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Bicompleteness of the fine quasi-uniformity

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Abstract

A characterization of the topological spaces that possess a bicomplete fine quasi-uniformity is obtained. In particular we show that the fine quasi-uniformity of each sober space, of each first-countable T_1-space and of each quasi-pseudo-metrizable space is bicomplete. Moreover we give examples of T_1-spaces that do not admit a bicomplete quasi-uniformity.

We obtain several conditions under which the semi-continuous quasi-uniformity of a topological space is bicomplete and observe that the well-monotone covering quasi-uniformity of a topological space is bicomplete if and only if the space is quasi-sober.

1. *Introduction*

It is known that the Pervin quasi-uniformity of a topological space X is bicomplete if and only if X is a hereditarily compact quasi-sober space ([**14**], essentially corollary 3·2). Naturally this result suggests the question under which conditions other well-known canonical quasi-uniformities $\mathscr{U}$ (defined on appropriate classes of topological spaces) are bicomplete. In this note we wish to consider this problem in the case that $\mathscr{U}$ is the fine quasi-uniformity or the semi-continuous quasi-uniformity of an arbitrary topological space.

In the first part of this note we prove that the fine quasi-uniformity of each quasi-pseudo-metrizable space and of each sober space is bicomplete. We remark that, on the other hand, while it seems to be unknown whether the fine quasi-uniformity of each quasi-pseudo-metrizable space is complete, it is shown in [**17**] that the fine quasi-uniformity of some well-known normal Hausdorff spaces is not complete (see also [**19**, **21**] for related results). As a by-product of our investigations we get the result that the sobrification of a topological (T_0)-space X can be obtained by constructing the bicompletion of the well-monotone covering quasi-uniformity of X. In [**13**] the analogy between the separated Cauchy completion of a uniform space and the sobrification of a topological space is studied from the point of view of category theory. Our observation may help to explain the similarity between these two constructions from a different point of view.

In the second part of this note we try to determine familiar conditions under which the semi-continuous quasi-uniformity of a topological space is bicomplete. In particular we show that the semi-continuous quasi-uniformity of a sober hereditarily countably metacompact space is bicomplete if and only if the space is hereditarily closed-complete.

We refer the reader to [**8**] for the basic facts about quasi-uniformities and to [**14**] for the basic facts about sober spaces. In particular we will use the following definitions and conventions.

A quasi-uniform space $(X,\mathscr{U})$ is said to be *bicomplete* (see [**8**], p. 61) if the uniformity $\mathscr{U}^*$ is complete. (As usual, $\mathscr{U}^*$ will denote the coarsest uniformity finer than both $\mathscr{U}$ and $\mathscr{U}^{-1}$.)

A non-empty subspace A of a topological space X is called *irreducible* if each pair of non-empty A-open subsets has a non-empty intersection. We will say that a closed irreducible subset of a topological space is *non-trivial* if it is not the closure of a singleton. A topological space is called *quasi-sober* (see e.g. [**14**], p. 154) if it does not have any non-trivial closed irreducible subsets. A quasi-sober T_0-space is called *sober* (see [**14**], p. 145). The *b-topology* ([**24**], p. 38) of a topological space Z is the topology $\mathscr{T}(\mathscr{P}^*)$ where $\mathscr{P}$ denotes the Pervin quasi-uniformity of Z (see [**3**, **14**] and [**22**], p. 238).

All separation axioms used in this paper are explicitly mentioned. In particular we recall that a topological space X is a T_D*-space* if for each point $x \in X$ there exists an open neighbourhood G of x such that $\{x\} = G \cap \mathrm{cl}\{x\}$ (see [**26**], p. 92).

By $\mathbb{N}$ we will denote the set of positive integers. The set of the limit points (cluster points) of a filter $\mathscr{F}$ will be denoted by $\lim \mathscr{F}$ ($\mathrm{adh}\, \mathscr{F}$).

2. *Preliminary results*

The proof of our first proposition is based on two auxiliary results. Since these results seem to be of independent interest, we state them in a form more general than needed in the following.

LEMMA 1. *Let $\mathscr{U}$ be a quasi-uniformity on a set X and let $\mathscr{F}$ be a $\mathscr{U}^*$-Cauchy filter on X.*

(*a*) *Then each $\mathscr{T}(\mathscr{U})$-cluster point of $\mathscr{F}$ is a $\mathscr{T}(\mathscr{U})$-limit point of $\mathscr{F}$.*

(*b*) *If $\mathscr{F}$ is $\mathscr{T}(\mathscr{U}^*)$-convergent to $x \in X$, then $adh_{\mathscr{T}(\mathscr{U})} \mathscr{F} = \mathrm{cl}_{\mathscr{T}(\mathscr{U})}\{x\}$.*

Proof. (*a*) Let $p \in \mathrm{adh}_{\mathscr{T}(\mathscr{U})} \mathscr{F}$. We have to show that $V(p) \in \mathscr{F}$ for each $V \in \mathscr{U}$. There are $W \in \mathscr{U}$ with $W^2 \subseteq V$ and $A \in \mathscr{F}$ with $A \times A \subseteq W \cap W^{-1}$. Since $p \in \mathrm{cl}_{\mathscr{T}(\mathscr{U})} A$, we have an $a \in A \cap W(p)$. Then, for any $x \in A$, we get $(p,x) \in W^2$, whence $x \in V(p)$, so that $A \subseteq V(p)$, and $V(p) \in \mathscr{F}$. (*b*) See [**22**], lemma 1.

LEMMA 2. *Let X be a topological space and let $\mathscr{V}$ be a compatible quasi-uniformity on X that is finer than the Pervin quasi-uniformity $\mathscr{P}$ of X.*

(*a*) *Let $\mathscr{F}$ be a $\mathscr{V}^*$-Cauchy filter on X, and let $x \in X$. Then $\mathscr{F}$ converges to x with respect to the topology $\mathscr{T}(\mathscr{V}^*)$ if and only if x is a $\mathscr{T}(\mathscr{V})$-cluster point of $\mathscr{F}$ and $\mathrm{cl}_{\mathscr{T}(\mathscr{V})}\{x\}$ belongs to $\mathscr{F}$.*

(*b*) *The topologies $\mathscr{T}(\mathscr{V}^*)$ and $\mathscr{T}(\mathscr{P}^*)$ are equal.*

(*c*) *If X is a T_D-space, the quasi-uniformity $\mathscr{V}$ is bicomplete and $\mathscr{W}$ is an arbitrary (possibly not compatible) quasi-uniformity finer than $\mathscr{V}$ on X, then $\mathscr{W}$ is bicomplete, too.*

Proof. (*a*) Assume that $x \in X$ is a $\mathscr{T}(\mathscr{V})$-cluster point of $\mathscr{F}$ such that $\mathrm{cl}_{\mathscr{T}(\mathscr{V})}\{x\}$ belongs to $\mathscr{F}$. Then we have $\{U(x) : U \in \mathscr{V}\} \subseteq \mathscr{F}$ by Lemma 1 (*a*). Clearly

$$\mathrm{cl}_{\mathscr{T}(\mathscr{V})}\{x\} \subseteq U^{-1}(x)$$

whenever $U \in \mathscr{V}$. Since $\mathrm{cl}_{\mathscr{T}(\mathscr{V})}\{x\} \in \mathscr{F}$, we deduce that the filter $\mathscr{F}$ converges to x in

$(X, \mathscr{T}(\mathscr{V}^*))$. (Note that we do not need that $\mathscr{V}$ is finer than $\mathscr{P}$ in this part of the proof.)

On the other hand assume that $\mathscr{F}$ converges to the point $x \in X$ with respect to the topology $\mathscr{T}(\mathscr{V}^*)$. Clearly x is a $\mathscr{T}(\mathscr{V})$-cluster point of $\mathscr{F}$. Since

$$V = (\mathrm{cl}_{\mathscr{T}(\mathscr{V})}\{x\} \times X) \cup ((X \backslash \mathrm{cl}_{\mathscr{T}(\mathscr{V})}\{x\}) \times (X \backslash \mathrm{cl}_{\mathscr{T}(\mathscr{V})}\{x\})) \in \mathscr{P} \subseteq \mathscr{V},$$

we see that $V^{-1}(x) = \mathrm{cl}_{\mathscr{T}(\mathscr{V})}\{x\}$ belongs to $\mathscr{F}$.

(*b*) This is well known and obvious.

(*c*) We have $\mathscr{T}(\mathscr{V}^*) = \mathscr{T}(\mathscr{W}^*)$, because $\mathscr{T}(\mathscr{V}^*)$ is discrete (see [**3**], proposition 4·1). Hence $\mathscr{W}^*$ is complete, because $\mathscr{V}^* \subseteq \mathscr{W}^*$ and $\mathscr{V}^*$ is complete.

We recall that a family $\mathscr{L}$ of subsets of a topological space X is called *well-monotone* (see [**16**], p. 20) provided that the partial order $\subseteq$ of set inclusion is a well-order on $\mathscr{L}$. The compatible quasi-uniformity $\mathscr{M}$ on a topological space X which has as a subbase the set of all binary relations that are associated with well-monotone open covers of X is called the *well-monotone covering* quasi-uniformity of X ([**16**], p. 21). (We will say that a binary relation T is *associated* with a well-monotone open cover $\mathscr{H}$ of X if $T = \bigcup\{\{x\} \times (\bigcap\{G : x \in G \in \mathscr{H}\}) : x \in X\}$.)

We note that it is easy to see that the restriction of $\mathscr{M}$ to an arbitrary subspace Y of X is the well-monotone covering quasi-uniformity of Y and that $\mathscr{M}$ is finer than the Pervin quasi-uniformity of X. Our first proposition collects some further useful observations on the well-monotone covering quasi-uniformity. They will be crucial in the following.

Proposition 1. *Let $(X, \mathscr{S})$ be a topological space and let $\mathscr{W}$ be a compatible quasi-uniformity on X that is finer than the well-monotone covering quasi-uniformity $\mathscr{M}$ of X.*

(*a*) *Let $\mathscr{F}$ be a $\mathscr{W}^*$-Cauchy filter on X. Then $\mathrm{adh}_{\mathscr{S}}\mathscr{F} \in \mathscr{F}$ and $\mathrm{adh}_{\mathscr{S}}\mathscr{F}$ is $\mathscr{S}$-irreducible. The filter $\mathscr{F}$ converges to some point $x_0 \in X$ with respect to the topology $\mathscr{T}(\mathscr{W}^*)$ on X if and only if $\mathrm{adh}_{\mathscr{S}}\mathscr{F} = \mathrm{cl}_{\mathscr{S}}\{x_0\}$.*

(*b*) *Let $\mathscr{H}$ be a minimal $\mathscr{W}^*$-Cauchy filter on X. Then $\mathscr{H}$ is the filter generated by the filterbase $\{G \cap \mathrm{adh}_{\mathscr{S}}\mathscr{H} : G \cap \mathrm{adh}_{\mathscr{S}}\mathscr{H} \neq \emptyset$ and $G \in \mathscr{S}\}$ on X.*

(*c*) *Let F be a closed irreducible subset of X and let $\mathscr{G}$ be the filter generated by the filter base $\{G \cap F : G \cap F \neq \emptyset$ and $G \in \mathscr{S}\}$ on X. Then $\mathscr{G}$ is a $\mathscr{W}^*$-Cauchy filter on X if and only if for each $V \in \mathscr{W}$ there exists an $x \in F$ such that $F \subseteq V^{-1}(x)$. In particular $\mathscr{G}$ is an $\mathscr{M}^*$-Cauchy filter on X.*

Proof. (*a*) Consider an arbitrary $\mathscr{W}^*$-Cauchy filter $\mathscr{F}$ on X. Assume that $\mathrm{adh}_{\mathscr{S}}\mathscr{F} \notin \mathscr{F}$. Then there exists a minimal (infinite) cardinal number m so that there is a subcollection $\mathscr{E}$ of $\mathscr{F}$ consisting of $\mathscr{S}$-closed subsets of X such that card $(\mathscr{E}) = m$ and $\bigcap \mathscr{E} \notin \mathscr{F}$. We can assume that $\mathscr{E} = \{F_\alpha : \alpha < m\}$. For each $\beta < m$ let $E_\beta = \bigcap\{F_\alpha : \alpha < \beta\}$. (In particular let $E_0 = X$.) Set

$$\mathscr{C} = \{X \backslash E_\beta : \beta < m\} \cup \{X\} \quad \text{and} \quad T(x) = \bigcap\{D : x \in D \in \mathscr{C}\}$$

whenever $x \in X$. We have $T = \bigcup\{\{x\} \times T(x) : x \in X\} \in \mathscr{W}$ and $E_\beta \in \mathscr{F}$ for each $\beta < m$. Since $\mathscr{F}$ is a $\mathscr{W}^*$-Cauchy filter on X, there is an $x \in X$ such that $(T \cap T^{-1})(x) \in \mathscr{F}$. If $x \in \bigcap \mathscr{E}$, then $T^{-1}(x) = \bigcap \mathscr{E} \in \mathscr{F}$, a contradiction. Therefore $x \notin \bigcap \mathscr{E}$. Hence there is a $\beta < m$ such that $x \in (X \backslash E_\beta)$. Then $T(x) \subseteq (X \backslash E_\beta)$ and $(X \backslash E_\beta) \cap E_\beta \in \mathscr{F}$. We have reached another contradiction and we conclude that $\mathrm{adh}_{\mathscr{S}}\mathscr{F} \in \mathscr{F}$.

Since $\mathrm{adh}_{\mathscr{S}}\mathscr{F} \in \mathscr{F}$, it follows from Lemma 1(*a*) that $\mathrm{adh}_{\mathscr{S}}\mathscr{F}$ is $\mathscr{S}$-irreducible.

Assume that $\mathrm{adh}_{\mathscr{S}}\mathscr{F} = \mathrm{cl}_{\mathscr{S}}\{x_0\}$. Then $\mathrm{cl}_{\mathscr{S}}\{x_0\} \in \mathscr{F}$. Hence the filter $\mathscr{F}$ converges to x_0 with respect to the topology $\mathscr{T}(\mathscr{W}^*)$ by Lemma 2(*a*). The converse follows from Lemma 1(*b*).

(*b*) Let $\mathscr{H}$ be a minimal $\mathscr{W}^*$-Cauchy filter (cf. [**8**], proposition 3·30) on X. By part (*a*), $\mathrm{adh}_{\mathscr{S}}\mathscr{H} \in \mathscr{H}$ and $\mathrm{adh}_{\mathscr{S}}\mathscr{H}$ is a closed irreducible subset of $(X, \mathscr{S})$. Let $\mathscr{D}$ be the filter generated by the filter base

$$\{G \cap \mathrm{adh}_{\mathscr{S}}\mathscr{H} : G \cap \mathrm{adh}_{\mathscr{S}}\mathscr{H} \neq \varnothing \text{ and } G \in \mathscr{S}\}$$

on X. Consider an arbitrary $G \in \mathscr{S}$. Since $((X\backslash G) \times X) \cup (G \times G) \in \mathscr{W}$ and $\mathscr{H}$ is a $\mathscr{W}^*$-Cauchy filter on X, we conclude that $G \in \mathscr{H}$ or $X\backslash G \in \mathscr{H}$. Hence $\mathscr{D} \subseteq \mathscr{H}$. Let $U \in \mathscr{W}$. There exist an $\mathscr{S}$-open neighbournet V ([**8**], p. 4) such that $V \subseteq U$ and an $H \in \mathscr{W}$ such that $H^2 \subseteq V$. Since $\mathscr{H}$ is a $\mathscr{W}^*$-Cauchy filter on X, there exists an $x \in X$ such that $(H \cap H^{-1})(x) \in \mathscr{H}$. Clearly we have that

$$\mathrm{adh}_{\mathscr{S}}\mathscr{H} \subseteq \mathrm{cl}_{\mathscr{S}}H^{-1}(x) \subseteq V^{-1}(x).$$

Moreover $\mathrm{adh}_{\mathscr{S}}\mathscr{H} \cap V(x) \neq \varnothing$, because $\mathrm{adh}_{\mathscr{S}}\mathscr{H} \in \mathscr{H}$ and $H(x) \in \mathscr{H}$. Hence

$$(U \cap U^{-1})(x) \in \mathscr{D}$$

by the definition of $\mathscr{D}$. We conclude that $\mathscr{D}$ is a $\mathscr{W}^*$-Cauchy filter on X. Therefore $\mathscr{D} = \mathscr{H}$.

(*c*) Before giving the proof of this assertion we note that in the stated criterion it suffices to assume that $\mathscr{W}$ is a compatible quasi-uniformity on X.

Assume that $\mathscr{G}$ is a $\mathscr{W}^*$-Cauchy filter on X. Let $V \in \mathscr{W}$ and let $W \in \mathscr{W}$ be such that $W^3 \subseteq V$. Since $\mathscr{G}$ is a $\mathscr{W}^*$-Cauchy filter on X, there exists an $x \in X$ such that $(W \cap W^{-1})(x) \in \mathscr{G}$. Choose $y \in (W \cap W^{-1})(x) \cap F$. Then

$$(W \cap W^{-1})(x) \subseteq (W^2 \cap W^{-2})(y) \in \mathscr{G}$$

and

$$F = \mathrm{adh}_{\mathscr{S}}\mathscr{G} \subseteq \mathrm{cl}_{\mathscr{S}}W^{-2}(y) \subseteq W^{-3}(y) \subseteq V^{-1}(y).$$

Hence F satisfies the condition stated in the proposition. In order to prove the converse, assume that F satisfies this condition and let $V \in \mathscr{W}$. By our assumption there is an $x \in F$ such that $F \subseteq V^{-1}(x)$. Since $((\mathrm{int}_{\mathscr{S}} V(x)) \cap V^{-1}(x)) \in \mathscr{G}$ by the definition of $\mathscr{G}$, we see that $\mathscr{G}$ is a $\mathscr{W}^*$-Cauchy filter on X.

To show that $\mathscr{G}$ is an $\mathscr{M}^*$-Cauchy filter, we use the criterion just established. Given $V \in \mathscr{M}$, we have finitely many well-monotone open covers $\mathscr{A}_1, \ldots, \mathscr{A}_n$ of X such that $\bigcap\{T_j : 1 \leqslant j \leqslant n\} \subseteq V$, where

$$T_j = \bigcup\{\{x\} \times (\bigcap\{G : x \in G \in \mathscr{A}_j\}) : x \in X\}.$$

Let A_j be the least member of $\mathscr{A}_j$ which intersects F. Since F is irreducible, there exists an $x_0 \in F \cap A_1 \cap \ldots \cap A_n$. It follows that $F \subseteq T_j^{-1}(x_0)$ for each j. Thus $F \subseteq V^{-1}(x_0)$, and $\mathscr{G}$ is an $\mathscr{M}^*$-Cauchy filter on X.

COROLLARY 1. *A topological space admits a bicomplete quasi-uniformity if and only if its fine quasi-uniformity is bicomplete.*

Proof. Let $(X, \mathscr{S})$ be a topological space admitting a bicomplete quasi-uniformity $\mathscr{U}$. We have to show that its fine quasi-uniformity $\mathscr{FN}$ is bicomplete. Let $\mathscr{F}$

be an $\mathscr{F}\mathscr{N}^*$-Cauchy filter on X. Then $\mathscr{F}$ is a $\mathscr{U}^*$-Cauchy filter on X. Moreover $\text{adh}_{\mathscr{S}}\mathscr{F} = \text{cl}_{\mathscr{S}}\{x\}$ for some $x \in X$ by Lemma 1 (*b*), since $\mathscr{U}$ is bicomplete. By Proposition 1 (*a*) $\mathscr{F}$ converges on X with respect to the topology $\mathscr{T}(\mathscr{F}\mathscr{N}^*)$. Thus $\mathscr{F}\mathscr{N}$ is bicomplete.

We are now ready to formulate our characterization of the topological spaces that have a bicomplete fine quasi-uniformity. Let us note that the topological T_0-spaces admitting a bicomplete *totally bounded* quasi-uniformity are characterized in [**22**] as the strongly sober locally compact spaces.

PROPOSITION 2. *Let $(X, \mathscr{S})$ be a topological space and let $\mathscr{W}$ be a compatible quasi-uniformity on X that is finer than the well-monotone covering quasi-uniformity of X. Then $\mathscr{W}$ is* not *bicomplete if and only if X has a non-trivial closed irreducible subset F such that for each entourage $V \in \mathscr{W}$ there is an $x \in F$ satisfying $F \subseteq V^{-1}(x)$.*

Proof. Assume that $\mathscr{W}$ is not bicomplete. Then there exists a minimal $\mathscr{W}^*$-Cauchy filter $\mathscr{F}$ on X without $\mathscr{T}(\mathscr{W}^*)$-limit point in X. By Proposition 1 (*a*), $\text{adh}_{\mathscr{S}}\mathscr{F}$ is a non-trivial closed irreducible subset of $(X, \mathscr{S})$. Moreover, by Proposition 1 (*b*) and 1 (*c*), for each entourage $V \in \mathscr{W}$ there is an $x \in \text{adh}_{\mathscr{S}}\mathscr{F}$ satisfying $\text{adh}_{\mathscr{S}}\mathscr{F} \subseteq V^{-1}(x)$.

On the other hand, if X has a non-trivial closed irreducible subset F such that for any entourage $V \in \mathscr{W}$ there is an $x \in F$ satisfying $F \subseteq V^{-1}(x)$, then by Proposition 1 (*c*) and 1 (*a*) there exists a $\mathscr{W}^*$-Cauchy filter on X without $\mathscr{T}(\mathscr{W}^*)$-limit point in X. Hence $\mathscr{W}$ is not bicomplete.

As an application of Proposition 2 we wish to establish three results mentioned next in the Abstract. (Note that, of course, each Hausdorff space is quasi-sober.)

PROPOSITION 3. *The fine quasi-uniformity of each quasi-pseudo-metrizable space is bicomplete.*

Proof. Let $(X, \mathscr{S})$ be a topological space the topology of which is induced by a quasi-pseudo-metric d on X. Assume that the fine quasi-uniformity $\mathscr{F}\mathscr{N}$ of X is not bicomplete. By Proposition 2 there exists a non-trivial closed irreducible subset F of X such that whenever $V \in \mathscr{F}\mathscr{N}$ there is an $x \in F$ satisfying $F \subseteq V^{-1}(x)$. Let $n \in \mathbb{N}$. Set

$$V_n = \{(x, y) \in X \times X : d(x, y) < 2^{-n}\}.$$

Since $V_n \in \mathscr{F}\mathscr{N}$, we can choose an $x_n \in F$ such that $F \subseteq V_n^{-1}(x_n)$. Assume first that

$$\bigcap\{V_n(x_n) : n \in \mathbb{N}\} \cap F = \varnothing.$$

Let

$$\mathscr{C} = \{(X \backslash F) \cup (\bigcap\{V_k(x_k) : k = 1, \ldots, n\}) : n \in \mathbb{N}\} \cup \{X\}$$

and $T(x) = \bigcap\{D : x \in D \in \mathscr{C}\}$ for each $x \in X$. Then

$$T = \bigcup\{\{x\} \times T(x) : x \in X\} \in \mathscr{F}\mathscr{N}$$

by our assumption. Using that $\bigcap\{V_k(x_k) : k = 1, \ldots, n\} \cap F \neq \varnothing$ whenever $n \in \mathbb{N}$, we note that there does not exist any $x \in F$ such that $F \subseteq T^{-1}(x)$, a contradiction. We conclude that our last assumption was incorrect and that we can find a $y \in \bigcap\{V_n(x_n) : n \in \mathbb{N}\} \cap F$. Then

$$F \subseteq V_n^{-1}(x_n) \subseteq V_n^{-2}(y) \subseteq V_{n-1}^{-1}(y)$$

for each $n \in (\mathbb{N}\setminus\{1\})$. Thus we see that

$$y \in \mathrm{cl}_{\mathcal{S}}\{y\} \subseteq F \subseteq \bigcap\{V_n^{-1}(y) : n \in \mathbb{N}\} = \mathrm{cl}_{\mathcal{S}}\{y\}.$$

Hence F is not a non-trivial closed irreducible subset of X. We have reached another contradiction. We deduce that the fine quasi-uniformity of X is bicomplete.

Problem. We remark that in the proof of Proposition 3 we have only used that $(X, \mathcal{S})$ has a normal (see [**8**], p. 5) sequence $(V_n)_{n\in\mathbb{N}}$ of (open) neighbournets such that $\mathrm{cl}_{\mathcal{S}}\{x\} = \bigcap\{V_n^{-1}(x) : n \in \mathbb{N}\}$ whenever $x \in X$. Let us also note that a slight modification of the proof given above shows that the fine transitive quasi-uniformity of each non-archimedeanly quasi-pseudo-metrizable space is bicomplete. The value of this result is not clear however, since it is unknown (cf. [**8**], problem P, p. 155) whether the fine transitive quasi-uniformity and the fine quasi-uniformity of each non-archimedeanly quasi-pseudo-metrizable space are equal. Furthermore we do not know whether the fine transitive quasi-uniformity of a topological space is bicomplete whenever its fine quasi-uniformity is bicomplete.

COROLLARY 2. *The fine transitive quasi-uniformity of a topological space X is bicomplete if and only if each closed irreducible subspace F of X that is not a point-closure has an (in the subspace F) interior-preserving collection of non-empty F-open sets with an empty intersection.*

Proof. Suppose that the fine transitive quasi-uniformity $\mathcal{FT}$ of X is bicomplete. Let F be an arbitrary closed irreducible subspace of X that is not a point-closure. By Proposition 2 there exists a transitive entourage $T \in \mathcal{FT}$ such that $F \nsubseteq T^{-1}(x)$ for all $x \in F$. Set

$$\mathcal{C} = \{T(x) \cap F : x \in F\}.$$

Then $\mathcal{C}$ is an (in F) interior-preserving collection of non-empty F-open sets with an empty intersection.

In order to prove the converse assume that each closed irreducible subspace F of X which is not a point-closure has an (in F) interior-preserving collection $\mathcal{C}_F$ of non-empty F-open sets with an empty intersection. Set

$$\mathcal{D}_F = \{G \cup (X\setminus F) : G \in \mathcal{C}_F\} \cup \{X\} \quad \text{and} \quad T = \bigcup\{\{x\} \times (\bigcap\{H : x \in H \in \mathcal{D}_F\}) : x \in X\}.$$

Then T belongs to the fine transitive quasi-uniformity $\mathcal{FT}$ of X (see [**8**], corollary 2·6), but there does not exist an $x \in F$ such that $F \subseteq T^{-1}(x)$. By Proposition 2 we conclude that $\mathcal{FT}$ is bicomplete.

COROLLARY 3. *The fine transitive quasi-uniformity of any countable space or any first-countable T_1-space is bicomplete.*

Proof. In such a space each closed irreducible subspace F that is not a point-closure has a sequence $(G_n)_{n\in N}$ of non-empty F-open sets with an empty intersection.

PROPOSITION 4. *Let X be a topological space and let $\mathcal{W}$ be a compatible quasi-uniformity on X that is finer than the well-monotone covering quasi-uniformity $\mathcal{M}$ of X.*

(*a*) *If X is quasi-sober, then $(X, \mathcal{W})$ is bicomplete.*

(*b*) *The quasi-uniform space $(X, \mathcal{M})$ is bicomplete if and only if X is quasi-sober.*

Proof. (*a*) Since X is quasi-sober, it does not have any non-trivial closed irreducible subsets. The result follows from Proposition 2.

(*b*) Assume that $(X, \mathscr{M})$ is bicomplete. Let F be a closed irreducible subset of X. By Proposition 1 (*c*), for each $V \in \mathscr{M}$ there exists an $x \in F$ such that $F \subseteq V^{-1}(x)$. Hence F is not non-trivial by Proposition 2, since $\mathscr{M}$ is bicomplete. We conclude that X is quasi-sober. The converse follows from part (*a*). The result just proved will be studied further in the next section.

3. *The sobrification as bicompletion*

In this section we show that the sobrification of a topological (T_0)-space X can be obtained by constructing the bicompletion of the well-monotone covering quasi-uniformity of X. We also try to understand why, in general, a compatible quasi-uniformity $\mathscr{U}$ on a topological T_0-space X cannot be extended to a compatible quasi-uniformity on the sobrification of X, although such an extension is always possible if $\mathscr{U}$ is totally bounded (see [**22**], lemma 6).

In the following the bicompletion of a quasi-uniform T_0-space $(X, \mathscr{U})$ will be denoted by $(\tilde{X}, \tilde{\mathscr{U}})$. The construction is described in detail in [**8**], chapter 3·2. We will use the same notation as in [**8**], theorem 3·33. By $\tilde{X}$ we denote the set of the minimal $\mathscr{U}^*$-Cauchy filters on X. (We observe that this notation does not show that $\tilde{X}$ depends on $\mathscr{U}$. The reader is warned that, given two quasi-uniformities $\mathscr{U}$ and $\mathscr{V}$ on a set X, the ground sets of the quasi-uniform spaces $(\tilde{X}, \tilde{\mathscr{U}})$ and $(\tilde{X}, \tilde{\mathscr{V}})$ may be different.) Moreover $\tilde{\mathscr{U}}$ will denote the quasi-uniformity on $\tilde{X}$ that is generated by all the sets $\tilde{U}$ where U belongs to $\mathscr{U}$. Recall that

$$\tilde{U} = \{(\mathscr{F}, \mathscr{G}) \in \tilde{X} \times \tilde{X} : \text{there exist } F \in \mathscr{F} \text{ and } G \in \mathscr{G} \text{ such that } F \times G \subseteq U\}$$

whenever $U \in \mathscr{U}$. As usual we will sometimes identify a $\mathscr{T}(\mathscr{U}^*)$-convergent minimal $\mathscr{U}^*$-Cauchy filter $\mathscr{F} \in \tilde{X}$ with its limit point in $(X, \mathscr{U}^*)$ and, using this identification, think of $(X, \mathscr{U})$ as a subspace of $(\tilde{X}, \tilde{\mathscr{U}})$ (see [**8**], theorem 3·33).

Let us still observe that the theory of the bicompletion of a quasi-uniform space is used in [**18**] to study the so-called Fell compactification of a locally compact topological space.

A construction of the sobrification $({}^sX, \mathscr{B})$ of a topological space X is given in [**14**], p. 145. For our purpose it will suffice to know that ${}^sX$ is the set of the closed irreducible subsets of X, that the topology $\mathscr{B}$ on ${}^sX$ is equal to

$$\{[G] : G \text{ is open in } X\}$$

where $[G] = \{F \in {}^sX : F \cap G \neq \emptyset\}$ whenever G is an open subset of X, and that the map $f: X \to {}^sX$ defined by $f(x) = \mathrm{cl}\{x\}$ for each $x \in X$ is a topological embedding onto a b-dense subspace of ${}^sX$ provided that X is a T_0-space.

Proposition 4 suggests that one should look for connections between the two constructions under consideration (see also [**13**]).

PROPOSITION 5. *Let $(X, \mathscr{S})$ be a topological T_0-space and let $\mathscr{W}$ be a compatible quasi-uniformity on X that is finer than the well-monotone covering quasi-uniformity $\mathscr{M}$ of X. Then $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{W}}))$ is a subspace of the sobrification of X. In particular $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{M}}))$ is the sobrification of X.*

Proof. Let $\tilde{X}$ be the set of the minimal $\mathscr{W}^*$-Cauchy filters on X and let ${}^sX$ be the set of the closed irreducible subsets of X. Set $h(\mathscr{F}) = \mathrm{adh}_{\mathscr{S}} \mathscr{F}$ for each $\mathscr{F} \in \tilde{X}$.

Obviously the map $h:\tilde{X}\to {}^sX$ is well-defined and one-to-one by Proposition 1(*a*) and 1(*b*).

Next we show that h is onto if $\mathscr{W}=\mathscr{M}$. Indeed, let F be a closed irreducible subset of X and let $\mathscr{G}$ be the filter on X generated by $\{G\cap F: G\cap F\neq\varnothing \text{ and } G\in\mathscr{S}\}$. By Proposition 1(*c*) $\mathscr{G}$ is an $\mathscr{M}^*$-Cauchy filter on X. Clearly $\mathrm{adh}_{\mathscr{S}}\mathscr{G}=F$. Let $\mathscr{H}$ be the minimal $\mathscr{M}^*$-Cauchy filter on X contained in $\mathscr{G}$ (see [**8**], proposition 3·30). Since $\mathscr{H}\subseteq\mathscr{G}$, we have

$$\lim_{\mathscr{S}}\mathscr{H}\subseteq\lim_{\mathscr{S}}\mathscr{G}\subseteq\mathrm{adh}_{\mathscr{S}}\mathscr{G}\subseteq\mathrm{adh}_{\mathscr{S}}\mathscr{H}.$$

Applying Lemma 1(*a*) to $\mathscr{H}$ we obtain $\mathrm{adh}_{\mathscr{S}}\mathscr{H}=F$. Hence h is surjective.

Observe that the minimal $\mathscr{W}^*$-Cauchy filter representing the point x of X in $\tilde{X}$ corresponds to that point of ${}^sX$ that is represented by the set $\mathrm{cl}_{\mathscr{S}}\{x\}$. Therefore it is clear that the set $\tilde{X}$ of the minimal $\mathscr{W}^*$-Cauchy filters can be considered a subset of the sobrification of X. For each $G\in\mathscr{S}$ set

$$[G]=\{\mathscr{F}\in\tilde{X}:\mathrm{adh}_{\mathscr{S}}\mathscr{F}\cap G\neq\varnothing\}.$$

Of course in view of Proposition 1(*a*) we have $[X]=\tilde{X}$ and $[G_1\cap G_2]=[G_1]\cap[G_2]$ whenever $G_1,G_2\in\mathscr{S}$.

It remains to show that the two topologies $\mathscr{D}=\{[G]:G\in\mathscr{S}\}$ and $\mathscr{T}(\tilde{\mathscr{W}})$ on $\tilde{X}$ are equal. Let $\mathscr{F}\in\tilde{X}$ and $\mathscr{F}\in[G]$ for some $G\in\mathscr{S}$. Note that $G\in\mathscr{F}$ by Proposition 1(*b*). Set

$$H=((X\backslash G)\times X)\cup(G\times G).$$

Then $H\in\mathscr{W}$. Moreover $\tilde{H}(\mathscr{F})\subseteq[G]$, because whenever $\mathscr{G}\in\tilde{H}(\mathscr{F})$, then $G\in\mathscr{G}$ by the definition of $\tilde{H}$ and, thus, $\mathscr{G}\in[G]$ by Proposition 1(*b*). Hence $\mathscr{D}\subseteq\mathscr{T}(\tilde{\mathscr{W}})$.

On the other hand let $\mathscr{F}\in\tilde{X}$ and $V\in\mathscr{W}$. Choose an open neighbournet H (see [**8**], p. 4) belonging to $\mathscr{W}$ such that $H^2\subseteq V$. Since $\mathscr{F}$ is a $\mathscr{W}^*$-Cauchy filter on X, there is an $x\in X$ such that $(H^{-1}\cap H)(x)\in\mathscr{F}$. By Proposition 1(*b*) we have $\mathscr{F}\in[H(x)]$. Let $\mathscr{G}\in[H(x)]$. Since $H(x)\in\mathscr{G}$ by Proposition 1(*b*) and since $(H^{-1}(x)\times H(x))\subseteq V$, it is clear that $\mathscr{G}\in\tilde{V}(\mathscr{F})$. Hence $\mathscr{F}\in[H(x)]\subseteq\tilde{V}(\mathscr{F})$. It follows that $\mathscr{D}$ and $\mathscr{T}(\tilde{\mathscr{W}})$ are equal. We have shown that we can think of $(\tilde{X},\mathscr{T}(\tilde{\mathscr{W}}))$ as a subspace of the sobrification of X. In particular, if $\mathscr{W}=\mathscr{M}$ then we can think of $(\tilde{X},\mathscr{T}(\tilde{\mathscr{M}}))$ as the sobrification of X.

Remark 1. It is well-known that a topological space is hereditarily compact if and only if each strictly increasing sequence of open sets is finite (see e.g. [**25**], theorem 1). Hence a topological space is hereditarily compact if and only if its well-monotone covering quasi-uniformity coincides with its Pervin quasi-uniformity (compare [**8**], propositions 2·7 and 2·8). Therefore the last statement of Proposition 5 generalizes that part of theorem 3 in [**22**] considerably which says that if $\mathscr{P}$ is the Pervin quasi-uniformity of a hereditarily compact T_0-space X, then $(\tilde{X},\mathscr{T}(\tilde{\mathscr{P}}))$ is the sobrification of X.

It is well-known that the Pervin quasi-uniformity is the finest compatible totally bounded quasi-uniformity on a topological space (see [**8**], p. 28). Thus on an arbitrary topological space X each compatible totally bounded quasi-uniformity $\mathscr{V}$ is coarser than the well-monotone covering quasi-uniformity of X. It follows from Proposition 1(*c*) that each of the filters of the form as defined in Proposition 1(*c*) is a $\mathscr{V}^*$-Cauchy filter on X. Using this fact one can give another proof of the result (established in [**22**], corollary on p. 239) that whenever $\mathscr{V}$ is a compatible totally bounded quasi-uniformity on a T_0-space X, then the space $(\tilde{X}, \mathcal{T}(\tilde{\mathcal{V}}))$ contains the sobrification of X as a subspace.

In order to get more information about the structure of the quasi-uniformity $\tilde{\mathcal{W}}$ mentioned in Proposition 5 we need some further preparation.

Let X be a b-dense subspace of a topological space Y and let G be an open subset of X. The unique open set H of Y such that $H \cap X = G$ will be denoted by $[G]$ in the following proof.

LEMMA 3. *Let X be a b-dense subspace of a topological space $(Y, \mathcal{S})$.*

(a) Let H_2, H_1, V_2 and V_1 be open neighbournets of Y such that

$$H_2^2 \subseteq H_1, H_1 \cap (X \times X) \subseteq V_2 \cap (X \times X)$$

and $V_2^2 \subseteq V_1$. Then $H_2 \subseteq V_1$.

(b) If S is a transitive neighbournet of X, then there is at most one transitive neighbournet T of Y such that $S = T \cap (X \times X)$.

(c) Let $\mathcal{U}$ and $\mathcal{V}$ be two compatible quasi-uniformities on Y such that the restriction $\mathcal{U}|X$ is coarser than the restriction $\mathcal{V}|X$. Then $\mathcal{U} \subseteq \mathcal{V}$.

Proof. (a) Let $y \in Y$. There exists an $x \in V_2(y) \cap \mathrm{cl}_{\mathcal{S}}\{y\} \cap X$, because X is b-dense in Y. Since $x \in \mathrm{cl}_{\mathcal{S}}\{y\} \subseteq H_2^{-1}(y)$, we have

$$H_2(y) \subseteq H_2^2(x) \subseteq H_1(x) = [H_1(x) \cap X] \subseteq [V_2(x) \cap X] = V_2(x) \subseteq V_2^2(y) \subseteq V_1(y).$$

Thus $H_2 \subseteq V_1$.

(b) Let T_1 and T_2 be transitive neighbournets of Y such that $T_1 \cap (X \times X) = T_2 \cap (X \times X)$. Set

$$H_2 = H_1 = T_2 \quad \text{and} \quad V_2 = V_1 = T_1.$$

Then $T_2 \subseteq T_1$ by (a) and thus $T_2 = T_1$ by symmetry.

(c) Let $V \in \mathcal{U}$. There exist open neighbournets $V_1, V_2 \in \mathcal{U}$ such that $V_1 \subseteq V$ and $V_2^2 \subseteq V_1$. By our assumption $V_2 \cap (X \times X) \in \mathcal{V}|X$. Hence there exist open neighbournets $H_1, H_2 \in \mathcal{V}$ such that

$$H_1 \cap (X \times X) \subseteq V_2 \cap (X \times X) \quad \text{and} \quad H_2^2 \subseteq H_1.$$

By (a) we have $H_2 \subseteq V_1$. Therefore $V \in \mathcal{V}$ and, thus, $\mathcal{U} \subseteq \mathcal{V}$.

Lemma 3(c) shows that if X is a b-dense subspace of a topological space Y and $\mathcal{U}$ is a compatible quasi-uniformity on X, then there exists at most one compatible quasi-uniformity $\mathcal{V}$ on Y such that $\mathcal{V}|X = \mathcal{U}$. The following result answers the natural question under which condition such an extension $\mathcal{V}$ of $\mathcal{U}$ from X to Y exists.

PROPOSITION 6. *Let X be a b-dense subspace of a topological space $(Y, \mathcal{S})$ and let $\mathcal{U}$ be a compatible quasi-uniformity on the subspace X of Y. Then $\mathcal{U}$ can be extended to a compatible quasi-uniformity $\mathcal{V}$ on Y if and only if for each $y \in Y$ and each $U \in \mathcal{U}$ there exists an $x \in \mathrm{cl}_{\mathcal{S}}\{y\} \cap X$ such that $(\mathrm{cl}_{\mathcal{S}}\{y\} \cap X) \subseteq U^{-1}(x)$.*

We leave the proof of this result to the reader, since we will not use the result in the following and since the argument is quite similar to the proof of a closely related (see [**4**]) result on quasi-pseudo-metrics that we wish to present instead.

We will call a quasi-pseudo-metric p on a topological space $(X, \mathcal{S})$ *admissible* if

$B_n^p(x) \in \mathscr{S}$ whenever $n \in \mathbb{N}$ and $x \in X$. (Here, as usual, $B_n^p(x) = \{y \in X : p(x,y) < 2^{-n}\}$.) Note that the fine quasi-uniformity of a topological space X is the supremum quasi-uniformity of all quasi-pseudo-metric quasi-uniformities induced by bounded admissible quasi-pseudo-metrics on X (see [**4**], p. 157 and [**8**], p. 3).

LEMMA 4. *Let X be a b-dense subspace of a topological space $(Y, \mathscr{S})$ and let p be a bounded admissible (compatible) quasi-pseudo-metric on X. Then p can be extended to an admissible (compatible) quasi-pseudo-metric q on Y if and only if for each $y \in Y$ and each $n \in \mathbb{N}$ there exists an $x \in \mathrm{cl}_{\mathscr{S}}\{y\} \cap X$ such that $p(f,x) < 2^{-n}$ whenever $f \in \mathrm{cl}_{\mathscr{S}}\{y\} \cap X$. If such an admissible extension of p to Y exists, then it is unique.*

Proof. For each $x, y \in Y$ set

$$r(x,y) = \sup\{p(x', \mathrm{cl}_{\mathscr{S}}\{y\} \cap X) : x' \in \mathrm{cl}_{\mathscr{S}}\{x\} \cap X\}.$$

(Here, as usual,

$$p(x', \mathrm{cl}_{\mathscr{S}}\{y\} \cap X) = \inf\{p(x', y') : y' \in \mathrm{cl}_{\mathscr{S}}\{y\} \cap X\}.)$$

It is easy to check that r is a quasi-pseudo-metric on Y (cf. [1], p. 337) and that $r(x,y) = p(x,y)$ for each $x, y \in X$. However, in general r is not admissible on Y, even if p is compatible on X (see Example 2(*a*) below).

Assume now that p can be extended to an admissible quasi-pseudo-metric q on Y. We will show that $q = r$. Let $x, y \in Y$. Consider an arbitrary $n \in \mathbb{N}$. Since X is b-dense in Y, there is a $c \in B_n^q(y) \cap \mathrm{cl}_{\mathscr{S}}\{y\} \cap X$. Furthermore by the definition of r there exists an $a \in \mathrm{cl}_{\mathscr{S}}\{x\} \cap X$ such that $r(x,y) \leqslant p(a, \mathrm{cl}_{\mathscr{S}}\{y\} \cap X) + 2^{-n}$. Hence

$$\begin{aligned} r(x,y) \leqslant p(a,c) + 2^{-n} = q(a,c) + 2^{-n} &\leqslant q(a,x) + q(x,y) + q(y,c) + 2^{-n} \\ &\leqslant 0 + q(x,y) + 2^{-n+1}. \end{aligned}$$

Therefore $r(x,y) \leqslant q(x,y)$. Assume for a moment that there is an $m \in \mathbb{N}$ such that $2^{-m+1} + r(x,y) < q(x,y)$. Since X is b-dense in Y, there is a $d \in B_m^q(x) \cap \mathrm{cl}_{\mathscr{S}}\{x\} \cap X$. Moreover it is clear that there is an $e \in \mathrm{cl}_{\mathscr{S}}\{y\} \cap X$ such that

$$p(d,e) \leqslant 2^{-m} + p(d, \mathrm{cl}_{\mathscr{S}}\{y\} \cap X).$$

Thus

$$\begin{aligned} q(x,y) &\leqslant q(x,d) + q(d,e) + q(e,y) \leqslant 2^{-m} + p(d,e) + 0 \leqslant 2^{-m+1} + p(d, \mathrm{cl}_{\mathscr{S}}\{y\} \cap X) \\ &\leqslant 2^{-m+1} + r(x,y) < q(x,y), \end{aligned}$$

a contradiction. Hence $q(x,y) = r(x,y)$. We conclude that $q = r$. It follows that if p can be extended to an admissible quasi-pseudo-metric on Y, then such an extension is unique.

Next we want to show that the condition formulated in the lemma is satisfied whenever p can be extended to an admissible quasi-pseudo-metric q on Y. Let $y \in Y$ and $n \in \mathbb{N}$. Since X is b-dense in Y, there exists an $x \in B_n^q(y) \cap \mathrm{cl}_{\mathscr{S}}\{y\} \cap X$. Consider an arbitrary $f \in \mathrm{cl}_{\mathscr{S}}\{y\} \cap X$. Then

$$p(f,x) = q(f,x) \leqslant q(f,y) + q(y,x) = q(y,x) < 2^{-n}.$$

This proves the assertion.

In order to prove the converse, assume that p is an admissible quasi-pseudo-metric on X satisfying the condition stated in the lemma. Define a quasi-pseudo-metric r on

Y as above. We wish to show that r is admissible on Y. Let $y \in Y$ and $n \in \mathbb{N}$. By our assumption there is an $x \in \mathrm{cl}_{\mathscr{S}}\{y\} \cap X$ such that $x \in B^p_{n+2}(f)$ whenever $f \in \mathrm{cl}_{\mathscr{S}}\{y\} \cap X$. Let G be the unique open subset of Y such that $G \cap X = B^p_{n+2}(x)$. Note that $y \in G$, because $x \in \mathrm{cl}_{\mathscr{S}}\{y\} \cap G$. Consider an arbitrary $z \in G$ and choose an $a \in \mathrm{cl}_{\mathscr{S}}\{z\} \cap G \cap X$. Then $p(x,a) < 2^{-(n+2)}$, and therefore $p(f,a) < 2^{-(n+1)}$ whenever $f \in \mathrm{cl}_{\mathscr{S}}\{y\} \cap X$. Hence

$$r(y,z) = \sup\{p(f, \mathrm{cl}_{\mathscr{S}}\{z\} \cap X): f \in \mathrm{cl}_{\mathscr{S}}\{y\} \cap X\} \leqslant \sup\{p(f,a): f \in \mathrm{cl}_{\mathscr{S}}\{y\} \cap X\} < 2^{-n}$$

and $z \in B^r_n(y)$. We conclude that $y \in G \subseteq B^r_n(y)$. It follows that $B^r_m(c)$ is open in Y whenever $c \in Y$ and $m \in \mathbb{N}$. We have shown that r is admissible on Y.

Finally we prove that if p is compatible on X and r (defined as above) is admissible on Y, then r is compatible on Y. To this end let $y \in Y$ and let G be an arbitrary open neighbourhood of y in Y. Choose $z \in \mathrm{cl}_{\mathscr{S}}\{y\} \cap G \cap X$. Then there exists an $n \in \mathbb{N}$ such that

$$B^p_n(z) = B^r_n(z) \cap X \subseteq G \cap X.$$

We deduce that $y \in B^r_n(z) \subseteq G$, since r is admissible on Y and X is b-dense in Y. It follows that r is compatible on Y.

We note that these results help to explain the condition appearing in Proposition 2 and Proposition 1 (c). In particular Lemma 4 confirms the natural conjecture that for any T_0-space X (with fine quasi-uniformity $\mathscr{F}\mathscr{N}$) the set of the points of the sobrification ${}^sX$ of X to which all bounded admissible quasi-pseudo-metrics on X can be extended in an admissible way is exactly the set of the points of ${}^sX$ corresponding to the minimal $\mathscr{F}\mathscr{N}^*$-Cauchy filters on X. (In order to verify our assertion use the ideas of Propositions 1 and 5 and note that $\{\mathrm{cl}\{y\} \cap X : y \in {}^sX\}$ is exactly the set of the closed irreducible subsets of X.)

Our next two results should be compared with theorem 3 of [**22**] where the corresponding question is studied for the Pervin quasi-uniformity.

COROLLARY 4. *Let $\mathscr{M}$ be the well-monotone covering quasi-uniformity of a topological T_0-space X. Then $\tilde{\mathscr{M}}$ is the well-monotone covering quasi-uniformity of the topological space $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{M}}))$.*

Proof. Let $\mathscr{N}$ denote the well-monotone covering quasi-uniformity of $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{M}}))$. Then $\mathscr{N}|X = \mathscr{M} = \tilde{\mathscr{M}}|X$. Moreover (e.g. by [**13**], section 3·1·1 (b2)) X is a b-dense subspace of its sobrification $(\tilde{X}, \mathscr{T}(\tilde{\mathscr{M}}))$ (see Proposition 5). From Lemma 3 (c) it follows that $\mathscr{N} = \tilde{\mathscr{M}}$.

COROLLARY 5. *Let X be a topological T_0-space and let $\mathscr{F}\mathscr{N}$ be its fine quasi-uniformity. Then $\widetilde{\mathscr{F}\mathscr{N}}$ is the fine quasi-uniformity of the space $(\tilde{X}, \mathscr{T}(\widetilde{\mathscr{F}\mathscr{N}}))$.*

Proof. Let $\mathscr{F}\mathscr{E}$ denote the fine quasi-uniformity of $(\tilde{X}, \mathscr{T}(\widetilde{\mathscr{F}\mathscr{N}}))$. Clearly $\widetilde{\mathscr{F}\mathscr{N}} \subseteq \mathscr{F}\mathscr{E}$. On the other hand we have

$$\mathscr{F}\mathscr{E}|X \subseteq \mathscr{F}\mathscr{N} = \widetilde{\mathscr{F}\mathscr{N}}|X.$$

Since X is b-dense in $(\tilde{X}, \mathscr{T}(\widetilde{\mathscr{F}\mathscr{N}}))$ by Proposition 5 (see section 3·1·1 (b2) of [**13**]), it follows that $\mathscr{F}\mathscr{E} \subseteq \widetilde{\mathscr{F}\mathscr{N}}$ by Lemma 3 (c). We conclude that $\mathscr{F}\mathscr{E} = \widetilde{\mathscr{F}\mathscr{N}}$.

Remark 2 (compare e.g. [**14**]). It is easy to see that each continuous map $f: X \to Y$ from a topological space X to a topological space Y is quasi-uniformly continuous with respect to the well-monotone covering quasi-uniformities $\mathscr{M}(X)$ of X and $\mathscr{M}(Y)$ of Y. Hence in case that X and Y are T_0-spaces there exists a unique quasi-uniformly continuous extension $\tilde{f}:(\tilde{X}, \widetilde{\mathscr{M}(X)}) \to (\tilde{Y}, \widetilde{\mathscr{M}(Y)})$ of f (see [**8**], theorem 3·29 and proposition 1·14). By Corollary 4 and Proposition 5 it is clear that this only means that $\tilde{f}$ is a continuous map from the sobrification of X to the sobrification of Y (extending f).

4. *Examples*

This short section is devoted to the study and construction of several examples illustrating the results presented so far.

Example 1. Each Hausdorff space that does not admit a complete quasi-uniformity (such examples are given in [**17**, **19**]) shows that a bicomplete fine quasi-uniformity need not be complete.

Example 2. (*a*) Let $\alpha \neq 0$ be a limit ordinal and let $\mathscr{T} = \{\varnothing\} \cup \{[x,a) : x \in \alpha\}$. Set $d(x,y) = 0$ if $x, y \in \alpha$ and $x \leqslant y$, and set $d(x,y) = 1$ if $x,y \in \alpha$ and $y < x$. Then the compatible quasi-pseudo-metric d on the space $(\alpha, \mathscr{T})$ cannot be extended to an admissible quasi-pseudo-metric on the sobrification of $(\alpha, \mathscr{T})$. (Note that by Lemma 4 the quasi-pseudo-metric d cannot be extended in an admissible way to the point β of the sobrification of $(\alpha, \mathscr{T})$ corresponding to the closed irreducible subset α of α, because we have $\mathrm{cl}\{\beta\} \cap \alpha = \alpha$, but $\sup\{d(f,x) : f \in \alpha\} = 1$ for each $x \in \alpha$.) We observe that the fine quasi-uniformity $\mathscr{F}\mathscr{N}$ of $(\alpha, \mathscr{T})$ is bicomplete, since $\mathscr{F}\mathscr{N}^*$ is discrete; indeed, $\mathscr{F}\mathscr{N}$ is generated by the one basic entourage $\{(x,y) \in \alpha \times \alpha : x \leqslant y\}$.

(*b*) Let X be a countably infinite set equipped with the cofinite topology. Then the fine quasi-uniformity of X is bicomplete, but X is not quasi-sober (see [**22**], example 1).

PROPOSITION 7. (*a*) *A topological space X admits only bicomplete quasi-uniformities if and only if X is quasi-sober and hereditarily compact.* (*b*) *The unique compatible quasi-uniformity* (*namely the Pervin quasi-uniformity*) *of a topological space X admitting a unique quasi-uniformity is bicomplete if and only if X is quasi-sober.*

Proof. The two assertions follow immediately from the following three known facts. The Pervin quasi-uniformity of a topological space is bicomplete if and only if the space is hereditarily compact and quasi-sober ([**14**], essentially corollary 3·2), each hereditarily compact quasi-sober space admits a unique quasi-uniformity ([**22**], proposition 3(*a*)), and each topological space admitting a unique quasi-uniformity is hereditarily compact (see [**8**], theorem 2·36 or [**20**]).

In view of Proposition 7 each topological space that admits a unique quasi-uniformity, but is not quasi-sober (e.g. the cofinite topology on an uncountable set (see [**8**], example 2·37 or [20]) is an example of a topological space that does not admit a bicomplete quasi-uniformity, although the unique quasi-uniformity that it admits is complete ([**8**], p. 50).

Before giving a more general method to construct topological spaces the fine quasi-uniformity of which is not bicomplete, we record the following fact. Essentially it shows that it suffices to consider irreducible topological spaces in the remaining paragraphs of this section.

Remark 3. A topological space X has a bicomplete fine quasi-uniformity if and only if each closed irreducible subspace of X has this property.

Proof. Let $\mathscr{FN}$ be the fine quasi-uniformity of the topological space X. Assume that $\mathscr{FN}$ is bicomplete. Then each b-closed (i.e. $\mathscr{T}(\mathscr{FN}^*)$-closed) subspace of the uniform space $(X, \mathscr{FN}^*)$ is complete. Since $\mathscr{T}(\mathscr{FN}) \subseteq \mathscr{T}(\mathscr{FN}^*)$, we deduce that each closed irreducible subspace F of X admits the bicomplete quasi-uniformity $\mathscr{FN}|F$. Hence F has a bicomplete fine quasi-uniformity by Corollary 1. (In fact the fine quasi-uniformity of the closed subspace F of X is $\mathscr{FN}|F$ by [**8**], chart on p. 55.)

In order to prove the converse, assume that each closed irreducible subspace of X has a bicomplete fine quasi-uniformity. Suppose that F is a non-trivial closed irreducible subspace of X. (Note that we are done if X does not have any subspace of this kind, since then the fine quasi-uniformity $\mathscr{FN}$ of X is bicomplete by Proposition 2.) Applying Proposition 2 to the subspace F of X, we see that there is an entourage V belonging to the fine quasi-uniformity of the subspace F of X such that for each $x \in F$ we have $F \nsubseteq V^{-1}(x)$. Set $W = V \cup (X \times (X \backslash F))$. Then $W \in \mathscr{FN}$ (by [**8**], proposition 2·19) and $F \nsubseteq W^{-1}(x)$ whenever $x \in F$. We conclude by Proposition 2 that the fine quasi-uniformity $\mathscr{FN}$ of X is bicomplete.

Example 3. We describe a class of topological spaces that do not admit a bicomplete quasi-uniformity. Let m be an infinite cardinal number and let X be a set such that card $(X) > m$. Set $\mathscr{S} = \{X \backslash A : \text{card } (A) < m\} \cup \{\varnothing\}$. It follows from the next lemma that the fine quasi-uniformity of the topological space $(X, \mathscr{S})$ is not bicomplete.

LEMMA 5. *Let $(X, \mathscr{S})$ be a topological T_1-space without isolated points whose density is strictly smaller than the minimal number of closed nowhere dense subsets of X needed to cover X (e.g. take the usual topology on the reals or any other separable T_1-space without isolated points that is non-meagre in itself). Then the set X equipped with the T_1-topology*

$$\mathscr{R} = \{G \in \mathscr{S} : G \text{ is } \mathscr{S}\text{-dense in } X\} \cup \{\varnothing\}$$

does not have a bicomplete fine quasi-uniformity.

Proof. Let $\mathscr{FN}$ be the fine quasi-uniformity of $(X, \mathscr{R})$ and let $V, W \in \mathscr{FN}$ be such that $W^2 \subseteq V$. Choose an $\mathscr{S}$-dense subset D of X of minimal cardinality. We have $\bigcup\{W^{-1}(d) : d \in D\} = X$, since D is $\mathscr{R}$-dense in X. If $\text{cl}_{\mathscr{R}}\, W^{-1}(d) \neq X$ whenever $d \in D$, then for each $d \in D$ the set $\text{cl}_{\mathscr{R}}\, W^{-1}(d)$ is a closed nowhere dense subset of $(X, \mathscr{S})$. However this contradicts our assumption about $(X, \mathscr{S})$. We conclude that there is an $e \in D$ such that

$$X = \text{cl}_{\mathscr{R}}\, W^{-1}(e) \subseteq W^{-2}(e) \subseteq V^{-1}(e).$$

Since $(X, \mathscr{R})$ is a T_1-space, the assertion follows by applying Proposition 2 to the closed irreducible subset X of $(X, \mathscr{R})$.

5. *Bicompleteness of the semi-continuous quasi-uniformity*

Since a completely regular Hausdorff space X in which each closed discrete subspace has non-measurable cardinality admits a complete fine uniformity if and only if the uniformity $\mathscr{C}(X)$ is complete (see [**10**], theorem 15·20 and corollary 15·14), it seems natural to look for canonical compatible quasi-uniformities (on appropriate classes of topological spaces) the bicompleteness of which is equivalent to the bicompleteness of the fine quasi-uniformity. In this connection we note that if $\mathscr{U}$ and $\mathscr{V}$ are two quasi-uniformities on a given topological space X satisfying $\mathscr{U} \subseteq \mathscr{V}$ and both belonging to the Pervin quasi-proximity class of X, then the uniformity $\mathscr{V}^*$ is complete whenever the uniformity $\mathscr{U}^*$ is complete, because the topologies $\mathscr{T}(\mathscr{U}^*)$ and $\mathscr{T}(\mathscr{V}^*)$ are equal (e.g. we can argue that the fine quasi-uniformity of a quasi-sober topological space is bicomplete, since its well-monotone covering quasi-uniformity is bicomplete). In particular, because of the results on uniformities cited above, it seems natural to wonder under which conditions the semi-continuous quasi-uniformity of a topological space is bicomplete.

The contents of this section can be summarized as follows. First, before starting our investigations of the semi-continuous quasi-uniformity, we describe an example indicating that the general problem mentioned in the beginning of this section may be rather involved. Then, after recalling some pertinent facts and definitions, we try to characterize those topological spaces that possess a bicomplete semi-continuous quasi-uniformity. Several examples illustrate our main results, which we formulate in Propositions 9 and 10.

It may be helpful to recall already now the following simple consequence of Lemma 2(*c*): if $(X, \mathscr{S})$ is a T_D-space the semi-continuous quasi-uniformity of which is bicomplete and $\mathscr{S}'$ is a topology finer than $\mathscr{S}$ on X, then the semi-continuous quasi-uniformity of the space $(X, \mathscr{S}')$ is also bicomplete.

Example 4. Let κ be an uncountable regular cardinal number (equipped with the order topology $\mathscr{S}$) and let $\mathscr{E}_\kappa$ be the supremum quasi-uniformity of all quasi-pseudo-metric quasi-uniformities $\mathscr{U}_p$ on κ such that p is an admissible quasi-pseudo-metric on κ and the induced pseudo-metric p^* on κ generates a topology with density strictly smaller than κ. (Here, as usual, p^* is defined by

$$p^*(x,y) = \max\{p(x,y), p(y,x)\}$$

for all $x, y \in \kappa$.) Clearly $\mathscr{E}_\kappa$ is a compatible quasi-uniformity on κ. We want to show that $\mathscr{E}_\kappa$ is not bicomplete.

Let $\mathscr{F}$ be the filter on κ generated by the closed unbounded subsets of κ and let V be an arbitrary open neighbournet of κ belonging to $\mathscr{E}_\kappa$. Choose $W \in \mathscr{E}_\kappa$ such that $W^2 \subseteq V$. By the definition of $\mathscr{E}_\kappa$ there are a $\beta < \kappa$ and a cover $\{A_\alpha : \alpha < \beta\}$ of κ such that $A_\alpha \times A_\alpha \subseteq W$ whenever $\alpha < \beta$. Since the intersection of less than κ closed unbounded subsets of κ is closed unbounded (by [**15**], lemma 7·4), there is a $\gamma < \beta$ such that A_γ is stationary (i.e. intersects each closed unbounded subset of κ). Choose an $x \in A_\gamma$. Since

$$A_\gamma \subseteq W(x) \subseteq V(x),$$

the set $\kappa \setminus V(x)$ is bounded. Hence there is a $\delta \in \kappa$ such that

$$\{\epsilon : \delta \leqslant \epsilon < \kappa\} \subseteq V(x).$$

Since $\{\epsilon : \delta \leqslant \epsilon < \kappa\}$ is $\mathscr{S}$-closed, we have $V(x) \in \mathscr{F}$. Moreover $V^{-1}(x) \in \mathscr{F}$, because

$$\mathrm{cl}_{\mathscr{S}} A_\gamma \subseteq W^{-2}(x) \subseteq V^{-1}(x)$$

and A_γ is unbounded. Hence $\mathscr{F}$ is an $\mathscr{E}_\kappa^*$-Cauchy filter on κ. Since $\mathscr{F}$ does not have any $\mathscr{S}$-cluster point in κ, we conclude that $\mathscr{E}_\kappa$ is not bicomplete.

Let $\mathcal{Q}$ be the quasi-uniformity on the set $\mathbb{R}$ of real numbers generated by the base $\{Q_\epsilon : \epsilon > 0\}$ where

$$Q_\epsilon = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x - y < \epsilon\}$$

whenever $\epsilon > 0$. By definition, the *semi-continuous quasi-uniformity* $\mathscr{SC}$ of a topological space X is the coarsest quasi-uniformity on X for which each continuous function $f: X \to (\mathbb{R}, \mathcal{Q})$ is quasi-uniformly continuous (see [**8**], p. 32). In our context a different description of $\mathscr{SC}$ seems to be more convenient. Recall that an *open spectrum* $\mathscr{A}$ in a topological space X is a family $\{A_n : n \in \mathbb{Z}\}$ of open sets of X (indexed by the set $\mathbb{Z}$ of integers) such that for each $n \in \mathbb{Z}$ we have

$$A_n \subseteq A_{n+1}, \quad \bigcap\{A_n : n \in \mathbb{Z}\} = \varnothing \quad \text{and} \quad \bigcup\{A_n : n \in \mathbb{Z}\} = X$$

(see [**8**], p. 33). We will say that $U_{\mathscr{A}} = \bigcup\{(A_n \backslash A_{n-1}) \times A_n : n \in \mathbb{Z}\}$ is the *neighbournet associated with the open spectrum* $\mathscr{A}$ in X. It is known that the semi-continuous quasi-uniformity $\mathscr{SC}$ of a topological space X is the (compatible) quasi-uniformity on X generated by all neighbournets associated with open spectra in X (see [**8**], theorem 2·12). Clearly $\mathscr{SC}$ is finer than the Pervin quasi-uniformity of X. The following simple observation will be useful in this section.

LEMMA 6. *Let $\mathscr{SC}$ be the semi-continuous quasi-uniformity of a topological space X. Then each $\mathscr{SC}^*$-Cauchy filter $\mathscr{F}$ on X has the properties that*

(*i*) *each countable sequence of open elements of $\mathscr{F}$ has a non-empty intersection, and*

(*ii*) *each countable sequence of closed elements of $\mathscr{F}$ has a non-empty intersection.*

Proof. Consider an arbitrary $\mathscr{SC}^*$-Cauchy filter $\mathscr{F}$ on X. Let $(G_n)_{n \in \mathbb{N}}$ be a sequence of open elements of $\mathscr{F}$ and let $(F_n)_{n \in \mathbb{N}}$ be a sequence of closed elements of $\mathscr{F}$.

Assume that $\bigcap\{G_n : n \in \mathbb{N}\} = \varnothing$. Set $G'_{-n} = \bigcap\{G_k : k = 1, \ldots, n\}$ whenever $n \in \mathbb{N}$ and $G'_n = X$ whenever $n \in \mathbb{N} \cup \{0\}$. Then $\mathscr{A} = \{G'_n : n \in \mathbb{Z}\}$ is an open spectrum in X. Clearly for the neighbournet $T \in \mathscr{SC}$ associated with $\mathscr{A}$ there does not exist any x in X such that $T^{-1}(x)$ belongs to $\mathscr{F}$, a contradiction.

Assume that $\bigcap\{F_n : n \in \mathbb{N}\} = \varnothing$. Set $G'_n = X \backslash \bigcap\{F_k : k = 1, \ldots n\}$ whenever $n \in \mathbb{N}$ and $G'_{-n} = \varnothing$ whenever $n \in \mathbb{N} \cup \{0\}$. Then $\mathscr{A} = \{G'_n : n \in \mathbb{Z}\}$ is an open spectrum in X so that for the neighbournet $T \in \mathscr{SC}$ associated with $\mathscr{A}$ there does not exist any x in X such that $T(x) \in \mathscr{F}$, a contradiction. This proves the assertion.

Next we use Lemma 6 to characterize the topological spaces that have a bicomplete semi-continuous quasi-uniformity. Proposition 8 should be compared with the characterization of the topological spaces having a bicomplete fine transitive quasi-uniformity given in Corollary 2.

PROPOSITION 8. *Let $(X, \mathscr{T})$ be a topological space and let $\mathscr{A}$ be the smallest field of sets containing $\mathscr{T}$ and contained in the power set of X. Then the semi-continuous quasi-uniformity $\mathscr{SC}$ of X is bicomplete if and only if each maximal filter $\mathscr{G}$ over the field $\mathscr{A}$ which satisfies*

(*i*) *each countable sequence of open elements of $\mathscr{G}$ has a non-empty intersection, and*

(*ii*) *each countable sequence of closed elements of $\mathscr{G}$ has a non-empty intersection, is fixed* (*i.e. has a non-empty intersection*).

Proof. (Since Proposition 8 does not seem to simplify our arguments essentially, we will not use the result in the following. Hence we only outline its proof.) The assertion is an immediate consequence of the following three facts the proofs of which are left to the reader.

Let $\mathscr{G}$ be a maximal filter over the field $\mathscr{A}$ with the properties that (i) each countable sequence of open elements of $\mathscr{G}$ has a non-empty intersection and that (ii) each countable sequence of closed elements of $\mathscr{G}$ has a non-empty intersection. Then the filter generated by the filter base $\mathscr{G}$ on X is a minimal $\mathscr{S}\mathscr{C}^*$-Cauchy filter on X.

Let $\mathscr{F}$ be an $\mathscr{S}\mathscr{C}^*$-Cauchy filter on X. Then $\mathscr{F} \cap \mathscr{A}$ is a maximal filter over the field $\mathscr{A}$ with the properties that (i) each countable sequence of open elements of $\mathscr{F} \cap \mathscr{A}$ has a non-empty intersection and that (ii) each countable sequence of closed elements of $\mathscr{F} \cap \mathscr{A}$ has a non-empty intersection.

Let $\mathscr{G}$ be a maximal filter over $\mathscr{A}$. Then the filter base $\mathscr{G}$ on X converges with respect to the b-topology $\mathscr{T}(\mathscr{S}\mathscr{C}^*)$ if and only if $\bigcap \mathscr{G} \neq \varnothing$.

We will call a topological space X a *quasi-fc-space* (cf. [**23**], p. 71) if the point-closures are the only (non-empty) closed irreducible subsets of X with the *FCI-property*. A subset $A \subseteq X$ is said to have the *FCI*-property if an open filter $\mathscr{G}$ on X satisfies $G \cap A \neq \varnothing$ for all $G \in \mathscr{G}$ only if $\mathscr{G}$ has the countable intersection property. Of course each quasi-sober space is a quasi-*fc*-space. An easy consequence of Corollary 2 is the following observation.

COROLLARY 6. *The fine transitive quasi-uniformity of each quasi-fc-space is bicomplete.*

Proof. Let X be a quasi-*fc*-space and let F be a non-trivial closed irreducible subspace of X. Since X is a quasi-*fc*-space, F does not have the *FCI*-property. Hence there is an open filter $\mathscr{G}$ on X satisfying $G \cap F \neq \varnothing$ for each $G \in \mathscr{G}$, but containing a countable collection $\{S_n : n \in \mathbb{N}\}$ with an empty intersection. Considering the collection $\{(\bigcap_{k=1}^{n} S_k) \cap F : n \in \mathbb{N}\}$ we see by Corollary 2 that the fine transitive quasi-uniformity of X is bicomplete.

Before stating our next proposition we still recall that a topological space X is said to be *closed-complete* (= a-real-compact [**2**, **5**]) if each maximal closed filter on X with the countable intersection property has a non-empty intersection.

PROPOSITION 9. *A topological space X whose semi-continuous quasi-uniformity $\mathscr{S}\mathscr{C}$ is bicomplete is a quasi-fc-space and hereditarily closed-complete.*

Proof. Let A be a non-empty subspace of X, let $\mathscr{F}$ be a maximal closed filter with the countable intersection property on the subspace A of X and let $\mathscr{G}$ be the filter generated by the filter base $\mathscr{F}$ on X. We wish to show that $\mathscr{G}$ is an $\mathscr{S}\mathscr{C}^*$-Cauchy filter on X. Let $\mathscr{A} = \{G_n : n \in \mathbb{Z}\}$ be an open spectrum in X and let T denote the transitive neighbournet of X associated with $\mathscr{A}$. Observe first that there exists an $n \in \mathbb{Z}$ such that $(A \backslash G_n) \notin \mathscr{F}$, because $\mathscr{F}$ has the countable intersection property and $\mathscr{A}$ is an open spectrum in X. Note then that for each $n \in \mathbb{Z}$ such that $A \backslash G_n \notin \mathscr{F}$ there is an $F_n \in \mathscr{F}$ such that $F_n \subseteq A \cap G_n$, because $\mathscr{F}$ is a maximal closed filter on the subspace A of X. In particular we see that there is a minimal $n_0 \in \mathbb{Z}$ such that $A \backslash G_{n_0} \notin \mathscr{F}$, since $\mathscr{F}$ has the countable intersection property and $\mathscr{A}$ is an open spectrum in X. Choose an $x \in (A \backslash G_{n_0-1}) \cap G_{n_0}$. It follows that

$$(X \backslash G_{n_0-1}) \cap G_{n_0} = (T^{-1}(x) \cap T(x)) \in \mathscr{G},$$

because $A \backslash G_{n_0-1} \in \mathscr{F}$ and $F \subseteq A \cap G_{n_0}$ for some $F \in \mathscr{F}$. Since $(T^{-1}(x) \times T(x)) \subseteq T$, we can conclude that for an arbitrary entourage $U \in \mathscr{S}\mathscr{C}$ there exists an $M \in \mathscr{G}$ such that $M \times M \subseteq U$. Hence $\mathscr{G}$ is an $\mathscr{S}\mathscr{C}^*$-Cauchy filter on X. Since $\mathscr{S}\mathscr{C}$ is bicomplete, $\mathscr{G}$ converges to some point y in X with respect to the b-topology $\mathscr{T}(\mathscr{S}\mathscr{C}^*)$ of X. Thus

$$\emptyset \neq A \cap \mathrm{cl}_{\mathscr{T}(\mathscr{S}\mathscr{C})}\{y\} \subseteq \mathrm{adh}_{\mathscr{T}(\mathscr{S}\mathscr{C})}\mathscr{G}$$

and $\mathscr{F}$ has a cluster point in A. We have shown that the subspace A of X is closed-complete. Hence X is hereditarily closed-complete.

It remains to prove that X is a quasi-*fc*-space. Let A be a closed irreducible subset of X with the *FCI*-property and let $\mathscr{F}$ be the filter on X generated by the filter base $\{G \cap A : G \text{ is open in } X \text{ and } G \cap A \neq \emptyset\}$. We want to show that $\mathscr{F}$ is an $\mathscr{S}\mathscr{C}^*$-Cauchy filter on X. To this end let $\{G_n : n \in \mathbb{Z}\}$ be an open spectrum in X. Since the non-empty subset A of X has the *FCI*-property, there is a minimal $n \in \mathbb{Z}$ such that $G_n \in \mathscr{F}$. Then by the definition of $\mathscr{F}$ the set $G_n \backslash G_{n-1}$ belongs to $\mathscr{F}$. As above it follows that $\mathscr{F}$ is an $\mathscr{S}\mathscr{C}^*$-Cauchy filter on X. Since $\mathscr{S}\mathscr{C}$ is bicomplete, $A = \mathrm{cl}_{\mathscr{T}(\mathscr{S}\mathscr{C})}\{x\}$ for some $x \in X$ by Lemma 1(*b*). We have shown that X is a quasi-*fc*-space.

Example 5. The semi-continuous quasi-uniformity of $w_1 + 1$ (equipped with the order topology) is not bicomplete, because the noncompact countably compact subspace w_1 of $w_1 + 1$ is not closed-complete. Of course, the semi-continuous quasi-uniformity of the compact space $w_1 + 1$ is complete.

PROPOSITION 10. *The semi-continuous quasi-uniformity $\mathscr{S}\mathscr{C}$ of a hereditarily countably metacompact, hereditarily closed-complete, quasi-fc-space X is bicomplete.*

Proof. Let $\mathscr{F}$ be an $\mathscr{S}\mathscr{C}^*$-Cauchy filter on X and let A be an arbitrary open subset of X belonging to $\mathscr{F}$. There exists a maximal closed filter $\mathscr{M}$ on the subspace A of X such that

$$\{F \cap A : F \in \mathscr{F} \text{ and } F \text{ is closed in } X\} \subseteq \mathscr{M}.$$

We want to show that $\mathscr{M}$ has the countable intersection property. Assume the contrary. Let $(F_n)_{n \in \mathbb{N}}$ be a decreasing sequence of closed subsets of X such that $F_n \cap A \in \mathscr{M}$ whenever $n \in \mathbb{N}$ and such that

$$\bigcap\{F_n : n \in \mathbb{N}\} \cap A = \emptyset.$$

Since A is countably metacompact, there is a sequence $(G_n)_{n \in \mathbb{N}}$ of open subsets of X such that $F_n \cap A \subseteq G_n \cap A$ whenever $n \in \mathbb{N}$ and such that $\bigcap\{G_n : n \in \mathbb{N}\} \cap A = \emptyset$ (by [11], theorem 2·2). Let $n \in \mathbb{N}$. If $X \backslash G_n \in \mathscr{F}$, then $A \backslash G_n \in \mathscr{M}$, a contradiction, because $F_n \cap A \in \mathscr{M}$. We conclude that $G_n \in \mathscr{F}$, because $((X \backslash G_n) \times X) \cup (G_n \times G_n)$ belongs to $\mathscr{S}\mathscr{C}$ and $\mathscr{F}$ is an $\mathscr{S}\mathscr{C}^*$-Cauchy filter on X. Hence $\{G_n : n \in \mathbb{N}\} \cup \{A\}$ is a countable open subcollection of $\mathscr{F}$ with an empty intersection. However this is impossible by Lemma 6. We deduce that $\mathscr{M}$ has the countable intersection property.

There is an $x \in \bigcap \mathscr{M} \subseteq A$, since A is closed-complete. Because $x \in \mathrm{adh}_{\mathscr{T}(\mathscr{S}\mathscr{C})}\mathscr{F}$ by the definition of $\mathscr{M}$, we see that A cannot be equal to $X \backslash \mathrm{adh}_{\mathscr{T}(\mathscr{S}\mathscr{C})}\mathscr{F}$. Since we have assumed that A is an arbitrary open member of $\mathscr{F}$, it follows that $(X \backslash \mathrm{adh}_{\mathscr{T}(\mathscr{S}\mathscr{C})}\mathscr{F}) \notin \mathscr{F}$ and, thus, $\mathrm{adh}_{\mathscr{T}(\mathscr{S}\mathscr{C})}\mathscr{F} \in \mathscr{F}$, because $\mathscr{F}$ is an $\mathscr{S}\mathscr{C}^*$-Cauchy filter on X. By Lemma 1(*a*) each $\mathscr{T}(\mathscr{S}\mathscr{C})$-cluster point of $\mathscr{F}$ is a $\mathscr{T}(\mathscr{S}\mathscr{C})$-limit point of $\mathscr{F}$. Since $\mathrm{adh}_{\mathscr{T}(\mathscr{S}\mathscr{C})}\mathscr{F} \in \mathscr{F}$, we conclude that $\mathrm{adh}_{\mathscr{T}(\mathscr{S}\mathscr{C})}\mathscr{F}$ is an irreducible subspace of X.

We want to show next that $\mathrm{adh}_{\mathscr{T}(\mathscr{S}\mathscr{C})}\mathscr{F}$ has the *FCI*-property. Let $\mathscr{G}$ be an open filter on X such that $G \cap \mathrm{adh}_{\mathscr{T}(\mathscr{S}\mathscr{C})}\mathscr{F} \neq \emptyset$ whenever $G \in \mathscr{G}$. Since $\mathscr{F}$, as an

$\mathscr{SC}^*$-Cauchy filter on X, contains G or $X\backslash G$ whenever G is an open subset of X, we see that $\mathscr{G} \subseteq \mathscr{F}$. By Lemma 6 it follows that $\mathscr{G}$ has the countable intersection property. Hence $\operatorname{adh}_{\mathscr{T}(\mathscr{SC})} \mathscr{F}$ has the *FCI*-property. There is an $x \in X$ such that $\operatorname{adh}_{\mathscr{T}(\mathscr{SC})} \mathscr{F} = \operatorname{cl}_{\mathscr{T}(\mathscr{SC})}\{x\}$, because X is a quasi-*fc*-space. Since

$$\operatorname{adh}_{\mathscr{T}(\mathscr{SC})} \mathscr{F} = \operatorname{cl}_{\mathscr{T}(\mathscr{SC})}\{x\} \in \mathscr{F},$$

the filter $\mathscr{F}$ converges to x with respect to the topology $\mathscr{T}(\mathscr{SC}^*)$ by Lemma 2 (*a*). We have shown that the semi-continuous quasi-uniformity $\mathscr{SC}$ of X is bicomplete.

Remark 4. If we set $A = X$ in the first part of the proof of Proposition 10 we obtain the result that in a countably metacompact closed-complete space X each $\mathscr{SC}^*$-Cauchy filter $\mathscr{F}$ converges to some point $x \in X$ with respect to the topology $\mathscr{T}(\mathscr{SC})$ (use Lemma 1 (*a*)). Observe that it suffices then to assume that $\operatorname{cl}_{\mathscr{T}(\mathscr{SC})}\{x\}$ is a G_δ-set in X in order that $\mathscr{F}$ converges to x even with respect to the topology $\mathscr{T}(\mathscr{SC}^*)$. (*Proof*. By our assumption there is a sequence $(G_n)_{n\in\mathbb{N}}$ of open subsets of X such that $\operatorname{cl}_{\mathscr{T}(\mathscr{SC})}\{x\} = \bigcap\{G_n : n \in \mathbb{N}\}$. Since $G_n \in \mathscr{F}$ whenever $n \in \mathbb{N}$ and since

$$(X\backslash \operatorname{cl}_{\mathscr{T}(\mathscr{SC})}\{x\}) \cap (\bigcap\{G_n : n \in \mathbb{N}\}) = \varnothing,$$

we conclude by Lemma 6 that $\operatorname{cl}_{\mathscr{T}(\mathscr{SC})}\{x\}$ belongs to the $\mathscr{SC}^*$-Cauchy filter $\mathscr{F}$ on X. Hence $\mathscr{F}$ converges to x in $(X, \mathscr{T}(\mathscr{SC}^*))$ by Lemma 2 (*a*).) It follows e.g. that the semi-continuous quasi-uniformity of each first-countable countably metacompact closed-complete T_1-space is bicomplete.

Example 6. Let $\mathscr{S}$ be the topology $\{\varnothing, w\} \cup \{[0, \beta] : \beta \in w+1\}$ on the set $X = w+1$. Since the space $(X, \mathscr{S})$ is strongly sober and locally compact, the coarsest compatible quasi-uniformity on X is bicomplete (see [**22**], example 2). Obviously each open spectrum in X is a finite cover of X, because X is compact and each strictly decreasing sequence of open subsets of X is finite. Hence the semi-continuous quasi-uniformity of X coincides with the Pervin quasi-uniformity $\mathscr{P}$ of X. Since $\mathscr{S}$ is not hereditarily compact, $\mathscr{P}$ is not bicomplete (cf. [**14**], corollary 3·2). On the other hand the fine quasi-uniformity of X is bicomplete by Proposition 4, because X is sober. Since $\mathscr{S}$ is countable, it is clear that X is hereditarily closed-complete. Hence the condition 'hereditarily countably metacompact' cannot be omitted in Proposition 10.

In an attempt to weaken the assumption 'hereditarily countably metacompact' in the proof of Proposition 10, the authors discovered the following variant.

PROPOSITION 11. *The semi-continuous quasi-uniformity $\mathscr{SC}$ of a hereditarily real-compact completely regular space X is bicomplete.*

Proof. Let $\mathscr{F}$ be an $\mathscr{SC}^*$-Cauchy filter on X. The quasi-uniform space $(X, \mathscr{SC})$ is (convergence) complete by proposition 2·2 of [**19**] (see also [**7**]), because X is a completely regular real-compact space. Therefore the filter $\mathscr{F}$ has a $\mathscr{T}(\mathscr{SC})$-limit point x in X, since $\mathscr{F}$ is an $\mathscr{SC}$-Cauchy filter on X. Set $A = X\backslash \operatorname{cl}_{\mathscr{T}(\mathscr{SC})}\{x\}$ and

$$\mathscr{M} = \{F \cap A : F \in \mathscr{F} \text{ and } F \cap A \text{ is a zero-set of the subspace } A \text{ of } X\}.$$

Note that we have $\bigcap \mathscr{M} = \varnothing$, because $\mathscr{F}$ converges to x in the completely regular space $(X, \mathscr{T}(\mathscr{SC}))$.

First assume that $A \in \mathscr{F}$. We wish to show that in this case $\mathscr{M}$ is a maximal filter of zero-sets on the subspace A of X having the countable intersection property.

Clearly $\mathscr{M}$ is a filter of zero-sets on the subspace A of X. Let us prove that $\mathscr{M}$ has the countable intersection property. To this end suppose that $Z_n \in \mathscr{M}$ whenever $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ there is a sequence $(C_{n,m})_{m\in\mathbb{N}}$ of co-zero sets of the subspace A of X such that $Z_n = \bigcap\{C_{n,m} : m \in \mathbb{N}\}$. Let $n, m \in \mathbb{N}$. There exists an open subset $G_{n,m}$ of X such that $G_{n,m} \cap A = C_{n,m}$. Since $Z_n \in \mathscr{M}$, we have $A \setminus C_{n,m} \notin \mathscr{M}$ and, thus, $X \setminus G_{n,m} \notin \mathscr{F}$. Therefore $G_{n,m} \in \mathscr{F}$, because $\mathscr{F}$ is an $\mathscr{SC}^*$-Cauchy filter on X. It follows that if $\bigcap\{Z_n : n \in \mathbb{N}\} = \varnothing$, then $\{G_{n,m} : n, m \in \mathbb{N}\} \cup \{A\}$ is a countable open subcollection of $\mathscr{F}$ with an empty intersection. However this is impossible by Lemma 6. We conclude that $\mathscr{M}$ has the countable intersection property.

Next we are going to show that $\mathscr{M}$ is maximal. Let Z be a zero-set of the subspace A of X that meets every member of $\mathscr{M}$. There is a sequence $(G_n)_{n\in\mathbb{N}}$ of open subsets of X such that $Z = \bigcap\{G_n : n \in \mathbb{N}\} \cap A$ and such that $G_n \cap A$ is a co-zero set of the subspace A of X whenever $n \in \mathbb{N}$. Moreover there is a closed subset F of X such that $F \cap A = Z$. Let $n \in \mathbb{N}$. Note that G_n is an element of the $\mathscr{SC}^*$-Cauchy filter $\mathscr{F}$, because the assumption that $X \setminus G_n \in \mathscr{F}$ yields the contradiction that the zero-set $(X \setminus G_n) \cap A = A \setminus G_n$ of the subspace A belongs to $\mathscr{M}$, although $Z \cap (A \setminus G_n) = \varnothing$. Since $\{X \setminus F, A\} \cup \{G_n : n \in \mathbb{N}\}$ is a countable open collection of X with an empty intersection, we conclude by Lemma 6 that $(X \setminus F) \notin \mathscr{F}$. Hence $F \in \mathscr{F}$, because $\mathscr{F}$ is an $\mathscr{SC}^*$-Cauchy filter on X. By the definition of $\mathscr{M}$ we get that $F \cap A = Z \in \mathscr{M}$. We have shown that $\mathscr{M}$ is a maximal filter of zero-sets on the subspace A of X that has the countable intersection property, but satisfies $\bigcap \mathscr{M} = \varnothing$. Since A is a real-compact subspace of X, we have reached a contradiction (see [**10**], lemma 8·12).

Therefore our assumption that A belongs to $\mathscr{F}$ was incorrect. Since $\mathscr{F}$ is an $\mathscr{SC}^*$-Cauchy filter, we conclude that $\mathrm{cl}_{\mathscr{T}(\mathscr{SC})}\{x\} \in \mathscr{F}$. By Lemma 2(*a*) it is clear that the filter $\mathscr{F}$ converges to x in the space $(X, \mathscr{T}(\mathscr{SC}^*))$. Hence the semi-continuous quasi-uniformity $\mathscr{SC}$ of X is bicomplete.

Example 7. The semi-continuous quasi-uniformity $\mathscr{SC}$ of the well-known topological space Ψ (see [**10**], 5*I*) is bicomplete, but not complete.

Proof. It is known that Ψ is a Moore space (see [**2**], p. 39). Since a Moore space of non-measurable cardinality is closed-complete (by [**2**], corollary 4·3) and countably metacompact (by [**12**], theorem 7·8), the semi-continuous quasi-uniformity $\mathscr{SC}$ of Ψ is bicomplete by Remark 4 (or Proposition 10). Since the completely regular Hausdorff space Ψ is pseudo-compact, but not compact, it is not almost real-compact (see [**9**, **27**] and e.g. [**6**], theorem 3·10·23 and problem 3·12·5*a*)). Hence the quasi-uniformity $\mathscr{SC}$ of Ψ is not complete, because the semi-continuous quasi-uniformity of a completely regular Hausdorff space X is complete if and only if X is almost real-compact (see [**19**], proposition 2·2).

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