+1}(z)|/a,$$

for $p=0,1,\ldots,n-2$, we see that

$$|z|^2<K_{12}|f^n(0)/Df^{n-1}(f(z))|$$

Re-applying corollary 2.4 to 1 and $f(z)$, we see that

$$\begin{aligned}|z|^2&<K_{12}^2|f^n(0)/Df^{n-1}(1)|\\&\le K_{13}\alpha^{1-\frac{3}{2}\tau+1+4\tau}<\alpha^{2+2\tau}/4.\end{aligned}$$

We have discussed all cases and have thus proved the proposition. □

COROLLARY 2.9. *Under the assumptions of proposition* 2.8 *we have the bound*

$$\left|\frac{Df^n(x)}{Df^n(z)}-1\right|<\tfrac{1}{2}n\alpha^{\tau/4}.$$

Proof. This is, apart from converting $e^{\pm\delta}$ to $1\pm2\delta$ in bounds when δ is small, a direct reformulation of the steps of the proof of proposition 2.8 leading to the equations 2.8.4, 2.8.5 and 2.8.6. □

The next lemma treats the exceptional case of proposition 2.8. It shows that if z is small, then so are x and x'.

LEMMA 2.10. *Assume that z is an n-pre-image of zero and x, x' are n, α-close to z, $n_\alpha \geq n > n'_\alpha$.*

If

$$|f^p(z)| > \alpha^{1+\tau} \qquad \text{for } p = 1, \ldots, n - n'_\alpha$$

but

$$|z| \leq \alpha^{1+\tau}$$

then

$$|x| < 2\alpha^{1+\tau} \quad \text{and} \quad |x'| < 2\alpha^{1+\tau}.$$

Proof. We rely heavily on the proof of proposition 2.8. Note that the condition $|z| > \alpha^{1+\tau}$ was only used in case 5 at the end of the proof. Hence 2.8.7 holds, and we have

$$|x| \leq |x - z| + |z|,$$

$$|x - z| \leq L_7 \frac{|f^n(x) - f^n(z)|}{|Df^n(z)|} \exp(n\alpha^{\tau/3})$$

$$\leq \frac{\alpha}{C_1} 2L_7 \exp(n\alpha^{\tau/3} - n'_\alpha\theta) \leq \alpha^{1+\tau},$$

by the definition of τ and n'_α. The proof for x' is the same. □

Our next result complements proposition 2.8 in that it again gives bounds on $|x - x'|$ in terms of $|f^n(x) - f^n(x')|$, but this time for $n \geq n_\alpha$. In this case we have to assume that the orbit of z does not come close to zero too fast. We shall see later that this requirement is implied essentially by (H4) for a vast majority of homtervals.

PROPOSITION 2.11. *Assume z is an n-pre-image of 0 and x, x' are n, α-close to z, α small.*

If

$$|f^l(z)| > \sigma_{n-l,\alpha} \qquad \text{for } l \leq n - n'_\alpha, \tag{1}$$

then

$$|x - x'| < C_2 |f^n(x) - f^n(x')| \exp(-n\theta/2) \tag{2}$$

Proof. Consider first the case $n < n_\alpha$. Then corollary 2.9 leads to the conclusion that

$$|Df^n(x) Df^n(x')|^{\frac{1}{2}} \geq \exp(-n\alpha^{\tau/4}) |Df^n(z)|. \tag{3}$$

Let $y = f^n(x)$, $y' = f^n(x')$; by lemma 2.6 and (3),

$$|y - y'| \geq |Df^n(x) Df^n(x')|^{\frac{1}{2}} |x - x'| \tag{4}$$

$$\geq \exp(-n\alpha^{\tau/4}) |Df^n(z)| \, |x - x'|. \tag{5}$$

From (H4), we have $|Df^n(z)| > C_1 \exp(n\theta)$, so that (5) implies

$$|x - x'| \leq C_1^{-1} |y - y'| \exp[-n(\theta - \alpha^{\tau/4})] < C_2 |y - y'| \exp(-n\theta/2),$$

which is (2).

Assume next that we have shown the assertion for all numbers up to $n - 1$, $n \geq n_\alpha$. We proceed inductively to n.

If z, x, x' satisfy the assumptions of the proposition for n, then $f(z)$, $f(x)$, $f(x')$ satisfy it for $n-1$. Namely (1) becomes

$$|f^{l-1}(f(z))|>\sigma_{n-l,\alpha} \qquad \text{for } l\le n-n'_\alpha, \tag{6}$$

which means

$$|f^{k}(f(z))|>\sigma_{n-1-k,\alpha} \qquad \text{for } k\le (n-1)-n'_\alpha.$$

Hence (2), for $n-1$, implies

$$|f(x)-f(x')|<C_2|y-y'|\exp(-(n-1)\theta/2). \tag{7}$$

By lemma 2.2, $|f'(w)|>2L|w|$ and since x, x' must have the same sign,

$$|f(x)-f(x')|>2L\int_x^{x'}|w|\,dw=L|x-x'|^2. \tag{8}$$

Combining (7) and (8) we get

$$|x-x'|<K_2|y-y'|^{\frac{1}{2}}\exp(-n\theta/4).$$

By (1), we have $|z|>\exp(-n\theta/20)$, so that

$$\frac{|x-x'|}{|z|}<K_2|y-y'|^{\frac{1}{2}}\exp(-n\theta/5).$$

We proceed now with the case $x'=z$, and deal below with $x'\ne z$. For $j=0,\ldots,n-n_\alpha$ we find as above

$$\frac{|f^j(x)-f^j(z)|}{|f^j(z)|}<K_2|y-y'|^{\frac{1}{2}}\exp[-(n-j)\theta/5] \tag{9}$$

and

$$\frac{|f^j(x)-f^j(x')|}{|f^j(z)|}<K_2|y-y'|^{\frac{1}{2}}\exp[-(n-j)\theta/5]. \tag{9'}$$

Using (9) in lemma 2.3 we see that

$$\left|\frac{f'(f^j(x))}{f'(f^j(z))}\right|>\exp\{-K_3\alpha^{\frac{1}{2}}\exp[-(n-j)\theta/5]\}, \qquad j=0,\ldots,n-n_\alpha.$$

Hence, we have from corollary 2.9,

$$\begin{aligned}\left|\frac{Df^n(x)}{Df^n(z)}\right| &= \left|\frac{Df^{n_\alpha}(f^{n-n_\alpha}(x))Df^{n-n_\alpha}(x)}{Df^{n_\alpha}(f^{n-n_\alpha}(z))Df^{n-n_\alpha}(z)}\right| \\ &\ge \exp(-n_\alpha\alpha^{7/4})\exp\left[-K_3\alpha^{\frac{1}{2}}\sum_{p=0}^{n-n_\alpha}\exp(-p\theta/5)\right]>\tfrac{1}{4},\end{aligned} \tag{10}$$

provided α is small. Applying again lemma 2.6, and (10), we see that

$$\begin{aligned}|f^n(x)-f^n(z)| &\ge |Df^n(x)Df^n(z)|^{\frac{1}{2}}|x-z| \\ &\ge \tfrac{1}{4}|Df^n(z)|\,|x-z|.\end{aligned} \tag{11}$$

From (H4), we then find

$$|x-z|\le 4C_1^{-1}|f^n(x)-f^n(z)|\exp(-n\theta). \tag{12}$$

This completes the proof when $x'=z$. If $x'\neq z$, we proceed to (9′). From the inductively proved eq. (2) for n and $x'=z$ we have

$$\left|\frac{f^j(x)}{f^j(z)}\right|\le 1+\frac{|f^j(x)-f^j(z)|}{|f^j(z)|}<2, \qquad \text{if } j\le n-n_\alpha, \tag{13}$$

so that (9′) implies

$$\frac{|f^j(x)-f^j(x')|}{|f^j(x)|}<2K_2|y-y'|^{\frac{1}{2}}\exp[-(n-j)\theta/5]. \tag{9''}$$

Thus we get as before:

$$\begin{aligned}|f^n(x)-f^n(x')|&\ge|Df^n(x)||x-x'|/4\\ &\ge|Df^n(z)||x-x'|/16,\end{aligned} \tag{11'}$$

by (10). The desired inequality (2) now follows from (H4). □

Our next lemma deals with the situation where an orbit comes close to $f(1)$. We describe the general picture (cf. figure 1). Assume f^n is monotone on $[f(1), x]$, and

$$|f^n(f(1))|\le\alpha, |f^n(x)|=\alpha.$$

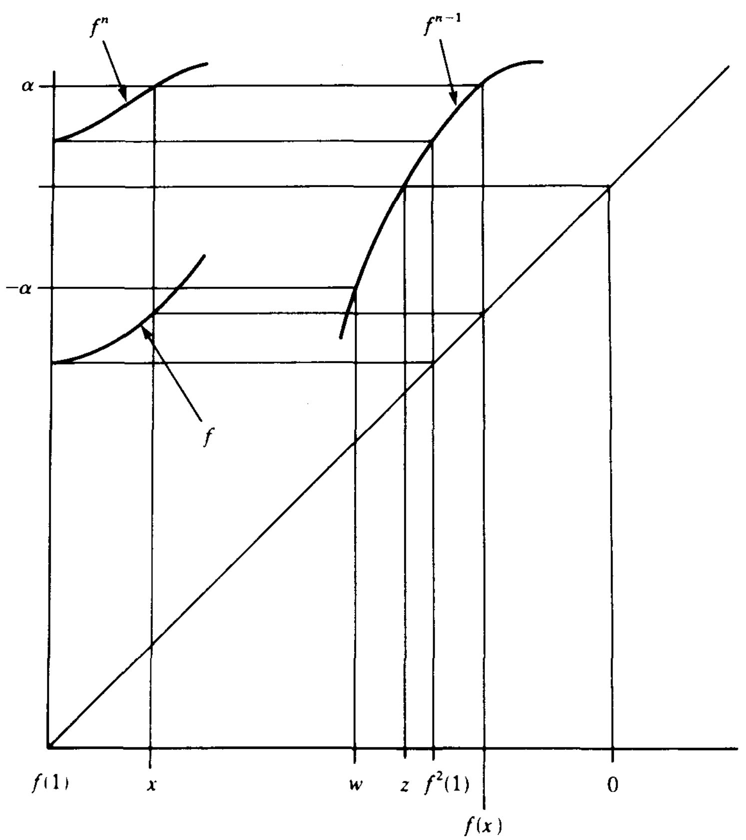

FIGURE 1

Note that we do not assume there is a y with $f^n(y)=0$ in $[f(1), x]$. Consider the maximal connected component of $f^{-(n-1)}([-\alpha, \alpha])$ containing $f^2(1)$, call it J. Assume furthermore $f^{(n-1)}$ is monotone on J and denote the endpoints of J by w and $f(x)$. Let $z = f^{-(n-1)}(0) \cap J$.

LEMMA 2.12. *Under the above assumptions, the following holds:*

(1) *if the above situation applies, then* $n > n'_\alpha$;

(2) *if* $n < n_\alpha$, *then* $|x - f(1)| \le \alpha^{2+\eta}$ *for some universal constant* $\eta > 0$.

Remark. If we were only to use the quadratic nature of f near $x=0$ then we would get $|x - f(1)| \le \alpha^2$. Thus the lemma states that $|f(1) - x|$ is very short compared to $|f^n(x)|^2$.

Proof. By lemma 2.7, and the definition of n'_α,

$$n > \frac{C_8 + |\log \varepsilon|}{\log L_1 - \theta} > n'_\alpha, \tag{1}$$

if α is small.

We use (H4) and (1) to get

$$\begin{aligned}|Df^{n+1}(1)| = \frac{|Df^{n+2}(1)|}{|f'(f^{n+1}(1))|} &\ge \frac{C_1 \exp[(n+2)\theta]}{2U\alpha} \\ &\ge K_3 \alpha^{-1-\eta_1},\end{aligned} \tag{2}$$

for some $\eta_1 > 0$.

The general assumptions of the lemma imply that there is a $t > 0$ such that $f^2([0, t]) = [f(1), x]$. By lemma 2.2,

$$t^2 \le \frac{1}{L}|1 - f(t)|. \tag{3}$$

We now apply lemma 2.6 to $f^{n+1}|_{[f(t),1]}$. Its conditions are fulfilled since f^n is monotone on $[f(1), x]$ (or else we apply lemma 2.10 directly). Hence

$$\begin{aligned}L^{-1}|1 - f(t)| \le L^{-1}&|Df^{n+1}(1) Df^{n+1}(f(t))|^{-\frac{1}{2}} \\ &\cdot |f^{n+2}(0) - f^{n+2}(t)|.\end{aligned} \tag{4}$$

Next we apply corollary 2.9, to $f^{n-1}|_{[w, f(x)]}$. Its conditions are fulfilled. Note that $n \le n_\alpha$. Hence

$$\left|\frac{Df^{n-1}(f^2(1))}{Df^{n-1}(f(x))} - 1\right| \le \tfrac{1}{2} n_\alpha \alpha^{\tau/4}.$$

Therefore

$$\begin{aligned}0 \le \frac{Df^{n+1}(1)}{Df^{n+1}(f(t))} &= \frac{Df^{n-1}(f^2(1)) f'(1) f'(f(1))}{Df^{n-1}(f(x)) f'(f(t)) f'(x)} \\ &\le K_4 \exp(n_\alpha \alpha^{\tau/4}),\end{aligned} \tag{5}$$

by the continuity of the derivative and since $f(t)$ and x are away from 0.

Combining (3), (4), (5), we get, by (2)

$$t^2 \le \frac{1}{L} K_5 |Df^{n+1}(1)|^{-1} \cdot |f^n(f(1)) - f^n(x)| \le K_6 \alpha^{1+\eta_1} \alpha.$$

The result follows. □

The next proposition deals with the difference of reciprocals of derivatives, and this will of course be crucial to the study of the continuity of h_n.

PROPOSITION 2.13. *Let z be an n-pre-image of 0 and assume x and x' are n, α-close to z. If $|f^p(z)|>\sigma_{n-p,\alpha}$ for $p\le n-n'_\alpha$, then*

$$\left|\frac{1}{|Df^n(x)|}-\frac{1}{|Df^n(x')|}\right|\le\frac{|f^n(x)-f^n(x')|^{\tau/3}}{|Df^n(x)|}. \tag{1}$$

Proof. The assumptions imply that $Df^n(x)$ and $Df^n(x')$ have the same sign. Therefore

$$\left|\frac{1}{|Df^n(x)|}-\frac{1}{|Df^n(x')|}\right|=\frac{1}{|Df^n(x)|}\cdot\left|1-\frac{Df^n(x)}{Df^n(x')}\right|. \tag{2}$$

We rewrite

$$\frac{Df^n(x)}{Df^n(x')}=\prod_{j=0}^{n-1}\left\{1+\left(\frac{f'(f^j(x))}{f'(f^j(x'))}-1\right)\right\}. \tag{3}$$

In order to apply lemma 2.3, we bound $[f^j(x)-f^j(x')]/f^j(x)$. We have, for $n-j\le n_\alpha$, by proposition 2.8,

$$\left|\frac{f^j(x)}{f^j(z)}\right|\ge 1-\frac{|f^j(x)-f^j(z)|}{|f^j(z)|}>(1+\alpha^{\tau/2})^{-1} \tag{4}$$

and applying (4) and proposition 2.8,

$$\begin{aligned}\frac{|f^j(x)-f^j(x')|}{|f^j(x)|}&<(1+\alpha^{\tau/2})\frac{|f^j(x)-f^j(x')|}{|f^j(z)|}\\&<\alpha^{\tau/13}|f^n(x)-f^n(x')|^{\tau/3}.\end{aligned} \tag{5}$$

If $n-j\ge n_\alpha$, then by prop. 2.11 and the bound on $f^p(z)$,

$$\begin{aligned}\left|\frac{f^j(x)}{f^j(z)}\right|&\ge 1-\frac{|f^j(x)-f^j(z)|}{|f^j(z)|}\\&\ge 1-C_2\alpha\exp[-(n-j)\theta/2]\exp[(n-j)\theta/5]>(1+\alpha^{\tau/2})^{-1}.\end{aligned} \tag{6}$$

Combining (6) with prop. 2.11, we get

$$\begin{aligned}\frac{|f^j(x)-f^j(x')|}{|f^j(x)|}&<(1+\alpha^{\tau/2})\frac{|f^j(x)-f^j(x')|}{|f^j(z)|}\\&<C_2(1+\alpha^{\frac{1}{2}})|f^n(x)-f^n(x')|\exp[-(n-j)\theta/2]\exp[(n-j)\theta/5]\\&<\alpha^{\frac{1}{7}}|f^n(x)-f^n(x')|^{\frac{1}{3}}\exp[-(n-j)\theta/5].\end{aligned} \tag{7}$$

Now applying lemma 2.3 to (3), (5) and (7), we get

$$\frac{Df^n(x)}{Df^n(x')}=\prod_{j=0}^{n-1}(1+\Delta_j),$$

with Δ_j small, so that

$$\prod_{j=0}^{n-1}\exp(-2|\Delta_j|)\le\frac{Df^n(x)}{Df^n(x')}\le\prod_{j=0}^{n-1}\exp(|\Delta_j|),$$

with

$$\sum_{j=0}^{n-1} |\Delta_j| \le \sum_{j=n-n_\alpha+1}^{n-1} |f^n(x)-f^n(x')|^{\tau/3}\alpha^{\tau/13}$$
$$+ \sum_{j=0}^{n-n_\alpha} |f^n(x)-f^n(x')|^{\frac{1}{3}}\alpha^{\frac{1}{7}} \exp[-(n-j)\theta/5]$$
$$< \alpha^{\tau/14}|f^n(x)-f^n(x')|^{\tau/3}.$$

Since $e^x < 1+2x$ for small $x > 0$, the result follows. □

3. *The invariant measure near the critical point*

In this section we are going to prove theorems 1.2 and 1.3. We use extensively the interplay between orbits getting close to 0, 1 or $f(1)$, and those orbits never getting close to these points. The first case will be called singular, the second case regular. We proceed now to the detailed definitions. These definitions will only be used in this section.

Let J be a maximal connected component of $f^{-m}([-\varepsilon, \varepsilon])$, $\varepsilon > 0$ small. The interval J will be called *m, ε-regular* if

(R1) $0 \notin f^k(J)$ for $k = 0, 1, \ldots, m-1$,

(R2) $1 \notin J$,

(R3) $f(1) \notin J$,

(R4) Let $\{z\} = f^{-m}(0) \cap J$. Then

$$|f^j(z)| \ge \sigma_{m-j,\varepsilon} \qquad \text{for } j = 0, 1, \ldots, m-n'_\varepsilon.$$

Remark. Conditions (R1), (R2), (R3) imply that $f^m|_J$ is strictly monotone and maps *onto* $[-\varepsilon, \varepsilon]$. Thus (R4) makes sense (see figure 2).

If J is m, ε-regular, then $f(J)$ is $m-1$, ε-regular. If J is m, ε-regular, we call any subset of J m, ε-regular.

Assume now J is not m, ε-regular, but $f(J)$ is $m-1$, ε-regular. Then we call J *m, ε-singular* (and only then). Since J is a maximal connected component of $f^{-m}([-\varepsilon, \varepsilon])$, we must have:

Either (S1) $f(1) \in J$

or (S2) $f(1) \notin J$ and $f^{-m}(0) \cap J \neq \emptyset$, and $z = f^{-m}(0) \cap J$ satisfies $|z| < \sigma_{m,\varepsilon}$ (and thus $m \ge n'_\varepsilon$).

Note that these are the only two possibilities for singularity to occur. For if $1 \in J$ then $f(1) \in f(J)$ which means that $f(J)$ is not regular, and similarly when $0 \in J$. (We have included the cases (R2) and (R1) for $k = 0, 1$, for convenience of formulation in the definition of m, ε-regularity.)

We now define sets $H_{k,\varepsilon}$ which are similar to the sets $E^\varepsilon_{m,\gamma}$, $F^\varepsilon_{m,\gamma}$, $G^\varepsilon_{m,\gamma}$ introduced in section 1.

Definition. $H_{k,\varepsilon} = \{x \in \Omega \mid x \text{ is } k, \varepsilon\text{-singular and } |f^k(x)| \le \varepsilon\}$.

We have now the following interesting relations between the set $H_{k,\varepsilon}$ and small intervals around 0.

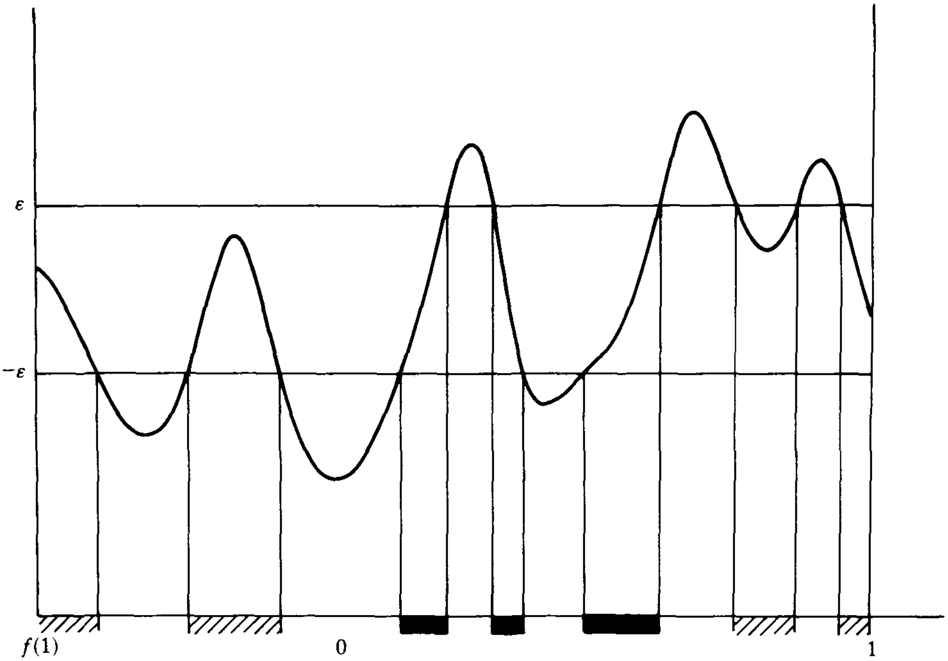

FIGURE 2. Regular and non-regular sets: ■ regular; ▨, non-regular.

LEMMA 3.1. *If* $0<\varepsilon<\varepsilon_0$ *then*

(1) $H_{k,\varepsilon}=\varnothing$ *if* $k<n'_\varepsilon$;

(2) $H_{k,\varepsilon}\subset f^2(\{y\,|\,|y|\le \varepsilon^{1+\tau}\})\cup\{y\,|\,|y|<2\varepsilon^{1+\tau}\}$, *if* $n'_\varepsilon\le k<n_\varepsilon$;

(3) $H_{k,\varepsilon}\subset f^2(\{y\,|\,|y|\le \exp(-k\theta/30)\}\cup\{y\,|\,|y|\le \exp(-k\theta/30)\}$ *if* $k\ge n_\varepsilon$.

Proof. Assume $x\in H_{k,\varepsilon}$ and assume it is k, ε-singular because of (S1). The set of all such x forms a maximal interval J of $f^{-k}([-\varepsilon,\varepsilon])$, namely the one containing $f(1)$. Suppose $J=[f(1),t]$. If $k<n_\varepsilon$ we apply lemma 2.12. Then we see that we must have had $k\ge n'_\varepsilon$ and

$$|f(1)-t|\le\varepsilon^{2+\eta}.$$

On the other hand, $[f(1),t]$ is an image under f^2 of an interval $[0,t']$, which by lemma 2.2 and the boundedness away from zero of f' near 1 implies the assertions (1), (2) when (S1) applies.

Still in the case when (S1) applies, but when $k\ge n_\varepsilon$, we apply proposition 2.11 to $f^{k-1}|_{[f^2(1);f(t)]}$. Its hypotheses are fulfilled by the construction of $H_{k,\varepsilon}$. Hence

$$|f^2(1)-f(t)|\le 2C_2\varepsilon\exp(-k\theta/2)$$

so, by the fact that f' is bounded away from zero near $f(1)$,

$$|f(1)-t|\le K_1\varepsilon\exp(-k\theta/2).$$

The set of t satisfying this inequality is contained in

$$f^2(\{y\,|\,|y|\le\exp(-k\theta/30)\}.$$

This ends the analysis of the case when (S1) applies.

Case (S2) can only occur for $k \geq n'_\varepsilon$. Since $f^k(z)=0$, and by the definition of $H_{k,\varepsilon}$, every point $x \in H_{k,\varepsilon}$ has the following property: $f(x)$ is $k-1$, ε-close to $f(z)$. Thus we can apply prop. 2.11. (Strictly speaking, we have not proved it for $k=n'_\alpha-1$, but the extension to this case is obvious by a slight redefinition of n'_α.) Hence

$$|f(z)-f(x)| \leq C_2\varepsilon \exp(-k\theta/2), \qquad \text{when } k \geq n_\varepsilon - 1.$$

The case $n_\varepsilon \geq k \geq n'_\varepsilon$ is covered by lemma 2.10.
Since x and z have the same sign, we find, using lemma 2.2,

$$\begin{aligned}|x| &\leq |x-z|+|z| \leq L^{-\frac{1}{2}}|f(x)-f(z)|^{\frac{1}{2}}+|z|\\ &\leq K_2\exp(-k\theta/4)+\exp(-k\theta/20).\end{aligned}$$

The assertion of the lemma is proved. □

We now split $f^{-n}(y)$ as follows. Define

$$\mathcal{S}^{(n)}_{k,\varepsilon}(y)=f^{-n}(y)\cap f^{-k}(H_{n-k,\varepsilon}), \qquad k=0,\ldots,n.$$

In fact, by lemma 3.1

$$\mathcal{S}^{(n)}_{k,\varepsilon}(y)=\varnothing \qquad \text{if } n-k<n'_\varepsilon.$$

In words, $\mathcal{S}^{(n)}_{k,\varepsilon}(y)$ are those x for which $f^n(x)=y$, and $f^{n-k}(x)$ is k,ε-singular. By lemma 3.1, we can write

$$f^{-n}(y)=\mathcal{R}_{n,\varepsilon}(y)\cup\Big(\bigcup_{k\leq n-n'_\varepsilon}\mathcal{S}^{(n)}_{k,\varepsilon}(y)\Big), \tag{1}$$

with $\mathcal{R}_{n,\varepsilon}(y)$ disjoint from the $\mathcal{S}^{(n)}_{k,\varepsilon}(y)$. We could also write

$$\mathcal{R}_{n,\varepsilon}(y)=\{x\,|\,f^n(x)=y, \text{ and for } k=0,\ldots,n, \quad f^k(x) \text{ is } n-k,\ \varepsilon\text{-regular}\}.$$

Note that if $x\in\mathcal{R}_{n,\varepsilon}(y)$ then x is n,ε-close to a unique z for which $f^n(z)=0$, and this z satisfies

$$|f^k(z)|>\sigma_{n-k,\varepsilon} \qquad \text{for } k\leq n-n'_\varepsilon.$$

We now define the regular part of h_n, for $|y|\leq\varepsilon$ by

$$h^{\text{reg}}_{n,\varepsilon}(y)=\sum_{x\in\mathcal{R}_{n,\varepsilon}(y)}\frac{1}{|Df^n(x)|}.$$

In the sequel, α denotes a sufficiently small number.

LEMMA 3.2. *If* $0<|\eta|<\alpha/2$ *and* $0<\varepsilon<\alpha$, *then*

$$\int_{|y|<\alpha/2}|h^{\text{reg}}_{n,\varepsilon}(y)-h^{\text{reg}}_{n,\varepsilon}(y+\eta)|\,dy\leq|\eta|^{\tau/4}.$$

Proof. By lemma 2.13 and the definition of $\mathcal{R}_{n,\varepsilon}(y)$,

$$|h^{\text{reg}}_{n,\varepsilon}(y)-h^{\text{reg}}_{n,\varepsilon}(y+\eta)|\leq\tfrac{1}{2}|\eta|^{\tau/4}h^{\text{reg}}_{n,\varepsilon}(y).$$

But

$$0\leq h^{\text{reg}}_{n,\varepsilon}(y)\leq h_n(y) \quad\text{and}\quad \int_{|y|<\alpha/2}h_n(y)\,dy\leq 2.$$

The result follows. □

THEOREM 3.3 (= theorem 1.2). *If* α *is sufficiently small, if* $0<\varepsilon<\alpha/4$ *and* $0<\sigma<\alpha/4$, *then*

$$\int_{|y|<\sigma} h_{n,\varepsilon}^{\mathrm{reg}}(y)\,dy \le 4(\sigma^{\tau/4}+\sigma/\alpha).$$

Proof. Set

$$\rho(t)=\int_{|s|<t} h_{n,\varepsilon}^{\mathrm{reg}}(s)\,ds.$$

For $0<t<\alpha/2$ we have

$$\begin{aligned}\rho(t)&=\int_{|y|\le t/2} h_{n,\varepsilon}^{\mathrm{reg}}(y)\,dy+\int_{t/2\le|y|\le t} h_{n,\varepsilon}^{\mathrm{reg}}(y)\,dy\\ &=2\int_{|y|<t/2} h_{n,\varepsilon}^{\mathrm{reg}}(y)\,dy+\int_{t/2<y<t}[h_{n,\varepsilon}^{\mathrm{reg}}(y)-h_{n,\varepsilon}^{\mathrm{reg}}(y-t/2)]\,dy\\ &\quad+\int_{-t/2>y>-t}[h_{n,\varepsilon}^{\mathrm{reg}}(y)-h_{n,\varepsilon}^{\mathrm{reg}}(y+t/2)]\,dy.\end{aligned}$$

Thus, by lemma 3.2,

$$\rho(t)\ge 2\rho(t/2)-2(t/2)^{\tau/4},$$

i.e.

$$\rho(t/2)\le \tfrac{1}{2}\rho(t)+(t/2)^{\tau/4},$$

or if $2^{-k-1}\alpha\le\sigma<2^{-k}\alpha$,

$$\begin{aligned}\rho(\sigma)&\le\tfrac{1}{2}\rho(2\sigma)+\sigma^{\tau/4}\\ &\le\tfrac{1}{4}\rho(4\sigma)+\tfrac{1}{2}(2\sigma)^{\tau/4}+\sigma^{\tau/4}\\ &\le\cdots\\ &\le\frac{1}{2^k}\rho(2^k\sigma)+\sigma^{\tau/4}\sum_{j=0}^{k-1}2^{-\frac{3}{4}j}\\ &\le 2^{-k}+\sigma^{\tau/4}/(1-2^{-\frac{3}{4}})\\ &<\frac{2\sigma}{\alpha}+\sigma^{\tau/4}\cdot 4,\ \text{as asserted.}\end{aligned}$$

□

THEOREM 3.4 (= theorem 1.3). *For* $\varepsilon>0$ *sufficiently small we have*:

$$\int_{|y|<\varepsilon} h_n(y)\,dy\le\varepsilon^{\tau/40}.$$

Proof. If λ denotes Lebesgue measure and E is a measurable set, then it is well known that

$$\int_E h_n(y)\,dy=\int_{f^{-1}(E)} h_{n-1}(y)\,dy=\int_{f^{-n}(E)} dy=\lambda(f^{-n}(E)).$$

Hence, by the discussion on p.33, eq. (1)

$$\int_{|y|\le\varepsilon} h_n(y)\,dy\le\int_{|y|\le\varepsilon} h_{n,\varepsilon}^{\mathrm{reg}}(y)\,dy+\sum_{k\ge n'_\varepsilon}\lambda(f^{-(n-k)}(H_{k,\varepsilon})). \qquad (2)$$

We now bound $\int_{|y|<\varepsilon} h_n(y)\,dy$ by recursion.

By lemma 3.1, since $H_{p,\varepsilon} = \varnothing$ for $p < n'_\varepsilon$, we have for $p < n'_\varepsilon$,

$$\int_{|y|<\varepsilon} h_p(y)\,dy = \int_{|y|<\varepsilon} h^{\text{reg}}_{p,\varepsilon}(y)\,dy \leq \varepsilon^{\tau/5},$$

provided ε is sufficiently small, by theorem 3.3.

Hence the theorem is proved for $n < n'_\varepsilon$. Assume now it has been shown for some $n-1 \geq n'_\varepsilon - 1$. We proceed to n. By (2), lemma 3.1 and theorem 3.3,

$$\int_{|y|\leq\varepsilon} h_n(y)\,dy \leq \varepsilon^{\tau/5} + \sum_{n_\varepsilon \geq k \geq n'_\varepsilon - 2} \lambda[f^{-(n-k)}(f^2(\{|x| \leq K_1\varepsilon^{1+\tau}\}))]$$
$$+ \sum_{k \geq n_\varepsilon - 2} \lambda[f^{-(n-k)}(\{|x| \leq K_1 \exp(-k\theta/30)\})].$$

We have used the fact that if E is a small interval $E = \{x \mid |x| < \delta\}$ then

$$f^{-2}(f^2(E)) \subset \{|x| < \tfrac{1}{2}K_1\delta\},$$

for some K_1 which depends only on f (use lemma 2.2).

Hence, recursively, we get, for small ε,

$$\lambda[f^{-n}(\{|y| \leq \varepsilon\})] \leq \varepsilon^{\tau/5} + 2|\log\varepsilon| \cdot (1000/\theta)\varepsilon^{(1/40+\tau/40)\tau}$$
$$+ \sum_{k \geq (999/\theta)|\log\varepsilon|} 2K_1^{1/40} \exp(-\tau k\theta/800)$$
$$\leq \varepsilon^{\tau/5} + \varepsilon^{\tau(1/40+\tau/80)} + K_2 \exp(-\tfrac{999}{800}\tau|\log\varepsilon|)$$
$$< \varepsilon^{\tau/40}.$$

This completes the proof. □

4. *Absolute continuity*

This section is devoted to the proof of theorem 1.4, see § 1 for the definitions. We start with a few preliminary estimates.

LEMMA 4.1. *Let z be a q-pre-image of zero and assume f^n is monotone on $[f(1), z]$. Then*

$$|f(1) - z| \leq C_4 \exp(-2n\theta/3).$$

Note. We may assume $q < n$.

Proof. By lemma 2.6,

$$|f(1) - z| \leq |f^{q+1}(1)|\,|Df^q(f(1))Df^q(z)|^{-\frac{1}{2}}. \tag{1}$$

On the other hand, by lemmas 2.2 and 2.6 and the monotonicity of $f^n|_{[f(1),z]}$,

$$|f^{q+1}(1)| \leq L^{-1}|f^{q+1}(1)|^{-1}|f^{q+2}(1) - 1|$$
$$\leq (2/L)|f^{q+1}(1)|^{-1} \cdot |Df^{n-q-2}(f^{q+2}(1))Df^{n-q-2}(1)|^{-\frac{1}{2}}, \tag{2}$$

provided $q < n-1$. The case $q = n-1$ is an easy variant of (2) and is left to the reader.

Case 1. $|f^{q+1}(1)| \leq \exp(-2n\theta/3)$. Then, by (1) and (H4)

$$|f(1) - z| \leq K_3 \exp(-2n\theta/3) \exp(-q\theta) \leq C_4 \exp(-2n\theta/3).$$

Case 2. $|f^{q+1}(1)| > \exp(-2n\theta/3)$.
Then we combine (1), (2) with (H4) and lemma 2.2 to get

$$\begin{aligned}|f(1)-z| &\le K_1|Df^q(f(1))Df^q(z)Df^{n-q-2}(f^{q+2}(1))Df^{n-q-2}(1)|^{-\frac{1}{2}}\\ &\quad\cdot|f^{q+1}(1)|^{-1}|f'(f^{q+1}(1))|^{-\frac{1}{2}}|f'(f^{q+1}(1))|^{\frac{1}{2}}\\ &= K_1\frac{|f'(f^{q+1}(1))|^{\frac{1}{2}}}{|f^{q+1}(1)|}|Df^q(z)Df^{n-q-2}(1)Df^{n-1}(f(1))|^{-\frac{1}{2}}\\ &\le K_2\exp(n\theta/3)\exp(-q\theta/2)\exp[-(n-q-2)\theta/2]\exp[-(n-1)\theta/2]\\ &\le C_4\exp(-2n\theta/3).\end{aligned}$$

□

LEMMA 4.2. *For sufficiently small* ε, *if* f^n *is monotone on* $[f(1), x]$ *and* $n < 7|\log\varepsilon|^{\frac{1}{2}}$ *and*

$$|f^{n-p}(x)| > \exp(-|\log\varepsilon|^{\frac{1}{3}}) \qquad \textit{for } p = 0, 1, \ldots, n$$

and

$$|f^{n+1}(1) - f^n(x)| < \varepsilon,$$

then

$$|f(1) - x| \le \exp(-|\log\varepsilon|^{\frac{1}{3}}).$$

Proof. For all $0 \le l \le n$, lemma 2.2 and the assumption of the lemma imply

$$|f'(f^l(x))| \ge K_1\exp(-|\log\varepsilon|^{\frac{1}{3}}).$$

Therefore

$$|Df^n(x)| \ge \exp(-2|\log\varepsilon|^{\frac{1}{3}}n) \ge \exp(-14|\log\varepsilon|^{\frac{5}{6}})$$

By (H4),

$$|Df^n(f(1))| \ge C_1\exp(n\theta),$$

so that by (H1), (H2) and the theory of the Schwarzian derivative (see e.g. [**3**, § 4]),

$$|Df^n(t)| \ge \exp(-14|\log\varepsilon|^{\frac{5}{6}})$$

for all $t \in [f(1), x]$. Thus

$$|f(1) - x| \le \varepsilon\exp(14|\log\varepsilon|^{\frac{5}{6}}) \le \exp(-|\log\varepsilon|^{\frac{1}{3}}),$$

for $\varepsilon < \varepsilon_0$. □

COROLLARY 4.3 *If* f^n *is monotone on* $[f(1), x]$ *and*

$$|f^p(x)| \ge \rho_{n-p,\varepsilon} \quad \textit{and} \quad |f^{n+1}(1) - f^n(x)| < \varepsilon$$

then

$$|f(1) - x| \le \rho_{n-p,\varepsilon} \qquad \textit{for } 0 \le p \le n.$$

Proof. Combine lemmas 4.1, 4.2 and the definition of $\rho_{m,\varepsilon}$ cf. page 17. □

Our next lemma shows that for $n > 7|\log\varepsilon|^{\frac{1}{2}}$, either homtervals are close to the boundary of Ω or the derivative of f^n on that homterval is relatively large in its middle part.

Let $[z_1; z_2]$ be a maximal interval of monotonicity of f^n and assume $z_1, z_2 \notin \{f(1), 1\}$. Then we have

$$f^{q_1}(z_1) = f^{q_2}(z_2) = 0, \quad \text{with } q_1, q_2 < n,$$

and we assume $q_2 \ge q_1$ (in fact equality cannot occur). Assume $x \in [z_1, z_2]$.

LEMMA 4.4. *Assume the above situation applies for x, z_1, z_2, n. There is an $\varepsilon_0>0$ such that if $\varepsilon<\varepsilon_0$ and if*

$$|f^p(x)|>\rho_{n-p,\varepsilon} \qquad \text{for } p=0,1,\ldots,n-1,$$

then the following is true:

(1) *if $n>|\log\varepsilon|^{\frac{1}{2}}$ and $q_2\le q_1+4|\log\varepsilon|^{\frac{1}{3}}/\theta$ then*

$$|Df^{n-l}(f^l(x))|\ge \exp((n-l)\theta/2-|\log\varepsilon|^{\frac{1}{3}}C_5)$$

for $l\le q_1$;

(2) *if $n>|\log\varepsilon|^{\frac{1}{2}}$ and $q_2>q_1+4|\log\varepsilon|^{\frac{1}{3}}/\theta$ then*

$$|f(1)-f^{q_1+2}(x)|\le C_6\exp[-(n-q_1)\theta/3].$$

Proof. From (H2) and the theory of functions with negative Schwarzian derivative we have

$$|Df^{q_1}(x)|\ge \min\{|Df^{q_1}(z_1)|, |Df^{q_1}(z_2)|\}$$

and also

$$|Df^{q_2-q_1-1}(f^{q_1+1}(x))|\ge \min\{|Df^{q_2-q_1-1}(1)|, |Df^{q_2-q_1-1}(f^{q_1+1}(z_2))|\}$$

and

$$|Df^{n-q_2-1}(f^{q_2+1}(x))|\ge \min\{|Df^{n-q_2-1}(f^{q_2-q_1}(1))|, |Df^{n-q_2-1}(1)|\}$$

since $f^{q_1+1}(z_1)=f^{q_2+1}(z_2)=1$. Write

$$L_1=\sup_{x\in\Omega}|f'(x)|.$$

By (H4),

$$|Df^{n-q_2-1}(f^{q_2-q_1}(1))|=\left|\frac{Df^{n-q_2-1}(1)}{Df^{q_2-q_1}(1)}\right| \ge K_2\exp[(n-q_2-1)\theta]L_1^{-(q_2-q_1)},$$

and similarly

$$|Df^{q_1}(z_2)|=\left|\frac{Df^{q_2}(z_2)}{Df^{q_2-q_1}(f^{q_1}(z_2))}\right|\ge K_3\exp(q_2\theta)L_1^{-(q_2-q_1)},$$

and, using (H4) and $f^{q_2}(z_2)=0$,

$$|Df^{q_2-q_1-1}(f^{q_1+1}(z_2))|\ge C_1\exp[(q_2-q_1)\theta]. \tag{1}$$

Combining these six bounds, we get

$$\begin{aligned}|Df^n(x)|&=|Df^{q_1}(x)f'(f^{q_1}(x))Df^{q_2-q_1-1}(f^{q_2+1}(x))\\&\quad\cdot f'(f^{q_2}(x))Df^{n-q_2-1}(f^{q_2+1}(x))|\\&\ge K_5\exp(q_1\theta)L_1^{-(q_2-q_1)}\exp[(q_2-q_1-1)\theta]\\&\quad\cdot\exp[(n-q_2-1)\theta]L_1^{-(q_2-q_1)}\cdot|f'(f^{q_1}(x))f'(f^{q_2}(x))|.\end{aligned} \tag{2}$$

Using the condition $|f^p(x)|>\rho_{n-p,\varepsilon}$ and lemma 2.2, we find

$$\begin{aligned}|Df^n(x)|&\ge|K_6\exp(n\theta)K_7^{-(q_2-q_1)}\rho_{n-q_2,\varepsilon}\rho_{n-q_1,\varepsilon}\\&\ge K_8\exp(2n\theta/3)K^{|\log\varepsilon|^{\frac{1}{3}}}\exp(-n\theta/10)\\&\ge\exp(n\theta/2).\end{aligned}$$

This proves (1) for $l=0$. For $l>0$, the result can be read off different variants of eq. (2). The most pessimistic estimate is

$$|Df^{n-l}(f^l(x))|\geq K_9 \exp[(n-l)\theta]K_1^{-2(q_2-q_1)}\rho_{n-l-q_1,\varepsilon}\rho_{n-l-q_2,\varepsilon}$$
$$\geq K_9 \exp[(n-l)\theta]K_{10}^{|\log\varepsilon|^{\frac{1}{3}}}\exp[-(n-l)\theta/10],$$

from which the assertion follows.

Proof of (2) *when* $q_2>n-4|\log\varepsilon|^{\frac{1}{3}}/\theta$. Let $y\in[f^{q_1+1}(z_2),1]$. Since f^n is monotone on $[z_1,z_2]$, $f^{q_2-q_1-1}$ is monotone on $[f^{q_1+1}(z_2),1]$, since $1=f^{q_1+1}(z_2)$. The hypotheses of lemma 2.6 are thus fulfilled for $f^{q_2-q_1-1}$ and hence, using eq. (1) we find

$$|1-y|\leq|1-f^{q_1+1}(z_2)|$$
$$\leq 2|Df^{q_2-q_1-1}(1)Df^{q_2-q_1-1}(f^{q_1+1}(z_2))|^{-\frac{1}{2}}$$
$$\leq K_{11}\exp[-(q_2-q_1)\theta]\leq K_{11}\exp[-(n-q_1)\theta/3],$$

since $q_2-q_1>n-q_2$.

Proof of (2) *when* $q_2\leq n-4|\log\varepsilon|^{\frac{1}{3}}/\theta$. In this case, we note that

$$|f(1)-f^{q_1+2}(x)|\leq|f(1)-f^{q_1+2}(z_2)|$$

and the assertion follows at once from lemma 4.1. □

In the next theorem we use the definitions from § 1.

THEOREM 4.5 (= theorem 1.4(1)). *For* $0\leq\varepsilon<\varepsilon_0$, *we have for all* $y\in\mathbb{R}$,

$$|h^R_{n,\varepsilon}(y)-h^R_{n,\varepsilon}(y+\varepsilon)|\leq C_7\varepsilon^{\frac{1}{10}}[h^R_{n,\varepsilon}(y)+h^R_{n,\varepsilon}(y+\varepsilon)]$$
$$+h^T_{n,\varepsilon}(y)+h^T_{n,\varepsilon}(y+\varepsilon).$$

Proof. The main ingredient of the proof is the following analysis of distances of pre-images. Write $y'=y+\varepsilon$. Consider f^n and let $H=[z_1,z_2]$ be a homterval. Let

$$x=f^{-n}(y)\cap H,\qquad x'=f^{-n}(y')\cap H.$$

(There is at most one x and one x'.) There is now a large variety of possibilities:

(P1) There is no x and no x'. Then the homterval H does not contribute to either $h^R_{n,\varepsilon}(y)$ or $h^R_{n,\varepsilon}(y')$.

(P2) There is an x but no x' (or an x' but no x which is totally analogous). We shall show that $\Omega^T_{n,\varepsilon}(x)=1$, so that this contribution to $h^R_{n,\varepsilon}(y)$ is bounded by the corresponding term in $h^T_{n,\varepsilon}(y)$.

(P3) Both x and x' are present, but $\Omega^R_{n,\varepsilon}(x)=1$ and $\Omega^R_{n,\varepsilon}(x')=0$ (or vice-versa, being handled analogously). Then we show that $\Omega^T_{n,\varepsilon}(x)=\Omega^T_{n,\varepsilon}(x')=1$ and argue as in the case (P2).

(P4) Both x and x' are present and $\Omega^R_{n,\varepsilon}(x)=\Omega^R_{n,\varepsilon}(x')=1$. Then we shall argue that

$$||Df^n(x)|^{-1}-|Df^n(x')|^{-1}|\leq C_7\varepsilon^{\frac{1}{10}}(|Df^n(x)|^{-1}+|Df^n(x')|^{-1}).$$

This clearly proves the proposition.

Analysis of (P2). If $y\in\Omega$ but $y'\notin\Omega$, then either $|y-f(1)|<\varepsilon$ or $|y-1|<\varepsilon$ so that $\Omega^T_{n,\varepsilon}(x)=1$ for all $x\in f^{-n}(y)$. So the assertion is shown in this case. Consider now the homterval $H=[z_1,z_2]$ and assume $y,y'\in\Omega$ and $f^{-n}(y)\cap H=x$ but $f^{-n}(y')\cap H=\varnothing$. Then there is a smallest $n\geq p>0$ such that $f^{-(n-p)}(y)\cap f^p(H)=f^p(x)$ and

$f^{-(n-p)}(y') \cap f^p(H) = U \neq \varnothing$. The set U has the property

$$f^{-1}(U) \cap f^{p-1}(H) = \varnothing,$$

while

$$f^{-1}(f^p(x)) \cap f^{p-1}(H) = f^{p-1}(x).$$

The only way in which this can happen is that $f^p(x)$ lies to the right of $f^2(1)$ while U lies to the left of $f^2(1)$, if $f^2(1) < 0$ (and conversely if $f^2(1) > 0$).

Using the fact that $f^p(H)$ is a homterval, one easily derives that a boundary point of $f^{p-1}(H)$ lies between $f^{p-1}(x)$ and $f(1)$. If this boundary point is $f(1)$ then f^{n-p+1} is monotone on $[f(1), f^{p-1}(x)]$ and hence corollary 4.3 implies $|1 - f^{p-1}(x)| \leq \rho_{n-p,\varepsilon}$, thus $\Omega^T_{n,\varepsilon}(x) = 1$.

Since $f^p(H)$ is a homterval, we see that no point other than $f(1)$ could be a left boundary point of $f^p(H)$. Thus all cases are covered.

Analysis of (P3). Since $\Omega^R_{n,\varepsilon}(x') = 0$, we have $f^p(x') \in G^\varepsilon_{n-p,1}$ for some p and/or $f^n(x') \in E^\varepsilon_{0,1}$ and/or $f^n(x') \in E^\varepsilon_{1,1}$ and/or $f^n(x') \in F^\varepsilon_{0,1}$.

The second and third cases are handled easily, since, e.g. $f^n(x') \in E^\varepsilon_{0,1}$ means $y' \in E^\varepsilon_{0,1}$, and hence $|y - y'| < \varepsilon$ implies $y \in E^\varepsilon_{0,3}$, hence $\Omega^T_{n,\varepsilon}(y) = 1$. If $f^n(x') \in F^\varepsilon_{0,1}$, then

$$|y' - 1| < \exp(-|\log \varepsilon|^{\frac{1}{3}})$$

and hence

$$|y - 1| \leq \varepsilon + |y' - 1| < 3 \exp(-|\log \varepsilon|^{\frac{1}{3}}),$$

i.e. $y \in F^\varepsilon_{0,3}$. So we now assume

$$y' = f^n(x') \notin E^\varepsilon_{0,1} \cup E^\varepsilon_{1,1} \cup F^\varepsilon_{0,1}.$$

Let p be the largest value for which $f^p(x') \in G^\varepsilon_{n-p,1}$ and write $u = f^p(x)$, $u' = f^p(x')$.

Case 1. $n - p \leq 7|\log \varepsilon|^{\frac{1}{2}}$. By (H2), and the theory of Schwarzian derivative, since f^{n-p} has no critical point in $(u; u')$, we have for all $t \in (u; u')$

$$|f^l(t)| \geq \rho_{n-l,\varepsilon} = \exp(-|\log \varepsilon|^{\frac{1}{3}}), \qquad l = 0, \dots, n-p-1.$$

since this condition is fulfilled for u and u' by the choice of p. By lemma 2.2, this implies

$$|f'(f^l(t))| \geq K_1 \exp(-|\log \varepsilon|^{\frac{1}{3}}),$$

and hence

$$|Df^{n-p}(t)| \geq K_2 \exp(-7|\log \varepsilon|^{\frac{1}{3}}|\log \varepsilon|^{\frac{1}{2}}) K_1^{7|\log \varepsilon|^{\frac{1}{2}}}.$$

Thus

$$\begin{aligned} |u - u'| &\leq |y - y'| / \inf_{t \in (u; u')} |Df^{n-p}(t)| \\ &\leq \varepsilon K_3 \exp(|\log \varepsilon|^{\frac{5}{6}}) \leq \exp(-|\log \varepsilon|^{\frac{1}{2}}). \end{aligned}$$

Since $|u' - 1| \leq \rho_{n-p,\varepsilon}$, we find $|u - 1| \leq (\gamma_0 + 1)\rho_{n-p,\varepsilon}$, for a suitable γ_0 and hence $\Omega^T_{n,\varepsilon}(x) = 1$ as asserted.

Case 2. $n - p > 7|\log \varepsilon|^{\frac{1}{2}}$, and f^{n-p} has a critical point z_1 in $(f(1), u)$, (see Figure 3).

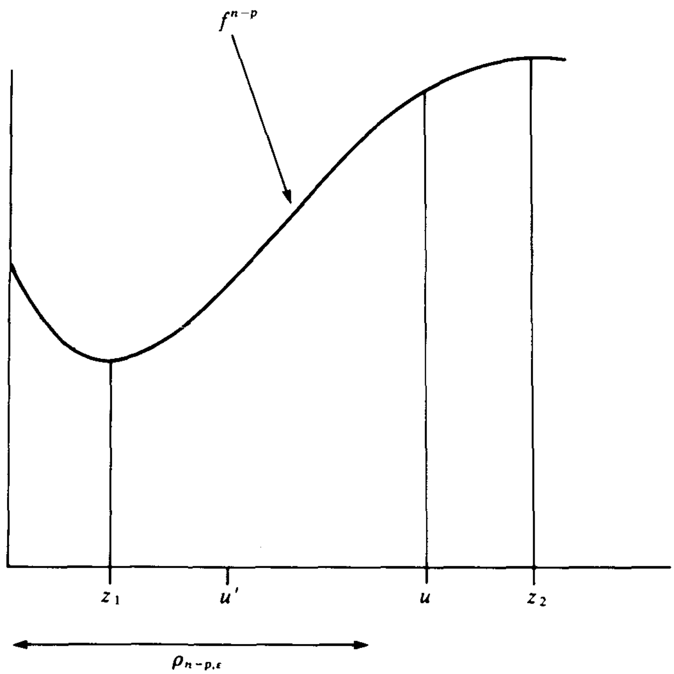

FIGURE 3

Suppose $f^{q_1}(z_1)=f^{q_2}(z_2)=0$, where z_2 is the other end of the homterval of f^{n-p} containing u and u'. We distinguish two subcases. Note that $n-p>q_1, q_2$.

Case 2a. $|q_1-q_2|\le 4|\log\varepsilon|^{\frac{1}{3}}/\theta$. In this case lemma 4.4 applies and we get

$$|Df^{n-p}(t)|\ge \exp[(n-p)\theta/2-C_5|\log\varepsilon|^{\frac{1}{3}}],$$

for all $t\in(u,u')$, hence

$$\begin{aligned}|u-u'| &\le |y-y'|/\inf_{t\in(u;u')}|Df^{n-p}(t)|\\ &\le \varepsilon K_3\exp[C_5|\log\varepsilon|^{\frac{1}{3}}-(n-p)\theta/2]\le\rho_{n-p,\varepsilon}.\end{aligned}$$

Thus we find

$$\begin{aligned}|f(1)-f^p(x)| &= |f(1)-u|\le|f(1)-u'|+|u'-u|\\ &\le(1+\gamma_0)\rho_{n-p,\varepsilon}\end{aligned}$$

i.e. $\Omega^T_{n,\varepsilon}(x)=1$.

Case 2b. $|q_1-q_2|>4|\log\varepsilon|^{\frac{1}{3}}/\theta$. If $n-p-q_1>7|\log\varepsilon|^{\frac{1}{2}}$, then (assuming $q_1<q_2$ for definiteness), we have by lemma 4.4,

$$|f(1)-f^{q_1}(u)|\le C_6\exp[-(n-p-q_1)\theta/3]\le\rho_{n-p-q_1,\varepsilon}.$$

If $n-p-q_1\le 7|\log\varepsilon|^{\frac{1}{2}}$, then from

$$n-p-q_1>q_2-q_1>\frac{4}{\theta}|\log\varepsilon|^{\frac{1}{3}}$$

we find by lemma 4.4

$$|f(1)-f^{q_1}(u)|\le C_6\exp(-|\log\varepsilon|^{\frac{1}{3}})\le\rho_{n-p-q_1,\varepsilon}.$$

Thus $|f(1)-f^{p+q_1}(x)|\le\rho_{n-p-q_1,\varepsilon}$ in both cases and hence

$$\Omega^T_{n,\varepsilon}(x)=1$$

as before.

Case 3. $n-p>7|\log\varepsilon|^{\frac12}$ and f^{n-p} has no critical point in $(f(1),u)$. Then lemma 4.1 implies

$$|f(1)-f^p(u)|<\exp[-(n-p)\theta/3]\le\rho_{n-p,\varepsilon},$$

i.e. $\Omega^T_{n,\varepsilon}(x)=1$.

This completes the analysis of (P3).

Analysis of (P4). It is in this case that strong cancellations occur for $|Df^n(x)|^{-1}-|Df^n(x')|^{-1}$.

Assume first $n<7|\log\varepsilon|^{\frac12}$. For every $t\in(x;x')$ we have, since $\Omega^R_{n,\varepsilon}(x)=\Omega^R_{n,\varepsilon}(x')=1$ and since $(x;x')$ contains no critical point of f^n,

$$|f^l(t)|\ge\rho^{\frac12}_{n-l,\varepsilon}\qquad\text{for } l=1,\dots,n.$$

Hence

$$|Df^{n-l}(t)|\ge K_4^{n-l}\prod_{j=0}^{n-l-1}\rho^{\frac12}_{j,\varepsilon}>\exp[-14(n-l)|\log\varepsilon|^{\frac13}].$$

Thus $|f^l(x)-f^l(x')|\le|y-y'|\exp(14|\log\varepsilon|^{\frac56})\le\varepsilon^{\frac12}$ and

$$\frac{|f^l(x)-f^l(x')|}{|f^l(x)|}\le\varepsilon^{\frac12}\exp(|\log\varepsilon|^{\frac12})\le\varepsilon^{\frac13}. \tag{1}$$

We can now repeat the end of the proof of proposition 2.11:

$$\left|\frac{Df^n(x)}{Df^n(x')}\right|=\prod_{j=0}^{n-1}\left\{1+\left(\frac{f'(f^j(x))}{f'(f^j(x'))}-1\right)\right\}.$$

By lemma 2.3 and (1),

$$\left|\frac{Df^n(x)}{Df^n(x')}\right|=\prod_{j=0}^{n-1}(1+\Delta_j)$$

with $|\Delta_j|\le L_3\varepsilon^{\frac13}$, so that

$$\left|\frac{1}{|Df^n(x)|}-\frac{1}{|Df^n(x')|}\right|=\frac{1}{|Df^n(x)|}\left|1-\frac{|Df^n(x)|}{|Df^n(x')|}\right|$$
$$\le\frac{1}{|Df^n(x)|}(\exp[(1+\gamma_0)\varepsilon^{\frac13}|\log\varepsilon|^{\frac12}]-1),$$

from which the assertion follows.

Next assume $n\ge7|\log\varepsilon|^{\frac12}$. We must have

$$|q_2-q_1|\le\frac{4}{\theta}|\log\varepsilon|^{\frac13},$$

since otherwise lemma 4.4.2 implies $\Omega^R_{n,\varepsilon}(x)=0$ or $\Omega^R_{n,\varepsilon}(x')=0$. But then lemma 4.4.1 implies

$$|Df^{n-l}(f^l(x))|\ge\exp[(n-l)\theta/2-C_5|\log\varepsilon|^{\frac13}],\qquad\text{for } l=0,\dots,q_1.$$

Thus, as before, we find

$$\frac{|f^l(x)-f^l(x')|}{|f^l(x)|} \le \varepsilon \exp[(n-l)\theta/20+|\log\varepsilon|^{\frac{1}{2}}-(n-l)\theta/2+C_5|\log\varepsilon|^{\frac{1}{3}}]$$
$$\le \varepsilon^{\frac{1}{3}} \exp[-(n-l)\theta/3]. \tag{1}$$

for $l=0,\ldots,q_1$. We next show that $n-q_1<7|\log\varepsilon|^{\frac{1}{2}}$. If we assume the contrary, then lemma 4.4.1 implies

$$|Df^{n-q_1-1}(f^{q_1+1}(x))| \ge K_4 \exp[(n-q_1)\theta/2-|\log\varepsilon|^{\frac{1}{3}}C_5]\cdot|f'(f^{q_1}(x))|$$
$$\ge \exp[(n-q_1)\theta/3-|\log\varepsilon|^{\frac{1}{3}}K_5],$$

since $|f^{q_1}(x)|>\rho_{n-q_1,\varepsilon}$. Therefore, by lemma 2.6

$$|1-f^{q_1+1}(x)| \le \frac{|f^{n-q_1-1}(1)-f^n(x)|}{|Df^{n-q_1-1}(1)Df^{n-q_1-1}(f^{q_1+1}(x))|^{\frac{1}{2}}}$$
$$\le \exp(K_6|\log\varepsilon|^{\frac{1}{3}})\exp[-(n-q_1)\theta/3]\le \rho_{n-1-q_1,\varepsilon}.$$

Thus $\Omega^R_{n,\varepsilon}(x)=0$, a contradiction. Now if $l\le q_1$, we have

$$|f^l(x)|\ge \rho_{n-l,\varepsilon}\ge \exp(-|\log\varepsilon|^{\frac{1}{3}}),$$

and

$$|Df^{n-l}(f^l(x))|\ge \exp[-2(n-l)|\log\varepsilon|^{\frac{1}{3}}]\ge \exp[-14|\log\varepsilon|^{\frac{5}{6}}].$$

This implies by lemma 2.6,

$$|f^l(x)-f^l(x')|\le |f^n(x)-f^n(x')|\cdot|Df^{n-l}(f^l(x))Df^{n-l}(f^l(x'))|^{-\frac{1}{2}}$$
$$\le \varepsilon \exp(14|\log\varepsilon|^{\frac{5}{6}}),$$

and hence

$$\frac{|f^l(x)-f^l(x')|}{|f^l(x)|}\le \varepsilon \exp(15|\log\varepsilon|^{\frac{5}{6}}). \tag{2}$$

Combining (1), (2) and corollary 2.4, we get

$$\frac{Df^n(x)}{Df^n(x')}=1+\Delta,$$

with $|\Delta|\le \exp(\varepsilon^{\frac{1}{4}})-1$, from which the assertion follows. □

LEMMA 4.6 (= theorem 4.4.2).

$$\int_{\mathbb{R}} h^S_{n,\varepsilon}(y)\,dy \le \int_{\mathbb{R}} h^T_{n,\varepsilon}(y)\,dy \le \exp(-|\log\varepsilon|^{\frac{1}{4}}).$$

Proof. The first inequality is an obvious consequence of

$$1-\Omega^R_{n,\varepsilon}(x)\le \Omega^T_{n,\varepsilon}(x).$$

Now, as in the proof of theorem 3.4,

$$\int h^T_{n,\varepsilon}(y)\,dy \le \sum_{p=0}^{n}\int_{|f(1)-y|\le(1+\gamma_0)\rho_{n-p,\varepsilon}} h_{n-p,\varepsilon}(y)\,dy$$
$$+\int_{y\in E^\varepsilon_{0,1}\cup F^\varepsilon_{0,1}} h_0(y)\,dy+\int_{y\in E^\varepsilon_{1,1}} h_1(y)\,dy.$$

The last two integrals are bounded by $\varepsilon^{\frac{1}{3}}$, by a direct application of the definition of h_n, for $n=0, 1$.

Since the set $\{y \| f(1)-y| \le (1+y_0)\rho^{\frac{1}{2}}_{n-p,\varepsilon}\}$ is contained in the second image of

$$\mathscr{C} = \{y \| y| \le K_1 \rho^{\frac{1}{2}}_{n-p,\varepsilon}\},$$

we have

$$\int_{|f(1)-y|\le(1+\gamma_0)\rho_{n-p,\varepsilon}} h_{n-p}(y)\,dy \le \int_{\mathscr{C}} h_{n-p-2}(y)\,dy$$
$$\le K_2 \rho^{\tau/80}_{n-p-2,\varepsilon},$$

by theorem 3.4, if $n-p \ge 2$. Hence the assertion follows, since for $j=0, 1, 2$

$$\int_{|f(1)-y|\le(1+\gamma_0)\rho_{j,\varepsilon}} h_j(y)\,dy \le \varepsilon^{\frac{1}{1000}},$$

by inspection. □

Appendix

We show here that, in the case of unimodal maps, the conditions of Misiurewicz' theorem [**7**] imply (H4). His conditions are (H1)–(H3) and

(M) The orbit of the critical point avoids a neighbourhood of the critical point. To be specific, let us assume that $|f^n(1)| > A > 0$ for all $n \ge 0$.

LEMMA A.1. *There is an* $\varepsilon > 0$ *such that for every* $|x| < \varepsilon$ *there is an* $n = n(x)$ *for which*

(i) $|Df^n(x)| > \dfrac{2}{A} \exp(n\theta/3)$;

(ii) $|f^n(x)| > \dfrac{A}{2}$.

Proof. It is obvious from Misiurewicz' work that (H1)–(H3) and (M) imply (H4.1), (see theorem 1.3 in [**7**]). We shall work with a constant $\eta > 0$ to be fixed later. Define $n = n(x)$ to be the smallest integer for which $|f^n(1) - f^n(f(x))| > \eta$. Then, for $0 \le j < n$ we have

$$\frac{|f^j(1) - f^j(f(x))|}{|f^j(1)|} \le \eta/A, \tag{1}$$

and choosing η small, lemma 2.3 implies

$$\exp(-K_1\eta n) \le \frac{Df^{n-1}(f(x))}{Df^{n-1}(1)} \le \exp(K_1\eta n). \tag{2}$$

Moreover, since $|f^{n-1}(1) - f^{n-1}(f(x))| \le \eta$ we have

$$|f^n(x)| > A - K_2\eta > A/2 \qquad \text{(which is (ii))}$$

and thus eq. (2) implies

$$K_3^{-1} \exp(-K_1\eta n) \le \frac{Df^n(f(x))}{Df^n(1)} \le K_3 \exp(K_1\eta n). \tag{3}$$

The argument leading to (3) can be repeated for every $t \in [f(x), 1]$ and we get

$$K_3^{-1} \exp(-K_1\eta n) \le \frac{Df^n(t)}{Df^n(1)} \le K_3 \exp(K_1\eta n). \tag{4}$$

From (4) and (H4.1), we deduce

$$|Df^n(t)| > K_4 \exp[n(\theta - \eta K_1)]. \tag{5}$$

We next show $n = n(x)$ is large. From $|f^n(1) - f^n(f(x))| > \eta$ we deduce

$$|1 - f(x)| \sup_{t \in [f(x), 1]} |Df^n(t)| > \eta,$$

so that (4) implies

$$|1 - f(x)| > K_5 \eta \exp(-K_1 \eta n) |Df^n(1)|^{-1}.$$

By lemma 2.2, we find

$$|x| > \eta^{\frac{1}{2}} K_6 \exp(-K_1 \eta n/2) |Df^n(1)|^{-\frac{1}{2}}. \tag{6}$$

Again by lemma 2.2, $|f'(x)| > K_7 |x|$, so that by (H4.1), (4) and (6),

$$|Df^{n+1}(x)| = |f'(x)| |Df^n(f(x))| \geq \eta^{\frac{1}{2}} K_8 \exp[n(\theta/2 - K_9 \eta)]. \tag{7}$$

On the other hand, $|f'|$ is bounded (by L_1). So we get:

$$\eta \leq |f^n(f(x)) - f^n(1)| \leq \sup_{t \in [f(x), 1]} |Df^n(t)| |1 - f(x)| \leq L_1^n K_{11} \varepsilon^2,$$

and hence

$$n = n(x) > K_{12} \log(\eta/\varepsilon),$$

provided $\eta > \varepsilon > 0$, η is small and ε is small relative to η. Substituting into (7), we can show that for small ε/η

$$\eta^{\frac{1}{2}} K_8 \exp[n(\theta/2 - K_9 \eta)] > \frac{2}{A} \exp(n\theta/3)$$

from which (i) follows. □

LEMMA A.2. *Let ε be defined as in lemma* A.1. *There exists an N_ε and an $\alpha > 1$ such that*

(i) *if $|x| > \varepsilon$ and $|f^n(x)| \leq \varepsilon$ for $n < N_\varepsilon$ then*

$$|Df^n(x)| > A/2;$$

(ii) *if $|x| > \varepsilon$ and $|f^j(x)| > \varepsilon$ for all $j < n$, $n \geq N_\varepsilon$ then*

$$|Df^n(x)| > \alpha^n \frac{A}{2}.$$

Proof. By the aforementioned Misiurewicz' result, there is an N_ε such that if

$$|f^j(x)| > \varepsilon \qquad \text{for every } j \leq N_\varepsilon,$$

then $|Df^{N_\varepsilon}(x)| > \gamma > 1$. Now (ii) is an easy consequence of (i). Write

$$n = qN_\varepsilon + r, \qquad 0 \leq r < N_\varepsilon,$$

then

$$|Df^n(x)| \geq \gamma^q |Df^r(f^{qN_\varepsilon}(x))| \geq \gamma^q A/2.$$

Adapting the constants, the assertion (ii) follows.

We now prove (i). Let z_1 and z_2 be the pre-images of zero of order less than n closest to x, for which $z_1 < x < z_2$. By the conditions (H1)–(H3) we have

$$|Df^n(t)| \leq |Df^n(x)| \qquad \text{for } t \in [z_1, x],$$

(or for $t \in [x, z_2]$, being handled analogously).

On the other hand,

$$A-\varepsilon \le |f^n(z_1)-f^n(x)|$$

$$\le \sup_{t\in[z_1,x]} |Df^n(t)||z_1-x|$$

$$\le |Df^n(x)|,$$

since x and z_1 are on the same side of zero. Therefore $|Df^n(x)|>A/2$. The proof is complete. □

THEOREM A.3. *If f satisfies* (H1)–(H3) *and* (M), *then there are constants* C_A, ε *and* $\rho>1$ *such that for all x for which* $|f^n(x)|<\varepsilon$ *we have*

$$|Df^n(x)|>C_A\rho^n.$$

Remark. This is stronger than (H4.2).

Proof. Define two increasing sequences:

$$0=m_0<n_1<m_1<n_2<m_2<\cdots<n_l<m_l=n$$

with $|f^{n_i}(x)|\le\varepsilon$,

$$|f^j(x)|>\varepsilon \qquad \text{for } m_i<j<n_{i+1},$$

and $m_i-n_i=n(f^{n_i}(x))$ defined by lemma A.1. (The discussion when we 'stop' with n_l is similar.) Now we find

$$|Df^n(x)|=\prod_{j=1}^{l} |Df^{m_j-n_j}(f^{n_j}(x))Df^{n_j-m_{j-1}}(f^{m_{j-1}}(x))|,$$

and each of these factors satisfies, by our previous estimates,

$$\begin{aligned}&|Df^{m_j-n_j}(f^{n_j}(x))Df^{n_j-m_{j-1}}(f^{m_{j-1}}(x))|\\ &\ge\frac{2}{A}\exp[(m_j-n_j)\theta/3]\cdot\begin{cases}\alpha^{n_j-m_{j-1}}A/2 & \text{if } n_j-m_{j-1}\ge N_\varepsilon\\ A/2 & \text{if } n_j-m_{j-1}<N_\varepsilon\end{cases}\\ &\ge\rho^{m_j-m_{j-1}},\end{aligned}$$

where ρ is defined by

$$\rho=\min(\alpha,\exp(\theta/4),\exp[K_{13}/(\log|\eta/\varepsilon|N_\varepsilon)])>1.$$

We have used lemma A.1, from which

$$m_j-n_j>\mathcal{O}|\log\eta/\varepsilon|.$$

This completes the proof. □

This work was supported in part by the Swiss National Science Foundation.

REFERENCES

[1] R. Bowen. Invariant measures for Markov maps of the interval. *Comm. Math. Phys.* **69** (1979), 1–17.
[2] P. Collet & J.-P. Eckmann. On the abundance of aperiodic behaviour for maps on the interval. *Comm. Math. Phys.* **73** (1980), 115–160.
[3] P. Collet and J.-P. Eckmann. *Iterated Maps on the Interval as Dynamical Systems.* Birkhäuser: Basel, Boston, Stuttgart, 1980.
[4] M. Dunford & J. T. Schwartz. *Linear Operators.* Interscience: New York, 1958.

[5] M. Jakobson. Construction of invariant measures absolutely continuous with respect to dx for some maps of the interval. *Comm. Math. Phys.* **81** (1981), 39–95.
[6] A. Lasota & J. A. Yorke. On the existence of invariant measures for piecewise monotonic transformations. *Trans. AMS* **183** (1979), 418–485.
[7] M. Misiurewicz. Absolutely continuous measures for certain maps of an interval. *Publ. Math. IHES* **53** (1981).
[8] D. Ruelle. Applications conservant une mesure absolument continue par rapport à dx sur $[0, 1]$. *Comm. Math. Phys.* **55** (1977), 47–51.
[9] D. Ruelle, Sensitive dependence on initial conditions and turbulent behavior of dynamical systems. In *Bifurcation Theory and its Applications in Scientific Disciplines. New York Acad. of Sci.* **316** (1979).
[10] W. Szlenk. Some dynamical properties of certain differentiable mappings of an interval. I and II. *Preprint IHES M*/80/30, *M*/80/41 (1980), to appear.
[11] S. M. Ulam & J. von Neumann. On combinations of stochastic and deterministic processes. *Bull. Amer. Math. Soc.* **53** (1947), 1120.
[12] P. Walters. Invariant measures and equilibrium states for some mappings which expand distances. *Trans. Amer. Math. Soc.* **236** (1975), 121–153.
[13] K. Yosida. *Functional Analysis.* Springer-Verlag: Berlin, Heidelberg, New York.