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TRANSFINITE DEPENDENT CHOICE AND ω-MODEL REFLECTION

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Abstract. In this paper we present some metapredicative subsystems of analysis. We deal with reflection principles, ω-model existence axioms (limit axioms) and axioms asserting the existence of hierarchies. We show several equivalences among the introduced subsystems. In particular we prove the equivalence of Σ^1_1 transfinite dependent choice and Π^1_2 reflection on ω-models of Σ^1_1-DC.

§1. Introduction. The formal system of classical analysis is second order arithmetic with full comprehension principle. It was called classical analysis, since classical mathematical analysis can be formalized in it. Often, subsystems of classical analysis suffice as formal framework for particular parts of mathematical analysis. During the last decades a lot of such subsystems have been isolated and proof-theoretically investigated. The subsystems of analysis introduced in this paper belong to *metapredicative* proof-theory. Metapredicative systems have proof-theoretic ordinals beyond Γ_0 but can still be treated by methods of predicative proof-theory only. Recently, numerous interesting metapredicative systems have been characterized. For previous work in metapredicativity the reader is referred to Jäger [3], Jäger, Kahle, Setzer and Strahm [4], Jäger and Strahm [5, 6], Kahle [7], Rathjen [8] and Strahm [11, 12, 13].

Metapredicative subsystems of analysis are for instance: ATR (proof-theoretic ordinal Γ_{ε_0}, e.g., [5]), ATR $+\Sigma^1_1$-DC, $\mathsf{ATR}_0+\Sigma^1_1$-DC (proof-theoretic ordinal $\varphi 1\varepsilon_0 0$, $\varphi 1\omega 0$ respectively, [5]) and FTR, FTR_0 (proof-theoretic ordinal $\varphi 20\varepsilon_0$, $\varphi 200$ respectively, [12]). We introduce in this paper a lot of subsystems of analysis with proof-theoretic ordinals between $\varphi 200$ and $\varphi\varepsilon_0 00$.

Three concepts are of central importance in this paper: ω-models, reflections and hierarchies. Each subsystem, which we shall introduce, deals with one of these concepts. We shall prove equivalences of some subsystems and determine the proof-theoretic ordinal of some of them. To prove these equivalences, we use the method of "pseudohierarchies" (cf. [10]).

In order to define ω-models within subsystems of analysis we have to formalize the notion of an ω-model. This leads to the notion of countable coded ω-model, cf. e.g., [10]. We say that M satisfies φ or that M is an ω-model of φ iff M reflects φ, i.e., iff φ^M holds. For instance, if Ax_{ACA} is a finite axiomatization of (ACA), then M is an ω-model of ACA iff $(Ax_{\mathsf{ACA}})^M$ holds. In the following we

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shall assert the existence of ω-models by so-called *limit axioms*. They are of the form $(\forall X)(\exists M)(X \dot{\in} M \land \varphi^M)$ for a given φ. For instance, if $Ax_{\Sigma^1_1\text{-AC}}$ is a finite axiomatization of $(\mathsf{ACA}) + (\Sigma^1_1\text{-AC})$, then the axiom

$$(1) \qquad (\forall X)(\exists M)(X \dot{\in} M \land (Ax_{\Sigma^1_1\text{-AC}})^M)$$

says that for each set X there exists an ω-model of Σ^1_1-AC containing X. In a certain sense a limit axiom is a special kind of a reflection principle. For any given formula the limit axiom asserts the existence of a set M reflecting this formula. On the other hand, a reflection principle asserts that for each formula φ belonging to a given class of formulas, there exists a set M reflecting the formula φ. For instance, the axiom scheme $(\Pi^1_2\text{-RFN})$ is of this type.

$(\Pi^1_2\text{-RFN})$ For all $\mathscr{L}_2$ formulas $\varphi[\vec{x}, \vec{Z}]$ in Π^1_2:
$\varphi[\vec{x}, \vec{Z}] \rightarrow (\exists X)(\vec{Z} \dot{\in} X \land (Ax_{\mathsf{ACA}})^X \land \varphi^X[\vec{x}, \vec{Z}])$.

We shall now discuss hierarchies. Fix a well-ordering Z. By a hierarchy along Z we mean a set Y, such that for each a in $\mathit{field}(Z)$ we have $\varphi(Y_{Za}, Y_a)$ for a given formula φ. Y_{Za} is the disjoint union of all stages Y_b such that b is Z-less than a. The formula φ characterizes the hierarchy and typically for b Z-less than a Y_a is in some sense more complex and contains more information than Y_b. A typical example is the hyperarithmetical hierarchy. We discuss two kinds of axioms which assure the existence of hierarchies. On the one hand we have axioms which claim directly the existence of the hierarchy. For instance (ATR) and (FTR) are of this form.

(ATR) For all arithmetic $\mathscr{L}_2$ formulas $\varphi(x, X)$:
$WO(Z) \rightarrow (\exists Y)(\forall a \in \mathit{field}(Z))(\forall x)(x \in Y_a \leftrightarrow \varphi(x, Y_{Za}))$.

As a second group we discuss axioms of the form "If we can build steps, then the corresponding stairs exist". For example $(\Sigma^1_1\text{-DC})$ is of this form. The resulting hierarchies are along ω.

(FTR) For all X-positive, arithmetic $\mathscr{L}_2$ formulas $\varphi(X, Y, x, a)$:
$WO(Z) \rightarrow (\exists X)(\forall a \in \mathit{field}(Z))(\forall x)(x \in X_a \leftrightarrow \varphi(X_a, X_{Za}, x, a))$.

The main questions addressed in this paper are

1. Are there equivalences among limit axioms, reflection principles and hierarchy existence axioms? For instance, what ω-models can be built given the existence of certain hierarchies and vice versa?
2. Are there further natural metapredicative subsystems of analysis?

An example of the equivalence described in point 1 is the equivalence of $(\Sigma^1_1\text{-DC})$ and $(\Pi^1_2\text{-RFN})$ over ACA_0 (cf. [10, theorem VIII.5.12]) or the equivalence of (ATR) and (1) (cf. [10, theorem VIII.4.20] — the difficult direction is shown there). We prove in this paper a lot of other such equivalences and we shall see that point 1 and point 2 mutually elucidate each other. In particular we shall prove the following equivalences over ACA_0.

I. Σ^1_1 transfinite dependent choice is equivalent to
Π^1_2 reflection on ω-models of Σ^1_1-DC.

II. Existence of fixed point hierarchies is equivalent to
existence of ω-models of $\mathsf{ATR} + \Sigma^1_1$-DC and to

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existence of hierarchies of ω-models of Σ^1_1-AC and to
existence of hierarchies of ω-models of ATR.

Since the proof-theoretic ordinal of FTR_0 is $\varphi 200$ (cf. [12]), the corresponding theories to the axioms listed in II have proof-theoretic ordinal $\varphi 200$, too. In [9] it is proved that $\varphi\omega 00$ is the proof-theoretic ordinal of the theories corresponding to the axioms listed in I[1]. Moreover, the proof-theoretic analysis of these theories uses methods of predicative proof-theory only. Hence they are metapredicative.

§2. Preliminaries. In this section we fix notations and abbreviations and introduce some well-known subsystems of analysis.

We let $\mathscr{L}_2$ denote the language of second order arithmetic. $\mathscr{L}_2$ includes *number variables* (denoted by small letters, except r, s, t), *set variables* (denoted by capital letters), symbols for all primitive recursive functions and relations, the symbol $\in$ for elementhood between numbers and sets, as well as equality in the first sort. The *number terms* r, s, t of $\mathscr{L}_2$ and the formulas $\varphi, \psi, \theta, \dots$ of $\mathscr{L}_2$ are defined as usual.

An $\mathscr{L}_2$ formula is called *arithmetic*, if it does not contain bound set variables (but possibly free set variables). For the collection of these formulas we write Π^1_0. Σ^1_1 is the collection of all arithmetic formulas and of all $\mathscr{L}_2$ formulas $\exists X \varphi(X)$ with $\varphi(X)$ from Π^1_0. Analogously Σ^1_k and Π^1_k are defined.

In the following $\langle \dots \rangle$ denotes a primitive recursive coding function for n-tuples $\langle t_1, \dots, t_n \rangle$ with associated projections $(\cdot)_1, \dots, (\cdot)_n$. Seq_n is the primitive recursive set of sequence numbers of length n. We write $\vec{X}$ for $X_1, \dots, X_n$, $s \in X_t$ for $\langle s, t \rangle \in X$ and $r \in X_{t,s}$ for $\langle\langle r, s\rangle, t\rangle \in X$. Occasionally we use the following abbreviations.

$$
\begin{aligned}
X = Y &:= (\forall x)(x \in X \leftrightarrow x \in Y),\\
X \neq Y &:= \neg(X = Y),\\
X \mathrel{\dot\in} Y &:= (\exists k)(\forall x)(x \in X \leftrightarrow \langle x, k\rangle \in Y),\\
(\exists Y \mathrel{\dot\in} Z)\varphi(Y) &:= (\exists k)\varphi(Y_k),\\
(\forall Y \mathrel{\dot\in} Z)\varphi(Y) &:= (\forall k)\varphi(Y_k),\\
\vec{X} \mathrel{\dot\in} Y &:= X_1 \mathrel{\dot\in} Y \wedge \dots \wedge X_n \mathrel{\dot\in} Y,\\
WO(X) &:= \text{formalization of}\\
&\qquad \text{“}X\text{ codes a (non-reflexive) well-ordering”},\\
x \in field(X) &:= (\exists y)(\langle x, y\rangle \in X \vee \langle y, x\rangle \in X),\\
x \in Y_{Za} &:= Seq_2 x \wedge x \in Y \wedge \langle (x)_1, a\rangle \in Z.
\end{aligned}
$$

Y_{Za} is the disjoint union of all projections Y_b with $\langle b, a\rangle \in Z$. For a well-ordering Z we let 0_Z denote the Z-least element in $field(Z)$ and for $a \in field(Z)$ we let $a +_Z 1$ denote the Z-least element in $field(Z)$ Z-greater than a, when it exists. Sometimes we write aZb for $\langle a, b\rangle \in Z$.

Furthermore, we write $\varphi[\vec{x}, \vec{X}]$ if all free number variables of φ are among $\vec{x}$ and all free set variables of φ are among $\vec{X}$. We write $\varphi[\vec{x}\backslash\vec{t}, \vec{X}\backslash\vec{Y}]$ for the formula φ

[1]In this paper we are chiefly interested in the proofs of the equivalences listed in I and II and not in the proof-theoretic analysis of the corresponding theories. Hence we do not define the terms “proof-theoretic ordinal” and “notation system”, but refer to [4, 6, 9].

where all occurrences of x_i are substituted by t_i and all occurrences of X_i by Y_i. Often we write directly $\varphi[\vec{t}, \vec{Y}]$ for $\varphi[\vec{x}\backslash\vec{t}, \vec{X}\backslash\vec{Y}]$.

All subsystems of analysis needed in this paper are based on the usual axioms and rules for the two-sorted predicate calculus. The theory ACA includes defining axioms for all primitive recursive functions and relations, the induction schema for arbitrary formulas of $\mathscr{L}_2$ and the schema (ACA) for arithmetical comprehension. Σ^1_1-AC extends ACA by the schema

(Σ^1_1-AC) For all $\mathscr{L}_2$ formulas $\varphi(x, X)$ in Σ^1_1:
$$(\forall x)(\exists X)\varphi(x, X) \rightarrow (\exists X)(\forall x)\varphi(x, X_x).$$

ATR extends ACA by (ATR), Σ^1_1-DC by (Σ^1_1-DC) and Π^1_2-RFN by (Π^1_2-RFN). Notice that we adopt the standard notation φ^X for the relativization of the $\mathscr{L}_2$ formula φ to X (for example $(\forall Y\varphi(Y))^X := (\forall Y \dot{\in} X)\varphi^X(Y)$). T_0 denotes the theory T with set-induction instead of the induction schema for arbitrary formulas.

§3. Σ^1_1 transfinite dependent choice and a principle of Π^1_2 reflection. In this section we introduce the basic subsystems of analysis studied in this paper: Σ^1_1 transfinite dependent choice and Π^1_2 reflection on ω-models of Σ^1_1-DC. Later on, in sections 4 and 5, we shall prove the equivalence of these two principles.

The theory Σ^1_1-TDC is the theory ACA extended by the schema of Σ^1_1 Transfinite Dependent Choice.

(Σ^1_1-TDC) For all Σ^1_1 formulas φ:
$$(\forall X)(\exists Y)\varphi(X, Y) \wedge WO(Z) \rightarrow (\exists Y)(\forall a \in \mathit{field}(Z))\varphi(Y_{Za}, Y_a).$$

Observing that φ can have free number and set variables, we have the following result, whose proof consists of a combination of folklore arguments and, which, therefore, will be omitted.

LEMMA 1. *Σ^1_1-TDC proves for each Σ^1_1 formula φ*
$$(\forall a)(\forall X)(\exists Y)\varphi(X, Y, Z, a) \wedge WO(Z)$$
$$\rightarrow (\exists Y)\big[Y_{0_Z} = Q \wedge (\forall a)(0_Z Za \rightarrow \varphi(Y_{Za}, Y_a, Z, a))\big].$$

The theory Σ^1_1-TDC is metapredicative. Its proof-theoretic strength is $\varphi\varepsilon_0 00$. The proof-theoretic ordinal of Σ^1_1-TDC_0 is $\varphi\omega 00$ (cf. [9]). Moreover Σ^1_1-TDC contains many relevant subsystems of analysis, in particular all subsystems of analysis introduced in this paper, with proof-theoretic strength less than or equal to $\varphi\varepsilon_0 00$. Notice that Σ^1_1-TDC is much stronger than *weak* Σ^1_1-TDC, where *weak* Σ^1_1-TDC is the theory ACA extended by the schema

(*weak* Σ^1_1-TDC) For all Σ^1_1 formulas φ:
$$(\forall X)(\exists! Y)\varphi(X, Y) \wedge WO(Z)$$
$$\rightarrow (\exists Y)(\forall a \in \mathit{field}(Z))\varphi(Y_{Za}, Y_a).$$

Clearly (Σ^1_1-TDC) implies (*weak* Σ^1_1-TDC) over ACA_0. But the opposite direction does not hold. In fact, the proof-theoretic ordinal of *weak* Σ^1_1-TDC is Γ_{ε_0}. (It is immediate that (*weak* Σ^1_1-TDC) implies (ATR) over ACA_0. But it is an open problem whether the opposite direction holds. A result in this connection is [9, lemma 49], where it is proved that ATR + Σ^1_1-IND implies (*weak* Σ^1_1-TDC).)

In the next lemma we collect some obvious implications which we shall often use tacitly. The proof uses standard arguments only and we omit it.

LEMMA 2. Σ^1_1-TDC_0 *proves* (Σ^1_1-AC), (Σ^1_1-DC) *and* (ATR).

Let us now introduce the reflection principle. The theory $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$ extends ACA by the schema

$((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$ For all Π^1_2 formulas $\varphi[\vec{z},\vec{Z}]$:

$$\varphi[\vec{z},\vec{Z}] \rightarrow (\exists M)\left[\vec{Z} \,\dot{\in}\, M \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M \wedge \varphi^M\right]$$

where we have written $Ax_{\Sigma^1_1\text{-}\mathsf{DC}}$ for a finite axiomatization of (Σ^1_1-DC) + (ACA). We call to mind the theory Π^1_2-RFN, where for each Π^1_2 formula there is an ω-model of ACA which reflects this Π^1_2 formula. In $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$ now, there is for each Π^1_2 formula even an ω-model of Σ^1_1-DC which reflects this formula. We mention that these ω-models are — so-called — countable coded ω-models, a notion important in [10].

A central point of this paper is to prove the equivalence of (Σ^1_1-TDC) and $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$ over ACA_0. In the proof of this equivalence, it is crucial that we can reflect on ω-models of Σ^1_1-DC and not only on ω-models of, e.g., Σ^1_1-AC. (It is worth mentioning that Π^1_0-DC would suffice too.) In [5] it is shown that with the aid of (Σ^1_1-DC) we can extend hierarchies, also hierarchies which do not have to be unique, i.e., which can have different initial sections. In contrast, using (Σ^1_1-AC) we can build only unique hierarchies. We need this property of (Σ^1_1-DC) for the construction of the desired hierarchy. Let us illustrate this difference of (Σ^1_1-DC) and (Σ^1_1-AC) by introducing the theory $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{AC}}$. It extends ACA by the schema

$((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{AC}})$ For all Π^1_2 formulas $\varphi[\vec{z},\vec{Z}]$:

$$\varphi[\vec{z},\vec{Z}] \rightarrow (\exists M)\left[\vec{Z} \,\dot{\in}\, M \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^M \wedge \varphi^M\right]$$

where we have written $Ax_{\Sigma^1_1\text{-}\mathsf{AC}}$ for a finite axiomatization of (Σ^1_1-AC) + (ACA). $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{AC}}$ is equivalent to $\mathsf{ATR} + \Sigma^1_1$-$\mathsf{DC}$ (cf. [9]). Hence the proof-theoretic ordinal is $\varphi 1\varepsilon_0 0$ (cf. [5]), in contrast to $\varphi\varepsilon_0 00$, the proof-theoretic ordinal of $(\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}$.

§4. Π^1_2 reflection on ω-models of Σ^1_1-DC implies Σ^1_1-TDC. We apply and extend the methods presented in [5] for building fixed point hierarchies with the aid of (Σ^1_1-DC), in order to prove the following lemma.

LEMMA 3. ACA_0 *proves*

$$((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}) \rightarrow (\Sigma^1_1\text{-}\mathsf{TDC}).$$

PROOF. Sometimes we work extremely informally in the following. For example, we often simply write "$\mathscr{F}$" or "$\mathscr{G}$" for "F is a function" or "G is a function" and use the notation "$\mathscr{F}(n)$" for the unique m with $\langle n,m\rangle \in F$. We first collect some basic facts.

1. The existence of fundamental sequences is provable in ATR_0. (Of course we do not need the full strength of ATR_0.) To be more precise: There is an arithmetic formula $\varphi(c,n,X,Z)$ such that ATR_0 proves

$$\begin{aligned}(2)\qquad & WO(Z) \wedge (\forall b \in \mathit{field}(Z))(\exists c) bZc \\ & \rightarrow (\exists! \mathscr{F})\big[(\forall c \in \mathit{field}(Z))(\forall n)(\varphi(c, n, \mathscr{F}_{<n}, Z) \leftrightarrow \mathscr{F}(n) = c) \wedge \\ & \qquad (\forall n)(\mathscr{F}(n) \in \mathit{field}(Z) \wedge \langle \mathscr{F}(n), \mathscr{F}(n+1)\rangle \in Z) \wedge \\ & \qquad (\forall b \in \mathit{field}(Z))(\exists n)\langle b, \mathscr{F}(n)\rangle \in Z\big].\end{aligned}$$

We now sketch the proof of the statement above. Fix a well-ordering without greatest element Z. We construct — reasoning in ATR_0 — the function $\mathscr{F}$ as follows: for any n let $\mathscr{F}(n)$ be the least (with respect to the standard ordering of natural number) element of $\mathit{field}(Z)$ which is Z-greater than every $\mathscr{F}(k)$ with $k < n$. There is an arithmetic formula $\varphi(c, n, X, Z)$ formalizing this construction. Then (ATR) implies the existence of a function $\mathscr{F}$ such that

$$(\forall c \in \mathit{field}(Z))(\forall n)(\varphi(c, n, \mathscr{F}_{<n}, Z) \leftrightarrow \mathscr{F}(n) = c).$$

It is easy to prove the remaining properties.

2. In general it is not possible to get a total operation "$+_Z$" on an arbitrary well-ordering Z. Therefore, we first define a well-ordering $\prec_{\omega^Z}$ which is closed under addition and such that Z is isomorphic to an initial section of $\prec_{\omega^Z}$ (notice that ATR_0 implies comparability of countable well-orderings, cf. [10, lemma V.2.9]).

We define in ACA_0 $\mathit{field}(\prec_{\omega^Z})$ as the set of all sequence numbers of the form (cf. [2])

$$\langle \langle b_0, k_0\rangle, \langle b_1, k_1\rangle, \dots, \langle b_n, k_n\rangle\rangle$$

such that (1) for all $i \leq n$, $b_i \in \mathit{field}(Z)$ and $0 \neq k_i$, and (2) whenever $i < j \leq n$, we have $\langle b_j, b_i\rangle \in Z$. We define $\prec_{\omega^Z}$ as the ordering with $\mathit{field}(\prec_{\omega^Z})$, ordered lexicographically. In particular, suppose that c and d are distinct elements of $\mathit{field}(\prec_{\omega^Z})$. If c extends d, then $d \prec_{\omega^Z} c$. If j is the least integer such that $\langle b_j, k_j\rangle = (s)_j \neq (d)_j = \langle b'_j, k'_j\rangle$ and either $\langle b'_j, b_j\rangle \in Z$ or both $b'_j = b_j$ and $k'_j < k_j$, then $d \prec_{\omega^Z} c$. Otherwise $c \prec_{\omega^Z} d$. Intuitively, $\langle \langle b_0, k_0\rangle, \langle b_1, k_1\rangle, \dots, \langle b_n, k_n\rangle\rangle$ correspond to the Cantor normal form $\omega^{b_0} \cdot k_0 + \cdots + \omega^{b_n} \cdot k_n$.

We observe that ACA_0 proves $WO(Z) \rightarrow WO(\prec_{\omega^Z})$ and define in ACA_0 ordinal addition on $\mathit{field}(\prec_{\omega^Z}) \times \mathit{field}(\prec_{\omega^Z})$ analogous to the addition of two codes of ordinals in Cartan normal form (cf. e.g., [2]). We write $+_{\prec_{\omega^Z}}$ for this operation.

3. In ATR_0 we can compare the well-orderings Z and $\prec_{\omega^Z}$. Thus, there is a comparison map $\mathscr{F}$ (cf. [10, section V.2]), which is an isomorphism from Z onto some initial section of $\prec_{\omega^Z}$.

4. Fix a well-ordering Z and let ℓ be a limit number in $\mathit{field}(\prec_{\omega^Z})$. Let $Z_\ell = \{a \in \mathit{field}(\prec_{\omega^Z}) : a \prec_{\omega^Z} \ell\}$ and apply fact 1 to get a (unique) $\mathscr{F}_\ell$ for the well-ordering $(Z_\ell, \prec_{\omega^Z})$. Write $\ell[n]$ in place of $\mathscr{F}_\ell(n)$, so that $\ell[n] \prec_{\omega^Z} \ell[n+1]$ and $\ell[n] \overset{n\to\infty}{\rightarrow} \ell$. Using the properties of the operation $+_{\prec_{\omega^Z}}$, for each ℓ and n there exists a unique $\ell^-[n] \in \mathit{field}(\prec_{\omega^Z})$ such that $\ell[n] +_{\prec_{\omega^Z}} \ell^-[n] = \ell[n+1]$. All this can be carried out within ATR_0.

5. For $a, b \in \mathit{field}(\prec_{\omega^Z})$ and any formula ψ we define

$$\begin{aligned} \mathit{Hier}_\psi(a, Y) &:= (\forall c \prec_{\omega^Z} a)\psi(Y_{\prec_{\omega^Z} c}, Y_c), \\ H_\psi(a, b, X, Y) &:= \mathit{Hier}_\psi(a, X) \rightarrow [\mathit{Hier}_\psi(a +_{\prec_{\omega^Z}} b, Y) \wedge \\ &\qquad (\forall c \prec_{\omega^Z} a)(X_c = Y_c)]. \end{aligned}$$

The same line of arguments which leads to lemma 2 in [5] proves in Σ^1_1-DC_0 the following statement:

$$\begin{aligned}&WO(Z) \wedge (\forall n)(\forall X)(\exists Y)H_\psi(a +_{\prec_\omega z} \ell[n], \ell^-[n], X, Y)\\ &\to (\forall X)(\exists Y)\big[Hier_\psi(a +_{\prec_\omega z} \ell[0], X)\\ &\qquad \to (Hier_\psi(a +_{\prec_\omega z} \ell, Y) \wedge (\forall b \prec_{\omega z} (a +_{\prec_\omega z} \ell[0]))(X_b = Y_b))\big].\end{aligned}$$

6. Lemma VIII.1.5 and its proof and theorem V.6.8 in [10] imply the existence of a finite Π^1_2 axiomatization of ATR_0. We write Ax_{ATR} for this axiomatization.

The stage is set to prove the claim. Reasoning in $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$ we will derive (Σ^1_1-TDC). Without loss of generality we may assume that the formula φ for which we need to prove (Σ^1_1-TDC) is arithmetic. We assume

$$(\forall X)(\exists Y)\varphi(X, Y) \qquad \text{and} \qquad WO(E)$$

and have to prove the existence of a set Y such that

$$(\forall a \in field(E))\varphi(Y_{Ea}, Y_a).$$

Furthermore, we assume that all parameters of φ are among $\vec{P}$ and set

$$\psi := (\forall X)(\exists Y)\varphi(X, Y) \wedge Ax_{\mathsf{ATR}}.$$

Since ψ is equivalent to a Π^1_2 formula (fact 6), we can apply $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$ in order to obtain a set M such that

$$(Ax_{\Sigma^1_1\text{-}\mathsf{DC}})^M \wedge \psi^M \wedge E, \vec{P} \mathrel{\dot\in} M.$$

The aim is to prove the claim by transfinite induction on the well-ordering E. But since we need a step more than E, we extend E to a well-ordering Z in M such that E is isomorphic to an initial section of Z. We now prove by transfinite induction on the well-ordering $\prec_{\omega z}$

$$(\forall b \in field(\prec_{\omega z}))(\forall a \in field(\prec_{\omega z}))(\forall X \mathrel{\dot\in} M)(\exists Y \mathrel{\dot\in} M)H_\varphi(a, b, X, Y).$$

We distinguish three cases. Since we have $((\forall X)(\exists Y)\varphi)^M$, the cases $b = 0_{\prec_\omega z}$ and $b +_{\prec_\omega z} 1$ are proved by standard arguments; and using fact 5 the limit case can be proved immediately. Thus, for each a in $field(\prec_{\omega z})$ there is a hierarchy Y such that $(\forall c \prec_{\omega z} a)\varphi(Y_{\prec_\omega z c}, Y_c)$. Since E is isomorphic to a proper initial section of $\prec_{\omega z}$ we obtain the claim. □

§5. **Σ^1_1-TDC implies Π^1_2 reflection on ω-models of Σ^1_1-DC.** To prove this implication we use the method of pseudohierarchies described in [10]. In a certain sense the proof presented here is an extension and combination of lemma VIII.4.19 and theorem VIII.5.12 in [10]

Lemma 4. ACA_0 *proves*

$$(\Sigma^1_1\text{-}\mathsf{TDC}) \to ((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}}).$$

Proof. We reason in Σ^1_1-TDC_0. Suppose that we have

$$(\forall X)(\exists Y)\varphi(X, Y)$$

with φ arithmetic. We assume that all set parameters of φ are among $\vec{P}$. Then we have to show the existence of a set M such that

$$(Ax_{\Sigma_1^1\text{-DC}})^M \wedge (\forall X \dot{\in} M)(\exists Y \dot{\in} M)\varphi \wedge \vec{P} \dot{\in} M.$$

Since the proof is long, we divide it into several steps.

1. Existence of hierarchies.

Suppose that Z is a well-ordering. We prove the existence of hierarchies Y of the form

- $Y_{0_Z,0}$ is an ω-model of Σ_1^1-AC containing $\vec{P}, Z$.
- If $a \in \mathit{field}(Z)$ and $a \neq 0_Z$ then
  - $Y_{a,0}$ is an ω-model of Σ_1^1-AC containing Y_{Za}.
  - $(\forall e)\varphi(Y_{a,0,e}, Y_{a,1,e})$ and in particular, for each set U in $Y_{a,0}$ there is a set V in $Y_{a,1}$ such that $\varphi(U, V)$.

We now give the formalization of the construction of these hierarchies.

$$\begin{aligned}
&\theta(X, Y, Z, a) := \\
&\quad a \in \mathit{field}(Z) \wedge a = 0_Z \rightarrow Z, \vec{P} \dot{\in} Y_0 \wedge (Ax_{\Sigma_1^1\text{-AC}})^{Y_0} \wedge \\
&\qquad\qquad\qquad a \neq 0_Z \rightarrow [X \dot{\in} Y_0 \wedge (Ax_{\Sigma_1^1\text{-AC}})^{Y_0} \wedge (\forall e)\varphi(Y_{0,e}, Y_{1,e})].
\end{aligned}$$

In order to conclude that such hierarchies exist, we have to prove that for all X there exists a set Y with $\theta(X, Y, Z, a)$. Hence we fix X. Recalling that we have (ATR), hence we have (cf. [10, lemma VIII.4.20])

$$(\exists Y)(X \dot{\in} Y \wedge (Ax_{\Sigma_1^1\text{-AC}})^Y) \tag{3}$$

and we conclude immediately that $(\exists Y)\theta(X, Y, Z, 0_Z)$. Suppose now $0_Z Za$. Using again (3) we obtain a set U such that $X \dot{\in} U$ and $(Ax_{\Sigma_1^1\text{-AC}})^U$. We know that $(\forall X)(\exists Y)\varphi(X, Y)$ holds and therefore we have $(\forall e)(\exists Y)\varphi(U_e, Y)$. An application of ($\Sigma_1^1$-AC) yields a set V such that $(\forall e)\varphi(U_e, V_e)$. We set $Y_0 := U$ and $Y_1 := V$ and conclude that $\theta(X, Y, Z, a)$. Now, we apply (Σ_1^1-TDC) and obtain

$$(\exists Y)(\forall a \in \mathit{field}(Z))\theta(Y_{Za}, Y_a, Z, a).$$

Hence we have shown

$$(\forall Z)(WO(Z) \rightarrow (\exists Y)(\forall a \in \mathit{field}(Z))\theta(Y_{Za}, Y_a, Z, a)).$$

2. Existence of pseudohierarchies.

We refer to [10, lemma VIII.4.18]. In this lemma the existence of pseudo jump hierarchies is proved. Using the same line of arguments, we can prove the existence of sets Y, Z, M^* such that

$$\begin{aligned}
&\neg WO(Z) \wedge Z, Y, \vec{P} \dot{\in} M^* \wedge (Ax_{\mathrm{ACA}})^{M^*} \wedge (WO(Z))^{M^*} \wedge \\
&(\forall a \in \mathit{field}(Z))\theta(Y_{Za}, Y_a, Z, a).
\end{aligned}$$

i.e., Y is the desired pseudohierarchy. Notice that $(WO(Z))^{M^*}$ implies that Z has a minimum, i.e., 0_Z exists.

3. Definition of the ω-model M.

Since Z is not a well-ordering, there exists a function $\mathscr{F}$ such that

$$(\forall n)(\langle \mathscr{F}(n+1), \mathscr{F}(n) \rangle \in Z).$$

Let I be a set which contains the elements beneath $\{\mathscr{F}(n) : n \in \omega\}$, i.e., $I = \{c : (\forall n)(\langle c, \mathscr{F}(n) \rangle \in Z\}$. I has the following straightforward (see [9] for details) properties which we shall often use tacitly.

$$I \dot{\notin} M^* \wedge I \neq \emptyset \wedge (\forall b \in I)(cZb \rightarrow c \in I) \wedge (\forall b \in I)(\exists c \in I)bZc.$$

The stage is set to define our ω-model M. M contains all sets in $Y_{b,0}$ with b in I; thus

$$M := \{\langle x, \langle e, b \rangle \rangle : b \in I \wedge \langle x, e \rangle \in Y_{b,0}\}.$$

4. $\vec{P} \dot{\in} M$ and M is an ω-model of (ACA).

We know that $\vec{P} \dot{\in} Y_{0_Z,0}$. Since $0_Z \in I$ holds, we have $\vec{P} \dot{\in} M$. Secondly, we show that M satisfies arithmetical comprehension. We choose an arithmetic formula $\psi[x, \vec{z}, \vec{Q}]$ and prove

$$\vec{Q} \dot{\in} M \rightarrow (\exists X \dot{\in} M)(\forall x)(x \in X \leftrightarrow \psi[x, \vec{z}, \vec{Z}]).$$

Notice that we have $U \dot{\in} Y_{c,0}$ and $cZb \rightarrow U \dot{\in} Y_{b,0}$, since we have $Y_{Zb} \dot{\in} Y_{b,0}$. Thus, there is a $b \in I$ such that each element of $\vec{Q}$ is in $Y_{b,0}$. Applying arithmetical comprehension in $Y_{b,0}$ we obtain a set C in $Y_{b,0}$ such that $(\forall x)(x \in C \leftrightarrow \psi[x, \vec{z}, \vec{Q}])$. Since C is in $Y_{b,0}$, C is in M too.

5. M satisfies $(\forall X)(\exists Y)\varphi(X, Y)$.

Suppose that $X \dot{\in} M$. We have to prove that there exists a $Y \dot{\in} M$ such that $\varphi(X, Y)$. We choose $b \in I$ such that $X \dot{\in} Y_{b,0}$ and e such that $X = Y_{b,0,e}$. Since $(\forall e)\varphi(Y_{b,0,e}, Y_{b,1,e})$ and $Y_{Z(b+_Z 1)} \dot{\in} Y_{b+_Z 1,0}$ we have $Y_{b,1,e} \dot{\in} Y_{b+_Z 1,0}$. Notice that $b +_Z 1$ is in I and thus $Y_{b,1,e} \dot{\in} M$.

6. M is an ω-model of $(\Sigma^1_1\text{-DC})$.

The proof of this statement is closely related to the corresponding part in the proof of lemma VIII.4.19 [10]. Suppose that

$$\vec{P} \dot{\in} M \wedge (\forall X \dot{\in} M)(\exists Y \dot{\in} M)\nu[\vec{z}, X, Y, \vec{P}]$$

holds. Without loss of generality we may assume that ν is arithmetic. We fix Q in M and have to prove that there exists a set V in M such that

$$V_0 = Q \wedge (\forall u)\nu[\vec{z}, V_u, V_{u+1}, \vec{P}]. \tag{4}$$

Fix $b_0 \in I$ and e_0 such that all parameters P_i belong to $Y_{b_0,0}$ and such that $Q = Y_{b_0,0,e_0}$. Reasoning within M^*, choose a sequence p of 2-tuples as follows. We set $(p)_0 = \langle e_0, b_0 \rangle$. Given $(p)_n = \langle e_n, b_n \rangle$ let b_{n+1} be the Z-least $c \in \mathit{field}(Z)$ such that $b_n Zc$, and such that $Y_{c,0}$ satisfies $(\exists U)\nu[\vec{z}, Y_{b_n,0,e_n}, U, \vec{P}]$. It is $c \in I$. Then let e_{n+1} be the $<$-least g such that $\nu[\vec{z}, Y_{b_n,0,e_n}, Y_{b_{n+1},0,g}, \vec{P}]$. Set $(p)_{n+1} := \langle e_{n+1}, b_{n+1} \rangle$. Let $\psi(n, p)$ be the arithmetic formula formalizing this construction, i.e.,

$$\psi(n, p) \leftrightarrow p \text{ is a sequence of length } n+1 \text{ according to the construction described above.}$$

By induction on n we can prove

$$(\forall n)[(\exists! p)\psi(n,p) \wedge (\psi(n,c) \to (\forall k \leq n)((((c)_k)_1 \in I))].$$

Put

$$J := \{c : c \in \mathit{field}(Z) \wedge (\exists n, k)(\psi(n,k) \wedge \langle c, ((k)_n)_1\rangle \in Z)\}.$$

Then $J \subset I$. Moreover, since J is arithmetic in $Z, Y, \vec{P}$, we have $J \dot{\in} M^*$, hence $J \neq I$. Since M^* satisfies $WO(Z)$, there must exists $b^* \in \mathit{field}(Z)$ such that $J = \{c : \langle c, b^*\rangle \in Z\}$. It is $b^\star \in I$. We set

$$H_{b^*} := \{\langle x, \langle e, b\rangle\rangle : \langle b, b^*\rangle \in Z \wedge \langle x, e\rangle \in Y_{Zb^*,b,0}\}$$

and can express $\psi(n,p)$ as a formula arithmetic in $Z, H_{b^*}, \vec{P}$. Since all these parameters are in M, the set $\{\langle n,k\rangle : \psi(n,k)\}$ is in M too. Thus,

$$V := \{\langle x, n\rangle : (\exists k)(\psi(n,k) \wedge \langle x, ((k)_n)_0\rangle \in Y_{Zb^*,((k)_n)_1,0})\}$$

is in M and satisfies (4). $\square$

The main result of section 4 and 5 is the following theorem, which follows immediately from lemmas 3 and 4.

Theorem 5. *The following two principles are equivalent over* ACA_0.

a) $(\Sigma^1_1\text{-}\mathsf{TDC})$,
b) $((\Pi^1_2\text{-}\mathsf{RFN})^{\Sigma^1_1\text{-}\mathsf{DC}})$.

§6. ω-models of $\mathsf{ATR}+\Sigma^1_1$-DC and hierarchies. In sections 3, 4 and 5 we have discussed a hierarchy and a reflection principle. The corresponding theories, $\Sigma^1_1\text{-}\mathsf{TDC}_0$ and $(\Pi^1_2\text{-}\mathsf{RFN})_0^{\Sigma^1_1\text{-}\mathsf{DC}}$, have proof-theoretic ordinal $\varphi\omega 00$ [9]. In this section, we deal with some special kinds of hierarchies. The corresponding theories will have proof-theoretic ordinal $\varphi 200$.

First, we mention a result of Avigad. In [1] he introduces the theory FP_0. It extends ACA_0 by

(FP) For all X-positive, arithmetic $\mathscr{L}_2$ formulas $\varphi(x, X)$:
$(\exists X)(\forall x)(x \in X \leftrightarrow \varphi(x, X))$.

(FP) asserts the existence of fixed points of positive arithmetic operators. Avigad proves in [1] the equivalence of (ATR) and (FP) over ACA_0. Furthermore, we remind the reader that (ATR) and (3) are equivalent over ACA_0. Hence, we have the equivalence of (FP) and (3) over ACA_0. In other words: the existence of fixed points is equivalent to the existence of ω-models of Σ^1_1-AC. The aim is to extend this result to hierarchies of fixed points and hierarchies of ω-models of Σ^1_1-AC. We recall the theory FTR_0, introduced in [12] (and section 1). (FTR) demands the existence of fixed point hierarchies along arbitrarily given well-orderings. Strahm shows in [12] that FTR_0 has proof-theoretic strength $\varphi 200$.

Analogous to fixed point hierarchies we introduce the theory $Ax_{\Sigma^1_1\text{-}\mathsf{AC}}\mathsf{TR}_0$. It extends ACA_0 by

$$(Ax_{\Sigma^1_1\text{-}\mathsf{AC}}\mathsf{TR}) \quad (\forall X, Z)(WO(Z) \to (\exists Y)(\forall a \in \mathit{field}(Z))(Y_{Za} \dot{\in} Y_a \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^{Y_a} \wedge X \dot{\in} Y_a)).$$

($Ax_{\Sigma^1_1\text{-AC}}\mathsf{TR}$) demands the existence of hierarchies Y along arbitrary given well-orderings, such that each stage of Y is an ω-model of Σ^1_1-AC and such that each stage is included in the succeeding stages. Remembering the equivalence of (FP) and $(\forall X)(\exists Y)(X \mathrel{\dot\in} Y \wedge (Ax_{\Sigma^1_1\text{-AC}})^Y)$, the question arises whether (FTR) and ($Ax_{\Sigma^1_1\text{-AC}}\mathsf{TR}$) are equivalent too. We shall show that these two principles are, in fact, equivalent.

Secondly, we look for ω-models corresponding to these hierarchies. Towards that goal we let $Ax_{\mathsf{ATR}+\Sigma^1_1\text{-DC}}$ denote a finite axiomatization of (ATR) + (Σ^1_1-DC) + (ACA). Then $Ax_{\mathsf{ATR}+\Sigma^1_1\text{-DC}}\mathsf{RFN}_0$ is defined as the theory ACA_0 extended by

$$(Ax_{\mathsf{ATR}+\Sigma^1_1\text{-DC}}\mathsf{RFN}) \qquad (\forall X)(\exists M)(X \mathrel{\dot\in} M \wedge (Ax_{\mathsf{ATR}+\Sigma^1_1\text{-DC}})^M).$$

We have not found a better name for this theory, which stemmed from the theory Π^1_2-RFN_0, where for each Π^1_2 sentence an ω-model of ACA exists which reflects this Π^1_2 sentence. Analogously in $Ax_{\mathsf{ATR}+\Sigma^1_1\text{-DC}}\mathsf{RFN}_0$ for each finite axiomatization of (ATR) + (Σ^1_1-DC) there is an ω-model of ACA which reflects this axiomatization.

We shall show that ($Ax_{\mathsf{ATR}+\Sigma^1_1\text{-DC}}\mathsf{RFN}$) is equivalent to ($Ax_{\Sigma^1_1\text{-AC}}\mathsf{TR}$) and (FTR). In the proof of this equivalence we shall often use the following lemma tacitly. We omit the proof since it follows immediately from the equivalence of (FP), $(\forall X)(\exists Y)(X \mathrel{\dot\in} Y \wedge (Ax_{\Sigma^1_1\text{-AC}})^Y)$ and (ATR) over ACA_0.

Lemma 6. *The theories* FTR_0, $Ax_{\Sigma^1_1\text{-AC}}\mathsf{TR}_0$ *and* $Ax_{\mathsf{ATR}+\Sigma^1_1\text{-DC}}\mathsf{RFN}_0$ *prove each instance of* (ATR).

§7. **Equivalence of $Ax_{\mathsf{ATR}+\Sigma^1_1\text{-DC}}\mathsf{RFN}_0$, $Ax_{\Sigma^1_1\text{-AC}}\mathsf{TR}_0$ and FTR_0.** In this section we prove the equivalence of the mentioned principles. Again, we shall use the method of pseudohierarchies. We begin with the proof that ($Ax_{\Sigma^1_1\text{-AC}}\mathsf{TR}$) implies ($Ax_{\mathsf{ATR}+\Sigma^1_1\text{-DC}}\mathsf{RFN}$).

Lemma 7. ACA_0 *proves*

$$(Ax_{\Sigma^1_1\text{-AC}}\mathsf{TR}) \to (Ax_{\mathsf{ATR}+\Sigma^1_1\text{-DC}}\mathsf{RFN}).$$

Proof. Here we use pseudohierarchies. The proof is similar to the proof of lemma 4. Hence we give only a sketch. Fix a set P and assume that

$$WO(Z) \to (\exists Y)(\forall a \in \mathit{field}(Z))(Y_{Za} \mathrel{\dot\in} Y_a \wedge (Ax_{\Sigma^1_1\text{-AC}})^{Y_a} \wedge X \mathrel{\dot\in} Y_a)$$

holds. Then we have to prove the existence of a set M such that

$$P \mathrel{\dot\in} M \wedge (Ax_{\mathsf{ATR}+\Sigma^1_1\text{-DC}})^M.$$

Arguing as in [10, lemma VIII.4.18], we obtain sets Y, Z, M^* such that

$$\neg WO(Z) \wedge Z, Y, P \mathrel{\dot\in} M^* \wedge (Ax_{\mathsf{ACA}})^{M^*} \wedge (WO(Z))^{M^*} \wedge$$
$$(\forall a \in \mathit{field}(Z))((Ax_{\Sigma^1_1\text{-AC}})^{Y_a} \wedge Y_{Za} \mathrel{\dot\in} Y_a \wedge P, Z \mathrel{\dot\in} Y_a).$$

Again, we can choose an initial section I of Z such that

$$I \mathrel{\dot\notin} M^* \wedge I \neq \emptyset \wedge (\forall b \in I)(cZb \to c \in I) \wedge (\forall b \in I)(\exists c \in I)bZc.$$

Finally, we set

$$M := \{\langle x, \langle e, b\rangle\rangle : b \in I \wedge \langle x, e\rangle \in Y_b\}.$$

M is the desired ω-model. The same line of arguments as in the proof of lemma 4 yields that $P \dot{\in} M$ holds and that M satisfies (ACA) and $(\Sigma^1_1\text{-}\mathsf{DC})$. Moreover, the properties of Y immediately imply that M satisfies

$$(\forall X)(\exists Y)(X \dot{\in} Y \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^Y),$$

hence M satisfies (ATR) too. □

Next we prove the implication $(Ax_{\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}}\mathsf{RFN}) \to (\mathsf{FTR})$.

LEMMA 8. *For each arithmetic, X-positive formula $\varphi(X, Y, x, y)$ ACA_0 proves*

$$\begin{aligned}&(Ax_{\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}}\mathsf{RFN}) \wedge WO(Z)\\ &\quad\to (\exists X)(\forall a \in \mathit{field}(Z))(\forall x)(x \in X_a \leftrightarrow \varphi(X_a, X_{Za}, x, a)).\end{aligned}$$

PROOF. We adapt the methods developed in the proof of lemma 3 to the situation here. We give only a sketch. Fix an X-positive, arithmetic formula $\varphi(X, Y, x, y)$. Choose $\vec{P}$ such that all set parameters of φ are listed in $\vec{P}$ and fix a well-ordering E. Reasoning in $Ax_{\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}}\mathsf{RFN}_0$ we have to prove

$$(\exists X)(\forall a \in \mathit{field}(E))(\forall x)(x \in X_a \leftrightarrow \varphi(X_a, X_{Ea}, x, a)).$$

Applying $(Ax_{\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}}\mathsf{RFN})$, we obtain a set M such that $E, \vec{P} \dot{\in} M$ and $(Ax_{\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}})^M$. Again we choose a well-ordering Z extending E and proceed as in the proof of lemma 3. We define in M the well-ordering $\prec_{\omega^Z}$ and ordinal addition on $\mathit{field}(\prec_{\omega^Z})$ as well as fundamental sequences. Then, using the equivalence of (ATR) and (FP), we can prove by transfinite induction on $\prec_{\omega^Z}$

$$\begin{aligned}(\forall b \in \mathit{field}(\prec_{\omega^Z}))(\exists X \dot{\in} M)(\forall a)(\langle a, b\rangle \in \prec_{\omega^Z} \to\\ (\forall x)(x \in X_a \leftrightarrow \varphi(X_a, X_{\prec_{\omega^Z} a}, x, a)),\end{aligned}$$

where we argue in the limit case as in the proof of lemma 3. □

It remains to show the implication

$$(\mathsf{FTR}) \to (Ax_{\Sigma^1_1\text{-}\mathsf{AC}}\mathsf{TR}). \tag{5}$$

We first show how to build ω-models of Σ^1_1-AC using fixed points. Let us sketch the idea.

1. We build a fixed point X such that for each well-ordering recursive in Q with index a, $(X)_a$ is the jump hierarchy along the well-ordering a, starting with Q.
2. We build a fixed point Y such that $X \dot{\in} Y$ and such that Y is an ω-model of ACA.

Then it immediately follows from [10, lemma VIII.4.19], that there exists an ω-model of Σ^1_1-AC in Y; and we are done. We have already mentioned this lemma above. Since we need here the statement of this lemma and not only the proof idea, we give its formulation.

LEMMA 9 ([10, lemma VIII.4.19]). *The following is provable in* ACA_0. *Let X be such that*

$$\begin{aligned}&(\forall a)(a \textit{ is a } X\textit{-recursive well-ordering with index } a\\ &\quad\to \textit{the jump hierarchy along } a, \textit{ starting with } X, \textit{ exists})\end{aligned}$$

holds. Then there exists M such that $X \dot{\in} M$ and M satisfies Σ^1_1-DC_0 (hence also Σ^1_1-AC_0 and Δ^1_1-CA_0).

With respect to the construction presented above, there are two points worth mentioning. First, we again use the method of pseudohierarchies but now implicitly, via [10, lemma VIII.4.19]. Secondly, we can formalize the argument in the theory $\widehat{ID}_2$, obtaining an embedding of Σ^1_1-AC into $\widehat{ID}_2$.

Now let $\pi^0_1[x, X]$ be a complete Π^0_1 formula. In order to formalize the argument above, we write $\overset{+}{\pi}[x, X, Y]$ and $\overset{-}{\pi}[x, X, Y]$ for the X- and Y-positive, arithmetic formulas with the following properties (provable in ACA_0).

$$\pi^0_1[x, X] \leftrightarrow \overset{+}{\pi}[x, X, \neg X] \qquad \text{and} \qquad \neg\pi^0_1[x, X] \leftrightarrow \overset{-}{\pi}[x, X, \neg X].$$

We now define three formulas $\mathscr{A}, \mathscr{B}, \mathscr{C}$ which finally lead to the desired ω-model of Σ^1_1-AC. (Cf. section 2 for the definition of Seq_2.)

1. $\mathscr{A}(X, Q, \langle\langle\langle z, y\rangle, x\rangle, a\rangle)$ is the following X-positive formula. The intended interpretation is that for each fixed point X of $\mathscr{A}$ and for each well-ordering recursive in Q with index a, X_a is the jump hierarchy along a starting with Q.

$$\begin{aligned}
&\mathscr{A}(X, Q, \langle\langle\langle z, y\rangle, x\rangle, a\rangle) := \\
&\quad \text{“}a \text{ codes a } Q\text{-recursive linear ordering } \prec^Q_a \text{ and} \\
&\quad \text{there is a least element } 0_a\text{”} \wedge \\
&\quad \big[(x = 0_a \wedge y = 0 \wedge Seq_2 z \wedge (z)_1 = 0_a \wedge (z)_0 \in Q) \vee \\
&\quad (x = 0_a \wedge y = 1 \wedge [\neg Seq_2 z \vee (z)_1 \neq 0_a \vee (z)_0 \notin Q]) \vee \\
&\quad (0_a \prec^Q_a x \wedge y = 0 \wedge Seq_2 z \wedge (z)_1 \prec^Q_a x \ \wedge \overset{+}{\pi}[(z)_0, X_{a,(z)_1,0}, X_{a,(z)_1,1}]) \vee \\
&\quad (0_a \prec^Q_a x \wedge y = 1 \wedge [\neg Seq_2 z \vee \neg((z)_1 \prec^Q_a x) \\
&\quad \vee \overset{-}{\pi}[(z)_0, X_{a,(z)_1,0}, X_{a,(z)_1,1}]])\big].
\end{aligned}$$

2. $\mathscr{B}(Y, X, \langle\langle z, y\rangle, x\rangle)$ is the following Y-positive formula. The intended interpretation is that for each fixed point Y of $\mathscr{B}$, Y is the jump hierarchy along $<$ starting with X.

$$\begin{aligned}
&\mathscr{B}(Y, X, \langle\langle z, y\rangle, x\rangle) := \\
&\quad (x = 0 \wedge y = 0 \wedge z \in X) \vee (x = 0 \wedge y = 1 \wedge z \notin X) \vee \\
&\quad (0 < x \wedge y = 0 \wedge Seq_2 z \wedge (z)_1 < x \wedge \overset{+}{\pi}[(z)_0, Y_{(z)_1,0}, Y_{(z)_1,1}]) \vee \\
&\quad (0 < x \wedge y = 1 \wedge [\neg Seq_2 z \vee \neg((z)_1 < x) \vee \overset{-}{\pi}[(z)_0, Y_{(z)_1,0}, Y_{(z)_1,1}]]).
\end{aligned}$$

3. $\mathscr{C}(Z, Y, \langle m, \langle e, x\rangle\rangle)$ is the following Z-positive formula. The intended interpretation is that for each fixed point Z of $\mathscr{C}$, $Z_{\langle e,x\rangle}$ is a set recursive in $Y_{x,0}$, namely $\{m : \{e\}^{Y_{x,0}}(m) = 0\}$. Notice that the set variable Z do not occur in $\mathscr{C}(Z, Y, \langle m, \langle e, x\rangle\rangle)$. (Hence there is no need of resorting to a fixed point axiom here: straightforward arithmetic comprehension suffices. We resort here to a fixed point axiom for technical reasons.)

$$\mathscr{C}(Z, Y, \langle m, \langle e, x\rangle\rangle) := (\forall z)(\exists n)(\{e\}^{Y_{x,0}}(z) = n) \wedge \{e\}^{Y_{x,0}}(m) = 0.$$

The next lemma establishes that the listed formulas serve the purpose.

LEMMA 10. *We can prove in* ACA_0 *that if* X, Y, Z, Q *are such that*

$$(\forall z, y, x, a)(\langle\langle\langle z, y\rangle, x\rangle, a\rangle \in X \leftrightarrow \mathscr{A}(X, Q, \langle\langle\langle z, y\rangle, x\rangle, a\rangle)) \wedge$$
$$(\forall z, y, x)(\langle\langle z, y\rangle, x\rangle \in Y \leftrightarrow \mathscr{B}(Y, X, \langle\langle z, y\rangle, x\rangle)) \wedge$$
$$(\forall m, e, x)(\langle m, \langle e, x\rangle\rangle \in Z \leftrightarrow \mathscr{C}(Z, Y, \langle m, \langle e, x\rangle\rangle)),$$

then we can conclude

a) $(\forall x, z)[\langle\langle z, 0\rangle, x\rangle \in Y \leftrightarrow \langle\langle z, 1\rangle, x\rangle \notin Y]$,
b) $(Ax_{\mathsf{ACA}})^Z \wedge X \dot{\in} Z \wedge Q \dot{\in} Z$,
c) *Z satisfies "for each Q-recursive well-ordering a there exists the jump hierarchy along a, starting with Q"*,
d) $(\exists M \dot{\in} Z)(Q \dot{\in} M \wedge (Ax_{\Sigma^1_1\text{-}\mathsf{AC}})^M)$.

PROOF. a) $\{x : (\forall z)(\langle\langle z, 0\rangle, x\rangle \in Y \leftrightarrow \langle\langle z, 1\rangle, x\rangle \notin Y)\}$ is a set. Therefore, the claim can be proved by induction along $<$.

b) $(Y_{x,0})_{x\in\omega}$ is by definition the jump hierarchy along $<$ (cf. a)), and Z is the union of all sets recursive in some $Y_{x,0}$. Using standard arguments, it can be proved that in this situation we have $(Ax_{\mathsf{ACA}})^Z$ (cf. for instance [10, theorem VIII.1.13]). Furthermore, we know $Y_{0,0} = X$ and $X_{a,0_a,0,0_a} = Q$ for an appropriate a. Thus $Q \dot{\in} Z$ holds.

c) Choose a Q-recursive well-ordering a such that $(WO(a))^Z$ holds. That is, all subsets of *field*(a) in Z are well-founded with respect to a. We know $Q \dot{\in} Z$ and $(Ax_{\mathsf{ACA}})^Z$. The set

$$N := \{b : b \prec^Q_a a \wedge (\forall z)(\langle\langle\langle z, 0\rangle, b\rangle, a\rangle \in X \leftrightarrow \langle\langle\langle z, 1\rangle, b\rangle, a\rangle \notin X)\}$$

is arithmetic in Q, X. Therefore, $N \dot{\in} Z$. By transfinite induction along a we can prove that $(\forall b \prec^Q_a a)(b \in N)$.

Thus, $E := \{\langle z, b\rangle : b \prec^Q_a a \wedge \langle\langle\langle z, 0\rangle, b\rangle, a\rangle \in X\}$ is the jump hierarchy along a and $E \dot{\in} Z$.

d) Lemma 9 and a), b), c) immediately imply the claim. □

The next step is to iterate the whole construction along a given well-ordering. Since lemma 10 is the crucial step in the proof of the following lemma, while the iteration of that construction is straightforward and uses only standard arguments, we omit the proof of the following lemma and refer to [9] for details of the iteration.

LEMMA 11. *There is an arithmetic, X-positive formula $\varphi(X, Y, Z, Q, x, a)$ with set parameters X, Y, Z, Q such that* ACA_0 *proves*

$$WO(Z) \wedge (\exists X)(\forall a \in \mathit{field}(Z))(\forall x)(x \in X_a \leftrightarrow \varphi(X_a, X_{Za}, Z, Q, x, a)) \rightarrow (Ax_{\Sigma^1_1\text{-}\mathsf{AC}}\mathsf{TR}).$$

We collect the results of lemma 7, 8 and 11 in the following theorem.

THEOREM 12. *The following are equivalent over* ACA_0.

a) (FTR),
b) $(Ax_{\Sigma^1_1\text{-}\mathsf{AC}}\mathsf{TR})$,
c) $(Ax_{\mathsf{ATR}+\Sigma^1_1\text{-}\mathsf{DC}}\mathsf{RFN})$.

Using theorem 9 in [12], which computes the proof-theoretic ordinal of FTR_0 and FTR, we obtain the following proof-theoretic ordinals.

COROLLARY 13. *The following holds.*

a) $|Ax_{\Sigma_1^1\text{-AC}}\mathsf{TR}_0| = |Ax_{\mathsf{ATR}+\Sigma_1^1\text{-DC}}\mathsf{RFN}_0| = |\mathsf{FTR}_0| = \varphi 200$.

b) $|Ax_{\Sigma_1^1\text{-AC}}\mathsf{TR}| = |Ax_{\mathsf{ATR}+\Sigma_1^1\text{-DC}}\mathsf{RFN}| = |\mathsf{FTR}| = \varphi 20\varepsilon_0$.

We end this section with an extension of theorem 12. $Ax_{\Sigma_1^1\text{-AC}}\mathsf{TR}_0$ asserts the existence of hierarchies of ω-models of Σ_1^1-AC along a given well-ordering. What about hierarchies of ω-models of ATR? — We will prove that $Ax_{\Sigma_1^1\text{-AC}}\mathsf{TR}_0$ implies hierarchies of models of ATR, too. We begin with the definition of such theories. For each natural number n we introduce a predicate It_n.

$$\mathsf{It}_0(M) := (Ax_{\Sigma_1^1\text{-AC}})^M,$$
$$\mathsf{It}_{n+1}(M) := (\forall X \dot\in M)(\exists Y \dot\in M)(X \dot\in Y \wedge \mathsf{It}_n(Y) \wedge (Ax_{\mathsf{ACA}})^M).$$

Notice that each M such that $\mathsf{It}_1(M)$ holds is an ω-model of ATR. Each M such that $\mathsf{It}_2(M)$ holds is an ω-model of ω-models of ATR, and so on. The corresponding axioms are

$(\mathsf{It}_n\mathsf{TR})$ $$(\forall Z, X)(WO(Z) \rightarrow (\exists Y)(\forall a \in \mathit{field}(Z))(\mathsf{It}_n(Y_a) \wedge Y_{Za} \dot\in Y_a \wedge X \dot\in Y_a)).$$

We write $\mathsf{It}_n\mathsf{TR}_0$ for the theory ACA_0 extended by $(\mathsf{It}_n\mathsf{TR})$. In particular $\mathsf{It}_0\mathsf{TR}$ is the theory $Ax_{\Sigma_1^1\text{-AC}}\mathsf{TR}$. The mentioned result is stated in the following theorem.

THEOREM 14. *For each natural number n the following axioms are equivalent over* ACA_0.

a) $Ax_{\Sigma_1^1\text{-AC}}\mathsf{TR}$,

b) $\mathsf{It}_n\mathsf{TR}$.

PROOF. We reason in ACA_0 and first show the implication b)$\Rightarrow$a). For $n = 0$ the implication is trivial. Hence we assume $n > 0$. Using the fact that ATR_0 implies $(\Sigma_1^1\text{-AC})$ (cf. theorem V.8.3 in [10]), we can prove by meta-induction on n for each $k \leq n$ $\mathsf{It}_n(M) \rightarrow \mathsf{It}_k(M)$. This implies b)$\Rightarrow$a). For the converse direction we use meta-induction on n. The case $n = 0$ is trivial. Hence we assume $n > 0$. Furthermore, we fix a set X and a well-ordering Z. Then we have to prove

$$(\exists Y)(\forall a \in \mathit{field}(Z))(\mathsf{It}_n(Y_a) \wedge Y_{Za} \dot\in Y_a \wedge X \dot\in Y_a).$$

For technical reasons we introduce the well-ordering $Z \cdot \omega$.

$$Z \cdot \omega := \{\langle\langle x, m\rangle, \langle y, n\rangle\rangle : x, y \in \mathit{field}(Z) \wedge ((x = y \wedge m < n) \vee xZy)\}.$$

$Z \cdot \omega$ is a well-ordering such that between two successive elements of $\mathit{field}(Z)$ there is a copy of the well-ordering $<$. Since we assume $(\mathsf{It}_{n-1}\mathsf{TR})$, there is a hierarchy E along the well-ordering $Z \cdot \omega$ such that $\mathsf{It}_{n-1}(E_a)$ and $E_{(Z\cdot\omega)a} \dot\in E_a$ holds for all a in $\mathit{field}(Z \cdot \omega)$. Note that we can choose E such that $(X \oplus Z) \dot\in E_{0_{Z\cdot\omega}}$ holds. Then for each a in $\mathit{field}(Z)$ we build a set Y_a consisting of all projections of $E_{\langle b,k\rangle}$ with bZa or $b = a$, i.e.,

$$Y_a := \{\langle x, \langle e, \langle b, k\rangle\rangle\rangle : x \in E_{\langle b,k\rangle,e} \wedge (bZa \vee b = a)\}.$$

By transfinite induction on the well-ordering Z it can be proved that we have

$$(\forall a \in \mathit{field}(Z))(\mathsf{It}_n(Y_a) \wedge Y_{Za} \dot\in Y_a \wedge X \dot\in Y_a).$$

Hence Y is the desired hierarchy. □

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