WITT GROUPS OF THE PUNCTURED SPECTRUM OF A 3-DIMENSIONAL REGULAR LOCAL RING AND A PURITY THEOREM

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0. *Introduction*

Let A be a regular local ring with quotient field K. Assume that 2 is invertible in A. Let $W(A) \longrightarrow W(K)$ be the homomorphism induced by the inclusion $A \hookrightarrow K$, where $W(\,)$ denotes the Witt group of quadratic forms. If $\dim A \leqslant 4$, it is known that this map is injective [**6**, **7**]. A natural question is to characterize the image of $W(A)$ in $W(K)$. Let $\mathrm{Spec}^1(A)$ be the set of prime ideals of A of height 1. For $P \in \mathrm{Spec}^1(A)$, let π_P be a parameter of the discrete valuation ring A_P and $k(P) = A_P/PA_P$. For this choice of a parameter π_P, one has the *second residue homomorphism* $\partial_P : W(K) \longrightarrow W(k(P))$ [**9**, p. 209]. Though the homomorphism ∂_P depends on the choice of the parameter π_P, its kernel and cokernel do not. We have a homomorphism

$$\partial = (\partial_P) : W(K) \longrightarrow \bigoplus_{P \in \mathrm{Spec}^1(A)} W(k(P)).$$

A part of the so-called Gersten conjecture is the following question on 'purity'. Is the sequence

$$W(A) \longrightarrow W(K) \xrightarrow{\partial} \bigoplus_{P \in \mathrm{Spec}^1(A)} W(k(P))$$

exact? This question has an affirmative answer for $\dim(A) \leqslant 2$ [**1**; **3**, p. 277]. There have been speculations by Pardon and Barge-Sansuc-Vogel on the question of purity. However, in the literature, there is no proof for purity even for $\dim(A) = 3$. One of the consequences of the main result of this paper is an affirmative answer to the purity question for $\dim(A) = 3$.

We briefly outline our main result. For any scheme X let $W^\epsilon(X)$ denote the Witt group of ϵ-symmetric spaces on X, $\epsilon = \pm 1$ ($W^{+1}(X) = W(X)$ being the usual Witt group of symmetric spaces over X). Let A be a regular local ring of dimension 3 with maximal ideal m and $Y = \mathrm{Spec}(A)\backslash\{m\}$. We associate (§3) to an ϵ-symmetric space over Y a $(-\epsilon)$-symmetric space over a finite-length A-module. This assignment leads to a homomorphism $W^\epsilon(Y) \longrightarrow W^{-\epsilon}_{\mathrm{fl}}(A)$, where $W^\epsilon_{\mathrm{fl}}(A)$ is the Witt group of ϵ-symmetric spaces of finite-length A-modules (cf. §1). Then we prove (§4) that the sequence

$$0 \longrightarrow W^\epsilon(A) \longrightarrow W^\epsilon(Y) \longrightarrow W^{-\epsilon}_{\mathrm{fl}}(A) \longrightarrow 0$$

is exact, where the map $W^\epsilon(A) \longrightarrow W^\epsilon(Y)$ is induced by the restriction. Since $W^\epsilon_{\mathrm{fl}}(A) \simeq W^\epsilon(A/m)$, it follows that $W^{-1}_{\mathrm{fl}}(A) = 0$. Thus the map $W(A) \longrightarrow W(Y)$ is an isomorphism. This leads to the purity theorem for the Witt groups. On the other hand, since every skew-symmetric space over A is hyperbolic, $W^{-1}(A) = 0$ and we get an isomorphism $W^{-1}(Y) \simeq W(A/m)$. We observe the curious fact that if A is complete, $W^{\pm 1}(Y)$ is isomorphic to $W(A/m)$.

Received 15 August 1996.
1991 *Mathematics Subject Classification* 19G12.
J. London Math. Soc. (2) 59 (1999) 521–540

A crucial result used in our proof of the main theorem is a theorem of Horrocks [**2**] on vector bundles on the punctured spectrum $Y = \mathrm{Spec}(A) \backslash \{m\}$, where A is a regular local ring of dimension 3 and m is its maximal ideal. We use his theorem on the equivalence of the category of 'Φ-equivalence' classes of vector bundles on Y with the category of finite-length A-modules.

We would like to remark parenthetically that purity for dimension 3 was used in [**8**] while establishing the equivalence of the finite generation of Witt groups of affine real 3-folds and the finite generation of Chow groups of codimension 2 cycles modulo 2.

1. *ϵ-symmetric spaces reminisced*

Let A be a regular local ring of dimension 3 in which 2 is invertible. We recall the definition of ϵ-symmetric spaces on finite-length A-modules and their Witt groups. For A-modules M, N and $i \geqslant 0$, let $\mathrm{Ext}^i(M, N)$ denote the group of congruence classes of i-fold extensions of N by M [**4**, p. 84]. For any homomorphism $f\colon M \longrightarrow M'$ of A-modules, let $\mathrm{Ext}^i(N, f)\colon \mathrm{Ext}^i(N, M) \longrightarrow \mathrm{Ext}^i(N, M')$ be the induced homomorphisms defined as follows. Let

$$\zeta = 0 \longrightarrow M \xrightarrow{\alpha} Z_i \xrightarrow{\partial_i} Z_{i-1} \longrightarrow \cdots \longrightarrow Z_2 \xrightarrow{\partial_2} Z_1 \xrightarrow{\beta} N \longrightarrow 0$$

be an i-fold extension of N by M. Let $Z = (Z_i \oplus M')/(\{(\alpha(x), f(x)) \mid x \in M\})$ be the push-out of Figure 1 [**4**].

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & Z_i \\ \Big\downarrow{\scriptstyle f} & & \\ M' & & \end{array}$$

FIGURE 1.

Then

$$\mathrm{Ext}^i(N, f)(\zeta) = 0 \longrightarrow M' \xrightarrow{\alpha'} Z \xrightarrow{\partial'} Z_{i-1} \xrightarrow{\partial_{i-1}} \cdots \longrightarrow Z_2 \xrightarrow{\partial_2} Z_1 \xrightarrow{\beta} N \longrightarrow 0$$

where α' and ∂' are the natural homomorphisms induced by the push-out. Similarly, we define $\mathrm{Ext}^i(f, N)$ as the pull-back under f of an i-fold extension of N by M'. Let M be a finite-length A-module and $M^\vee = \mathrm{Ext}^3(M, A)$. If M, M' are two finite-length A-modules and $f\colon M \longrightarrow M'$ is an A-linear map, then we denote $\mathrm{Ext}^3(f, A)$ by $f^\vee$. Let

$$\mathscr{P} = 0 \longrightarrow P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\theta} M \longrightarrow 0$$

be a projective resolution of M. Since $\mathrm{Ext}^i(M, A) = 0$ for $i = 0, 1, 2$ [**5**, Theorem 18.1], by dualizing the above exact sequence we see that

$$\mathscr{P}^* = 0 \longrightarrow P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\partial_2^*} P_2^* \xrightarrow{\partial_3^*} P_3^* \xrightarrow{\theta'} M^\vee \longrightarrow 0$$

is a projective resolution of $M^\vee$, where $P_i^* = \mathrm{Hom}_A(P_i, A)$, ∂_i^* is induced by ∂_i and for any $f \in P_3^*$,

$$\theta'(f) = \mathrm{Ext}^3(f, M)(\mathscr{P}) \in M^\vee.$$

Throughout this paper, for any surjection $\theta: P_0 \longrightarrow M$ as above, θ' denotes the map defined as above. We define a canonical homomorphism $\mathscr{C}: M \longrightarrow M^{\vee\vee}$ as follows. Let $x \in M$. Choose $y \in P_0$ such that $\theta(y) = x$. We define

$$\mathscr{C}(x) = \mathrm{Ext}^3(-e_y, M^{\vee})(\mathscr{P}^*) \in M^{\vee\vee},$$

where, for $f \in P_0^*$, $e_y(f) = f(y)$. Then it is easy to see that $\mathscr{C}(x)$ is independent of the choice of y and Figure 2 is commutative, where $\mathscr{C}: P_i \longrightarrow P_i^{**}$ are the canonical isomorphisms.

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & P_3 & \xrightarrow{\partial_3} & P_2 & \xrightarrow{\partial_2} & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\theta} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathscr{C}} & & \downarrow{\scriptstyle \mathscr{C}} & & \downarrow{\scriptstyle \mathscr{C}} & & \downarrow{\scriptstyle \mathscr{C}} & & \downarrow{\scriptstyle -\mathscr{C}} & & \\
0 & \longrightarrow & P_3^{**} & \xrightarrow{\partial_3^{**}} & P_2^{**} & \xrightarrow{\partial_2^{**}} & P_1^{**} & \xrightarrow{\partial_1^{**}} & P_0^{**} & \xrightarrow{\theta''} & M^{\vee\vee} & \longrightarrow & 0
\end{array}$$

FIGURE 2.

Thus $\mathscr{C}: M \longrightarrow M^{\vee\vee}$ is an isomorphism and it is obvious that it is independent of the choice of the projective resolution. We use this isomorphism to identify M with $M^{\vee\vee}$. The choice of the negative sign at e_y in the definition of $\mathscr{C}$ is explained in the following. Let $m = (x_1, x_2, x_3)$ be the maximal ideal of A and

$$\zeta = 0 \longrightarrow A \xrightarrow{\delta_3} A^3 \xrightarrow{\delta_2} A^3 \xrightarrow{\delta_1} A \xrightarrow{\eta} A/m \longrightarrow 0$$

be the Koszul resolution of A/m with respect to (x_1, x_2, x_3). With respect to the standard basis $\{e_1, e_2, e_3\}$ of A^3, we have

$$\delta_1 = (x_1 \quad x_2 \quad x_3), \quad \delta_2 = \begin{pmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{pmatrix}, \quad \delta_3 = \begin{pmatrix} x_3 \\ -x_2 \\ x_1 \end{pmatrix}$$

and $\eta: A \longrightarrow A/m$ is the natural homomorphism. Let M be a finite-dimensional vector space over A/m. Then M is a finite-length A-module. Let $\tilde{M} = \mathrm{Hom}(M, A/m)$. The assignment $f \mapsto \mathrm{Ext}^3(f, A)(\zeta) \in M^{\vee}$ induces a homomorphism

$$\Phi_M: \tilde{M} \longrightarrow M^{\vee}.$$

The following lemmas are well known, but for the sake of completeness we give their proofs here.

LEMMA 1.1. *The homomorphism Φ_M is an isomorphism and Figure 3 is commutative, where $\iota: M \longrightarrow \tilde{\tilde{M}}$ is the canonical isomorphism.*

$$\begin{array}{ccc}
M & \xrightarrow{\mathscr{C}} & M^{\vee\vee} \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \Phi_M^{\vee}} \\
\tilde{\tilde{M}} & \xrightarrow{\Phi_{\tilde{M}}} & (\tilde{M})^{\vee}
\end{array}$$

FIGURE 3.

Proof. Since $M \simeq \bigoplus_1^n A/m$, $M^\vee \simeq \bigoplus_1^n (A/m)^\vee$ and $\tilde{M} \simeq \bigoplus_1^n \widetilde{A/m}$, it is enough to prove the lemma in the case when $M = A/m$. In this case it is easy to see that $\Phi_M \neq 0$. Since $M^\vee \simeq A/m$ [**5**, Theorem 18.1] and $\tilde{M} \simeq A/m$, Φ_M is an isomorphism. We now prove the commutativity of Figure 3. For all $x \in M$ and $f \in \tilde{M}$ we have $\imath(x)(f) = f(x)$ and

$$\Phi_{\tilde{M}}(\imath(x)) = 0 \longrightarrow A \xrightarrow{\delta_3} A^3 \xrightarrow{\delta_2} A^3 \xrightarrow{\delta_1} A \xrightarrow{\imath(x)^{-1}\eta} A/m \longrightarrow 0.$$

Let $y \in A$ be such that $\eta(y) = x$. Then we have

$$\mathscr{C}(x) = 0 \longrightarrow A \xrightarrow{-\delta_1^* e_y^{-1}} A^{3*} \xrightarrow{\delta_2^*} A^{3*} \xrightarrow{\delta_3^*} A^* \xrightarrow{\eta'} (A/m)^\vee \longrightarrow 0.$$

Since Φ_M is an isomorphism, we have

$$\Phi_M^\vee(\mathscr{C}(x)) = 0 \longrightarrow A \xrightarrow{-\delta_1^* e_y^{-1}} A^{3*} \xrightarrow{\delta_2^*} A^{3*} \xrightarrow{\delta_3^*} A^* \xrightarrow{\Phi_M^{-1}\eta'} (A/m)^\vee \longrightarrow 0.$$

Let $\{e_1^*, e_2^*, e_3^*\}$ be the dual basis of A^{3*}. For $i = 1, 2$, let $\theta_i : A^3 \longrightarrow A^{3*}$ be given by the following matrices, with respect to the bases $\{e_1, e_2, e_3)$ and $\{e_1^*, e_2^*, e_3^*\}$.

$$\theta_1 = \begin{pmatrix} 0 & 0 & -y^{-1} \\ 0 & y^{-1} & 0 \\ -y^{-1} & 0 & 0 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} 0 & 0 & y^{-1} \\ 0 & -y^{-1} & 0 \\ y^{-1} & 0 & 0 \end{pmatrix}.$$

Let $\theta_3 : A \longrightarrow A^*$ be the homomorphism defined by $\theta_3(1) = l_y$, where $l_y(a) = ay$ for all $a \in A$. It is easy to see that Figure 4 is commutative. Thus $\Phi_{\tilde{M}}\,\imath = \Phi_M^\vee \mathscr{C}$.

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & A & \xrightarrow{\delta_3} & A^3 & \xrightarrow{\delta_2} & A^3 & \xrightarrow{\delta_1} & A & \xrightarrow{\imath(x)^{-1}\eta} & (A/m)^\vee & \longrightarrow & 0 \\
 & & \big\downarrow \mathrm{id} & & \big\downarrow \theta_1 & & \big\downarrow \theta_2 & & \big\downarrow \theta_3 & & \big\downarrow \mathrm{id} & & \\
0 & \longrightarrow & A & \xrightarrow{-\delta_1^* e_y^{-1}} & A^{3*} & \xrightarrow{\delta_2^*} & A^{3*} & \xrightarrow{\delta_3^*} & A^* & \xrightarrow{\Phi_M^{-1}\eta'} & (A/m)^\vee & \longrightarrow & 0
\end{array}$$

FIGURE 4. □

LEMMA 1.2. *Let $\psi : M \longrightarrow \tilde{M}$ be a homomorphism and $\tilde{\psi} : \tilde{\tilde{M}} \longrightarrow \tilde{M}$ be the induced homomorphism. Then Figure 5 is commutative.*

$$\begin{array}{ccc}
\widetilde{\widetilde{M}} & \xrightarrow{\Phi_{\widetilde{M}}} & (\widetilde{M})^\vee \\
\big\downarrow \widetilde{\psi} & & \big\downarrow \psi^\vee \\
\widetilde{M} & \xrightarrow{\Phi_M} & M^\vee
\end{array}$$

FIGURE 5.

Proof. Let $f \in \tilde{\tilde{M}}$. Then $\Phi_{\tilde{M}}(f)$ is the pull-back of the Koszul resolution ζ under $f: \tilde{M} \longrightarrow A/m$ and $\psi^{\vee}(\Phi_{\tilde{M}}(f))$ is the pull-back of the extension $\Phi_{\tilde{M}}(f)$ under ψ. Thus $\psi^{\vee}(\Phi_{\tilde{M}}(f))$ is the pull-back of the Koszul resolution under the homomorphism $f\psi: M \longrightarrow A/m$. Since $\tilde{\psi}(f) = f\psi$, $\Phi_M(\tilde{\psi})(f)$ is the pull-back of the Koszul resolution under $f\psi$. Thus $\Phi_M \psi^{\vee} = \psi^{\vee}\Phi_{\tilde{M}}$. □

Lemmas 1.1 and 1.2 enable us to embed the category of ϵ-symmetric spaces on finite-dimensional A/m-vector spaces into the category of ϵ-symmetric spaces on finite-length A-modules (cf. Corollary 1.3)).

Let $\epsilon = \pm 1$. An *ϵ-symmetric space of finite length* is a pair (M, ψ) where M is a finite-length A-module and $\psi: M \longrightarrow M^{\vee} = \operatorname{Ext}^3(M, A)$ is an isomorphism with $\psi^{\vee}\mathscr{C} = \epsilon\psi$. Let ψ_1 and ψ_2 be two ϵ-symmetric spaces on finite-length A-modules M_1 and M_2 respectively. We say that ψ_1 is *isometric to* ψ_2 if there exists a homomorphism $\theta: M_1 \longrightarrow M_2$ such that $\psi_1 = \theta^{\vee}\psi_2\theta$. An ϵ-symmetric space ψ on M is called *metabolic* if there exists a submodule N of M such that

$$0 \longrightarrow N \xrightarrow{\;i\;} M \xrightarrow{\;i^{\vee}\psi\;} N^{\vee} \longrightarrow 0$$

is exact, where $i: N \longrightarrow M$ is the inclusion. The *Witt group* of ϵ-symmetric spaces of finite-length A-modules is defined as the quotient of the Grothendieck group of isometry classes of ϵ-symmetric spaces with the orthogonal sum as addition, modulo the subgroup generated by metabolic spaces. It is denoted by $W^{\epsilon}_{\mathrm{fl}}(A)$.

Corollary 1.3. *Let M be a finite-dimensional vector space over A/m. Let $\psi: M \longrightarrow M$ be an ϵ-symmetric space, that is, $\tilde{\psi}\iota = \psi$ and ψ is an isomorphism. Then $\Phi_M\psi: M \longrightarrow M^{\vee}$ is an ϵ-symmetric space.*

Proof. By Lemma 1.1, we have $(\Phi_M\psi)^{\vee}\mathscr{C} = \psi^{\vee}\Phi_M^{\vee}\mathscr{C} = \psi^{\vee}\Phi_{\tilde{M}}\iota$. Using Lemma 1.2, we get that $(\Phi_M\psi)^{\vee}\mathscr{C} = \Phi_M\tilde{\psi}\iota = \epsilon\Phi_M\psi$. Thus $\Phi_M\psi$ is an ϵ-symmetric space. □

We need the following lemma.

Lemma 1.4. *Let M be a finite-length A-module and $\psi: M \longrightarrow M^{\vee}$ be an ϵ-symmetric space. If (M, ψ) is stably metabolic, then it is metabolic.*

Proof. If M is an A/m-module, then the result follows from the corresponding result for ϵ-symmetric spaces over the field A/m. We reduce the general case to the above case by induction on the length of M. Assume that the length of M is at least 2. Let V be a maximal submodule of M which is an A/m-module. Suppose that ψ restricted to V is singular. Then there exists a non-zero submodule L of V such that

$$L \subset L^{\perp} = \ker(M \xrightarrow{\;i^{\vee}\psi\;} L^{\vee})$$

and ψ induces an ϵ-symmetric form $\bar{\psi}$ on $L^{\perp}/L$ which is Witt equivalent to (M, ψ). Suppose that (M, ψ) is stably metabolic. Then $(L^{\perp}/L, \bar{\psi})$ is stably metabolic. By induction there exists a submodule N_1 of $L^{\perp}/L$ such that

$$0 \longrightarrow N_1 \xrightarrow{\;i\;} L^{\perp}/L \xrightarrow{\;i^{\vee}\bar{\psi}\;} N_1^{\vee} \longrightarrow 0$$

is exact. Let N be the submodule of M containing L such that $N/L = N_1$. Then it is easy to see that the sequence

$$0 \longrightarrow N \xrightarrow{i} M \xrightarrow{i^\vee \psi} N^\vee \longrightarrow 0$$

is exact and (M, ψ) is metabolic. We may therefore assume that ψ restricted to V is non-singular. Then $(M, \psi) \simeq (V, \psi|_V) \perp (M_1, \psi_1)$. If $M_1 \neq 0$, then M_1 contains a non-zero submodule which is an A/m-module, contradicting the maximality of V. Thus $M_1 = 0$ and $M = V$ is an A/m-module. This completes the proof of the lemma. □

Let X be a scheme such that 2 is invertible in $\Gamma(X)$. Let $\mathscr{E}$ be a vector bundle over X of finite rank. An *ϵ-symmetric space* on $\mathscr{E}$ is an isomorphism $\mathbf{q}: \mathscr{E} \longrightarrow \mathscr{E}^* = \mathrm{Hom}(\mathscr{E}, \mathscr{O}_X)$ such that $\mathbf{q}^*\mathscr{C} = \epsilon\mathbf{q}$, where $\mathscr{C}: \mathscr{E} \longrightarrow \mathscr{E}^{**}$ is the canonical identification. Let $W^\epsilon(X)$ be the Witt group of ϵ-symmetric spaces on vector bundles over X [**3**, p. 144]. If $X = \mathrm{Spec}(A)$, then we denote $W^\epsilon(X)$ by $W^\epsilon(A)$.

Throughout this paper, by an A-module we mean a finitely generated A-module. We call an ϵ-symmetric space simply a *quadratic space* if $\epsilon = +1$ and a *symplectic space* if $\epsilon = -1$. We also denote $W^{+1}(X)$ by $W(X)$. For a vector bundle $\mathscr{E}$ over X, we denote the hyperbolic space on $\mathscr{E}$ by $\mathbb{H}(\mathscr{E})$ [**3**, p. 130].

2. *Reflexive modules*

Let A be a regular local ring of dimension 3 with 2 invertible. An A-module E is said to be *reflexive* if it is finitely generated and the canonical homomorphism $E \longrightarrow E^{**}$ is an isomorphism. For a reflexive A-module E we use the canonical isomorphism to identify E^{**} with E. It is well known that a reflexive module over a regular ring of dimension 3 has projective dimension at most 1. Let E be a reflexive A-module and $M = \mathrm{Ext}^1(E^*, A)$, where $E^* = \mathrm{Hom}_A(E, A)$. Since reflexive modules over regular rings of dimension at most 2 are projective, M is a finite-length A-module. We define a homomorphism $\beta_E: \mathrm{Ext}^1(E, A) \longrightarrow M^\vee = \mathrm{Ext}^3(M, A)$ as follows. Let

$$0 \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} E^* \longrightarrow 0 \tag{2.1}$$

be a projective resolution of E^*. Then by dualizing, we get an exact sequence

$$0 \longrightarrow E \xrightarrow{\partial_0^*} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta} \mathrm{Ext}^1(E^*, A) = M \longrightarrow 0$$

where δ is defined by push-outs. We have the following lemma.

LEMMA 2.1. *The Yoneda composition* [**4**, p. 82] $\beta_E: \mathrm{Ext}^1(E, A) \longrightarrow M^\vee$ *given by*

$$\beta_E(0 \longrightarrow A \xrightarrow{\eta} Z \xrightarrow{\eta'} E \longrightarrow 0) = (0 \longrightarrow A \xrightarrow{\eta} Z \xrightarrow{\partial_0^*\eta'} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta} M \longrightarrow 0)$$

is an isomorphism and is independent of the choice of the projective resolution (2.1) *of* E^*.

Proof. Consider the long exact sequence of cohomology associated to the short exact sequences

$$0 \longrightarrow E \xrightarrow{\partial_0^*} P_0^* \xrightarrow{\partial_1^*} \ker(\delta) \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \ker(\delta) \hookrightarrow P_1^* \xrightarrow{\delta} M \longrightarrow 0.$$

Since $\mathrm{Ext}^i(M,A)=0$ for $i\leqslant 2$ and P_1^* is a projective module, $\mathrm{Ext}^1(\ker(\delta),A)=0$ and the connecting homomorphisms

$$\mathrm{Ext}^1(E,A)\longrightarrow \mathrm{Ext}^2(\ker(\delta),A) \quad \text{and} \quad \mathrm{Ext}^2(\ker(\delta),A)\longrightarrow \mathrm{Ext}^3(M,A)$$

induced by the above short exact sequences are isomorphisms. Since β_E, up to sign, is the composition of these two connecting homomorphisms [**4**, Theorem 9.1, p. 97], β is an isomorphism.

Suppose that

$$0\longrightarrow F_1\xrightarrow{\partial_1'}F_0\xrightarrow{\partial_0'}E^*\longrightarrow 0$$

is another projective resolution of E^*. Then by lifting the identity map on E^*, we get homomorphisms $P_i\longrightarrow F_i$, $i=0,1$, such that Figure 6 is commutative.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\partial_0} & E^* & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & F_1 & \xrightarrow{\partial_1'} & F_0 & \xrightarrow{\partial_0'} & E^* & \longrightarrow & 0
\end{array}$$

FIGURE 6.

By dualizing this diagram we get a commutative diagram (Figure 7) where δ' is defined by push-outs.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & E & \xrightarrow{\partial_0'^*} & F_0^* & \xrightarrow{\partial_1'^*} & F_1^* & \xrightarrow{\delta'} & M & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & E & \xrightarrow{\partial_0^*} & P_0^* & \xrightarrow{\partial_1^*} & P_1^* & \xrightarrow{\delta} & M & \longrightarrow & 0
\end{array}$$

FIGURE 7.

This implies that

$$(0\longrightarrow E\xrightarrow{\partial_0'^*}F_0^*\xrightarrow{\partial_1'^*}F_1^*\xrightarrow{\delta'}M\longrightarrow 0)=(0\longrightarrow E\xrightarrow{\partial_0^*}P_0^*\xrightarrow{\partial_1^*}P_1^*\xrightarrow{\delta}M\longrightarrow 0)$$

in $\mathrm{Ext}^2(M,E)$. Thus the homomorphism β_E is independent of the choice of the projective resolution of E^*. □

LEMMA 2.2. (i) *For any reflexive A-module E we have*

$$\beta_{E^*}=-\beta_E^\vee\mathscr{C}.$$

(ii) *Let E and E' be reflexive A-modules. Then, for any isomorphism $f\colon E\longrightarrow E'$, we have*

$$\mathrm{Ext}^1(f^*)^\vee\beta_{E'}=\beta_E\,\mathrm{Ext}^1(f).$$

Proof. Let $0\longrightarrow P_1\xrightarrow{\partial_1}P_0\xrightarrow{\partial_0}E^*\longrightarrow 0$ and $0\longrightarrow F_1\xrightarrow{\partial_1'}F_0\xrightarrow{\partial_0'}E\longrightarrow 0$ be projective resolutions of E^* and E respectively. By dualizing these exact sequences, we get exact sequences

$$0 \longrightarrow E \xrightarrow{\partial_0^*} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta} \mathrm{Ext}^1(E^*, A) \longrightarrow 0$$

and

$$0 \longrightarrow E^* \xrightarrow{\partial_0'^*} F_0^* \xrightarrow{\partial_1'^*} F_1^* \xrightarrow{\delta_1} \mathrm{Ext}^1(E, A) \longrightarrow 0.$$

Let $\zeta = (0 \longrightarrow A \xrightarrow{\alpha} Z \xrightarrow{\beta} E^* \longrightarrow 0) \in \mathrm{Ext}^1(E^*, A)$. Since P_0 and P_1 are projective, there exist homomorphisms $f: P_1 \longrightarrow A$ and $g: P_0 \longrightarrow Z$ such that Figure 8 is commutative.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\partial_0} & E^* & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & A & \xrightarrow{\alpha} & Z & \xrightarrow{\beta} & E^* & \longrightarrow & 0 \end{array}$$

FIGURE 8.

By the definition of δ we have $\delta(f) = \zeta$. Since

$$0 \longrightarrow F_1 \xrightarrow{\partial_1'} F_0 \xrightarrow{\partial_0^* \partial_0'} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta} \mathrm{Ext}^1(E^*, A) \longrightarrow 0$$

is a projective resolution of $\mathrm{Ext}^1(E^*, A)$, by dualizing it we get an exact sequence

$$0 \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0'^* \partial_0} F_0^* \xrightarrow{\partial_1'^*} F_1^* \xrightarrow{\delta_1'} \mathrm{Ext}^1(E^*, A)^\vee \longrightarrow 0.$$

Thus $\mathscr{C}(\zeta) = -\xi$, where

$$\xi = (0 \longrightarrow A \xrightarrow{\alpha} Z \xrightarrow{\partial_0'^* \beta} F_0^* \xrightarrow{\partial_1'^*} F_1^* \xrightarrow{\delta_1'} \mathrm{Ext}^1(E^*, A)^\vee \longrightarrow 0).$$

From the definitions of δ, δ' and β_E, it follows that Figure 9 is commutative.

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & E^* & \xrightarrow{\partial_0'^*} & F_0^* & \xrightarrow{\partial_1'^*} & F_1^* & \xrightarrow{\delta_1} & \mathrm{Ext}^1(E, A) & \longrightarrow & 0 \\ & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow \mathrm{id} & & \downarrow \beta_E & & \\ 0 & \longrightarrow & E^* & \xrightarrow{\partial_0'^*} & F_0^* & \xrightarrow{\partial_1'^*} & F_1^* & \xrightarrow{\delta_1'} & \mathrm{Ext}^1(E^*, A)^\vee & \longrightarrow & 0 \end{array}$$

FIGURE 9.

It follows from the definition of $\beta_E^\vee$ that

$$\beta_E^\vee(\xi) = (0 \longrightarrow A \xrightarrow{\alpha} Z \xrightarrow{\partial_0'^* \beta} F_0^* \xrightarrow{\partial_1'^*} F_1^* \xrightarrow{\delta_1} \mathrm{Ext}^1(E, A) \longrightarrow 0).$$

On the other hand, we have

$$\beta_{E^*}(\zeta) = (0 \longrightarrow A \xrightarrow{\alpha} Z \xrightarrow{\partial_0'^* \beta} F_0^* \xrightarrow{\partial_1'^*} F_1^* \xrightarrow{\delta_1} \mathrm{Ext}^1(E, A) \longrightarrow 0) = \beta_E^\vee(\xi).$$

Thus $-\beta_E^\vee \mathscr{C} = \beta_{E^*}$.

Let $f\colon E \longrightarrow E'$ be an isomorphism. Then $0 \longrightarrow P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{f^{*-1}\partial_0} E'^* \longrightarrow 0$ is a projective resolution of E'^*. By dualizing it we get an exact sequence

$$0 \longrightarrow E' \xrightarrow{\partial_0^* f^{-1}} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta_2} \operatorname{Ext}^1(E'^*, A) \longrightarrow 0$$

with $\delta_2 = \operatorname{Ext}^1(f^*)\ \delta$. Let $\zeta = (0 \longrightarrow A \xrightarrow{\alpha} Z \xrightarrow{\beta} E' \longrightarrow 0) \in \operatorname{Ext}^1(E', A)$. Then

$$\beta_{E'}(\zeta) = (0 \longrightarrow A \xrightarrow{\alpha} Z \xrightarrow{\partial_0^* f^{-1}\beta} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta_2} \operatorname{Ext}^1(E'^*, A) \longrightarrow 0)$$

and

$$\operatorname{Ext}^1(f^*)^\vee \beta_{E'}(\zeta) = (0 \longrightarrow A \xrightarrow{\alpha} Z \xrightarrow{\partial_0^* f^{-1}\beta} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta} \operatorname{Ext}^1(E^*, A) \longrightarrow 0)$$

since $\delta = \operatorname{Ext}^1(f^*)^{-1}\delta_2$. On the other hand, we have

$$\operatorname{Ext}^1(f)(\zeta) = (0 \longrightarrow A \xrightarrow{\alpha} Z \xrightarrow{f^{-1}\beta} E \longrightarrow 0)$$

and

$$\begin{aligned}\beta_E \operatorname{Ext}^1(f)(\zeta) &= (0 \longrightarrow A \xrightarrow{\alpha} Z \xrightarrow{\partial_0^* f^{-1}\beta} P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\delta} \operatorname{Ext}^1(E^*, A) \longrightarrow 0)\\ &= \operatorname{Ext}^1(f^*)^\vee \beta_{E'}(\zeta).\end{aligned}$$

This proves the lemma. □

Let A be any local ring in which 2 is invertible and let m be its maximal ideal. Let E be a reflexive A-module. By an *ϵ-symmetric space* on E we mean an isomorphism $q\colon E \longrightarrow E^*$ such that $q^*\mathscr{C} = \epsilon q$, where $\mathscr{C}\colon E \longrightarrow E^{**}$ is the canonical isomorphism.

Let V be an A-module. By a *unimodular* element of V we mean an element $x \in V$ such that $f(x) = 1$ for some A-linear map $f\colon V \longrightarrow A$. For example, an element $(a_1, \ldots, a_n) \in A^n$ is unimodular if and only if $a_i \notin m$ for some i. Thus, if an A-module V has no unimodular elements and $\eta\colon V \longrightarrow A^n$ is an A-linear map, then $\eta(V) \subset mA^n$.

LEMMA 2.3. *Let E be a reflexive A-module and q be an ϵ-symmetric space on E. Suppose that $E = E_0 \oplus A^n$ with E_0 having no unimodular elements. Then there exist ϵ-symmetric spaces q_1 and q_2 over E_0 and A^n respectively such that*

$$(E, q) \simeq (E_0, q_1) \perp (A^n, q_2).$$

Proof. Let $E = E_0 \oplus A^n$ be such that E_0 has no unimodular elements. Then

$$q = \begin{pmatrix} q_1 & \eta \\ \epsilon\eta^* & q_1' \end{pmatrix}$$

for some $q_1\colon E_0 \longrightarrow E_0^*$, $q_1'\colon A^n \longrightarrow A^{n*}$ and $\eta\colon A^n \longrightarrow E_0^*$. Since E_0 has no unimodular elements, $\eta^*(E_0) \subset mA^{n*}$ and hence $\eta(A^n) \subset mE_0^*$. This implies that

$$q \equiv \begin{pmatrix} q_1 & 0 \\ 0 & q_1' \end{pmatrix} \bmod mE^*.$$

Since q is an isomorphism, q_1 and q_1' are isomorphisms. We have

$$\begin{pmatrix} 1 & 0 \\ -\epsilon\eta^* q_1^{-1} & 1 \end{pmatrix}\begin{pmatrix} q_1 & \eta \\ \epsilon\eta^* & q_1' \end{pmatrix}\begin{pmatrix} 1 & -q_1^{-1}\eta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} q_1 & 0 \\ 0 & -\eta^* q_1^{-1}\eta + q_1' \end{pmatrix}.$$

Let $q_2 = -\eta^* q_1^{-1}\eta + q_1' \colon A^n \longrightarrow A^{n*}$. Since $q_1^* = \epsilon q_1$, $(E, q) \simeq (E_0, q_1) \perp (A^n, q_2)$. □

3. *Spaces over the punctured spectrum and on finite-length modules*

We begin by recalling from a paper of Horrocks [**2**] an equivalence between the categories $\Phi\mathscr{P}$ of Φ-equivalence classes of vector bundles on the punctured spectrum of a regular local ring A of dimension 3 and the category $\mathscr{M}$ of finite-length A-modules. Let m be the maximal ideal of A and $Y = \mathrm{Spec}(A)\backslash\{m\}$. Let $\mathscr{E}$ be a vector bundle over Y and $E = \Gamma(\mathscr{E})$ be the module of sections of $\mathscr{E}$. Then E is a reflexive A-module [**2**, Theorem 4.1] and $M = \mathrm{Ext}^1(E^*, A)$, which is isomorphic to $H^1(Y, \mathscr{E})$ [**2**, § 5], is a finite-length A-module [**2**, Corollary 7.2.5]. The functor $T\colon \Phi\mathscr{P} \longrightarrow \mathscr{M}$ given by $T(\mathscr{E}) = \mathrm{Ext}^1(E^*, A)$ is an equivalence of categories [**2**, Corollary 7.2.5]. Let M be a finite-length A-module. The construction below gives a vector bundle $\mathscr{E}$ on Y such that $T(\mathscr{E}) = M$. Let, in fact,

$$0 \longrightarrow P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\eta} M \longrightarrow 0$$

be a projective resolution of M. Let $E = \ker(\partial_1)$. Then E is an A-module of projective dimension at most 1 and $\mathrm{Ext}^1(E^*, A) = M$. Since M is a finite-length module, for any prime ideal p of A, $p \neq m, M_p = 0$ and hence E_p is free. Thus $E = \Gamma(\mathscr{E})$ for some vector bundle $\mathscr{E}$ on Y [**2**, Theorem 4.1].

Let A be a regular local ring of dimension 3 in which 2 is invertible. Let $\mathscr{E}$ be a vector bundle over Y and $\mathbf{q}$ be an ϵ-symmetric space on $\mathscr{E}$. We associate to $(\mathscr{E}, \mathbf{q})$ a $(-\epsilon)$-symmetric space $\rho(\mathbf{q})$ of finite length. The ϵ-symmetric space $\mathbf{q}$ on $\mathscr{E}$ gives rise to an ϵ-symmetric space (E, q), where $E = \Gamma(\mathscr{E})$. Then $M = \mathrm{Ext}^1(E^*, A)$ is a finite-length A-module. The isomorphism $q\colon E \longrightarrow E^*$ induces an isomorphism $\mathrm{Ext}^1(q)\colon M = \mathrm{Ext}^1(E^*, A) \longrightarrow \mathrm{Ext}^1(E, A)$. Let $\rho(\mathbf{q}) = \beta_E \,\mathrm{Ext}^1(q)$. We have the following lemma.

LEMMA 3.1. *$\rho(\mathbf{q})\colon M \longrightarrow M^\vee$ is a $(-\epsilon)$-symmetric space.*

Proof. In Figure 10, clearly, all the squares except perhaps the top left one commute.

$$\begin{array}{ccccc}
M = \mathrm{Ext}^1(E^*, A) & \xrightarrow{\mathrm{Ext}^1(q)} & \mathrm{Ext}^1(E, A) & \xrightarrow{\beta_E} & \mathrm{Ext}^1(E^*, A)^\vee = M^\vee \\
\downarrow{\scriptstyle \beta_{E^*}} & & \downarrow{\scriptstyle \beta_E} & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{Ext}^1(E, A)^\vee & \xrightarrow{\epsilon \mathrm{Ext}^1(q)^\vee} & \mathrm{Ext}^1(E^*, A)^\vee & \xrightarrow{\mathrm{id}} & \mathrm{Ext}^1(E^*, A)^\vee \\
\downarrow{\scriptstyle \beta_E^{\vee -1}} & & \downarrow{\scriptstyle \epsilon \mathrm{Ext}^1(q)^{\vee -1}} & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{Ext}^1(E^*, A)^{\vee\vee} & \xrightarrow{\beta_E^\vee} & \mathrm{Ext}^1(E, A)^\vee & \xrightarrow{\epsilon \mathrm{Ext}^1(q)^\vee} & \mathrm{Ext}^1(E^*, A)^\vee
\end{array}$$

FIGURE 10.

Since $q^* = \epsilon q$, by Lemma 2.2 this square also commutes. By Lemma 2.2, the composition of maps on the first column is equal to $-\mathscr{C}$. Thus

$$\rho(q)^\vee \mathscr{C} = \mathrm{Ext}^1(q)^\vee \beta_E^\vee \mathscr{C} = -\epsilon\beta_E \mathrm{Ext}^1(q) = -\epsilon\rho(q). \qquad \square$$

LEMMA 3.2. *Let M be a finite-length A-module and ψ be an ϵ-symmetric form on M. Suppose that there exists an exact sequence*

$$N \xrightarrow{f} M \xrightarrow{f^\vee \psi} N^\vee$$

of finite-length A-modules. Then (M, ψ) is metabolic.

Proof. Since the map f factors as $N \longrightarrow N/\ker(f) \xrightarrow{\bar{f}} M$, we have an exact sequence

$$0 \longrightarrow N/\ker(f) \xrightarrow{\bar{f}} M \xrightarrow{\bar{f}^\vee \psi} (N/\ker(f))^\vee.$$

Since, the dimension of A being 3, $\mathrm{Ext}^4(L, A) = 0$, the map $\bar{f}^\vee \psi$ is surjective and hence (M, ψ) is metabolic. $\square$

LEMMA 3.3. If $(\mathscr{E}, \mathbf{q})$ *is metabolic, then* $(M, \rho(\mathbf{q}))$ *is metabolic.*

Proof. Suppose that $(\mathscr{E}, \mathbf{q})$ is metabolic. Let $\mathscr{F}$ be a subbundle of $\mathscr{E}$ such that the sequence

$$0 \longrightarrow \mathscr{F} \xrightarrow{i} \mathscr{E} \xrightarrow{i^*\mathbf{q}} \mathscr{F}^* \longrightarrow 0$$

is exact, where $\mathscr{F} \xrightarrow{i} \mathscr{E}$ is the inclusion. By taking global sections and then applying the Ext functor to the following exact sequence of bundles

$$0 \longrightarrow \mathscr{F} \xrightarrow{\mathbf{q}i} \mathscr{E}^* \xrightarrow{i^*} \mathscr{F}^* \longrightarrow 0$$

we get an exact sequence

$$\mathrm{Ext}^1(F^*, A) \longrightarrow \mathrm{Ext}^1(E^*, A) \longrightarrow \mathrm{Ext}^1(F, A)$$

of finite-length modules, where $F = \Gamma(\mathscr{F})$. Let $N = \mathrm{Ext}^1(F^*, A)$. Then the canonical identification of $\mathrm{Ext}^1(F, A)$ with $N^\vee$ gives an exact sequence

$$N \xrightarrow{f} M \xrightarrow{f^\vee \rho(\mathbf{q})} N^\vee.$$

Now the lemma follows from Lemma 3.2. $\square$

LEMMA 3.4. *The assignment* $(\mathscr{E}, \mathbf{q}) \mapsto (M, \rho(\mathbf{q}))$ *induces a homomorphism*

$$\rho\colon W^\epsilon(Y) \longrightarrow W_{\mathrm{fl}}^{-\epsilon}(A).$$

Proof. Since ρ is clearly additive, it is enough to show that ρ takes stably metabolic spaces to metabolic spaces. Let $(\mathscr{E}, \mathbf{q})$ be an ϵ-symmetric space over Y which is stably metabolic. Then there exists a metabolic space $(\mathscr{E}_1, \mathbf{q}_1)$ such that $(\mathscr{E}, \mathbf{q}) \perp (\mathscr{E}_1, \mathbf{q}_1)$ is metabolic. By Lemma 3.3, $\rho(\mathbf{q}_1)$ and $\rho(\mathbf{q} \perp \mathbf{q}_1) = \rho(\mathbf{q}) \perp \rho(\mathbf{q}_1)$ are metabolic. Thus $\rho(\mathbf{q})$ is stably metabolic. $\square$

We note that if $\mathscr{E}$ is a trivial bundle then $M = 0$. Thus, if $(\mathscr{E}, \mathbf{q})$ comes from an ϵ-symmetric space on A, then $\rho(\mathbf{q}) = 0$.

The proof of the following lemma is by straightforward verification; hence we omit it.

LEMMA 3.5. *Let R be a ring. Let $0 \longrightarrow N \xrightarrow{i} M \xrightarrow{j} N' \longrightarrow 0$ be an exact sequence of R-modules. Assume that the projective dimensions of N and N' are finite. Let*

$$0 \longrightarrow P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\alpha} N \longrightarrow 0$$

and

$$0 \longrightarrow Q_n \xrightarrow{\partial'_n} Q_{n-1} \xrightarrow{\partial'_{n-1}} \cdots \longrightarrow Q_1 \xrightarrow{\partial'_1} Q_0 \xrightarrow{\beta} N' \longrightarrow 0$$

be projective resolutions of N and N' respectively. Let, for $l \geqslant 1$, $\phi_l : Q_l \longrightarrow P_{l-1}$ and $\theta : Q_0 \longrightarrow M$ be R-linear homomorphisms. Let

$$\delta_l = \begin{pmatrix} \partial_l & (-1)^l \phi_l \\ 0 & \partial'_l \end{pmatrix}.$$

Then Figure 11 is commutative if and only if Figure 12 is commutative.

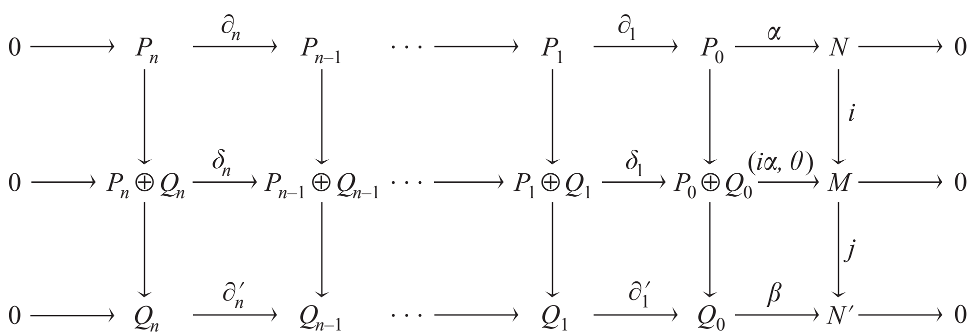

FIGURE 11.

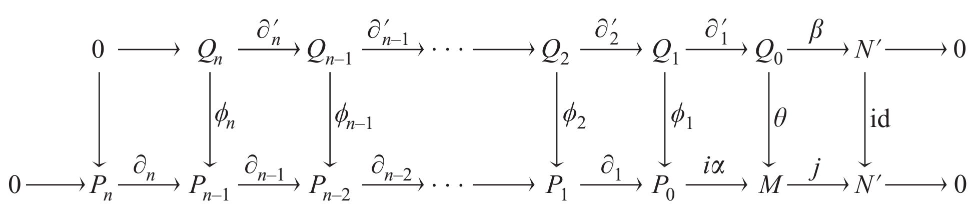

FIGURE 12.

PROPOSITION 3.6. *Let $(\mathscr{E}, \mathbf{q})$ be an ϵ-symmetric space over Y. Suppose that $E = \Gamma(\mathscr{E})$ has no unimodular elements and that $\rho(\mathbf{q})$ is metabolic. Then there exist ϵ-symmetric spaces $\mathbf{q}_1$ and $\mathbf{q}_2$ on $\mathscr{E}$ and $\mathscr{O}_Y^n$ respectively such that $\mathbf{q}_1 \perp \mathbf{q}_2$ is metabolic and $\rho(\mathbf{q}) \simeq \rho(\mathbf{q}_1)$.*

Proof. Let $M = \mathrm{Ext}^1(E^*, A)$ and $\rho(\mathbf{q})$ be the ϵ-symmetric space on M. Since $\rho(\mathbf{q})$ is metabolic, there exists an exact sequence

$$0 \longrightarrow N \xrightarrow{\;i\;} M \xrightarrow{\;i^\vee \rho(q)\;} N^\vee \longrightarrow 0.$$

Let

$$0 \longrightarrow Q_3 \xrightarrow{\;\partial_3\;} Q_2 \xrightarrow{\;\partial_2\;} Q_1 \xrightarrow{\;\partial_1\;} Q_0 \xrightarrow{\;\eta\;} N \longrightarrow 0$$

be a projective resolution of N. By dualizing this resolution, we get a projective resolution

$$0 \longrightarrow Q_0^* \xrightarrow{\;\partial_1^*\;} Q_1^* \xrightarrow{\;\partial_2^*\;} Q_2^* \xrightarrow{\;\partial_3^*\;} Q_3^* \xrightarrow{\;\eta'\;} N^\vee \longrightarrow 0$$

of $N^\vee$. By lifting the identity map of $N^\vee$, we obtain a commutative diagram (Figure 13).

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & Q_0^* & \xrightarrow{\partial_1^*} & Q_1^* & \xrightarrow{\partial_2^*} & Q_2^* & \xrightarrow{\partial_3^*} & Q_3^* & \xrightarrow{\eta'} & N^\vee & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \phi_3} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
Q_3 & \xrightarrow{\partial_3} & Q_2 & \xrightarrow{\partial_2} & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{i\eta} & M & \xrightarrow{i^\vee\rho(q)} & N^\vee & \longrightarrow & 0
\end{array}$$

FIGURE 13.

Let

$$\delta_1 = \begin{pmatrix} \partial_1 & -\phi_1 \\ 0 & \partial_3^* \end{pmatrix}, \quad \delta_2 = \begin{pmatrix} \partial_2 & \phi_2 \\ 0 & \partial_2^* \end{pmatrix}, \quad \delta_3 = \begin{pmatrix} \partial_3 & -\phi_3 \\ 0 & \partial_1^* \end{pmatrix}.$$

By Lemma 3.5, Figure 14 is commutative.

$$\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_3 & \xrightarrow{\partial_3} & Q_2 & \xrightarrow{\partial_2} & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{\eta} & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & Q_3 \oplus Q_0^* & \xrightarrow{\delta_3} & Q_2 \oplus Q_1^* & \xrightarrow{\delta_2} & Q_1 \oplus Q_2^* & \xrightarrow{\delta_1} & Q_0 \oplus Q_3^* & \xrightarrow{(i\eta,\theta)} & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle i^\vee\rho(q)} & & \\
0 & \longrightarrow & Q_0^* & \xrightarrow{\partial_1^*} & Q_1^* & \xrightarrow{\partial_2^*} & Q_2^* & \xrightarrow{\partial_3^*} & Q_3^* & \xrightarrow{\eta'} & N^\vee & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

FIGURE 14.

Since the first row, the last rows and all the columns in Figure 14 are exact and the diagram is commutative, from the long exact homology sequence [**4**, Theorem 4.1, p. 45] we get that the middle row is also exact. By dualizing Figure 14, we get the commutative diagram in Figure 15, with exact rows and columns.

$$
\begin{array}{ccccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q_3 & \xrightarrow{\partial_3} & Q_2 & \xrightarrow{\partial_2} & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{\eta''} & N^{\vee\vee} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \rho(q)^{\vee} i^{\vee\vee}} & & \\
0 & \longrightarrow & Q_3 \oplus Q_0^* & \xrightarrow{\delta_1^*} & Q_2 \oplus Q_1^* & \xrightarrow{\delta_2^*} & Q_1 \oplus Q_2^* & \xrightarrow{\delta_3^*} & Q_0 \oplus Q_3^* & \xrightarrow{(\mu,\nu)} & M^{\vee} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle i^{\vee}} & & \\
0 & \longrightarrow & Q_0^* & \xrightarrow{\partial_1^*} & Q_1^* & \xrightarrow{\partial_2^*} & Q_2^* & \xrightarrow{\partial_3^*} & Q_3^* & \xrightarrow{\eta'} & N^{\vee} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & & 0 & &
\end{array}
$$

FIGURE 15.

$$
\begin{array}{ccccccccccccc}
0 & \longrightarrow & Q_0^* & \xrightarrow{\partial_1^*} & Q_1^* & \xrightarrow{\partial_2^*} & Q_2^* & \xrightarrow{\partial_3^*} & Q_3^* & \xrightarrow{\eta'} & N^{\vee} & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \phi_1^*} & & \downarrow{\scriptstyle \phi_2^*} & & \downarrow{\scriptstyle \phi_3^*} & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
Q_3 & \xrightarrow{\partial_3} & Q_2 & \xrightarrow{\partial_2} & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{\rho(q)^{\vee} i^{\vee\vee} \eta''} & M^{\vee} & \xrightarrow{i^{\vee}} & N^{\vee} & \longrightarrow & 0
\end{array}
$$

FIGURE 16.

By Lemma 3.5, Figure 16 is commutative. From the definition of η'' and $\mathscr{C} : N \longrightarrow N^{\vee\vee}$ (cf. Figure 2) it follows that $\mathscr{C}\eta = -\eta''$. Since $\rho(q)^{\vee}\mathscr{C} = -\epsilon\rho(q)$, we have the commutative diagram in Figure 17, where $\nu' = \epsilon\rho(q)^{-1}\nu$.

$$
\begin{array}{ccccccccccccc}
0 & \longrightarrow & Q_0^* & \xrightarrow{\partial_1^*} & Q_1^* & \xrightarrow{\partial_2^*} & Q_2^* & \xrightarrow{\partial_3^*} & Q_3^* & \xrightarrow{\eta'} & N^{\vee} & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \phi_1^*} & & \downarrow{\scriptstyle \phi_2^*} & & \downarrow{\scriptstyle \phi_3^*} & & \downarrow{\scriptstyle \nu'} & & \downarrow{\scriptstyle \epsilon\mathrm{id}} & & \\
Q_3 & \xrightarrow{\partial_3} & Q_2 & \xrightarrow{\partial_2} & Q_1 & \xrightarrow{\partial_1} & Q_0 & \xrightarrow{i\eta} & M & \xrightarrow{i^{\vee}\rho(q)} & N^{\vee} & \longrightarrow & 0
\end{array}
$$

FIGURE 17.

From Figure 13 and Figure 17 we get maps $s_1 : Q_2^* \longrightarrow Q_1$ and $s_2 : Q_1^* \longrightarrow Q_2$ such that $\phi_2 - \epsilon\phi_2^* = \partial_2 s_2 - s_1 \partial_2^*$. Let $\phi = \phi_2 - \epsilon\partial_2 s_1^*$. Then we have

$$\begin{aligned}\partial_1 \phi &= \partial_1 \phi_2 \\ &= \partial_1(\phi_2 - \epsilon\phi_2^*) + \epsilon\partial_1\phi_2^* \\ &= \partial_1(\partial_2 s_2 - s_1 \partial_2^*) + \epsilon\partial_1 \phi_2^* \\ &= \epsilon\partial_1(\phi_2^* - \epsilon s_1 \partial_2^*) \\ &= \epsilon\partial_1 \phi^*.\end{aligned}$$

Let

$$\delta = \begin{pmatrix} \partial_2 & \dfrac{\phi + \epsilon\phi^*}{2} \\ 0 & \epsilon\partial_2^* \end{pmatrix}.$$

It is easy to see that Figure 18 commutes.

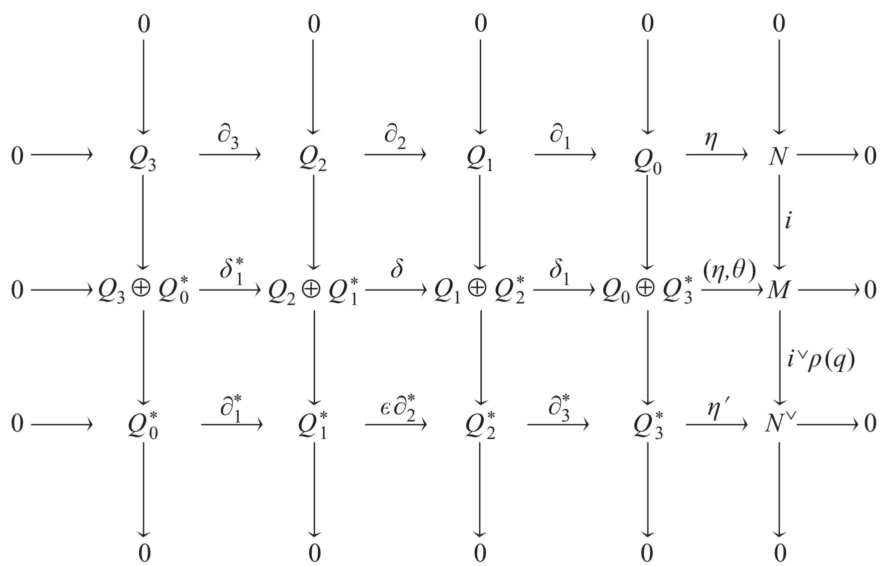

FIGURE 18.

Since the first row, the last row and all the columns are exact, the middle row is also exact. Let $E' = \ker(\delta_1)$. Since $\delta^*\mathscr{C} = \epsilon\delta$, from the middle row of Figure 18 it is easy to see that δ induces an ϵ-symmetric isomorphism $q' : E' \longrightarrow E'^*$. Let $(\mathscr{E}', \mathbf{q}')$ be the ϵ-symmetric space over Y with $\Gamma(\mathscr{E}') = E'$ and $\Gamma(\mathbf{q}') = q'$. Since $\mathrm{Ext}^1(E', A) \simeq M = \mathrm{Ext}^1(E^*, A)$ and E has no unimodular elements, by [**2**, Corollary 7.2.5, Lemma 7.1] we have $E' = E \oplus A^n$. Then by Lemma 2.3, $(E', q') \simeq (E, q_1) \perp (A^n, q_2)$ for some ϵ-symmetric spaces q_1 and q_2 on E and A^n respectively. Let $\mathbf{q}_1$ be the ϵ-symmetric space on $\mathscr{E}$ such that $\Gamma(\mathbf{q}_1) = q_1$. Let $F = \ker(\partial_1)$ and $\mathscr{F}$ be the vector bundle over Y with $\Gamma(\mathscr{F}) = F$. Then using Figure 18, it is easy to see that $\mathscr{F}$ is a Lagrangian for $(\mathscr{E}', \mathbf{q}') \simeq (\mathscr{E}, \mathbf{q}_1) \perp (\mathscr{O}_Y^n, \mathbf{q}_2)$, where $\Gamma(\mathbf{q}_2) = q_2$. Since the map $E' \longrightarrow F^*$ induced by Figure 18 induces $i^\vee \rho(\mathbf{q}) : M \longrightarrow N^\vee$, we have $i^\vee \rho(\mathbf{q}) = i^\vee \rho(\mathbf{q}_1)$. Thus, by Lemma 3.7 below, we have $\rho(\mathbf{q}) \simeq \rho(\mathbf{q}_1)$. □

LEMMA 3.7. *Let ψ_1 and ψ_2 be two ϵ-symmetric spaces on M. Suppose there exists a submodule N such that*

$$0 \longrightarrow N \xrightarrow{\;i\;} M \xrightarrow{\;i^\vee \psi_1\;} N^\vee \longrightarrow 0$$

is exact and $i^\vee \psi_1 = i^\vee \psi_2$. Then $\psi_1 \simeq \psi_2$.

Proof. Since $i^\vee \psi_1 = i^\vee \psi_2$, there exists a map $\theta: M \longrightarrow N$ such that $\psi_1^{-1}\psi_2 - 1 = i\theta$, that is, $\psi_1 i\theta = \psi_2 - \psi_1 = \theta^\vee i^\vee \psi_1$. We have

$$\begin{aligned}\frac{(1+\theta^\vee i^\vee)}{2}\psi_1\frac{(1+i\theta)}{2} &= \frac{(\psi_1+\theta^\vee i^\vee \psi_1)}{2}\frac{(1+i\theta)}{2}\\ &= \psi_1 + \psi_1\frac{i\theta}{2} + \frac{\theta^\vee i^\vee}{2}\psi_1 + \frac{\theta^\vee i^\vee}{2}\psi_1\frac{i\theta}{2}\\ &= \psi_1 + \frac{\psi_2-\psi_1}{2} + \frac{\psi_2-\psi_1}{2}\\ &= \psi_2.\end{aligned}$$

□

4. *The Witt groups of the punctured spectrum and purity*

Let A be a regular local ring of dimension 3 with 2 invertible. Let $Y = \operatorname{Spec}(A)\backslash\{m\}$, where m is the maximal ideal of A.

PROPOSITION 4.1. *Let $\mathscr{E}$ be a vector bundle on Y and $\mathbf{q}: \mathscr{E} \longrightarrow \mathscr{E}^*$ be an ϵ-symmetric isomorphism. Suppose that $\Gamma(\mathscr{E})$ has no unimodular elements. If $\rho(\mathbf{q})$ is isomorphic to a hyperbolic space, then $\mathbf{q}$ is in the image of $W^\epsilon(A)$.*

Proof. Let N be a finite-length A-module such that $(M, \rho(\mathbf{q}))$ is isomorphic to the hyperbolic space $\mathbb{H}(N)$. Let $\mathscr{F}$ be the vector bundle on Y with $\Gamma(\mathscr{F})$ having no unimodular elements and such that $H^1(Y, \mathscr{F}) \simeq N$ (cf. §3). Since

$$H^1(Y, \mathscr{E}) \simeq N \oplus N^\vee \simeq H^1(Y, \mathscr{F} \oplus \mathscr{F}^*)$$

with $\Gamma(\mathscr{E})$ and $\Gamma(\mathscr{F} \oplus \mathscr{F}^*)$ admitting no unimodular elements, by [**2**, Lemma 7.1, Corollary 7.2.5] we can and do identify $\mathscr{E}$ with $\mathscr{F} \oplus \mathscr{F}^*$. Let $\tilde{\psi}$ be an isometry of $\rho(\mathbf{q})$ with

$$\rho\left(\begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}\right).$$

Then by [**2**, Corollary 7.2.5] there exists an automorphism ψ of $\mathscr{E}$ such that $H^1(\psi) = \tilde{\psi}$. By the definition of ρ we have

$$\begin{aligned}\rho(\psi^*\mathbf{q}\psi) &= \beta_E \operatorname{Ext}^1(\Gamma(\psi^*)q\Gamma(\psi))\\ &= \beta_E \operatorname{Ext}^1(\Gamma(\psi^*)) \operatorname{Ext}^1(q) \operatorname{Ext}^1(\Gamma(\psi))\end{aligned}$$

where $E = \Gamma(\mathscr{E})$ and $q = \Gamma(\mathbf{q})$. By Lemma 2.2(ii), we have $\beta_E \operatorname{Ext}^1(\Gamma(\psi^*)) = \operatorname{Ext}^1(\Gamma(\psi))^\vee \beta_E$, so that

$$\begin{aligned}\rho(\psi^*\mathbf{q}\psi) &= \operatorname{Ext}^1(\Gamma(\psi))^\vee \beta_E \operatorname{Ext}^1(q) \operatorname{Ext}^1(\Gamma(\psi))\\ &= \operatorname{Ext}^1(\Gamma(\psi))^\vee \rho(\mathbf{q}) \operatorname{Ext}^1(\Gamma(\psi))\\ &= H^1(\psi)^\vee \rho(\mathbf{q}) H^1(\psi)\\ &= \tilde{\psi}\rho(\mathbf{q})\tilde{\psi}.\end{aligned}$$

We replace $\mathbf{q}$ by $\psi^*\mathbf{q}\psi$ and assume that

$$\rho(\mathbf{q}) = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}.$$

Let

$$\mathbf{q} = \begin{pmatrix} \alpha & \delta \\ \epsilon\delta^* & \beta \end{pmatrix}$$

with $\alpha : \mathscr{F} \longrightarrow \mathscr{F}^*$, $\beta : \mathscr{F}^* \longrightarrow \mathscr{F}$, $\delta : \mathscr{F}^* \longrightarrow \mathscr{F}^*$ maps such that $\alpha^* = \epsilon\alpha$, $\beta^* = \epsilon\beta$. Since

$$\rho(\mathbf{q}) = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix},$$

$H^1(\alpha) = 0$ and $H^1(\beta) = 0$. Therefore, by [**2**, Corollary 7.2.5], there exist

$$f_1 : \mathscr{F} \longrightarrow \mathscr{O}_Y^n, \quad f_2 : \mathscr{F}^* \longrightarrow \mathscr{O}_Y^n, \quad g_1 : \mathscr{O}_Y^n \longrightarrow \mathscr{F}^* \quad \text{and} \quad g_2 : \mathscr{O}_Y^n \longrightarrow \mathscr{F}$$

such that $\alpha = g_1 f_1$ and $\beta = g_2 f_2$. Let us consider the automorphism

$$\psi = \begin{pmatrix} 1 & 0 & -g_1/2 & f_1^* \\ 0 & 1 & -g_2/2 & f_2^* \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

of $\mathscr{F} \oplus \mathscr{F}^* \oplus \mathscr{O}_Y^n \oplus \mathscr{O}_Y^{n*}$. We have

$$\mathbf{q}' = \psi \begin{pmatrix} \alpha & \delta & 0 & 0 \\ \epsilon\delta^* & \beta & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \epsilon & 0 \end{pmatrix} \qquad \psi^* = \begin{pmatrix} 0 & X & \epsilon f_1^* & -g_1/2 \\ \epsilon X^* & 0 & \epsilon f_2^* & -g_2/2 \\ f_1 & f_2 & 0 & 1 \\ -\epsilon g_1^*/2 & -\epsilon g_2^*/2 & \epsilon & 0 \end{pmatrix}$$

where $X = \delta - \epsilon f_1^* g_2^*/2 - g_1 f_2/2$. Since $\Gamma(\mathscr{F})$ and $\Gamma(\mathscr{F}^*)$ admit no unimodular elements, $f_1 \equiv f_2 \equiv 0 \bmod m$. Hence X is an isomorphism. Since $\mathbf{q}'$ restricted to $\mathscr{F} \oplus \mathscr{F}^*$ is

$$\begin{pmatrix} 0 & X \\ \epsilon X^* & 0 \end{pmatrix}$$

with X an isomorphism, $\mathbf{q}'$ splits as

$$\begin{pmatrix} 0 & X \\ \epsilon X^* & 0 \end{pmatrix} \perp \mathbf{q}''$$

for some $\mathbf{q}''$ supported on a bundle $\mathscr{E}''$. Since $\mathscr{E} \oplus \mathscr{O}_Y^n \oplus \mathscr{O}_Y^{n^*} \simeq \mathscr{E} \oplus \mathscr{E}''$, by [**2**, Corollary 7.2.5], $\mathscr{E}''$ is a trivial bundle and hence $\mathbf{q}''$ is in the image of $W^\epsilon(A)$. Since

$$\mathbf{q} \perp H(\mathscr{O}_Y^n) \simeq \mathbf{q}' \simeq \begin{pmatrix} 0 & X \\ \epsilon X^* & 0 \end{pmatrix} \perp \mathbf{q}''$$

it follows that $\mathbf{q}$ is in the image of $W^\epsilon(A)$. □

LEMMA 4.2. *Let M be a finite-length A-module and $\psi : M \longrightarrow M^\vee$ be an ϵ-symmetric isomorphism. Then there exists a vector bundle $\mathscr{E}$ over Y with a $(-\epsilon)$-symmetric isomorphism $\mathbf{q} : \mathscr{E} \longrightarrow \mathscr{E}^*$ such that $\rho(\mathbf{q}) = \psi$.*

Proof. Let $\mathscr{E}$ be a vector bundle on Y such that $H^1(Y, \mathscr{E}) = M$ (cf. §3) and $E = \Gamma(\mathscr{E})$ has no unimodular elements. Then (cf. §3) the projective dimension of E is less than or equal to 1 and $\mathrm{Ext}^1(E^*, A) \simeq M$. Let

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow E \longrightarrow 0$$

and

$$0 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow E^* \longrightarrow 0$$

be projective resolutions of E and E^* respectively. By dualizing the projective resolution of E^* we get an exact sequence

$$0 \longrightarrow E \longrightarrow Q_0^* \longrightarrow Q_1^* \longrightarrow M \longrightarrow 0.$$

By taking the Yoneda composition of this exact sequence with the projective resolution of E we get a projective resolution

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow Q_0^* \longrightarrow Q_1^* \longrightarrow M \longrightarrow 0$$

of M. By dualizing this we get a projective resolution

$$0 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow M^\vee \longrightarrow 0$$

of $M^\vee$. By lifting the ϵ-symmetric isomorphism $\psi : M \longrightarrow M^\vee$, we get a commutative diagram of exact sequences (Figure 19).

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & Q_0^* & \longrightarrow & Q_1^* & \longrightarrow & M & \longrightarrow & 0 \\
 & & \Big\downarrow 3 & & \Big\downarrow 2 & & \Big\downarrow 1 & & \Big\downarrow 0 & & \Big\downarrow & & \\
0 & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & P_0^* & \longrightarrow & P_1^* & \longrightarrow & M^\vee & \longrightarrow & 0
\end{array}$$

FIGURE 19.

Since E has no unimodular elements it is easy to see, as in the proof of Proposition 3.6, that Figure 19 induces a $(-\epsilon)$-symmetric space $\mathbf{q}$ on $\mathscr{E}$ such that $\rho(\mathbf{q}) = \psi$. □

THEOREM 4.3. *Let A be a regular local ring of dimension 3 and let m be its maximal ideal. Assume that 2 is invertible in A. Let $Y = \mathrm{Spec}(A) \backslash \{m\}$. Then the complex*

$$0 \longrightarrow W^\epsilon(A) \xrightarrow{\ \iota\ } W^\epsilon(Y) \xrightarrow{\ \rho\ } W_{\mathrm{fl}}^{-\epsilon}(A) \longrightarrow 0$$

is exact, where ι is induced by the restriction.

Proof. If $\epsilon = 1$, then the injectivity of ι follows from the injectivity of the canonical homomorphism $W(A) \longrightarrow W(K)$ [**6**, Theorem 23], where K is the quotient field of A. If $\epsilon = -1$, ι is injective because $W^{-1}(A) = 0$.

We now prove the exactness in the middle. As we remarked in §3, $\rho\iota = 0$. Let $(\mathscr{E}, \mathbf{q})$ be an ϵ-symmetric space over Y such that $\rho(\mathbf{q})$ is zero in $W_{\mathrm{fl}}^{-\epsilon}(A)$. Then, by Lemma 1.4, $\rho(\mathbf{q})$ is metabolic. We show that $(\mathscr{E}, \mathbf{q})$ is in the image of ι. In view of Lemma 2.3, we assume that $\Gamma(\mathscr{E})$ has no unimodular elements. Then, by Proposition 3.6, there exist ϵ-symmetric spaces $\mathbf{q}_1$ and $\mathbf{q}_2$ supported respectively on $\mathscr{E}$ and $\mathscr{O}_Y^n$ for some integer n, such that $\mathbf{q}_1 \perp \mathbf{q}_2$ is metabolic and $\rho(\mathbf{q}) \simeq \rho(\mathbf{q}_1)$. Thus $\rho(\mathbf{q} \perp -\mathbf{q}_1)$ is isomorphic to a hyperbolic space. Since $\Gamma(\mathscr{E} \oplus \mathscr{E})$ has no unimodular elements, it follows from Proposition 4.1 that $\mathbf{q} \perp -\mathbf{q}_1$ is in the image of $\imath$. Since $\mathbf{q}_2$ is in the image of $\imath$ and $\mathbf{q}_1 \perp \mathbf{q}_2$ is metabolic, $\mathbf{q}_1$ and hence $\mathbf{q}$ is in the image of $\imath$.

The surjectivity of ρ follows from Lemma 4.2. □

Let A be any regular ring. Let $\mathrm{Spec}^1(A)$ denote the set of prime ideals of A of height 1. Then for any $P \in \mathrm{Spec}^1(A)$, the local ring A_P is a discrete valuation ring. Let $\partial_P : W(K) \longrightarrow W(A_P/PA_P)$ denote the second residue homomorphism with respect to some choice of a parameter of PA_P, where K is the quotient field of A.

Corollary 4.4. *Let A be a regular local ring of dimension 3, m be its maximal ideal and K be its quotient field. Assume that 2 is invertible in A. The sequence*

$$0 \longrightarrow W(A) \longrightarrow W(K) \xrightarrow{\oplus \partial_P} \bigoplus_{P \in \mathrm{Spec}^1(A)} W(A_P/PA_P)$$

is exact.

Proof. The injectivity of $W(A) \longrightarrow W(K)$ is proved in [**6**, Theorem 23]. Since $W^{-1}_{\mathrm{fl}}(A) \simeq W^{-1}(A/m) = 0$, by Theorem 4.3 we have $W(A) \simeq W(Y)$. Thus it is enough to prove that the complex

$$W(Y) \longrightarrow W(K) \xrightarrow{\oplus \partial_P} \bigoplus_{P \in \mathrm{Spec}^1(A)} W(A_P/PA_P)$$

is exact. Let q be a quadratic space over K such that $\partial_P(q) = 0$ for all height 1 prime ideals P of A. Since Y is a regular scheme of dimension 2, by [**1**, 2.5, p. 112], there exists a quadratic space $(\mathscr{E}, \mathbf{q})$ over $Y = \mathrm{Spec}(A) \setminus \{m\}$, such that its image in $W(K)$ under the restriction map is equal to q. This completes the proof. □

Using Corollary 4.4, one can prove the following theorem (cf. [**8**, Proposition 2.1]).

Corollary 4.5. *Let A be a regular ring of dimension 3 and K be its quotient field. Assume that 2 is invertible in A. The sequence*

$$0 \longrightarrow W(A) \longrightarrow W(K) \xrightarrow{\oplus \partial_P} \bigoplus_{P \in \mathrm{Spec}^1(A)} W(A_P/PA_P)$$

is exact.

We end this paper by giving a computation of $W^{-1}(Y)$ using Theorem 4.3.

Corollary 4.6. *Let A be a regular local ring of dimension 3 and m be its maximal ideal. Assume that 2 is invertible in A. Let $Y = \mathrm{Spec}(A) \setminus \{m\}$. Then $W^{-1}(Y) \simeq W(A/m)$.*

Proof. Since $W^{-1}(A) = 0$ and $W_{\mathrm{fl}}(A) \simeq W(A/m)$, the result follows from Theorem 4.3. □

Acknowledgements. We thank the referee for patiently pointing out errors, obscurities and misprints in the various versions of this paper.

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