TWO REMARKS ON THE HOMOLOGY OF GROUP EXTENSIONS

Dedicated to the memory of Hanna Neumann

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0. Introduction

In this note we apply a particular technique to obtain information on the homology homomorphism $\varepsilon_*: H_*(G; A) \to H_*(Q; A)$ associated with a group extension

(0.1) $$N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q,$$

and a Q-module A. The technique consists of using ε itself to pull-back (0.1); that is, we construct the pull-back extension induced from (0.1) by ε. This, however, is nothing but the semidirect product, $N \rtimes G$, of N and G, with G operating on the left on N by conjugation. Thus we obtain from (0.1) the commutative diagram

(0.2) $$\begin{array}{ccccc} N & \overset{\mu_0}{\rightarrowtail} & N \rtimes G & \overset{\varepsilon_0}{\twoheadrightarrow} & G \\ \| & & \downarrow \varepsilon_1 & & \downarrow \varepsilon \\ N & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & Q \end{array}$$

where ε_0 is the projection and ε_1 is the multiplication $\varepsilon_1(n,x) = nx, n \in N, x \in G$. We now apply the Lyndon-Hochschild-Serre spectral sequence functor to (0.2) and carry out computations in dimensions 2 and 3.

In Chapter 1 we are concerned to study the kernel of $\varepsilon_*: H_2(G;A) \to H_2(Q;A)$. Thus we seek to extend the standard 5-term exact sequence

(0.3) $$H_2(G; A) \to H_2(Q; A) \to N_{ab} \otimes_Q A \to H_1(G; A) \to H_1(Q; A) \to 0$$

one place to the left. We obtain, by the method outlined above, a generalization to arbitrary coefficients of a theorem proved by Nomura [6], by topological methods, for integer coefficients. The rest of Chapter 1 is concerned with refinements of the result and with the relation of the result to Ganea's extension of (0.3) in [5] in the case of a *central* extension (0.1) and integer coefficients.

In Chapter 2 we refer to the 8-term sequence obtained in [4] which extends (0.3) three places to the left, again in the case of a central extension and integer coefficients. The extended sequence (2.1) is then extended a further two places to the left, provided thet we replace $\varepsilon_*: H_3G \to H_3Q$, which is the initial homomorphism of the 8-term sequence, by an induced homomorphism

$$\bar{\varepsilon}: H_3G/B \to H_3Q,$$

where B is an explicitly described subgroup of H_3G. Once again, the technique is as described earlier, but we make decisive use, first, of the fact that the spectral sequence of a direct product collapses (if N is central, then the semi-direct product $N \rtimes G$ is just the direct product $N \times G$) and, second, of André's calculation [1] of the differential d_2 in the Lyndon-Hochschild-Serre spectral sequence.

The rest of Chapter 2 consists of a discussion of the subgroup B of H_3G and the associated quotient group H_3G/B. We remark that a similar extension of the 8-term sequence (2.1) was obtained in [2] by topological arguments, and that this extension also involved factoring out a certain subgroup B' of H_3G. It is always the case that $B' \subseteq B$, but we show by an example that, in general, $B' \neq B$. Of course, this difference in the third term of the two extensions of (2.1) (i.e (2.2) of this paper and (1.5) of [2]) is compensated by a complementary difference in their second term.

The fact that an exact sequence of the type of (2.2) must exist was first discovered by Gut.

1. Nomura's Theorem

1.1 Let $E: N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$ be an exact sequence of groups. Given any $\tau: P \to Q$ we form the pull-back

$$\begin{array}{ccc} G_\tau & \overset{\varepsilon_\tau}{\twoheadrightarrow} & P \\ \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \tau} \\ G & \overset{\varepsilon}{\twoheadrightarrow} & Q \end{array}$$

and hence an induced sequence

(1.1) $$\tau^*E: N \overset{\mu_\tau}{\rightarrowtail} G_\tau \overset{\varepsilon_\tau}{\twoheadrightarrow} P$$

together with a map

(1.2) $$(1, \kappa, \tau): \tau^*E \to E.$$

It follows from the pull-back property that any map $(\alpha, \beta, \tau): E' \to E$ of sequences,

$$(1.3)\qquad \begin{array}{ccccccc} E': & M & \overset{\mu'}{\rightarrowtail} & H & \overset{\varepsilon'}{\twoheadrightarrow} & P \\ \downarrow & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\tau} \\ E: & N & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & Q, \end{array}$$

factors uniquely through τ^*E as

$$(\alpha, \beta, \tau) = (1, \kappa, \tau) \circ (\alpha, \beta', 1)$$

for a uniquely determined $\beta': H \to G_\tau$.

Now consider, in particular, ε^*E. It is easy to see that ε^*E is just

$$(1.4)\qquad N \overset{\mu_0}{\rightarrowtail} N \rtimes G \overset{\varepsilon_0}{\twoheadrightarrow} G,$$

where $N \rtimes G$ is the semi-direct product of N and G, with G operating on the left on N by inner automorphism. Moreover, μ_0 is the canonical embedding, ε_0 the canonical projection; and, further, there is a canonical splitting $\lambda_0: G \to N \rtimes G$ of (1.4) such that $\varepsilon_0\lambda_0 = 1$, given by $\lambda_0(x) = (1, x), x \in G$. We regard the splitting λ_0 as part of the structure of the semi-direct product.

The map $\varepsilon^*E \to E$ is $(1, \varepsilon_1, \varepsilon)$,

$$(1.5)\qquad \begin{array}{ccccccc} \varepsilon^*E: & N & \overset{\mu_0}{\rightarrowtail} & N \rtimes G & \overset{\varepsilon_0}{\twoheadrightarrow} & G \\ \downarrow & \| & & \downarrow{\scriptstyle\varepsilon_1} & & \downarrow{\scriptstyle\varepsilon} \\ E: & N & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & Q, \end{array}$$

where $\varepsilon_1: N \rtimes G \to G$ is the multiplication map, given by $\varepsilon_1(n, x) = nx, n \in N, x \in G$.

Now, for any $\sigma: P \to G$, $\sigma^*\varepsilon^*E$ splits. Indeed we obtain

$$(1.6)\qquad \begin{array}{cccccc} N & \overset{\mu'}{\rightarrowtail} & H & \overset{\varepsilon'}{\twoheadrightarrow} & P \\ \| & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\sigma} \\ N & \overset{\mu_0}{\rightarrowtail} & N \rtimes G & \overset{\varepsilon_0}{\twoheadrightarrow} & G \end{array}$$

together with a splitting $\lambda: P \to H$ such that $\rho\lambda = \lambda_0\sigma$. We prove a strong converse of this.

PROPOSITION 1.1. *The sequence ε^*E is universal for splitting exact sequences over E. Precisely, if in* (1.3), *E' splits by $\lambda: P \to H$ then we have a unique factorization.*

$$(1.7)\qquad (\alpha, \beta, \tau) = (1, \varepsilon_1, \varepsilon) \circ (\alpha, \rho, \sigma),$$

where ρ satisfies

$$(1.8)\qquad \rho\lambda = \lambda_0\sigma.$$

PROOF. Set $\sigma = \beta\lambda: P \to G$. Then $\varepsilon\sigma = \varepsilon\beta\lambda = \tau\varepsilon'\lambda = \tau$.

Now $\varepsilon\sigma\varepsilon' = \tau\varepsilon' = \varepsilon\beta$. Thus, by the pull-back property, there exists a unique $\rho: H \to N \rfloor G$, given by $\varepsilon_0\rho = \sigma\varepsilon'$, $\varepsilon_1\rho = \beta$. One easily proves that $\rho\mu' = \mu_0$. Moreover, observing that $\varepsilon_1\lambda_0 = 1$, it follows readily that $\rho\lambda = \lambda_0\sigma$. Conversely, from (1.7) and (1.8) we infer that $\sigma = \varepsilon_1\rho\lambda = \beta\lambda$, and then ρ is determined by the equations $\varepsilon_0\rho = \sigma\varepsilon'$, $\varepsilon_1\rho = \beta$.

NOTE. In fact, ρ is given by $\rho x = (\beta x(\beta\lambda\varepsilon' x)^{-1}, \beta\lambda\varepsilon' x)$, $x \in H$.

COROLLARY 1.2. *If* $\alpha = 1$, *then* $E' = \sigma^*\varepsilon^* E$.

Finally, we remark that, if N is *central* in G, then (1.4) reduces to the direct product

$$N \rightarrowtail^{\mu_0} N \times G \xrightarrow{\varepsilon_0} G; \tag{1.9}$$

we will exploit this in Chapter 2.

1.2 In this section we prove the following theorem, generalizing (to arbitrary coefficient modules) a theorem proved by Nomura [6] by topological methods.

THEOREM 1.3. *Given the short exact sequence of groups*

$$E: N \xrightarrow{\mu} G \xrightarrow{\varepsilon} Q$$

and the Q-module A, there exists an exact sequence

$$\ker \varepsilon_{0*} \xrightarrow{\varepsilon_{1*}} H_2(G;A) \xrightarrow{\varepsilon_*} H_2(Q;A) \to N_{ab} \otimes_Q A \to H_1(G;A) \to H_1(Q;A) \to 0,$$

where $\varepsilon_0, \varepsilon_1$ *are as in* (1.5) *and* $\varepsilon_{0*}: H_2(N \rfloor G; A) \to H_2(G;A)$.

PROOF. Of course only exactness at $H_2(G;A)$ is in question. We prove this by considering the map of Lyndon-Hochschild-Serre (henceforth, L-HS) spectral sequences induced by (1.5). For $\varepsilon^* E$ we have a spectral sequence $\{\tilde{E}_r^{pq}\}$, such that

$$\tilde{E}_2^{pq} = H_p(G; H_q(N;A));$$

and there is a filtration $\tilde{F}_0 \subseteq \tilde{F}_1 \subseteq \tilde{F}_2$ of $H_2(N \rfloor G; A)$ such that

$$\tilde{F}_0 = \tilde{E}_\infty^{02},\ \tilde{F}_1/\tilde{F}_0 = \tilde{E}_\infty^{11},\ \tilde{F}_2/\tilde{F}_1 = \tilde{E}_\infty^{20},\ \tilde{F}_2 = H_2(N \rfloor G; A)$$

Moreover, $\tilde{F}_1 = \ker \varepsilon_{0*}: H_2(N \rfloor G; A) \to H_2(G;A)$.

Similarly, for the extension E we have a spectral sequence $\{E_r^{pq}\}$ such that

$$E_2^{pq} = H_p(Q; H_q(N;A));$$

and there is a filtration $F_0 \subseteq F_1 \subseteq F_2$ of $H_2(G;A)$ such that

$$F_0 = E_\infty^{02},\ F_1/F_0 = E_\infty^{11},\ F_2/F_1 = E_\infty^{20},\ F_2 = H_2(G;A).$$

Moreover, $F_1 = \ker \varepsilon_*: H_2(G;A) \to H_2(Q;A)$.

Thus we must prove that $(1, \varepsilon_1, \varepsilon): \varepsilon^* E \to E$ induces a surjection $\tilde{F}_1 \to F_1$. We have the diagram, with exact rows,

$$\begin{array}{ccccc} \tilde{E}_{\infty}^{02} & \rightarrowtail & \tilde{F}_1 & \twoheadrightarrow & \tilde{E}_{\infty}^{11} \\ \downarrow \gamma_0 & & \downarrow & & \downarrow \gamma_1 \\ E_{\infty}^{02} & \rightarrowtail & F_1 & \twoheadrightarrow & E_{\infty}^{11} \end{array} \tag{1.10}$$

so it suffices to show that $(1, \varepsilon_1, \varepsilon)$ induces surjections

$$\gamma_0 : \tilde{E}_{\infty}^{02} \to E_{\infty}^{02}, \qquad \gamma_1 : \tilde{E}_{\infty}^{11} \to E_{\infty}^{11}.$$

Now $\tilde{E}_2^{02} = H_2(N; A)_G$, $E_2^{02} = H_2(N; A)_Q$, so that $(1, \varepsilon_1, \varepsilon)$ induces an isomorphism $\tilde{E}_2^{02} \to E_2^{02}$. Since $\tilde{E}_{\infty}^{02}, E_{\infty}^{02}$ are quotients of $\tilde{E}_2^{02}, E_2^{02}$ respectively, it follows that γ_0 is surjective.

Again

$$\tilde{E}_2^{11} = H_1(G; H_1(N; A)), E_2^{11} = H_1(Q; H_1(N; A)),$$

so that $(1, \varepsilon_1, \varepsilon)$ induces a surjection $\tilde{E}_2^{11} \twoheadrightarrow E_2^{11}$. Since $\tilde{E}_{\infty}^{11}$, E_{∞}^{11} are quotients of $\tilde{E}_2^{11}$, E_2^{11} respectively, it follows that γ_1 is surjective. Thus the theorem is proved.

NOTES. (a) It may be observed from the proof that we may replace the first term of the exact sequence by

$$\ker \varepsilon_{\tau}{}^* \xrightarrow{\kappa_*} H_2(G; A),$$

for any surjective $\tau : P \twoheadrightarrow Q$, where (see (1.1), (1.2))

$$\begin{array}{lccccc} \tau^* E: & N & \overset{\mu_\tau}{\rightarrowtail} & G_\tau & \overset{\varepsilon_\tau}{\twoheadrightarrow} & P \\ & \| & & \downarrow \kappa & & \downarrow \tau \\ E\ : & N & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & Q. \end{array}$$

(b) Since (1.4) splits it follows that, in the spectral sequence $\{\tilde{E}_r^{pq}\}$, the differential $\tilde{d}_r : \tilde{E}_r^{p0} \to \tilde{E}_r^{p-r, r-1}$ is always zero. Thus $\tilde{E}_2^{11} = \tilde{E}_{\infty}^{11}$ and $\tilde{E}_3^{02} = \tilde{E}_{\infty}^{02}$. We therefore obtain from the top row of (1.10) the exact sequence

$$H_2(G; H_1(N; A)) \xrightarrow{\tilde{d}_2} H_2(N; A)_G \to \ker \varepsilon_{0*} \to H_1(G; H_1(N; A)) \to 0. \tag{1.11}$$

1.3 Nomura states in the introduction to [6] that the exact sequence of Theorem 1.3, with $A = \mathbb{Z}$, provides a generalization of Ganea's result [5] for central extensions. He does, in fact, reprove Ganea's result in [6], but Theorem 1.3 does not immediately yield that result. For if we suppose that N is central in G, then, as pointed out in 1.1, $N \wr G$ becomes the direct product $N \times G$ and $\ker \varepsilon_{0*}$ admits a natural direct sum decomposition

$$\ker \varepsilon_{0*} = H_2 N \oplus (N \otimes G_{ab}). \tag{1.12}$$

Moreover $\varepsilon_{1*}|H_2N$ is just μ_* and $\varepsilon_{1*}|(N \otimes G_{ab})$ is the *Ganea term* χ [3], which Ganea proved, by topological arguments, could be added to the left of the 5-term homology sequence, with integer coefficients. Explicitly, Theorem 1.3 yields, for central extensions, the exactness of

$$H_2N \oplus (N \otimes G_{ab}) \xrightarrow{\varepsilon_{1*} = \langle \mu_*, \chi \rangle} H_2G \to H_2Q \to \cdots, \tag{1.13}$$

while Ganea proved the exactness of

$$N \otimes G_{ab} \xrightarrow{\chi} H_2G \to H_2Q \to \cdots. \tag{1.14}$$

Thus, to deduce (1.14) from (1.13) one must prove

$$\mu_* H_2N \subseteq \chi(N \otimes G_{ab}). \tag{1.15}$$

Now naturality yields a commutative diagram

$$\begin{array}{ccc} N \otimes N & \xrightarrow{\bar\chi} & H_2N \\ \downarrow{\scriptstyle \mu_*} & & \downarrow{\scriptstyle \mu_*} \\ N \otimes G_{ab} & \xrightarrow{\chi} & H_2G \end{array} \tag{1.16}$$

where $\bar\chi$ is the Ganea term for the central extension $N \rightarrowtail N \twoheadrightarrow 1$. Thus (1.15) follows immediately from

PROPOSITION 1 4. *$\bar\chi: N \otimes N \to H_2N$ is surjective.*

NOTE. This proposition follows immediately if one *assumes* Ganea's result. However, to *deduce* Ganea's result from Theorem 1.3 we should provide an independent proof of the proposition.

PROOF. It is sufficient to consider the case where N is finitely generated, since we may then complete the proof by a direct limit argument. Since the case of N cyclic is trivial, the proof is completed by observing that, if $N = N_1 \oplus N_2$, then $\bar\chi|N_1 \otimes N_2$ maps $N_1 \otimes N_2$ identically onto $N_1 \otimes N_2 \subseteq H_2N$.

1.4. We return to the general case. In the light of (1.11) which may be regarded as a generalization of (1.12), it is reasonable to ask when we may replace the term $\ker \varepsilon_{0*} \xrightarrow{\varepsilon_{1*}} H_2G$ in Theorem 2.3 by $H_1(G; N_{ab}) \xrightarrow{\chi} H_2G$, for some suitably defined χ. We give below a sufficient condition which, of course, includes the case where N is central in G. However, we first draw a trivial inference from Theorem 1.3 and (1.11).

PROPOSITION 1.5. *If $\tilde d_2: H_2(G; H_1(N; A)) \to H_2(N; A)_G$ is surjective,* (e.g., if $H_2(N, A)_G = 0$), *we have an exact sequence*

$$H_1(G; H_1(N; A)) \to H_2(G; A) \to H_2(Q; A) \to \cdots.$$

To describe the sufficient condition referred to above, we *present* the extension $N \rightarrowtail G \twoheadrightarrow Q$ (see (2.5) of [3]). Thus if $R \overset{\iota}{\rightarrowtail} F \overset{\pi}{\twoheadrightarrow} G$ is a free presentation of G we set $S = \pi^{-1}N$, and

$$R \rightarrowtail S \twoheadrightarrow N,$$

$$S \rightarrowtail F \twoheadrightarrow Q$$

are free presentations of N, Q respectively.

The condition we impose is

$$[[F,S],S] \subseteq [F,R]. \tag{1.17}$$

We first prove

LEMMA 1.6. *The condition* (1.17) *is independent of the choice of presentation.*

PROOF. If also $R' \overset{\iota'}{\rightarrowtail} F' \overset{\pi'}{\twoheadrightarrow} G$ presents G, and $S' = \pi'^{-1}(N)$, there is a homomorphism $\phi: F \to F'$ such that

$$\pi'\phi = \pi.$$

Plainly $\phi S \subseteq S'$, $\phi R \subseteq R'$. Moreover, ϕ is determined modulo a function $F \to R'$. It follows that the homomorphism

$$\psi_0: [F,F] \to [F',F']/[F',R']$$

induced by ϕ is uniquely determined. We conclude, by standard arguments of a homological-algebraic type, that, in fact, ϕ induces an isomorphism

$$\psi: [F,F]/[F,R] \xrightarrow{\sim} [F',F']/[F',R'].$$

Now $\phi[[F,S],S] \subseteq [[F',S'],S']$. Suppose that $[[F',S'],S'] \subseteq [F',R']$ and let $x \in [[F,S],S]$. Then $\psi(x[F,R])$ is the neutral element of $[F',F']/[F',R']$ so that $x \in [F,R]$. Thus $[[F,S],S] \subseteq [F,R]$ and the lemma is proved.

We remark that the condition $[[F,S],S] \subseteq [F,R]$ is certainly satisfied by a central extension. However it is also plain that it is satisfied when N is commutative and $H_2N = 0$ (or $H_2G = 0$), so that it is more general than centrality.

We prove

THEOREM 1.7. *Let* $N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$ *be an extension satisfying* (1.17), *with* N *commutative. Then there is a natural exact sequence*

$$H_1(G;N) \xrightarrow{\chi} H_2G \xrightarrow{\varepsilon_*} H_2Q$$

PROOF. Since $\ker \varepsilon_* = (R \cap [F,S])/[F,R]$, it is sufficient to exhibit a natural surjection $\chi_0: H_1(G;N) \twoheadrightarrow (R \cap [F,S])/[F,R]$. Consider the diagram

$$\begin{array}{ccc}
H_1(G;N) & \overset{\chi_0}{\dashrightarrow} & (R \cap [F,S])/[F,R] \\
\Big\downarrow \rho & & \Big\downarrow \kappa \\
IG \otimes_G N & \xrightarrow{\ \theta\ } & [F,S]/[F,R] \\
\Big\downarrow \sigma & & \Big\downarrow \lambda \\
N & = & N
\end{array}$$

Here the columns are exact and $[F,S]/[F,R]$ is commutative by (1.17). Further θ is defined by

(1.18) $\qquad \theta((xR - 1) \otimes_G yR) = [x,y] \bmod [F,R], \ x \in F, \ y \in S.$

Now $[x, y_1 y_2] = [x,y_1][x,y_2]^{y_1} \equiv [x,y_1][x,y_2] \bmod [F,R]$, by (1.17). Also

$$[xz,y][z,y]^{-1} = [x, zyz^{-1}],$$

which shows that θ respects the defining relations involved in passing to the tensor product over G, and so is well-defined by (1.18). It is obvious that θ is surjective. It is also clear that $\lambda\theta = \sigma$, since $\sigma((xR-1) \otimes_G yR) = [x,y] \bmod R$. Thus θ induces $\chi_0\colon H_1(G;N) \to (R \cap [F,S])/[F,R]$ and χ_0 is surjective because θ is surjective.

That χ is canonical is proved as follows. If we define θ' as in (1.18), but with respect to the presentation $R' \rightarrowtail F' \twoheadrightarrow G$, then plainly $\theta' = \psi\theta$, where ψ is the isomorphism of the proof of Lemma 1.6. Thus $\chi_0' = \psi\chi_0$. However we use $\psi | (R \cap [F,S])/[F,R]$ to indentify the two kernels, $(R \cap [F,S])/[F,R]$ and $(R' \cap [F',S'])/[F',R']$, of $\varepsilon_*\colon H_2 G \to H_2 Q$. Thus χ is canonical. A similar type of argument shows χ to be natural.

We remark that the definition of θ makes it plain that the homomorphism χ of this theorem generalizes the Ganea map.

Note. It is easy to deduce from the L-HS spectral sequence that, if $H_2(N;A)_Q = 0$, then there is an exact sequence

$$H_3(G,A) \to H_3(Q;A) \to H_1(Q;H_1(N;A)) \to H_2(G;A) \to H_2(Q;A) \to \cdots.$$

The exact sequence of Proposition 1.5 then arises by composing the third homomorphism above with the surjection

$$H_1(G;H_1(N;A)) \twoheadrightarrow H_1(Q;H_1(N;A)).$$

2. The 10-term sequence

2.1 In the second chapter of our note we confine ourselves to the study of central extensions. We recall from [4] that given the central extension

$$N \overset{\mu}{\rightarrowtail} G \overset{\varepsilon}{\twoheadrightarrow} Q$$

there is an exact sequence

(2.1) $$H_3G \to H_3Q \to N \otimes G_{ab}/U \to H_2G \to H_2Q \to N \to G_{ab} \to Q_{ab} \to 0$$

Here (see (1.16)) the subgroup U of $N \otimes G_{ab}$ is defined as $\mu_*(\ker \bar{\chi})$ where $\bar{\chi}\colon N \otimes N \to H_2N$ is the Ganea-Pontryagin map and $\mu_*\colon N \otimes N \to N \otimes G_{ab}$ is induced by $\mu\colon N \to G$.

Here we will discuss the continuation to the left of the exact sequence (2.2). We note that such a discussion is already contained in [2]. However, whereas [2] uses topological methods, we proceed in a purely algebraic way. Our main result differs slightly from the main result of [2]; the precise deviation is discussed in 2.5.

Thus, our main result is as follows.

THEOREM 2.1. *Given the central extension $N \rightarrowtail G \twoheadrightarrow Q$, there is a natural exact sequence*

(2.2) $$H_4Q \to A \to H_3G/B \overset{\bar{\varepsilon}}{\to} H_3Q \to C \to H_2G \to H_2Q \to N \to G_{ab} \to Q_{ab} \to 0$$

The groups A, C appearing in (2.2) are defined as follows. We define $\mu_0\colon \ker \bar{\chi} \to N \otimes G_{ab}$ to be the restriction of $\mu_*\colon N \otimes N \to N \otimes G_{ab}$ to the kernel of $\bar{\chi}\colon N \otimes N \to H_2N$. Then

$$A = \ker \mu_0, \quad C = \operatorname{coker} \mu_0.$$

Thus, explicitly, $A = \ker \bar{\chi} \cap \ker \mu_*$, $C = (N \otimes G_{ab})/\mu_*(\ker \bar{\chi})$.

The subgroup B of H_3G is defined as follows. Denote by $\varepsilon_0\colon N \times G \to G$ the projection and by $\varepsilon_1\colon N \times G \to G$ the multiplication, as in (1.5). Then B is the image under $\varepsilon_{1*}\colon H_3(N \times G) \to H_3G$ of the group $\ker \varepsilon_{0*}\colon H_3(N \times G) \to H_3G$. Since $\varepsilon\varepsilon_0 = \varepsilon\varepsilon_1$, it is obvious that $\varepsilon_*\colon H_3G \to H_3Q$ annihilates B and thus induces $\bar{\varepsilon}\colon H_3G/B \to H_3Q$ in (2.2).

PROOF OF THEOREM 2.1. We have only to define the first two homomorphisms of the sequence (2.2) and prove exactness at A and H_3G/B. We will again exploit the map of L-HS spectral sequences induced by

$$\begin{array}{ccccc} N & \overset{\mu_0}{\rightarrowtail} & N \times G & \overset{\varepsilon_0}{\twoheadrightarrow} & G \\ \| & & \downarrow{\scriptstyle \varepsilon_1} & & \downarrow{\scriptstyle \varepsilon} \\ N & \overset{\mu}{\rightarrowtail} & G & \overset{\varepsilon}{\twoheadrightarrow} & Q \end{array}$$

Here we concentrate on dimension 3. Then there is a filtration

$$F_0 \subseteq F_1 \subseteq F_2 \subseteq F_3 = H_3G$$

such that

(2.4) $$F_0 = E_\infty^{03}, F_1/F_0 = E_\infty^{12}, F_2/F_1 = E_\infty^{21}, F_2 = \ker \varepsilon_*\colon H_3G \to H_3Q.$$

We note in particular that

$$\ker \bar{\varepsilon}_* = F_2/B. \tag{2.5}$$

Now, in the L-HS spectral sequence $\{\tilde{E}_r^{pq}\}$ associated with

$$N \rightarrowtail N \times G \twoheadrightarrow G$$

all differentials $\tilde{d}_r$, $r \geqq 2$, are trivial,

$$\tilde{E}_2 = \tilde{E}_\infty. \tag{2.6}$$

There are also relations analogous to (2.4).

We will study the homomorphisms δ_i: $\tilde{F}_i \to F_i$ induced by (2.3), $i = 0, 1, 2$,

$$\begin{array}{ccccc} \tilde{F}_0 & \subseteq & \tilde{F}_1 & \subseteq & \tilde{F}_2 \\ \downarrow{\scriptstyle \delta_0} & & \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_2} \\ F_0 & \subseteq & F_1 & \subseteq & F_2 \end{array} \tag{2.7}$$

Naturally, in this study, we utilize the spectral sequence maps γ_i: $\tilde{E}_\infty^{i,3-i} \to E_\infty^{i,3-i}$, $i = 0, 1, 2$, so that

$$\gamma_0 = \delta_0 \tag{2.8}$$

and there is a commutative diagram, with exact rows

$$\begin{array}{ccccc} \tilde{F}_{i-1} & \rightarrowtail & \tilde{F}_i & \twoheadrightarrow & \tilde{E}_\infty^{i,3-i} \\ \downarrow{\scriptstyle \delta_{i-1}} & & \downarrow{\scriptstyle \delta_i} & & \downarrow{\scriptstyle \gamma_i} \\ F_{i-1} & \rightarrowtail & F_i & \twoheadrightarrow & E_\infty^{i,3-i} \end{array}, \tag{2.9}$$

$i = 1, 2$.

Now $\tilde{E}_2^{03} = H_3N$, $E_2^{03} = H_3N$. Thus γ_0 ($= \delta_0$) is clearly surjective. Next, we observe that $\tilde{E}_2^{12} \to E_2^{12}$ is just $H_1(G; H_2N) \to H_1(Q; H_2N)$ and thus certainly surjective. It follows that γ_1 is surjective, so that, by (2.9), δ_1 is also surjective.

We now consider (2.9) with $i = 2$. Since δ_1 is surjective, we know that

$$\operatorname{coker} \delta_2 \cong \operatorname{coker} \gamma_2. \tag{2.10}$$

However, $B = \operatorname{im} \delta_2$, so that, by (2.5) and (2.10),

$$\ker \bar{\varepsilon}_* \cong \operatorname{coker} \gamma_2. \tag{2.11}$$

It remains to compute coker γ_2. We know that

$$\tilde{E}_\infty^{2,1} = \tilde{E}_2^{2,1} = H_2(G; N).$$

Also $E_\infty^{2,1} = E_3^{2,1}$. Thus we must study

(2.12)
$$\begin{array}{ccccc} E_2^{40} & \xrightarrow{d_2} & E_2^{21} & \xrightarrow{d_2} & E_2^{02} \\ \| & & \| & & \| \\ H_4Q & \xrightarrow{d_2} & H_2(Q;N) & \xrightarrow{d_2} & H_2N \end{array}$$

We now require to exploit the computation by André [1] of the differential d_2 in the L-HS spectral sequence. This computation, applied to the special case of a central extension and trivial coefficients, shows that

$$d_2\colon H_2(Q;N) \to H_2N$$

is the composite of the homomorphism β, appearing in the 5-term exact sequence with coefficient module N,

(2.13) $$H_2(G;N) \xrightarrow{\varepsilon_*} H_2(Q;N) \xrightarrow{\beta} N \otimes N \xrightarrow{\mu_*} N \otimes G_{ab} \to N \otimes Q_{ab} \to 0,$$

and the Ganea-Pontryagin map

$$\bar{\chi}\colon N \otimes N \to H_2N;$$

thus,

(2.14) $$d_2 = \bar{\chi}\beta\colon H_2(Q;N) \to H_2N.$$

If we regard ε_* in (2.13) as a map $\varepsilon_{**}\colon H_2(G;N) \to \ker d_2 = \beta^{-1}(\ker \bar{\chi})$, it is then plain that

(2.15) $$\operatorname{coker} \varepsilon_{**} = \beta^{-1}(\ker \bar{\chi})/\ker \beta \cong \ker \bar{\chi} \cap \ker \mu_* = A.$$

The proof of the theorem is then completed by appeal to (2.11) and the diagram

(2.16)
$$\begin{array}{ccccc} & & H_2(G;N) & = & \tilde{E}_\infty^{21} \\ & & \downarrow{\scriptstyle \varepsilon_{**}} & & \downarrow{\scriptstyle \gamma_2} \\ H_4(Q) & \longrightarrow & \ker d_2 & \twoheadrightarrow & E_\infty^{21} \\ \| & & \downarrow & & \downarrow \\ H_4(Q) & \longrightarrow & A & \twoheadrightarrow & \operatorname{coker} \gamma_2, \end{array}$$

which shows the bottom row to be exact.

2.2 In this section and the next it is convenient to write $G \times N$ instead of $N \times G$. Our concern is with the quotient group H_3G/B appearing in (2.2). By the Künneth Theorem one knows that $H_3(G \times N)$ fits into the natural short exact sequence

(2.17) $$H_3N \oplus (H_1G \otimes H_2N) \oplus (H_2G \otimes N) \oplus H_3G \rightarrowtail H_3(G \times N) \twoheadrightarrow \operatorname{Tor}(H_1G, N).$$

This sequence splits, but non-naturally in general. It follows that $\ker \varepsilon_{0*}$ fits into the exact sequence

(2.18) $$H_3N \oplus (H_1G \otimes H_2N) \oplus (H_2G \otimes N) \rightarrowtail \ker \varepsilon_{0*} \twoheadrightarrow \operatorname{Tor}(H_1G, N),$$

which also splits non-naturally.

We recall that the subgroup $B \subseteq H_3G$ is the image under $\varepsilon_{1*}: H_3(G \times N) \to H_3G$ of $\ker \varepsilon_{0*}$. Here $\varepsilon_1: G \times N \to G$ is the multiplication and $\varepsilon_0: G \times N \to G$ is the projection. It is the purpose of this section to show that

(2.19) $$\varepsilon_{1*}(H_1G \otimes H_2N) \subseteq \varepsilon_{1*}(H_2G \otimes N)$$

As a consequence we may write

(2.20) $$H_3G/B = H_3G/\varepsilon_{1*}(H_3N \oplus H_2(G;N)),$$

in the sense that one first factors out $\varepsilon_{1*}(H_3N \oplus (H_2G \otimes N))$, and then factors $\operatorname{Tor}(H_1G, N)$ out of the quotient.

For the proof of (2.19) we consider the diagram

(2.21) $$\begin{array}{ccc} G \times N \times N & \xrightarrow{1 \times m} & G \times N \\ \varepsilon_1 \times 1 \downarrow & & \downarrow \varepsilon_1 \\ G \times N & \xrightarrow{\varepsilon_1} & G \end{array} \qquad (m = \varepsilon_1 | N \times N)$$

which is obviously commutative, and the induced square

(2.22) $$\begin{array}{ccc} H_1G \otimes N \otimes N & \xrightarrow{1 \otimes \bar{\chi}} & H_1G \otimes H_2N \\ \varepsilon_{1*} \otimes 1 \downarrow & & \downarrow \varepsilon_{1*} \\ H_2G \otimes N & \xrightarrow{\varepsilon_{1*}} & H_3G \end{array}$$

Since $\bar{\chi}: N \otimes N \to H_2N$ is surjective, so is $1 \otimes \bar{\chi}$ and (2.19) now follows immediately.

2.3 We show by a counterexample that in equation (2.20) the term H_3N cannot be dropped.

Consider the diagram

(2.23) $$\begin{array}{ccccc} C_n & \rightarrowtail & C_n \times C_n & \xrightarrow{\varepsilon_0} \!\!\!\!\rightarrow & C_n \\ \| & & \downarrow \varepsilon_1 & & \downarrow\!\!\downarrow \varepsilon \\ C_n & \rightarrowtail & C_n & \xrightarrow{\varepsilon} \!\!\!\!\rightarrow & 1 \end{array}$$

where C_n denotes a cyclic group of order n. Of course we have $\varepsilon_{1*}(H_3C_n) = H_3C_n$. However we claim that, if n is even,

$$\varepsilon_{1*}(H_2(C_n; H_1C_n)) \neq H_3C_n.$$

We first show that $\varepsilon_{1*}|H_2(C_n; H_1C_n)$ is well-defined. Since $H_2C_n = 0$, we have the exact sequence (see (2.17))

(2.24) $$H_3C_n \oplus H_3C_n \rightarrowtail H_3(C_n \times C_n) \twoheadrightarrow \mathrm{Tor}(H_1C_n, H_1C_n).$$

Now the embeddings $H_3C_n \rightarrowtail H_3(C_n \times C_n)$ have natural left inverses induced by the projections $C_n \times C_n \to C_n$. Hence the splitting of (2.24) is canonical.

Next we use [2; Theorem 2.2] to show that under ε_{1*} the subgroup $\mathrm{Tor}(H_1C_n, H_1C_n) = \mathbb{Z}_n$ is mapped onto $2 \cdot H_3C_n = 2 \cdot \mathbb{Z}_n$. Thus, if n is even,

(2.25) $$\varepsilon_{1*}(H_2(C_n : H_1C_n)) = 2 \cdot H_3C_n \neq H_3C_n.$$

Finally, comparing sequence (2.2) with sequence (1.5) of [2], we see that in sequence (2.2) the group

(2.26) $$H_3G/\varepsilon_{1*}(H_3N \oplus H_2(G; N))$$

appears, whereas in sequence (1.5) of [2] we have the group

(2.27) $$H_3G/\varepsilon_{1*}(H_2(G; N)).$$

The above example (2.23) shows that these two groups do not in general agree, so that the two sequences are not in general the same. However, it is not at all difficult to deduce (2.2) from (1.5) of [2].

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