ESTIMATES FOR DIRICHLET EIGENFUNCTIONS

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Abstract

Estimates for the Dirichlet eigenfunctions near the boundary of an open, bounded set in euclidean space are obtained. It is assumed that the boundary satisfies a uniform capacitary density condition.

1. *Introduction*

Let D be an open, bounded set in euclidean space $\mathbb{R}^m$ $(m = 2, 3, \ldots)$ with boundary ∂D. Let $-\Delta_D$ be the Dirichlet laplacian for D. The spectrum of $-\Delta_D$ is discrete and consists of eigenvalues $\lambda_1 \leqslant \lambda_2 \leqslant \ldots$ with a corresponding orthonormal set of eigenfunctions $\{\phi_1, \phi_2, \ldots\}$. The behaviour of the eigenfunctions near the boundary ∂D of D has been investigated by several authors under a variety of assumptions on the geometry of D [**1**–**8**, **11**–**16**].

In this paper we obtain pointwise bounds on the eigenfunctions under the assumption that ∂D satisfies a uniform capacitary density condition. Denote by $\mathrm{Cap}(A)$ the newtonian capacity of a compact set $A \subset \mathbb{R}^m$ $(m = 3, 4, \ldots)$ or the logarithmic capacity of a compact set $A \subset \mathbb{R}^2$. For $x \in \mathbb{R}^m$ and $r > 0$ we define

$$B(x; r) = \{y \in \mathbb{R}^m : |y - x| \leqslant r\}, \tag{1.1}$$

and for a non-empty set $G \subset \mathbb{R}^m$

$$\mathrm{diam}(G) = \sup\{|x_1 - x_2| : x_1 \in G, x_2 \in G\}. \tag{1.2}$$

Definition 1.1. Let $D \subset \mathbb{R}^m$ $(m = 2, 3, \ldots)$ be an open set with boundary ∂D. Then ∂D satisfies an α-uniform capacitary density condition if for some $\alpha \in (0, 1]$

$$\mathrm{Cap}((\partial D) \cap B(x; r)) \geqslant \alpha\, \mathrm{Cap}(B(x; r)), \quad x \in \partial D, \quad 0 < r < \mathrm{diam}(D). \tag{1.3}$$

Condition (1.3) guarantees that all points of ∂D are regular, and in particular that $\lim_{x \to x_0} \phi_j(x) = 0$ for all $x_0 \in \partial D$. Definition 1.1 has been introduced in [**10**] in a study of the partition function of the Dirichlet laplacian on open sets with a non-smooth or fractal boundary.

Let $d : D \longrightarrow (0, \infty)$ denote the distance function

$$d(x) = \min\{|x - y| : y \in \partial D\}, \tag{1.4}$$

and let R be the inradius of D, defined by

$$R = \max_{x \in D} d(x). \tag{1.5}$$

The main results of this paper are the following.

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THEOREM 1.2. *Let D be an open, bounded set in* $\mathbb{R}^2$. *Suppose* ∂D *satisfies* (1.3) *for some* $\alpha > 0$. *Then for* $j = 1, 2, \ldots$ *and all* $x \in D$ *such that* $d(x) < \lambda_j^{-1/2}$

$$|\phi_j(x)| \leqslant \left\{ 6\lambda_j \log(2/\alpha^{2\pi}) \frac{-1}{\log(d(x)\lambda_j^{1/2})} \right\}^{1/2}. \tag{1.6}$$

THEOREM 1.3. *Let D be an open, bounded set in* $\mathbb{R}^m$ $(m = 3, 4, \ldots)$. *Suppose* ∂D *satisfies* (1.3) *for some* $\alpha > 0$. *Let* $j = 1, 2, \ldots$ *and suppose* $x \in D$ *satisfies*

$$d(x)\lambda_j^{1/2} \leqslant \left(\frac{\alpha^6}{2^{13}}\right)^{1+\gamma(m-1)/(m-2)}. \tag{1.7}$$

Then

$$|\phi_j(x)| \leqslant 2\lambda_j^{m/4}(d(x)\,\lambda_j^{1/2})^{(1/2)((1/\gamma)+(m-1)/(m-2))^{-1}}, \tag{1.8}$$

where

$$\gamma = \frac{3^{-m-1}\alpha}{\log(2(2/\alpha)^{1/(m-2)})}. \tag{1.9}$$

The bounds in (1.6) and (1.8) are being complemented by the following well-known estimate [**11**, Lemma 3.1].

THEOREM 1.4. *Let D be an open, bounded set in* $\mathbb{R}^m$ $(m = 2, 3, \ldots)$. *Then for* $j = 1, 2, \ldots$ *and* $x \in D$

$$|\phi_j(x)| \leqslant \lambda_j^{m/4}. \tag{1.10}$$

The bounds of Theorems 1.2 and 1.3 are in general not sharp. For example if D is open, bounded and ∂D is smooth then the eigenfunctions are comparable with $d(x)$. If D is open, bounded and simply connected in $\mathbb{R}^2$ then it was shown by Bañuelos [**3**, Corollary 2.3b] that ϕ_1 is comparable to the hyperbolic distance induced by the conformal map F from the unit disc onto D. By Koebe's 1/4 theorem [**17**] one then obtains that

$$\phi_1(x) \leqslant Cd(x)^{1/2} \tag{1.11}$$

for some constant C depending on D. We will use the ideas of [**3**] to prove (in Section 5) the following refinement of (1.11).

THEOREM 1.5. *Let D be an open, simply connected set in* $\mathbb{R}^2$ *with volume* $|D|$. *Then for* $j = 1, 2, \ldots$

$$|\phi_j(x)| \leqslant 2^{9/2}\pi^{1/4} j|D|^{1/4} R^{-2} d(x)^{1/2}. \tag{1.12}$$

The following example (see [**13**, 4.6.7]) shows that Theorem 1.5 is sharp.

EXAMPLE 1.6. Let $U \subset \mathbb{R}^m$ be the conical region in polar coordinates defined by

$$U = \{(r, \omega) : 0 < r < 1, \omega \in \Omega\}, \tag{1.13}$$

where Ω is an open subset of the unit sphere S^{m-1}. Then

$$\varphi_1(r, \omega) \asymp r^{\beta(\Omega)}, \tag{1.14}$$

where $\beta(\Omega)$ is the positive solution of

$$\beta(\beta + m - 2) = \lambda_1(\Omega), \tag{1.15}$$

and where $\lambda_1(\Omega)$ is the first eigenvalue of the Laplace–Beltrami operator on Ω with Dirichlet conditions of $\partial\Omega$. In particular if $m = 2$ and

$$\Omega_0 = \{\omega \in S^1 : 0 < \omega < 2\pi\}, \tag{1.16}$$

then

$$\lambda_1(\Omega_0) = \tfrac{1}{4} \tag{1.17}$$

and by (1.15)

$$\beta(\Omega_0) = \tfrac{1}{2}, \tag{1.18}$$

which shows that the exponent in (1.12) cannot be improved.

The main idea of the proofs of Theorems 1.2 and 1.3 goes back to Brossard and Carmona [**10**] who obtained estimates for the Dirichlet heat kernel $p_D(x, x; t)$ for x near D. We improve their lemma [**10**, Lemma 3.5] and its proof (see also [**9**]). In the proof of Theorem 1.3 we also require a refinement of Wiener's test [**18**].

Let $(B(s), s \geqslant 0; \mathbb{P}_x, x \in \mathbb{R}^m)$ be a brownian motion associated to $-\Delta + \partial/\partial t$. Let T_D denote the first exit time of the brownian motion from D:

$$T_D = \inf\{s \geqslant 0 : B(s) \in \partial D\}. \tag{1.19}$$

For a compact set K we also define the first entry time

$$\tau_K = \inf\{s > 0 : B(s) \in K\}. \tag{1.20}$$

Then

$$\mathbb{P}_x[T_D > t] = \int_D p_D(x, y; t)\, dy. \tag{1.21}$$

By the eigenfunction expansion of the heat kernel and by the semigroup property we have

$$e^{-t\lambda_j} \phi_j^2(x) \leqslant \sum_{j=1}^{\infty} e^{-t\lambda_j} \phi_j^2(x) = p_D(x, x; t) = \int_D p_D^2(x, y; t/2)\, dy. \tag{1.22}$$

Since the Dirichlet heat kernel is monotone in D

$$p_D(x, y; t/2) \leqslant p_{\mathbb{R}^m}(x, y; t/2) \leqq (2\pi t)^{-m/2}. \tag{1.23}$$

Hence

$$e^{-t\lambda_j} \phi_j^2(x) \leqslant (2\pi t)^{-m/2} \int_D p_D(x, y; t/2)\, dy = (2\pi t)^{-m/2}\, \mathbb{P}_x[T_D > t/2]. \tag{1.24}$$

The choice

$$t = 2\lambda_j^{-1} \tag{1.25}$$

yields for $m = 2, 3, \ldots$

$$|\phi_j(x)| \leqslant e(4\pi)^{-m/4} \lambda_j^{m/4} (\mathbb{P}_x[T_D > \lambda_j^{-1}])^{1/2} \leqslant \lambda_j^{m/4} (\mathbb{P}_x[T_D > \lambda_j^{-1}])^{1/2}. \tag{1.26}$$

This proves Theorem 1.4 since $\mathbb{P}_x[T_D > \lambda_j^{-1}] \leqslant 1$.

In Sections 2 and 3 we obtain the upper bounds for $\mathbb{P}_x[T_D > \lambda_j^{-1}]$ in the cases $m = 2$ and $m = 3, 4, \ldots$ respectively. In the proof of Theorem 1.3 we use the following modification of Wiener's test. See also [**20**, Theorem 4.7, p. 73] for related refinements of Wiener's test.

THEOREM 1.7. *Let D be an open, bounded set in $\mathbb{R}^m$ $(m = 3, 4, \ldots)$. Suppose ∂D satisfies* (1.3) *for some $\alpha > 0$. If $x \in D$ satisfies $d(x) \leqslant (\alpha^6/2^{13})\operatorname{diam}(D)$, then for any*

$$a \in \left[\frac{2^{13}}{\alpha^6}, \operatorname{diam}(D)/d(x)\right] \tag{1.27}$$

one has

$$\mathbb{P}_x[\tau_{(\partial D) \cap B(x; ad(x))} < \infty] \geqslant 1 - 2a^{-\gamma}, \tag{1.28}$$

where γ is given by (1.9).

The proof of Theorem 1.7 will be deferred to Section 4.

2. *Proof of Theorem* 1.2

Let $m = 2$ and define the Green function

$$g(x, y) = -(2\pi)^{-1} \log |x-y|. \tag{2.1}$$

The equilibrium measure on a compact set $K \subset \mathbb{R}^2$ is the unique probability measure μ_K concentrated on K for which

$$u_K(x) = \int_K g(x, y)\, \mu_K(dy) \tag{2.2}$$

is constant on the regular points of K. The function u_K is the equilibrium potential of K and its value $R(K)$ on the regular points of K is the Robin constant. The logarithmic capacity is defined by

$$\mathrm{Cap}(K) = e^{-R(K)}. \tag{2.3}$$

Then

$$\mathrm{Cap}(K) = \exp -\left\{ \inf_{\mu \in P(K)} \int_K \int_K g(x, y)\, \mu(dx)\, \mu(dy) \right\}, \tag{2.4}$$

where $P(K)$ is the set of all probability measures supported by K. Moreover

$$\mathrm{Cap}(B(x; r)) = r^{1/(2\pi)} \tag{2.5}$$

[**18**, Chapter 3, Proposition 4.11]. Define

$$B^{\circ}(x; r) = \{ y \in \mathbb{R}^m : |y-x| < r \}. \tag{2.6}$$

Let $a > 4$. Then

$$\begin{aligned} \mathbb{P}_x[T_D > t] &\leqslant \mathbb{P}_x[\tau_{(\partial D) \cap B(x; 2d(x))} > t] \\ &= \mathbb{P}_x[\tau_{(\partial D) \cap B(x; 2d(x))} > t, \tau_{(\partial D) \cap B(x; 2d(x))} > T_{B^{\circ}(x; ad(x))}] \\ &\quad + \mathbb{P}_x[\tau_{(\partial D) \cap B(x; 2d(x))} > t, \tau_{(\partial D) \cap B(x; 2d(x))} \leqslant T_{B^{\circ}(x; ad(x))}] \\ &\leqslant 1 - \mathbb{P}_x[\tau_{(\partial D) \cap B(x; 2d(x))} \leqslant T_{B^{\circ}(x; ad(x))}] + \mathbb{P}_x[T_{B^{\circ}(x; ad(x))} > t]. \end{aligned} \tag{2.7}$$

Let H be an open half space in $\mathbb{R}^2$ containing $B^{\circ}(x; ad(x))$ such that ∂H is tangent to $\partial B^{\circ}(x; ad(x))$. Then

$$\begin{aligned} \mathbb{P}_x[T_{B^{\circ}(x; ad(x))} > t] &\leqslant \mathbb{P}_x[T_H > t] \\ &= (\pi t)^{-1/2} \int_{[0, ad(x))} e^{-q^2/(4t)} dq \leqslant ad(x)\, (\pi t)^{-1/2}. \end{aligned} \tag{2.8}$$

For compact sets $K_1 \subseteq K_2$ we have by the variational formula (2.4)

$$\mathrm{Cap}(K_1) \leqslant \mathrm{Cap}(K_2). \tag{2.9}$$

Let $x_0 \in \partial D$ be such that $d(x) = |x - x_0|$. Then $B(x; 2d(x)) \supset B(x_0; d(x))$, and by (1.3), (2.5) and (2.9)

$$\begin{aligned} \mathrm{Cap}((\partial D) \cap B(x; 2d(x))) &\geqslant \mathrm{Cap}((\partial D) \cap B(x_0; d(x))) \\ &\geqslant \alpha\, \mathrm{Cap}(B(x_0; d(x))) = \alpha (d(x))^{1/(2\pi)}. \end{aligned} \tag{2.10}$$

By (2.3) and (2.10)

$$R((\partial D) \cap B(x; 2d(x))) \leqslant -(2\pi)^{-1} \log d(x) - \log \alpha. \tag{2.11}$$

By (2.3) and (2.9) we have for $K_1 \subseteq K_2$

$$R(K_1) \geqslant R(K_2). \tag{2.12}$$

Hence

$$R((\partial D) \cap B(x; 2d(x))) \geqslant R(B(x; 2d(x))) \geqslant -(2\pi)^{-1} \log(2d(x)). \tag{2.13}$$

Moreover, by (2.1) and (2.2)

$$\begin{aligned} &\sup\{u_{(\partial D) \cap B(x; 2d(x))}(y) : y \in \partial B(x; ad(x))\} \\ &\qquad \leqslant -(2\pi)^{-1} \log((a-2)\, d(x)) \int_{(\partial D) \cap B(x; 2d(x))} \mu_{(\partial D) \cap B(x; 2d(x))}(dy) \\ &\qquad = -(2\pi)^{-1} \log((a-2)\, d(x)). \end{aligned} \tag{2.14}$$

Following the proof of [**11**, Lemma 3.5] we define for $r > 0$

$$m(r) = -(2\pi)^{-1} \log((a-2)\, r), \tag{2.15}$$

and $h: \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$h(y) = (R((\partial D) \cap B(x; 2d(x))) - m(d(x)))^{-1} (u_{(\partial D) \cap B(x; 2d(x))}(y) - m(d(x))). \tag{2.16}$$

Now h is superharmonic, harmonic outside $(\partial D) \cap B(x; 2d(x))$, equal to one on $(\partial D) \cap B(x; 2d(x))$, and by (2.14) negative on $\partial B(x; ad(x))$. Hence

$$\mathbb{P}_x[\tau_{(\partial D) \cap B(x; 2d(x))} \leqslant T_{B^\circ(x; ad(x))}] \geqslant h(x) = \frac{u_{(\partial D) \cap B(x; 2d(x))}(x) - m(d(x))}{R((\partial D) \cap B(x; 2d(x))) - m(d(x))}. \tag{2.17}$$

But

$$u_{(\partial D) \cap B(x; 2d(x))}(x) \geqslant -\frac{1}{2\pi} \log(2d(x)), \tag{2.18}$$

so that by (2.11), (2.15), (2.17) and (2.18)

$$\mathbb{P}_x[\tau_{(\partial D) \cap B(x; 2d(x))} \leqslant T_{B^\circ(x; ad(x))}] \geqslant \frac{\log(a-2) - \log 2}{\log(a-2) - \log \alpha^{2\pi}}. \tag{2.19}$$

Hence by (2.7), (2.8) and (2.19)

$$\mathbb{P}_x[T_D > \lambda_j^{-1}] \leqslant \frac{\log(2/\alpha^{2\pi})}{\log((a-2)/\alpha^{2\pi})} + \pi^{-1/2} a d(x)\, \lambda_j^{1/2}. \tag{2.20}$$

We make the following choice for a:

$$a - 2 = \frac{2}{\lambda_j^{1/2} d(x)} \left(1 + \log\left(\frac{1}{\lambda_j^{1/2} d(x)}\right)\right)^{-1}. \tag{2.21}$$

Let $z = \lambda_j^{-1/2} d(x)^{-1}$. Then $d(x) \leqslant \lambda_j^{-1/2}$ implies $z \geqslant 1$, and $z \geqslant 1 + \log(z)$ implies $a \geqslant 4$. By (2.20) and (2.21)

$$\begin{aligned} \mathbb{P}_x[T_D > \lambda_j^{-1}] \leqslant &\left(\log \frac{2}{\alpha^{2\pi}}\right) \left(\log \frac{1}{\alpha^{2\pi}} + \log(2z) - \log(1 + \log z)\right)^{-1} \\ &+ 2\pi^{-1/2} z^{-1} + 2\pi^{-1/2} (1 + \log z)^{-1}. \end{aligned} \tag{2.22}$$

LEMMA 2.1. *For* $z \geqslant 1$

$$\log(2z) - \log(1 + \log z) \geqslant \tfrac{1}{2} \log z. \tag{2.23}$$

Proof. Inequality (2.23) is equivalent to

$$4z \geqslant (1+\log z)^2. \tag{2.24}$$

But (2.24) holds for $z = 1$. Moreover for $z \geqslant 1$

$$4 \geqslant 2(1+\log z)/z. \tag{2.25}$$

Integration of (2.25) over $[1, z]$ yields (2.24) and hence (2.23).

For $z \geqslant 1$, $1/z \leqslant (\log 1/z)^{-1}$. Hence by Lemma 2.1 and (2.22)

$$\begin{aligned} \mathbb{P}_x[T_D > \lambda_j^{-1}] &\leqslant \left(2\log\left(\frac{2}{\alpha^{2\pi}}\right)+\frac{4}{\pi^{1/2}}\right)(\log z)^{-1} \\ &\leqslant 6\left(\log\left(\frac{2}{\alpha^{2\pi}}\right)\right)(\log z)^{-1}. \end{aligned} \tag{2.26}$$

Theorem 1.2 follows from (1.26), (2.26) and by the definition of z. □

3. *Proof of Theorem* 1.3

We define for $m = 3, 4, \ldots$ the Green function on $\mathbb{R}^m$ by

$$g(x,y) = \frac{1}{c(m)}|x-y|^{2-m}, \tag{3.1}$$

where

$$c(m) = 4\pi^{m/2}(\Gamma((m-2)/2))^{-1}. \tag{3.2}$$

For a compact set $K \subset \mathbb{R}^m$ the equilibrium measure μ_K is the unique non-negative measure on K satisfying

$$\mathbb{P}_x[\tau_K < \infty] = \int g(x,y)\,\mu_K(dy). \tag{3.3}$$

The newtonian capacity of K is defined by

$$\mathrm{Cap}(K) = \mu_K(K). \tag{3.4}$$

The capacity of a ball is

$$\mathrm{Cap}(B(0;r)) = c(m)\,r^{m-2}. \tag{3.5}$$

Again, there is a variational description

$$\mathrm{Cap}(K) = \left\{\inf_{\mu\in P(K)} \int\int g(x,y)\,\mu(dx)\,\mu(dy)\right\}^{-1}, \tag{3.6}$$

where $P(K)$ is the set of all probability measures supported by K. If K_1, K_2 are compact sets with $K_1 \subseteq K_2$ then $\mathrm{Cap}(K_1) \leqslant \mathrm{Cap}(K_2)$. For these facts, see for example [**18**, Chapter 3].

To prove Theorem 1.3 we adapt [**10**, Lemma 3.5]. Let $b > a > 1$. Then by the strong Markov property

$$\begin{aligned} \mathbb{P}_x[T_D > t] &\leqslant 1-\mathbb{P}_x[\tau_{(\partial D)\cap B(x;ad(x))} \leqslant t] \\ &\leqslant 1-\mathbb{P}_x[\tau_{(\partial D)\cap B(x;ad(x))} \leqslant T_{B^\circ(x;bd(x))}]+\mathbb{P}_x[T_{B^\circ(x;bd(x))} > t] \\ &\leqslant 1-\mathbb{P}_x[\tau_{(\partial D)\cap B(x;ad(x))} < \infty] \\ &\quad +\mathbb{E}_x[\mathbb{P}_{B(T_{B^\circ(x;bd(x))})}[\tau_{(\partial D)\cap B(x;ad(x))} < \infty]]+\mathbb{P}_x[T_{B^\circ(x;bd(x))} > t]. \end{aligned} \tag{3.7}$$

To estimate the third term in the right-hand side of (3.7) we let y be such that $|y-x| = bd(x)$. Then

$$\mathbb{P}_y[\tau_{(\partial D)\cap B(x;ad(x))}] \leqslant \mathbb{P}_y[\tau_{B(x;ad(x))} < \infty] = \left(\frac{a}{b}\right)^{m-2}. \tag{3.8}$$

The fourth term in (3.7) is again estimated by (2.8). Hence

$$\mathbb{P}_x[T_D > t] \leqslant 1 - \mathbb{P}_x[\tau_{(\partial D)\cap B(x;ad(x))} < \infty] + \left(\frac{a}{b}\right)^{m-2} + bd(x)(\pi t)^{-1/2}. \tag{3.9}$$

Choose a and b as follows:

$$a = Ad(x)^{-\beta_1}, \tag{3.10}$$

$$b = B(d(x))^{\beta_2 - 1}, \tag{3.11}$$

where β_1, β_2, A and B are the solutions of

$$\beta_1 \gamma = (1-\beta_1-\beta_2)(m-2) = \beta_2, \tag{3.12}$$

$$A^{-\gamma} = \left(\frac{A}{B}\right)^{m-2} = B\lambda_j^{1/2}. \tag{3.13}$$

From (3.12) we obtain

$$\beta_2 = \beta_1\gamma = \left(\frac{1}{\gamma} + \frac{m-1}{m-2}\right)^{-1}, \tag{3.14}$$

and from (3.13) we obtain

$$B\lambda_j^{1/2} = A^{-\gamma} = \lambda_j^{(1/2)((1/\gamma)+(m-1)/(m-2))^{-1}}. \tag{3.15}$$

If we can show that (1.7) implies (1.27) and the requirement $b \geqslant a$, then by (3.9)–(3.11) and Theorem 1.7

$$\mathbb{P}_x[T_D > \lambda_j^{-1}] \leqslant 2A^{-\gamma} d(x)^{\beta_1\gamma} + \left(\frac{A}{B}\right)^{m-2} d(x)^{(1-\beta_1-\beta_2)(m-2)} + Bd(x)^{\beta_2}\lambda_j^{1/2}. \tag{3.16}$$

Substitution of the values of β_1, β_2, A and B respectively in (3.16) gives

$$\mathbb{P}_x[T_D > \lambda_j^{-1}] \leqslant 4A^{-\gamma} d(x)^{\beta_1\gamma} = 4(d(x)\,\lambda_j^{1/2})^{((1/\gamma)+(m-1)/(m-2))^{-1}}. \tag{3.17}$$

Estimate (1.8) follows from (1.26) and (3.17).

Note that $b \geqslant a$ is (by (3.10), (3.11)) equivalent to showing that

$$\frac{B}{A} \geqslant d(x)^{1-\beta_1-\beta_2}. \tag{3.18}$$

It follows from (3.14) that

$$0 < \beta_1 + \beta_2 = \frac{1+\gamma}{1+\gamma\dfrac{m-1}{m-2}} < 1. \tag{3.19}$$

Since (1.7) implies $d(x) \leqslant \lambda_j^{-1/2}$ it is (by (3.19)) sufficient to check that (3.18) holds for $d(x) = \lambda_j^{-1/2}$, that is,

$$\frac{B}{A} \geqslant \lambda_j^{-(1/2)(1-\beta_1-\beta_2)}. \tag{3.20}$$

We see by (3.15) and (3.19) that (3.20) holds with the equality sign.

It remains to check the validity of (1.27). Since D is bounded, D is contained in a hypercube of sidelength diam(D). By monotonicity of the Dirichlet eigenvalues [**19**, Chapter XIII.15, Proposition 4(a)]

$$\lambda_j \geqslant \lambda_1 = \frac{m\pi^2}{(\operatorname{diam}(D))^2} > \frac{1}{(\operatorname{diam}(D))^2}. \tag{3.21}$$

Hence the first inequality in (1.27) is satisfied if

$$\frac{1}{\lambda_j^{1/2} d(x)} \geqslant a. \tag{3.22}$$

By (3.10), (3.14) and (3.15)

$$a = (\lambda_j^{1/2} d(x))^{-(1/\gamma)((1/\gamma)+(m-1)/(m-2))^{-1}}. \tag{3.23}$$

Hence (3.22) is satisfied if

$$(d(x)\,\lambda_j^{1/2})^{1-(1/\gamma)((1/\gamma)+(m-1)/(m-2))^{-1}} \leqslant 1. \tag{3.24}$$

This is indeed the case because (1.7) implies $d(x)\,\lambda_j^{1/2} \leqslant 1$ and

$$1 - \frac{1}{\gamma}\left(\frac{1}{\gamma} + \frac{m-1}{m-2}\right)^{-1} > 0. \tag{3.25}$$

The second inequality in (1.27) follows directly from (3.23) and (1.7). □

4. *Proof of Theorem* 1.7

For $s > r > 0$ we define the annulus

$$B(x; r, s) = B(x; s) \backslash B^\circ(x; r), \tag{4.1}$$

and the set

$$\partial D_i(x) = (\partial D) \cap B(x; b^i, b^{i+1}), \tag{4.2}$$

where $b > 1$ will be specified later. Let $A_i(x)$ be the event

$$A_i(x) = \{\tau_{\partial D_i(x)} < \infty\}. \tag{4.3}$$

Define

$$N = \max\{k \in \mathbb{Z} : b^{k+1} \leqslant a(d(x))\, d(x)\}. \tag{4.4}$$

Then for any $n \leqslant N$

$$\{\tau_{(\partial D)\cap B(x; a(d(x)) d(x))} < \infty\} \supset \bigcup_{i=n}^{N} A_i(x). \tag{4.5}$$

If $b^{i+1} < d(x)$ then $A_i(x) = \varnothing$. We will choose

$$n = \min\{k \in \mathbb{Z} : b^k \geqslant 2d(x)\}. \tag{4.6}$$

We choose a 'spacing' $s \in \mathbb{Z}^+$, $s \geqslant 2$ (to be specified later) and use

$$\bigcup_{i=n}^{N} A_i(x) \supset \bigcup_{j=0}^{[(N-n)/s]} A_{n+js}(x), \tag{4.7}$$

to obtain

$$\mathbb{P}_x[\tau_{(\partial D)\cap B(x; a(d(x)) d(x))} < \infty] \geqslant 1 - \mathbb{P}_x\left[\bigcap_{j=0}^{[(N-n)/s]} A^c_{n+js}(x)\right]. \tag{4.8}$$

For technical reasons we replace $A_j(x)$ by $\bar{A}_j(x)$ which are defined by

$$\bar{A}_j(x) = \{\tau_{\partial D_j(x)} \leqslant \tau_{B(x;b^{j+s},\infty)}\}. \tag{4.9}$$

Note that

$$\mathbb{P}_x[\bar{A}_j(x) \backslash A_j(x)] = 0, \tag{4.10}$$

and therefore

$$\mathbb{P}_x[\tau_{(\partial D)\cap B(x;a(d(x))d(x))} < \infty] \geqslant 1 - \mathbb{P}_x\left[\bigcap_{j=0}^{[(N-n)/s]} \bar{A}^c_{n+js}(x)\right]. \tag{4.11}$$

Next we derive a lower bound for $\mathbb{P}_y(\bar{A}_j(x))$ for $|x-y| \leqslant b^j$.

LEMMA 4.1. *Let*

$$b = 2\left(\frac{2}{\alpha}\right)^{1/(m-2)}. \tag{4.12}$$

Then for $j \geqslant n$ *satisfying* $b(b^j + d(x)) \leqslant 2\,\mathrm{diam}(D)$ *and any* y *satisfying* $|y-x| \leqslant b^j$

$$\mathbb{P}_y[\bar{A}_j(x)] \geqslant 2 \cdot 3^{-m}\alpha. \tag{4.13}$$

Proof. Let $x_0 \in \partial D$ be such that $d(x) = |x - x_0|$. One easily checks that if $b^j \geqslant 2d(x)$

$$\partial D_j(x) \supset (\partial D) \cap B(x_0; r, br/2), \tag{4.14}$$

where

$$r = b^j + d(x). \tag{4.15}$$

From this we obtain, by the monotonicity and subadditivity of the newtonian capacity,

$$\begin{aligned} \mathrm{Cap}(\partial D_j(x)) &\geqslant \mathrm{Cap}((\partial D) \cap B(x_0; r, br/2)) \\ &\geqslant \mathrm{Cap}((\partial D) \cap B(x_0; br/2)) - \mathrm{Cap}(B(x_0; r)) \\ &\geqslant \alpha\,\mathrm{Cap}(B(x_0; br/2)) - \mathrm{Cap}(B(x_0; r)), \end{aligned} \tag{4.16}$$

since $br/2 \leqslant \mathrm{diam}(D)$ by assumption. By the choice of b and by (3.5)

$$\mathrm{Cap}(\partial D_j(x)) \geqslant \alpha c(m)\,(br/2)^{m-2} - c(m)\,r^{m-2} = c(m)\,r^{m-2} \geqslant c(m)\,b^{j(m-2)}. \tag{4.17}$$

For $z \in \partial D_j(x)$ and $|y-x| \leqslant b^j$ we have since $b \geqslant 2$

$$|y-z| \leqslant |z-x| + |x-y| \leqslant b^{j+1} + b^j \leqslant 3b^{j+1}/2. \tag{4.18}$$

Hence by (4.17) and (4.18)

$$\begin{aligned} \mathbb{P}_y[A_j(x)] &= \int_{\partial D_j(x)} \frac{c(m)^{-1}}{|y-z|^{m-2}} \mu_{\partial D_j(x)}(dz) \\ &\geqslant c(m)^{-1}(3/2)^{2-m} b^{(2-m)(j+1)}\,\mathrm{Cap}(\partial D_j(x)) \\ &\geqslant c(m)^{-1}(3/2)^{2-m} b^{(2-m)(j+1)} c(m)\,b^{j(m-2)} \\ &= (3b/2)^{2-m}. \end{aligned} \tag{4.19}$$

Furthermore

$$\mathbb{P}_y[\bar{A}_j(x)] \geqslant \mathbb{P}_y[A_j(x)] - \mathbb{P}_w[\tau_{B(x;b^{j+1})} < \infty], \tag{4.20}$$

where $|w-x| = b^{j+s}$. Hence

$$\mathbb{P}_y[\bar{A}_j(x)] \geqslant (3b/2)^{2-m} - b^{(m-2)(1-s)}. \tag{4.21}$$

From now on we choose $s = 3$. Then by (4.21) and (4.12)

$$\mathbb{P}_y[\bar{A}_j(x)] \geqslant 3^{2-m}\frac{\alpha}{2} - \alpha^2 2^{2(1-m)} \geqslant 2\cdot 3^{-m}\alpha. \tag{4.22}$$

□

Let

$$\mathscr{F}_j \equiv \sigma(B(t): t \leqslant \tau_{B(x;b^j,\infty)}). \tag{4.23}$$

By definition of $\bar{A}_j(x)$, we have that $\bar{A}_j(x)$ is $\mathscr{F}_{j+3}$-measurable. Let x be such that

$$N - n \geqslant 3. \tag{4.24}$$

Then for any $k \in \{1, 2, \ldots, [(N-n)/3]\}$ we have

$$\begin{aligned}\mathbb{P}_x\left[\bigcap_{j=0}^{k} \bar{A}^c_{n+3j}(x)\right] &= \mathbb{E}_x\left[\mathbb{P}_x[\bar{A}^c_{n+3k}(x) \mid \mathscr{F}_{n+3k}]; \bigcap_{j=0}^{k-1} \bar{A}^c_{n+3j}(x)\right] \\ &= \mathbb{E}_x\left[\mathbb{P}_{B(\tau_{B(x;b^{n+3k},\infty)})}(\bar{A}^c_{n+3k}); \bigcap_{j=0}^{k-1} \bar{A}^c_{n+3j}(x)\right] \\ &\leqslant (1 - 2\cdot 3^{-m}\alpha)\,\mathbb{P}_x\left[\bigcap_{j=0}^{k-1} \bar{A}^c_{n+3j}(x)\right].\end{aligned} \tag{4.25}$$

From this we finally obtain

$$\begin{aligned}\mathbb{P}_x\left[\bigcap_{j=0}^{[(N-n)/3]} \bar{A}^c_{n+3j}(x)\right] &\leqslant (1 - 2\cdot 3^{-m}\alpha)^{[(N-n)/3]} \\ &\leqslant \exp -\{[(N-n)/3]\, 2\cdot 3^{-m}\alpha\}.\end{aligned} \tag{4.26}$$

Since $N - n \geqslant 3$,

$$[(N-n)/3] \geqslant (N-n)/6, \tag{4.27}$$

and

$$\mathbb{P}_x\left[\bigcap_{j=0}^{[(N-n)/3]} \bar{A}^c_{n+3j}(x)\right] \leqslant \exp -\{(N-n)\, 3^{-m-1}\alpha\}. \tag{4.28}$$

By the choice of N and n we have

$$b^{N+2} > a(d(x))\, d(x) \geqslant b^{N+1}, \tag{4.29}$$

$$b^{n-1} < 2d(x) \leqslant b^n, \tag{4.30}$$

and hence

$$b^{N-n} \geqslant \frac{a(d(x))}{2b^3}. \tag{4.31}$$

By (4.28) and (4.31) we obtain

$$\begin{aligned}\mathbb{P}_x\left[\bigcap_{j=0}^{[(N-n)/3]} \bar{A}^c_{n+3j}(x)\right] &\leqslant \left(\frac{a(d(x))}{2b^3}\right)^{-\alpha 3^{-m-1}/\log b} \\ &= a(d(x))^{-\gamma} e^{\alpha 3^{-m} + \alpha 3^{-m-1}(\log 2)/\log b} \\ &\leqslant a(d(x))^{-\gamma} e^{4\alpha 3^{-m-1}} \leqslant 2a(d(x))^{-\gamma},\end{aligned} \tag{4.32}$$

by definition of γ and the fact that $b \geqslant 2$, $m \geqslant 3$ and $\alpha < 1$. Estimate (1.28) follows from (4.11) and (4.32) provided $x \in D$ is such that (i) $b(b^j + d(x)) \leqslant 2\,\mathrm{diam}(D)$ for $j = n, \ldots, N$, (ii) (4.24) holds. But (i) is satisfied if

$$b^{N+1} + 2b^{N-n+1} d(x) \leqslant 2\,\mathrm{diam}(D), \tag{4.33}$$

since $b \geqslant 2$. But $b^{N+1} \leqslant a(d(x))\, d(x)$ and $2b^{N-n+1} \leqslant a(d(x))$. So (4.33) and hence (i) are clearly satisfied if

$$a(d(x))\, d(x) \leqslant \mathrm{diam}(D). \tag{4.34}$$

For (4.24) to hold we have to have $b^{N+2-(n-1)} \geqslant b^6$. This is the case by (4.29) and (4.30) if

$$a(d(x)) \geqslant 2b^6. \tag{4.35}$$

But $b \leqslant 4/\alpha$ since $0 < \alpha < 1$ and $m = 3, 4, \ldots$. Hence (4.35) and (4.24) hold if

$$a(d(x)) \geqslant \frac{2^{13}}{\alpha^6}. \tag{4.36}$$

This completes the proof of Theorem 1.7. □

5. *Proof of Theorem* 1.5

Let $G_D(\cdot, \cdot)$ be the Green function for $-\Delta_D$. Then

$$G_D(x, y) = \int_0^\infty p_D(x, y; t)\, dt, \tag{5.1}$$

and any Dirichlet eigenfunction of $-\Delta_D$ satisfies

$$\varphi_j(x) = \lambda_j \int_D G_D(x, y)\, \varphi_j(y)\, dy. \tag{5.2}$$

By the Cauchy–Schwarz inequality

$$|\varphi_j(x)| \leqslant \lambda_j \left\{ \int_D G_D^2(x, y)\, dy \right\}^{1/2} \tag{5.3}$$

since $\|\varphi_j\|_2 = 1$.

Lemma 5.1. *For* $j = 1, 2, \ldots$

$$\lambda_j \leqslant 8\pi j R^{-2}. \tag{5.4}$$

Proof. By definition of R, D contains an open ball with radius R. Hence D contains an open square with sidelength $R\sqrt{2}$. Since the Dirichlet eigenvalues are monotone in D, λ_j is bounded from above by the jth eigenvalue of this square. The eigenvalues for this square are given by

$$\lambda_{k,l} = \pi^2(k^2 + l^2)/(2R^2), \quad k \in \mathbb{Z}^+, l \in \mathbb{Z}^+. \tag{5.5}$$

By definition

$$j = \#\{(k, l) : k^2 + l^2 \leqslant 2\lambda_j R^2/\pi^2\}. \tag{5.6}$$

Suppose $j \geqslant 4$. Then $k^2 + l^2 \geqslant 8$ since $j = 1$ corresponds to $(k, l) = (1, 1)$ and $j = 2, 3$ corresponds to $(k, l) = (2, 1)$ and $(k, l) = (1, 2)$. Hence

$$\lambda_j \geqslant \frac{4\pi^2}{R^2}, \quad j \geqslant 4. \tag{5.7}$$

But the right-hand side of (5.6) is equal to the number of lattice points in the first quadrant of the disc with radius $R(2\lambda_j/\pi^2)^{1/2}$. Hence by (5.6) and (5.7) we have for $j \geqslant 4$

$$j \geqslant \frac{\pi}{4}((2\lambda_j R^2/\pi^2)^{1/2} - 2^{1/2})^2 \geqslant \frac{\lambda_j R^2}{8\pi}. \tag{5.8}$$

This proves the lemma for $j \geqslant 4$. The case $j = 1, 2, 3$ is easily verified. □

Let F be the conformal map from the unit disc onto D with $F(0) = x$. Then by the results of [**3**, §1]

$$G_D(x,y) = \frac{1}{2\pi}\log\coth(\wp_D(x,y)), \tag{5.9}$$

where

$$\wp_D(x,y) = \inf_{\gamma}\int_0^1 \frac{|\gamma'(t)|}{|F'(0)|}\,dt, \tag{5.10}$$

and where the infimum is taken over all rectifiable curves γ in D with $\gamma(0) = x$, $\gamma(1) = y$, and where $F'(0)$ is evaluated at $\gamma(t)$. By Koebe's 1/4 theorem

$$d(\gamma(t)) \leqslant |F'(0)| \leqslant 4d(\gamma(t)). \tag{5.11}$$

Without loss of generality we may assume that γ has a parametrisation with constant speed c. Then for any such γ we have

$$d(\gamma(t)) \leqslant d(x) + tc. \tag{5.12}$$

By (5.10)–(5.12)

$$\wp_D(x,y) \geqslant \frac{1}{4}\int_0^1 \frac{c}{d(x)+tc}\,dt = \frac{1}{4}\log\left(1+\frac{c}{d(x)}\right). \tag{5.13}$$

Since $c \geqslant |x-y|$ we have by (5.9) and (5.13)

$$G_D(x,y) \leqslant \frac{1}{2\pi}\log\frac{(d(x)+|x-y|)^{1/2}+d(x)^{1/2}}{(d(x)+|x-y|)^{1/2}-d(x)^{1/2}}. \tag{5.14}$$

We note that the right-hand side of (5.14) is positive and strictly decreasing in $|x-y|$ for x fixed. Hence the square of the right-hand side of (5.14) is strictly decreasing in $|x-y|$ for x fixed. Let R_0 be defined by

$$\pi R_0^2 = |D|. \tag{5.15}$$

By spherical-symmetric rearrangement

$$\begin{aligned}\int_D G_D^2(x,y)\,dy &\leqslant \frac{1}{2\pi}\int_0^{R_0} r\,dr\left(\log\frac{(d(x)+r)^{1/2}+d(x)^{1/2}}{(d(x)+r)^{1/2}-d(x)^{1/2}}\right)^2 \\ &= \frac{2d(x)^2}{\pi}\int_{d(x)/R_0}^{\infty}\frac{dr}{r^3}(\log((1+r)^{1/2}+r^{1/2}))^2 \leqslant \frac{8d(x)\,R_0}{\pi},\end{aligned} \tag{5.16}$$

since $\log((1+r)^{1/2}+r^{1/2}) \leqslant 2r^{1/2}$. The theorem follows from (5.3), (5.4) and (5.16). □

COROLLARY 5.2. *Let D be open, simply connected in $\mathbb{R}^2$ with finite volume $|D|$. Then*

$$\mathbb{E}_x[T_D] \leqslant 2^{3/2}\pi^{-3/4}|D|^{3/4}d(x)^{1/2}. \tag{5.17}$$

Proof. By the Cauchy–Schwarz inequality

$$\mathbb{E}_x[T_D] = \int_D G_D(x,y)\,dy \leqslant |D|^{1/2}\left\{\int_D G_D^2(x,y)\,dy\right\}^{1/2} \tag{5.18}$$

and (5.17) follows from (5.18), (5.16) and (5.15). □

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