Felder, G. and A. P. Veselov. (2007) “Polynomial Solutions of the Knizhnik–Zamolodchikov Equations and Schur–Weyl Duality,”
International Mathematics Research Notices, Vol. 2007, Article ID rnm046, 21 pages.
doi:10.1093/imrn/rnm046

Polynomial Solutions of the Knizhnik–Zamolodchikov Equations and Schur–Weyl Duality

Giovanni Felder[1] and Alexander P. Veselov[2]

[1]Department of Mathematics, ETH Zurich, 8092 Zurich, Switzerland and
[2]Department of Mathematical Sciences, Loughborough University, Loughborough LE11 3TU, UK and Landau Institute for Theoretical Physics, Moscow, Russia

Correspondence to be sent to: Giovanni Felder, Department of Mathematics, ETH Zurich, 8092 Zurich, Switzerland. e-mail: giovanni.felder@math.ethz.ch

An integral formula for the solutions of Knizhnik–Zamolodchikov (KZ) equation with values in an arbitrary irreducible representation of the symmetric group S_N is presented for integer values of the parameter. The corresponding integrals can be computed effectively as certain iterated residues determined by a given Young diagram and give polynomials with integer coefficients. The derivation is based on Schur–Weyl duality and the results of Matsuo on the original $SU(n)$ KZ equation. The duality between the spaces of solutions with parameters m and $-m$ is discussed in relation with the intersection pairing in the corresponding homology groups.

1 Introduction

Let G be a finite Coxeter group, R be the corresponding root system, $m_\alpha, \alpha \in R$ be a system of multiplicities, which is a G-invariant function on R. Let W be an irreducible representation of G and define the *Knizhnik–Zamolodchikov equation* (KZ equation)

Received November 3, 2006; Revised April 16, 2007; Accepted May 22, 2007
Communicated by Igor Krichever

See http://www.oxfordjournals.org/our_journals/imrn/ for proper citation instructions.

related to W as the following system

$$\partial_\xi \psi = \sum_{\alpha \in R_+} m_\alpha \frac{(\alpha, \xi)}{(\alpha, z)} (s_\alpha + 1)\psi,$$

where s_α are the corresponding reflections acting on W-valued functions $\psi(z)$. If the multiplicities m_α are integers, then all the solutions of the corresponding systems are homogeneous polynomials (see [11, 3]) of degree equal to the value of the central element $\sum_{\alpha \in R_+} m_\alpha (s_\alpha + 1)$ in the irreducible representation W. The finding of these solutions is an important part of the description of the so-called m-harmonic polynomials [2]. In the paper [3] these solutions were found explicitly in the simplest case of the standard (reflection) representation of $G = S_N$.

The main result of the present paper is an explicit integral formula for the solutions of the corresponding KZ equation

$$\partial_i \psi = m \sum_{j \neq i}^{N} \frac{s_{ij} + 1}{z_i - z_j} \psi, \quad i = 1, \ldots, N \tag{1.1}$$

with values in an arbitrary irreducible representation of the symmetric group S_N for any positive integer m. Our approach is based on Schur–Weyl duality and the results of Matsuo, who found some integral formulas for the solutions of the original $SU(n)$ KZ equation [10] inspired by the work of Fateev and Zamolodchikov [13] and Christe and Flume [1].

The main construction is the following. Let λ be the *Young diagram* with N boxes and n rows of lengths $\lambda_1, \ldots, \lambda_n$ with $\lambda_1 \geq \cdots \geq \lambda_n > 0$. It is well known that for any such diagram one can construct an irreducible representation W^λ of the symmetric group S_N and any irreducible representation of S_N can be described in this way (see e.g. [6]).

The space W^λ has a basis v_T labeled by the set $\mathcal{T}(\lambda)$ of all *standard tableaux* on λ, which are the numberings $T\colon \lambda \to \{1, \ldots, N\}$ of the boxes of λ, increasing from left to right and from top to bottom.

A fundamental set of solutions of the KZ equation with values in W^λ can be also labeled by the set $\mathcal{T}(\lambda)$:

$$\psi_T(z_1, \ldots, z_N) = \sum_{T' \in \mathcal{T}(\lambda)} \psi_{T,T'}(z_1, \ldots, z_N) v_{T'}. \tag{1.2}$$

The components $\psi_{T,T'}(z_1,\dots,z_N)$ are known to be polynomial in $z_1,\dots,z_N$ [3]. We can give now an explicit formula for $\psi_{T,T'}(z_1,\dots,z_N)$ as an integral

$$\psi_{T,T'} = \int_{\sigma_T} \omega_{T'} \tag{1.3}$$

of some rational differential form $\omega_{T'}$ over a certain cycle σ_T in the top homology of the following configuration space $C_\lambda(z_1,\dots,z_N)$ related to Young diagram λ.

For given $\lambda = (\lambda_1,\dots,\lambda_n)$ let us define the integers $m_i, i = 1,\dots,n$ from the relation $\lambda = (m_0 - m_1, m_1 - m_2, \dots, m_{n-2} - m_{n-1}, m_{n-1})$. Explicitly we have

$$m_0 = \lambda_1 + \lambda_2 + \cdots + \lambda_n = N, m_1 = \lambda_2 + \cdots + \lambda_n, \dots, m_{n-2} = \lambda_{n-1} + \lambda_n, m_{n-1} = \lambda_n,$$

so that m_s is the number of boxes in the rows of λ strictly lower than s.

Consider n finite sets $X_0, X_1, \dots, X_{n-1}$ of points on the complex plane $\mathbb{C}$ consisting of $m_0,\dots,m_{n-1}$ points respectively with the condition that X_i and X_{i+1} have no common points for all $i = 1,\dots,n-2$. Let us denote the elements of X_0 as $z_1,\dots,z_N$ and fix them. The corresponding configuration space of all admissible $\{X_1,\dots,X_{n-1}\}$ is our $C_\lambda(z_1,\dots,z_N)$. It has the dimension

$$d_\lambda = \sum_{i=1}^{n-1} m_i = \sum_{r=1}^{n} (r-1)\lambda_r,$$

and can be described as the following subset in $\mathbb{C}^{d_\lambda}$:

$$C_\lambda(z_1,\dots,z_N) = \{t_s^b \in \mathbb{C}, b \in \lambda, 1 \le s \le r(b) - 1 : t_{s+1}^b \neq t_s^{b'}, t_1^b \neq z_k\}, \tag{1.4}$$

where we have denoted the elements of X_s as

$$X_s = \{t_s^b \in \mathbb{C}, b \in \lambda, r(b) > s\},$$

and $r(b)$ is the row which the corresponding box b belongs to.

On this space $C_\lambda(z_1,\dots,z_N)$ we have a natural action of the group $G_\lambda = S_{m_1} \times S_{m_2} \times \cdots \times S_{m_{n-1}}$. The cycles σ_T in the top homology group $H_{top}(C_\lambda(z_1,\dots,z_N))$ can be defined for any numbering T of λ, namely any bijection $T\colon \lambda \to \{1,\dots,N\}$. Consider first the product Γ_T of circles consecutively surrounding anti-clockwise z_k with the variables $t_1^b,\dots,t_{r(b)-1}^b, b = T^{-1}(k)$ located on these circles:

$$\Gamma_T = \{t_s^b \in \mathbb{C} : |t_s^b - z_k| = \epsilon s, b = T^{-1}(k)\} \tag{1.5}$$

for any real positive ϵ small enough. The corresponding cycle σ_T is the skew-symmetrization of Γ_T by the action of G_λ:

$$\sigma_T = \sum_{g\in G_\lambda} (-1)^g g_*(\Gamma_T), \tag{1.6}$$

where $(-1)^g$ denote the sign of g, which is the product of signs of the corresponding permutations in S_{m_i}.

The form ω_T has the form

$$\omega_T = \frac{1}{(2\pi i)^{d_\lambda}} \Phi_\lambda^m \phi_T dt, \tag{1.7}$$

where

$$\Phi_\lambda = \prod_{i<j}^N (z_i - z_j)^2 \prod_{s,b\neq b'} (t_s^b - t_s^{b'})^2 \prod_{s,b,b'} (t_{s+1}^b - t_s^{b'})^{-1} \prod_{k,b} (t_1^b - z_k)^{-1}, \tag{1.8}$$

$$\phi_T = \prod_{s,b} (t_{s+1}^b - t_s^b)^{-1} \prod_b (t_1^b - z_{T(b)})^{-1} \tag{1.9}$$

and $dt = \prod_{s,b} dt_s^b$ is the exterior product of the differentials of all the coordinates (the order is not essential since it is only changes sign).

Theorem 1.1. For any given positive integer m, the integral formulas (1.2), (1.3) with the cycles σ_T and forms $\omega_T, T \in \mathcal{T}(\lambda)$ defined above give a basis in the space of solutions of the KZ equation (1.1) with values in the irreducible S_N-module W^λ. The integral (1.3) can be effectively computed as an iterated residue and gives a polynomial in $z_1, \ldots, z_N$ with integer coefficients. □

We have derived these formulas from the results of Matsuo [10] using the Schur-Weyl duality. The fact that in such a way we get all the solutions of the corresponding KZ equation does not follow from [10] and needs to be proven independently. Using representation theory it turns out that it is sufficient to prove that one of the integrals $\psi_{T,T'}$ does not vanish identically. This we prove by evaluating $\psi_{T,T}$ for the tableau assigning k to the kth box, counted from left to right and from top to bottom, in the asymptotic region $0 \ll |z_1| \ll \cdots \ll |z_N|$. We find

$$\psi_{T,T}(z) \sim C \prod_{b\in\lambda} z_{T(b)}^{m(T(b)-1+c(b)-r(b))} + \cdots ,$$

for some integer $C \neq 0$, see Lemma 3.7, where $c(b), r(b)$ are the coordinates (column and row number) of the box $b \in \lambda$. As a by-product, we obtain a new derivation of the interesting formula, due to Frobenius [5] (see [9], exercise 7 in Chapter I and comment on p. 134) for the value $f_2(\lambda)$ of the central element $\sum_{i<j} s_{ij}$ in the representation W^λ:

$$f_2(\lambda) = \sum_{b\in\lambda}(c(b) - r(b)).$$

Theorem 1.1 applies to the case of positive integer m. The case of negative m can be reduced to it by using the isomorphism between the space of solutions $KZ(V, m)$ of the KZ equation with values in the representation V and parameter m and the space $KZ(V \otimes \mathrm{Alt}, -m)$, where $\mathrm{Alt} = \mathbb{C}\epsilon$ is the alternating representation. Indeed, it is easy to check that if $\psi \in KZ(V, m)$ then $\phi = \prod_{i>j}(z_i - z_j)^{-2m}\psi \otimes \epsilon \in KZ(V \otimes \mathrm{Alt}, -m)$. In particular it follows that for negative m all solutions are rational functions. In the last section we discuss also the duality between $KZ(V, m)$ and $KZ(V^*, -m)$ given by the canonical map

$$KZ(V, m) \otimes KZ(V^*, -m) \to \mathbf{C}$$

in relation with the intersection pairing in the corresponding homology groups.

The case we consider can be viewed as a very degenerate limit of the general theory of hypergeometric solutions of KZ equations associated with Kac–Moody algebras, see [12] and references therein. We should mention that similar integral formulas were found also by Kazarnovski-Krol [8] in the theory of zonal spherical functions of type A_n, but combinatorics of the corresponding configuration space is very different and not related to Young diagrams.

2 Schur–Weyl duality

We start with the classical Schur–Weyl duality between the representations of the general linear and symmetric groups.

Let V be an n-dimensional complex vector space. Then the symmetric group S_N on N letters acts on the N-fold tensor product $V^{\otimes N} = V \otimes \cdots \otimes V$ by permutations of factors and this action commutes with the diagonal action of $GL(V)$. The Schur–Weyl theorem states that, as a $GL(V) \times S_N$ module, $V^{\otimes N}$ has a decomposition into a direct sum

$$V^{\otimes N} \cong \oplus_\lambda M^\lambda \otimes W^\lambda \tag{2.1}$$

where M^λ are inequivalent irreducible $GL(V)$-modules and W^λ are inequivalent irreducible S_N-modules. The sum is over partitions of N into at most n parts, namely sequences of integers $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ with $\sum \lambda_i = N$.

Moreover if $n \geq N$ all irreducible S_N modules appear. Thus we can realize every irreducible S_N-module as

$$W^\lambda = \mathrm{Hom}_{GL(V)}(M^\lambda, V^{\otimes N}),$$

for any V of dimension $\geq N$.

An explicit description of W^λ is obtained from the description of M^λ as a highest weight module and depends on a choice of basis of V. Namely, let us fix a basis $e_1, \ldots, e_n$ of V and introduce the decomposition $\mathfrak{gl}(V) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ of the Lie algebra of $GL(V)$ into strictly lower triangular, diagonal and strictly upper triangular $n \times n$ matrices. For any $\mathfrak{gl}(V)$-module E and any $\mu \in \mathfrak{h}^* = \mathbb{C}^n$, denote by $E_\mu = \{v \in E | x \cdot v = \mu(x)v, \forall x \in \mathfrak{h}\}$ the weight subspace of weight μ. The space of *primitive vectors* of weight μ in E is

$$E_\mu^{\mathfrak{n}_+} = \{v \in M_\mu | a \cdot v = 0, \forall a \in \mathfrak{n}_+\}.$$

The irreducible module M^λ is uniquely characterized by having a nonzero primitive vector v_λ of weight λ, which is unique up to normalization. Moreover M^λ is generated over $U(\mathfrak{n}_-)$ by v_λ and thus

$$M^\lambda = \mathbb{C}v_\lambda \oplus \mathfrak{n}_- M^\lambda. \tag{2.2}$$

An element of $\mathrm{Hom}_{GL(V)}(M^\lambda, V^{\otimes N})$ is then uniquely determined by the image of the primitive vector and we obtain an isomorphism of S_N-modules

$$W^\lambda = (V^{\otimes N})_\lambda^{\mathfrak{n}_+}.$$

From this realization we obtain a basis of W^λ labeled by standard Young tableaux, making connection to the classical construction of W^λ as a Specht module, see [6]. Here is how it works: let λ also denote the Young diagram with N boxes with rows of lengths $\lambda_1, \ldots, \lambda_m$. To each numbering $T\colon \lambda \to \{1, \ldots, N\}$ of the boxes we associate a weight vector $e_T = e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_N} \in (V^{\otimes N})_\lambda$ so that $\alpha_k = i$ whenever $T^{-1}(k)$ is in the ith row. For example, if T is the numbering

3	4	1
2	5	
6		

of $\lambda = (3, 2, 1)$, then $e_T = e_1 \otimes e_2 \otimes e_1 \otimes e_1 \otimes e_2 \otimes e_3$.

The symmetric group S_N acts on the set of numberings of λ and we have $\sigma e_T = e_{\sigma T}$ for any $\sigma \in S_N$. For any numbering T of λ, the row group $R(T)$ is the subgroup of S_N consisting of permutations preserving the image of each row. Similarly, we have the column group $C(T) \subset S_N$. Two numberings T, T' give the same weight vector if and only $T' = \sigma T$ for some $\sigma \in R(T)$. In this case $R(T) = R(T')$ and we say that T and T' are row equivalent. Thus the vectors e_T are associated to row equivalence classes $\{T\}$ of numberings of λ, which are called *tabloids* on λ. Recall that a *standard tableau* on λ is a numbering of λ which is increasing from left to right and from top to bottom.

Proposition 2.1. Let λ be a partition of N and let $\dim(V) \geq N$.

1. The vectors e_T, where $\{T\}$ runs over tabloids on λ, form a basis of the weight space $(V^{\otimes N})_\lambda$.
2. The vectors $v_T = \sum_{\sigma \in C(T)} \operatorname{sign}(\sigma) e_{\sigma T}$, where T runs over the set $\mathcal{T}(\lambda)$ of standard tableaux on λ, form a basis of the S_N-module $W^\lambda = (V^{\otimes N})_\lambda^{\mathfrak{n}_+}$ of primitive vectors of weight λ. □

For proofs see, e.g., Chapter 9 of [7].

A dual realization of W^λ is also relevant. The symmetric bilinear form on V for which the e_i form an orthonormal basis induces a symmetric nondegenerate bilinear form $\langle\ ,\ \rangle$ on each weight space $(V^{\otimes N})_\lambda$. If λ is a partition of N, the tensors e_T, where $\{T\}$ runs over the set of tabloids on λ, are then an orthonormal basis of $(V^{\otimes N})_\lambda$. Let τ be the antiautomorphism of $\mathfrak{gl}(V)$ given by matrix transposition with respect to the basis e_i. Then $\langle x \cdot v, w\rangle = \langle v, \tau(x) \cdot w\rangle$, $x \in \mathfrak{gl}(V)$. Moreover the bilinear form is S_N-invariant: $\langle \sigma \cdot v, \sigma \cdot w\rangle = \langle v, w\rangle$, $\sigma \in S_N$, $v, w \in V^{\otimes N}$.

Proposition 2.2. Let λ be a partition of N and let $\dim(V) \geq N$.

(i) The form $\langle\ ,\ \rangle$ induces a nondegenerate S_N-invariant pairing

$$(V^{\otimes N}/\mathfrak{n}_- V^{\otimes N})_\lambda \otimes (V^{\otimes N})_\lambda^{\mathfrak{n}_+} \to \mathbb{C}.$$

Thus we can identify $(V^{\otimes N}/\mathfrak{n}_- V^{\otimes N})_\lambda$ with the dual S_N-module $(W^\lambda)^*$.

(ii) The basis dual to the basis v_T of W^λ is given by the classes of the vectors e_T, $T \in \mathcal{T}(\lambda)$ in $(W^\lambda)^* = (V^{\otimes N}/\mathfrak{n}_- V^{\otimes N})_\lambda$. □

Proof. It follows from (2.2) and the complete reducibility of $V^{\otimes N}$ into a direct sum of irreducible highest weight modules that

$$(V^{\otimes N})_\lambda = (V^{\otimes N})_\lambda^{\mathfrak{n}_+} \oplus (\mathfrak{n}_- V^{\otimes N})_\lambda.$$

Moreover this is an orthogonal direct sum with respect to the contravariant form, since τ maps $\mathfrak{n}_-$ to $\mathfrak{n}_+$. Therefore the pairing is well-defined and is nondegenerate. Since v_T $(T \in \mathcal{T}(\lambda))$ form a basis of W^λ and e_T occurs in v_T with coefficient 1, we get $\langle e_T \bmod \mathfrak{n}_-, v_S\rangle = \delta_{T,S}$, $T, S \in \mathcal{T}(\lambda)$. Thus the classes of e_T form the dual basis of the dual module. ■

Remark 2.3. Actually, S_N-modules are self-dual, $(W^\lambda)^* \cong W^\lambda$ but the expression of the isomorphism with respect to the bases labeled by tableaux is nontrivial. The space of cycles in our integral formulae are more naturally associated with $(W^\lambda)^*$.

3 Integral representation of solutions

3.1 The action of the symmetric group on the solution space

We fix a Young diagram λ and a positive integer m and keep the notations of the introduction.

The KZ operators $D_i = \partial_i - m\sum_{j\neq i}(s_{ij}+1)/(z_i - z_j)$ appearing in (1.1) are commuting first-order holomorphic differential operators acting on W^λ-valued functions on the configuration space $C_N = \mathbb{C}^N - \cup_{i<j}\{z_i = z_j\}$. Thus the space of holomorphic solutions on any connected open subset $U \subset C_N$ has dimension $\dim W^\lambda$. The symmetric group S_N acts on C_N and thus on the functions with values in the S_N-module W^λ by $(g \cdot \psi)(z) = g(\psi(g^{-1} \cdot z))$. The KZ operators obey $g \cdot D_i\psi = D_{g(i)} g \cdot \psi$ for all $g \in S_N$. Therefore $g \in S_N$ maps solutions on U to solutions on $g \cdot U$. In particular we have an action of S_N on the space of global solutions

$$KZ(\lambda, m) = \{\text{holomorphic functions}\psi \colon C_N \to W^\lambda \colon D_i\psi = 0, i = 1, \ldots, N\}.$$

In fact, all local solutions extend to global solutions:

Theorem 3.1. [11, 3] All local solutions of the Knizhnik–Zamolodchikov equation (1.1) with $m \in \mathbb{Z}_{>0}$ extend to homogeneous polynomials of degree $m(f_2(\lambda) + (N-1)N/2)$, where $f_2(\lambda)$ is the value of the central element $\sum_{i<j} s_{ij}$ of the group algebra of S_N in the representation W^λ. Moreover, the space of solutions $KZ(\lambda, m)$ is isomorphic to W^λ as an S_N-module. □

The homogeneity follows directly from the equations: if ψ is a solution then $\sum z_i D_i \psi = (\sum z_i \partial_i - m(f_2(\lambda) + (N-1)N/2))\psi = 0$.

3.2 Matsuo's integral formulae.

The configuration spaces $C_\lambda(z)$, $z \in C_N$, of (1.4) form a fiber bundle over C_N and the action of S_N on the base lifts canonically to an action on the bundle. Indeed, $C_\lambda(z_1, \ldots, z_N)$ does not depend on the ordering of the z_i. The forms ω_T of eqn. (1.7) are holomorphic differential forms on the total space that restrict to holomorphic top differential forms $\omega_T(z)$ on each fiber $C_\lambda(z)$. They are defined for any numberings T, not just for standard tableaux and, by construction, they obey $\omega_{gT}(g \cdot z) = \omega_T(z)$ for all $g \in S_N$, where $gT = g \circ T$ is the natural action on the set of numberings $T\colon \lambda \to \{1, \ldots, N\}$.

Let $H_{top}(C_\lambda(z))_s$, $z = (z_1, \ldots, z_N) \in C_N$, be the skew-symmetric part of the homology of the top degree d_λ under the action of $G_\lambda = S_{m_1} \times \cdots \times S_{m_{n-1}}$:

$$H_{top}(C_\lambda(z))_s = \{\sigma \in H_{top}(C_\lambda(z))\colon g_*\sigma = (-1)^g \sigma, \quad g \in G_\lambda\}.$$

Lemma 3.2. If $\sigma \in H_{top}(C_\lambda(z))_s$ then $\int_\sigma \omega_T(z)$ depends only on the tabloid $\{T\}$ of T. □

Indeed, if T and T' differ by an element h of the row group $R(T)$, inducing a permutation h_1 of the set of boxes of λ then $\omega_T(z) = (-1)^g g^* \omega_{T'}(z)$, where $g \in G^\lambda$ is the permutation $t_i^b \mapsto t_i^{h_1(b)}$ of the variables (the sign comes from the volume form dt).

The following result can be extracted from Matsuo's paper [10].

Theorem 3.3. (Matsuo [10]) Let $\sigma \in H_{top}(C_\lambda(z))_s$, $\psi_\sigma(z) = \sum_{\{T\}} \int_\sigma \omega_T(z) e_T \in (V^{\otimes N})_\lambda$, with summation over all tabloids $\{T\}$ on λ.

(i) $\psi_\sigma(z) \in W^\lambda = (V^{\otimes N})_\lambda^{\mathfrak{n}_+}$.

(ii) As z varies in some neighborhood of a point in C_N, $\psi_\sigma(z)$ is a solution of the KZ equation (1.1). □

Recall that homology groups of neighboring fibers of a fiber bundle are canonically identified, so (ii) makes sense.

Here are some details on how to deduce Theorem 3.3 from [10]: Matsuo considers the general Knizhnik–Zamolodchikov equation

$$\partial_i \varphi = m \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} \varphi, \tag{3.1}$$

for $\varphi(z_1, \dots, z_N)$ taking values in a tensor product $V^*_{\lambda_1} \otimes \cdots \otimes V^*_{\lambda_N}$ of Verma modules $V^*_{\lambda_i}$ over $\mathfrak{sl}_n$ of highest weight λ_i[1]. In this version of the KZ equation, Ω_{ij} denotes the action on the i-th and j-th factors of the tensor in $\mathfrak{sl}_n \otimes \mathfrak{sl}_n$ dual to the invariant bilinear form $(x, y) = \mathrm{tr}(xy)$ on $\mathfrak{sl}_n$. The Verma module V^*_λ is generated by a highest weight vector v^*_λ and is free over the Lie subalgebra of strictly lower triangular matrices $\mathfrak{n}_-$, so it has a Poincaré–Birkhoff–Witt basis

$$u^*_\lambda(\vec{p}) = \prod_{\alpha > \beta} \frac{E_{\alpha,\beta}^{p_{\alpha,\beta}}}{p_{\alpha,\beta}!} v^*_\lambda, \quad \vec{p} = (p_{\alpha,\beta})_{n \geq \alpha > \beta \geq 1}$$

where the product is defined for some choice of ordering of the standard basis $(E_{\alpha,\beta})_{\alpha>\beta}$, of $\mathfrak{n}_-$. Accordingly, we have a basis $u^*(\vec{p}) = \otimes_{a=1}^N u^*_{\lambda_a}(\vec{p}_a)$, $\vec{p} = (\vec{p}_1, \dots, \vec{p}_N)$, of the tensor product. Matsuo's integral formula for solutions has the form $\varphi = \sum I(\vec{p}) u^*(\vec{p})$ for some integrals $I(\vec{p})$ whose integrand is described explicitly in [10]. The sum is over all $\vec{p}$ such that $u^*(\vec{p})$ has a given weight λ. It is then shown in [10] that the solution takes values in the primitive vectors of weight λ.

For our purpose we need a special case of Matsuo's construction, for which the formulae simplify considerably. The vector representation V of $\mathfrak{sl}_n$ is the quotient of the Verma module $V^*_{\varpi_1}$ with fundamental highest weight ϖ_1 by its maximal proper submodule. Thus we take $\lambda_1 = \cdots = \lambda_N = \varpi_1$, and we have the tensor product $\pi_N \colon (V^*_{\varpi_1})^{\otimes N} \to V^{\otimes N}$ of canonical projections. The KZ equation (3.1) makes sense for the tensor product of any N modules; moreover any solution taking values in $(V^*_{\varpi_1})^{\otimes N}$ projects to a solution taking values in $V^{\otimes N}$. In the latter case we have $\Omega_{ij} = S_{ij} - \frac{1}{n}\mathrm{Id}$, where S_{ij} exchanges the i-th and the j-th factor of the tensor product and thus coincides with the action of s_{ij} on the S_N-module $V^{\otimes N}$. It follows that if φ is a solution of (3.1), then

$$\psi(z_1, \dots, z_N) = \prod_{i<j} (z_i - z_j)^{m(1+1/n)} \pi_N \circ \varphi(z_1, \dots, z_N)$$

[1] Actually, he uses lowest weights instead of highest weights, $N-1$ instead of N and $n+1$ instead of n, but it is easy to translate to our convention.

is a solution of (1.1). The canonical projection $\pi\colon V^*_{\varpi_1} \to V$ sends most PBW basis vectors to zero. The only remaining ones are the classes of

$$e_\alpha = E_{\alpha,\alpha-1} \dots E_{3,2}E_{2,1}v_{\varpi_1}, \quad \alpha = 1, \dots, n.$$

It is straightforward to check that we obtain our integral formulae, as described in the Introduction, by restricting the sum in Matsuo's formula to include only tensor products of vectors e_α.

Corollary 3.4.

(i) The solution ψ_σ of Theorem 3.3 can be written as

$$\psi_\sigma(z) = \sum_{T \in T(\lambda)} \int_\sigma \omega_T(z) v_T.$$

(ii) $\psi_\sigma(g \cdot z) = g\psi_\sigma(z)$, $\sigma \in H_{top}(C_\lambda(z))_s = H_{top}(C_\lambda(g \cdot z))_s$. □

Proof. (i) follows from Theorem 3.3, (i) and Proposition 2.1. For (ii) one uses the original expression for ψ_σ. ■

3.3 Completeness

We show here that all solutions are given as integrals over suitable cycles. The proof is in two parts: first we construct an S_N-equivariant map $(W^\lambda)^* \to KZ(\lambda, m)$, defined as the integral over the cycles σ_T. Since $(W^\lambda)^*$ is irreducible and of the right dimension, it then suffices to check that the map is nonzero, which can be done by an asymptotic analysis.

We start by describing the action of the symmetric group on cycles.

Lemma 3.5. For any numbering T of λ and $z = (z_1, \dots, z_N) \in C_N$, let $\sigma_T = \sigma_T(z) \in H_{top}(C_\lambda(z))_s$ be the homology class defined in the introduction as the skew-symmetrization of the image of the fundamental class by a map $(S^1)^{d_\lambda} \to C_\lambda(z)$.

(i) For all $g \in S_N$, $\sigma_{gT}(g \cdot z) = \sigma_T(z)$.

(ii) If T and T' are numberings of λ differing by a row permutation, then $\sigma_T(z) = \sigma_{T'}(z)$. Thus $\sigma_T(z)$ depends only on the tabloid of T. □

Proof. (i) holds by construction. The proof of (ii) is the same as the proof of Lemma 3.2. This time the sign comes from the change of orientation. ■

Thus we get a map

$$\Psi_m^\lambda\colon (V^{\otimes N})_\lambda \to KZ(\lambda, m)$$
$$e_T \mapsto \int_{\sigma_T} \omega = \sum_{T' \in T(\lambda)} \psi_{T,T'} v_{T'}.$$

It is well defined since σ_T, just as e_T, depends only on the tabloid of T.

Lemma 3.6. The map Ψ_m^λ is S_N-equivariant. □

This follows from Lemma 3.5 and Corollary 3.4.

To prove that the map is nonzero we will use the following key technical lemma. Let $(r(b), c(b))$ be the coordinates (row and column number) of the box b in the Young diagram λ.

Lemma 3.7. Let T be the standard tableau mapping the kth box, counted from left to right and top to bottom, to k. Then $\psi_{T,T}(z)$ is not identically zero. The leading term for $0 \ll |z_1| \ll |z_2| \ll \cdots \ll |z_N|$ is

$$\psi_{T,T}(z) \sim C \prod_{b \in \lambda} z_{T(b)}^{m(T(b)-1+c(b)-r(b))} + \cdots ,$$

for some $C \neq 0$. □

Proof. We show that, as $z_N \to \infty$,

$$\psi_{T,T}(z_1, \ldots, z_N) = C' z_N^{m(N-1+\lambda_n-n)} (\psi_{T',T'}(z_1, \ldots, z_{N-1}) + O(z_N)),$$

where T' is the standard tableau with $N-1$ boxes obtained from T by removing the last box and C' is a nonzero combinatorial factor. Since $\psi_{\boxed{1},\boxed{1}} = 1$ for the tableau with one box, this gives an inductive proof of the claim.

Let λ', the shape of T', be λ without the last box. Then

$$\Phi_\lambda = \Phi_{\lambda'} \prod_{k=1}^{N-1} (z_k - z_N)^2 (t_1^{\boxed{k}} - z_N)^{-1} \prod_{s=1}^{n-1} (t_s^{\boxed{N}} - t_{s-1}^{\boxed{N}})^{-1}$$
$$\prod_{s=1}^{n-1} \left(\prod_{\substack{k<N \\ r(\boxed{k})>s}} (t_s^{\boxed{k}} - t_s^{\boxed{N}})^2 \prod_{\substack{k<N \\ r(\boxed{k})>s+1}} (t_{s+1}^{\boxed{k}} - t_s^{\boxed{N}})^{-1} \prod_{\substack{k<N \\ r(\boxed{k})>s-1}} (t_s^{\boxed{N}} - t_{s-1}^{\boxed{k}})^{-1} \right),$$

where $\boxed{k} = T^{-1}(k)$ is the box of λ labeled by k and we set $t_0^{\boxed{k}} = z_k$. Also,

$$\phi_T = \phi_{T'} \prod_{s=1}^{n-1} (t_s^{\boxed{N}} - t_{s-1}^{\boxed{N}})^{-1}.$$

The leading term as $z_N \to \infty$ in $\psi_{T,T}$ is obtained when the variables $t_s^{\boxed{N}}$ run over circles around z_N. With the variable substitution $t_s^{\boxed{N}} = z_N + \tau_1 + \cdots + \tau_s$, the leading term as $z \to \infty$ of the integration of ω_T over the variables $t_s^{\boxed{N}}$ is

$$\pm\, \omega_{T'} z_N^{m(N-1)} \mathrm{res}_{\tau_{n-1}=0} \cdots \mathrm{res}_{\tau_1=0} \Omega,$$

where

$$\begin{aligned}\Omega = &\prod_{s=1}^{n-2} (z_N + \tau_1 + \cdots + \tau_s)^{m(2m_s - m_{s-1} - m_{s+1})} \tau_s^{-m-1} d\tau_s \\ &\times (z_N + \tau_1 + \cdots + \tau_{n-1})^{m(2m_{n-1} - m_{n-2} - 1)} \tau_{n-1}^{-m-1} d\tau_{n-1} \\ &= \prod_{s=1}^{n-2} (z_N + \tau_1 + \cdots + \tau_s)^{m(\lambda_{s+1} - \lambda_s)} \tau_s^{-m-1} d\tau_s \\ &\times (z_N + \tau_1 + \cdots + \tau_{n-1})^{m(\lambda_n - \lambda_{n-1} - 1)} \tau_{n-1}^{-m-1} d\tau_{n-1}.\end{aligned}$$

The residues can be computed explicitly. Such expressions give a nonzero result if the total power of all factors containing any given τ_s is *negative*. This power is $(\lambda_n - \lambda_s - 1)m$ which is indeed negative for all $s = 1, \ldots, n-1$. ■

Theorem 3.8. The map $\Psi_m^\lambda \colon (V^{\otimes N})_\lambda \to KZ(\lambda, m)$ is an epimorphism of S_N-modules with kernel $(\mathfrak{n}_- V^{\otimes N})_\lambda$ and therefore defines an isomorphism $(W^\lambda)^* \to KZ(\lambda, m)$. In particular, the images ψ_T of the basis vectors $[e_T]$, $T \in \mathcal{T}(\lambda)$, of $(W^\lambda)^* = (V^{\otimes N}/\mathfrak{n}_- V^{\otimes N})_\lambda$, form a basis of the space of solutions of the KZ equation. □

Proof. By Lemma 3.6 and Lemma 3.7, Ψ_m^λ is a nonzero homomorphism from the S_N-module $(V^{\otimes N})_\lambda$ to the S_N-module $KZ(\lambda, m)$. By Theorem 3.1 $KZ(\lambda, m)$ is isomorphic to the irreducible S_N-module W^λ. Since the image of a homomorphism is a submodule, it follows that the map is surjective. On the other hand, by (2.1), $W^\lambda \simeq (W^\lambda)^*$ occurs in $(V^{\otimes N})_\lambda$ with multiplicity $\dim M_\lambda^\lambda = 1$ and the claim follows from Proposition 2.2. ■

Another interesting corollary of Lemma 3.7 is the following classical formula.

Proposition 3.9. (Frobenius [5]) Let $f_2(\lambda)$ be the value of the central element $\sum_{i<j} s_{ij}$ in the representation W^λ. Then

$$f_2(\lambda) = \sum_{b\in\lambda}(c(b) - r(b)), \tag{3.2}$$

where as before $r(b)$ and $c(b))$ are respectively the row and column coordinates of the box b in the Young diagram λ. □

To prove this we recall that the degree of the polynomial solutions from $KZ(W, m)$ is equal to the value of the central element $\sum_{i<j} m(s_{ij} + 1)$ in the irreducible representation W (see [3]). Comparing this with the leading term of the solution from Lemma 3.7 and taking into account that $\sum_{i<j} 1 = \sum_{b\in\lambda}(T(b) - 1) = \frac{N(N-1)}{2}$ we come to Frobenius formula (3.2).

3.4 Integrality

It is well known (and clear from Proposition 2.1) that $(v_T)_{T\in\mathcal{T}(\lambda)}$ is an integral basis of W^λ, i.e., $W^\lambda_{\mathbb{Z}} = \oplus_{T\in\mathcal{T}(\lambda)}\mathbb{Z}v_T$ is a module over the group ring $\mathbb{Z}S_N$.

Theorem 3.10. The functions $\psi_{T,T'}$ are homogeneous polynomials in $z_1, \ldots, z_N$ with integer coefficients. Thus

$$\Psi^\lambda_m(z_1, \ldots, z_N) = \sum_{T,T'\in\mathcal{T}(\lambda)} \psi_{T,T'}(z_1, \ldots, z_N)v_T \otimes v_{T'} \in \mathbb{W}^\lambda_{\mathbb{Z}} \otimes_{\mathbb{Z}} W^\lambda_{\mathbb{Z}}[z_1, \ldots, z_N].$$

Moreover Ψ^λ_m is S_N-invariant: $\Psi^\lambda_m(g \cdot z) = g \otimes g\Psi^\lambda_m(z)$, for all $g \in S_N$. □

The invariance is a rephrasing of the homomorphism property of Ψ^λ_m of Theorem 3.8. The integrality follows from repeated application of the following elementary

Lemma 3.11. Let m_i be any integers and $w_1, \ldots, w_k$ distinct complex numbers. Then, for any contour γ in the complex plane not passing through $w_1, \ldots, w_k$,

$$\frac{1}{2\pi i}\oint_\gamma \prod_{i=1}^{k}(t - w_i)^{m_i}dt = \sum_{i<j} c_{ij}(w_i - w_j)^{\ell_{ij}},$$

for some integers c_{ij}, ℓ_{ij}. □

Theorems 3.8 and 3.10 imply the statements of Theorem 1.1.

Example 3.12. We demonstrate here our formulae in the simplest nontrivial example when $N = 3$ and $\lambda = (2, 1)$, which corresponds to the usual two-dimensional representation of S_3. In this case there are two standard tableaux:

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}\,, \qquad S = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}\,.$$

The corresponding primitive vectors are $v_T = \epsilon_3 - \epsilon_1$, $v_S = \epsilon_2 - \epsilon_1$.

The residues can be computed explicitly and one obtains a basis of $KZ(\lambda, m)$:

$$\psi_1(z_1, z_2, z_3) = z_{23}^{2m} \sum_{k=0}^{m} d_{m,k}\left((m-k)v_T + kv_S\right) z_{12}^{m-k} z_{13}^{k},$$

$$\psi_2(z_1, z_2, z_3) = z_{13}^{2m} \sum_{k=0}^{m} (-1)^{m-k} d_{m,k}\left((m-k)v_T - mv_S\right) z_{12}^{m-k} z_{23}^{k},$$

where we abbreviate $z_i - z_j = z_{ij}$ and

$$d_{m,k} = -\frac{1}{m}\binom{-m}{k}\binom{-m}{m-k}.$$

4 Duality $m \leftrightarrow -m$ and intersection pairing

To apply our results to the case of negative integers m we can use the isomorphism between the spaces of solutions $KZ(V, m)$ and $KZ(V \otimes \mathrm{Alt}, -m)$ mentioned in the Introduction.

Lemma 4.1. For any $\psi \in KZ(V, m)$ the function

$$\phi = \prod_{i<j} (z_i - z_j)^{-2m} \psi \otimes \epsilon$$

belongs to the space of solutions $KZ(V \otimes \mathrm{Alt}, -m)$. □

Indeed from the KZ equation (1.1) we have

$$\partial_i \phi = (-2m)\Big(\sum_{j \neq i}^{N} \frac{1}{z_i - z_j}\Big) \prod_{i<j} (z_i - z_j)^{-2m} \psi \otimes \epsilon + m \prod_{i<j} (z_i - z_j)^{-2m} \Big(\sum_{j \neq i}^{N} \frac{s_{ij} + 1}{z_i - z_j} \psi\Big) \otimes \epsilon$$

$$= (-m) \prod_{i<j} (z_i - z_j)^{-2m} \sum_{j \neq i}^{N} \frac{s_{ij} + 1}{z_i - z_j} (\psi \otimes \epsilon) = (-m) \sum_{j \neq i}^{N} \frac{s_{ij} + 1}{z_i - z_j} \phi,$$

since $s_{ij}(\phi \otimes \epsilon) = -(s_{ij}\phi) \otimes \epsilon$.

Remark 4.2. The pre-factor $\prod_{i<j}(z_i - z_j)^{-2m}$ will disappear if we consider the KZ equation in the form

$$\partial_i \psi = m \sum_{j \neq i}^{N} \frac{s_{ij}}{z_i - z_j} \psi, \quad i = 1, \ldots, N,$$

which in general has rational solutions. The duality between m and $-m$ for similar systems was used in [4] to explain the shift operator for the Calogero–Moser systems.

It is well known that the involution $V \to V \otimes \mathrm{Alt}$ corresponds to the *transposition* of the Young diagram $\lambda \to \lambda'$. Thus we have established an isomorphism

$$KZ(\lambda, m) \approx KZ(\lambda', -m). \tag{4.1}$$

Note that the configuration spaces $C_\lambda(z_1, \ldots, z_N)$ and $C_{\lambda'}(z_1, \ldots, z_N)$ in general are quite different (in particular, they have different dimensions), so the structure of the integral formulas for the solution of the KZ equations for a given Young diagram, which we get in this way, substantially depends on the sign of m.

It turns out that there is a link between the spaces of KZ solutions with the *same* Young diagram:

$$j : KZ(\lambda, m) \approx KZ(\lambda, -m)^*. \tag{4.2}$$

More precisely, there exists a natural pairing

$$KZ(V, m) \times KZ(V^*, -m) \to \mathbb{C}, \tag{4.3}$$

where V^* is the dual space to V with the natural action of S_N. It is defined by the following lemma.

Lemma 4.3. Let $\langle\ ,\ \rangle$ denote the canonical pairing between V and V^*. Then for any two solutions $\psi(z_1, \ldots, z_N) \in KZ(V, m)$ and $\phi(z_1, \ldots, z_N) \in KZ(V^*, -m)$ the product $\langle \psi(z_1, \ldots, z_N), \phi(z_1, \ldots, z_N) \rangle$ is independent of $z_1, \ldots, z_N$ and thus defines a nondegenerate canonical pairing (4.3). □

The proof is a straightforward check using the KZ equations (1.1):

$$\partial_i \langle \psi(z_1, \ldots, z_N), \phi(z_1, \ldots, z_N) \rangle = \langle m \sum_{j \neq i}^{N} \frac{s_{ij} + 1}{z_i - z_j} \psi, \phi \rangle + \langle \psi, (-m) \sum_{j \neq i}^{N} \frac{s_{ij} + 1}{z_i - z_j} \phi \rangle = 0,$$

since $\langle s_{ij}v, w\rangle = \langle v, s_{ij}w\rangle$ for any $v \in V, w \in V^*$. The same calculation holds of course for any Coxeter group.

Note that the S_N-module V^* is isomorphic to V and in the irreducible case the isomorphism is *almost canonical* in the sense that it is unique up to a scaling factor. This leads to an isomorphism (4.2) and allows us to find the solutions from $KZ(\lambda, -m)$ as follows.

Let us choose any basis $e_1, \dots, e_M$ in $V = W^\lambda$ and a basis of solutions $\psi_\alpha = \sum_{i=1}^M \psi_\alpha^i(z_1, \dots, z_N)e_i$, in $KZ(\lambda, m)$. Let $\Phi_{\lambda,m}(z_1, \dots, z_N) = \|\psi_\alpha^i(z_1, \dots, z_N)\|$ be the corresponding $M \times M$ *fundamental matrix* of $KZ(\lambda, m)$. Let $e^1, \dots, e^M$ be the dual basis in $V^* : \langle e^i, e_j\rangle = \delta_j^i$. We are looking now for a fundamental matrix $\Phi_{\lambda,-m}(z_1, \dots, z_N) = \|\phi_\beta^j(z_1, \dots, z_N)\|$ for $KZ(\lambda, -m)$, given by a basis of solutions $\phi^\beta = \sum_{j=1}^M \phi_j^\beta(z_1, \dots, z_N)e^j$. From Lemma 4.3 it follows that one can choose the basis of solutions in such a way that

$$\sum_{i=1}^{M} \psi_\alpha^i(z_1, \dots, z_N)\phi_i^\beta(z_1, \dots, z_N) = \delta_\alpha^\beta. \tag{4.4}$$

Proposition 4.4. A fundamental matrix for $KZ(\lambda, -m)$ can be found as the transposed inverse matrix to the fundamental matrix of $KZ(\lambda, m)$:

$$\Phi_{\lambda,-m}(z_1, \dots, z_N) = (\Phi_{\lambda,m}(z_1, \dots, z_N)^{-1})^T.$$

The determinant of the fundamental matrix is given by

$$\det \Phi_{\lambda,m}(z_1, \dots, z_N) = C \prod_{i<j} (z_i - z_j)^{2md_+(\lambda)}, \tag{4.5}$$

where $C = C(\lambda, m)$ is a nonzero constant and $d_+(\lambda) = \dim W_+^\lambda$ is the dimension of the fixed subspace of reflection s_{ij} acting in the representation W^λ. □

The first part is equivalent to (4.4). To prove the formula for the determinant we can use the standard fact that if matrix Φ satisfies the differential equation $\Phi' = A\Phi$ then its determinant satisfies the equation $\det \Phi' = trA \det \Phi$. Applying this to the KZ equation (1.1) and using the fact that $tr(s_{ij} + 1) = 2d_+(\lambda)$ we have the result. The formula (4.5) shows that the singularities of $\Phi_{\lambda,-m}(z_1, \dots, z_N)$ are located on the hyperplanes $z_i = z_j$, which of course follows also from the previous considerations.

In the rest of this section we discuss the topological interpretation of the duality (4.3) as intersection pairing between certain homology groups. It is based on the integral formula for the solutions of the KZ equation found in our previous work [3] (see section 4.5).

We restrict ourselves with the special case of the reflection representation of S_N, which is the standard $(N-1)$-dimensional irreducible representation on the hyperplane $x_1 + \cdots + x_n = 0$ in $\mathbb{C}^N$ defined by permutation of coordinates. This representation is isomorphic to W^λ with $\lambda = (N-1, 1)$. For positive m our integrals giving the solutions are one-dimensional and, in terms of the standard basis ϵ_b of $\mathbb{C}^N$, they have the form

$$\psi_a = \prod_{1\le i<j\le N} (z_i - z_j)^{2m} \mathrm{res}_{t=z_a} \prod_{i=1}^{N} (t - z_i)^{-m} \sum_{b=1}^{N} \frac{1}{t - z_b} \epsilon_b \, dt, \quad a = 1, \ldots, N.$$

They obey the relation $\psi_1 + \cdots + \psi_N = 0$ and $\psi_1, \ldots, \psi_{N-1}$ form a basis of the solution space $KZ(\lambda, m)$.

A different integral representation for the solution space $KZ(\lambda, -m)$ for positive m was found in [3], where it was shown that

$$\phi_a = \prod_{1\le i<j\le N} (z_i - z_j)^{-2m} \int_{z_a}^{z_N} \prod_{i=1}^{N} (t - z_i)^{m} \sum_{b=1}^{N} \frac{1}{t - z_b} \epsilon_b \, dt, \qquad a = 1, \ldots, N-1$$

give a basis in $KZ(\lambda, -m)$. In particular, in the case $N = 3$ we have after explicit evaluation of the integrals the following basis:

$$\phi_1(z_1, z_2, z_3) = z_{23}^{-2m} \sum_{k=0}^{m} (-1)^{m+k} d'_{m,k} \left((-m-k) v_T + k v_S\right) z_{12}^{-m-k} z_{13}^{k},$$

$$\phi_2(z_1, z_2, z_3) = z_{13}^{-2m} \sum_{k=0}^{m} d'_{m,k} \left((-m-k) v_T + m v_S\right) z_{12}^{-m-k} z_{23}^{k},$$

where

$$d'_{m,k} = \binom{m}{k} \frac{(m-1)!(m+k-1)!}{(2m+k)!}.$$

Thus, in a more invariant geometric terms, for $\lambda = (N-1, 1)$ and $m \in \mathbb{Z}_{>0}$, we have two maps

$$\psi \colon H_1(\mathbb{C} \smallsetminus \{z_1, \ldots, z_N\}) \to W^\lambda, \qquad \phi \colon H_1(\mathbb{C}, \{z_1, \ldots, z_N\}) \to W^\lambda,$$

sending horizontal sections for the Gauss–Manin connection to solutions in $KZ(\lambda, m)$ and in $KZ(\lambda, -m)$, respectively. The map ϕ induces an isomorphism between the complexification of the space of horizontal relative cycles and $KZ(\lambda, -m)$, whereas ψ has a one-dimensional kernel spanned by a cycle surrounding all the points z_i. This kernel is exactly the complexification of the left kernel of the intersection pairing

$$H_1(\mathbb{C} \smallsetminus \{z_1, \ldots, z_N\}) \times H_1(\mathbb{C}, \{z_1, \ldots, z_N\}) \to \mathbb{Z},$$

and the right kernel is trivial. Since the intersection pairing of cycles is preserved by the Gauss–Manin connection we obtain a nondegenerate S_N-invariant pairing $KZ(\lambda, m) \times KZ(\lambda, -m) \to \mathbb{C}$.

The claim is that this pairing is proportional to the one described in Lemma 4.3.

Proposition 4.5. Let $\lambda = (N-1, 1)$ and $m > 0$. The pairing (4.3) of solution spaces $KZ(\lambda, m)$ and $KZ(\lambda, -m)$ is proportional to the image of the intersection pairing $(\cdot)$. More precisely, let $\sigma \in H_1(\mathbb{C} \smallsetminus \{z_1, \ldots, z_N\}), \quad \tau \in H_1(\mathbb{C}, \{z_1, \ldots, z_N\})$ vary with the points z_i as horizontal sections, then

$$\langle \psi_\sigma(z_1, \ldots, z_N), \phi_\tau(z_1, \ldots, z_N) \rangle = C_N \frac{1}{m} (\sigma \cdot \tau),$$
$$\sigma \in H_1(\mathbb{C} \smallsetminus \{z_1, \ldots, z_N\}), \quad \tau \in H_1(\mathbb{C}, \{z_1, \ldots, z_N\}),$$

for some constant $C_N \neq 0$ depending on the normalization of the isomorphism $(W^\lambda)^* \to W^\lambda$. □

Proof. The proof follows from Schur's lemma, except for the determination of the constant of proportionality of the two pairings. To compute it, we consider two special cycles, namely a small circle around z_1 and a path from z_1 to z_N. These cycles have intersection number -1 and the corresponding solutions are ψ_1 and ϕ_1, respectively. It is sufficient to compute the pairing when $z_1 = 0$ in the limit $z_N \to 0$, where also ϕ_1 is regular. After the change of variable $t = z_N \tau$ we obtain

$$\psi_1(0, z_2, \ldots, z_{N-1}, 0) = F_m \operatorname{res}_{\tau=0} \tau^{-m} (\tau - 1)^{-m} \left(\frac{\epsilon_1}{\tau} + \frac{\epsilon_N}{\tau - 1} \right) d\tau,$$
$$\phi_1(0, z_2, \ldots, z_{N-1}, 0) = F_m^{-1} \int_0^1 \tau^m (\tau - 1)^m \left(\frac{\epsilon_1}{\tau} + \frac{\epsilon_N}{\tau - 1} \right) d\tau,$$

for some function F_m of $z_2, \ldots, z_{N-1}$. By integrating by parts we see that the coefficient of ϵ_N is minus the coefficient of ϵ_1 (as it should be since the solutions take values in primitive vectors). The result of the calculation is

$$\psi_1(0, z_2, \ldots, z_{N-1}, 0) = F_m (-1)^m \frac{(2m-1)!}{m!(m-1)!} (\epsilon_1 - \epsilon_N),$$
$$\phi_1(0, z_2, \ldots, z_{N-1}, 0) = F_m^{-1} (-1)^m \frac{m!(m-1)!}{(2m)!} (\epsilon_1 - \epsilon_N).$$

If we normalize the pairing $W^\lambda \times W^\lambda \to \mathbb{C}$ defining the isomorphism between W^λ and $(W^\lambda)^*$ so that the basis ϵ_a is orthonormal, we obtain $\langle\psi_1, \phi_1\rangle = 1/m$. ■

Note that since the cycles defining the bases ψ_a and ϕ_a are dual (up to sign) with respect to the intersection pairing, we deduce

$$\langle\psi_a(z_1, \ldots, z_N), \phi_b(z_1, \ldots, z_N)\rangle = -C_N \frac{1}{m}\, \delta_{a,b},$$

where $C_N = -1$ with the choice of normalization described in the proof.

Remark 4.6. We would like to mention that an extension of these results to the case of general representations should involve a suitable replacement of relative homology. The singularity as integration variables approach each other causes difficulties with the naive generalization.

Acknowledgments

We are grateful to A. Okounkov, who helped us to trace back the Frobenius formula, to O. Chalykh, who attracted our attention to Opdam's lecture notes [11], and to A. Chervov for stimulating discussions.

This work has been partially supported by the European Union through the FP6 Marie Curie RTN ENIGMA (Contract number MRTN-CT-2004-5652). It was made possible by a visit of A.V. to FIM at ETH, Zurich with the support of the MISGAM programme of the European Science Foundation, and a visit of the authors to the Newton Institute for Mathematical Sciences, Cambridge. We thank these institutions for their support and hospitality.

The work of A.V. was also partially supported by EPSRC (grant EP/E004008/1).

References

[1] Christe, P., and R. Flume. "The Four-Point Correlations of All Primary Operators of the $d = 2$ Conformally Invariant SU(2) σ-model with Wess-Zumino Term." *Nuclear Physics B* 282, no. 2 (1987): 466–494.

[2] Feigin, M., and A. P. Veselov. "Quasi-invariants of Coxeter Groups and m-harmonic Polynomials." *International Mathematics Research Notices* (2002): 521–545.

[3] Felder, G., and A. P. Veselov. "Action of Coxeter Groups On m-harmonic Polynomials and KZ Equations." *Moscow Mathematical Journal* 3, no. 4 (2003): 1269–1291.

[4] Felder, G., and A. P. Veselov. "Shift Operators for the Quantum Calogero-Sutherland Problems Via Knizhnik-Zamolodchikov Equation." *Communications in Mathematical Physics* 160 (1994): 259–273.

[5] Frobenius, F. G. *Über die Charaktere der symmetrischen Gruppe*, 516–534. Berlin: Sitz. König. Preuss. Akad. Wissenschaften zu Berlin, 1900.

[6] Fulton, W. *Young Tableaux, with Applications to Representation Theory and Geometry*: Cambridge, UK: Cambridge University Press, 1997.

[7] Goodman, R., and N. R. Wallach. *Representations and Invariants of the Classical Groups*: Cambridge, UK: Cambridge University Press, 1998.

[8] Kazarnovski-Krol, A. "Cycles for Asymptotic Solutions and the Weyl Group." Pp. 123–150 in *The Gelfand Mathematical Seminars, 1993–1995*. Boston, MA: Gelfand Math. Sem., 1996.

[9] Macdonald, I. G. *Symmetric Functions and Hall Polynomials*, 2nd edn. New York, NY: Oxford University Press, 1999.

[10] Matsuo, A. "An Application of Aomoto-Gelfand Hypergeometric Functions to the SU(n) Knizhnik-Zamolodchikov Equation." *Communications in Mathematical Physics* 134 (1990): 65–77.

[11] Opdam, E. "Complex Reflection Groups and Fake Degrees." In *Lectures at RIMS, Kyoto University (Japan) in 1997*. (1998): Preprint arXiv: math.RT/9808026.

[12] Varchenko, A. "Hypergeometric Functions and Representation Theory of Lie Algebras and Quantum Groups." In *Advanced Series in Mathematical Physics*, Vol. 21. River Edge, NJ: World Scientific, 1995.

[13] Zamolodchikov, A. B., and V. A. Fateev. "Operator Algebra and Correlation Functions in the Two-Dimensional $SU(2) \times SU(2)$ chiral Wess-Zumino Model." *Soviet Journal of Nuclear Physics* 43 (1986): 657–664.