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Scattering theory for Schrödinger operators with Bessel-type potentials

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Abstract. We show that for the Schrödinger operators on the semi-axis with Bessel-type potentials $\kappa(\kappa+1)/x^2$, $\kappa \in \left[-\frac{1}{2}, \frac{1}{2}\right)$, there exists a meaningful direct and inverse scattering theory. Several new phenomena not observed in the "classical case" of Faddeev–Marchenko potentials arise here; in particular, for $\kappa \neq 0$ the scattering function S takes two different values on the positive and negative semi-axes and is thus discontinuous both at the origin and at infinity.

1. Introduction

The main goal of this paper is to show that there exists a meaningful direct and inverse scattering theory for the Schrödinger operators H_κ generated by the differential expressions

$$\ell_\kappa(y) := -y'' + \frac{\kappa(\kappa+1)}{x^2}y$$

with Bessel-type potentials $\kappa(\kappa+1)/x^2$, where $\kappa \in \left[-\frac{1}{2}, \frac{1}{2}\right)$. For non-negative integer values of κ such operators arise in the decomposition in spherical harmonics of the three-dimensional Laplacian $-\Delta$, and then κ is the *angular momentum*, or *partial wave*. Operators of the form H_κ with non-integer values of κ arise in the study of scattering of waves and particles in conical domains (see, e.g., [8]), as well as in the study of the Aharonov–Bohm effect [2].

The scattering theory for the one-dimensional Schrödinger operators

$$H = -\frac{d^2}{dx^2} + q(x)$$

on the semi-axis relates the asymptotic behaviour of solutions $e^{-itH}\psi(0)$ of the corresponding Schrödinger equation $i\psi'(t) = H\psi(t)$ and the free evolution $e^{-itH_0}\phi(0)$ via the *scattering* *operator* S (also called the *scattering matrix* or the *scattering function* in our context). Some partial results in the inverse scattering problem of reconstruction of the potential q from the scattering function S appeared already in the late 1940-ies, but a systematic and successful theory was only developed in the works of Gelfand and Levitan [18], Krein [25], [26], and Marchenko [29], [30], see also the reviews [13], [16] and the books [7], [28], [31], [33]. This "classical theory" works for the set of real-valued potentials q in the space $L_1^1(\mathbb{R}_+)$, i.e., for potentials satisfying the condition

$$\int_0^\infty x|q(x)|\,dx < \infty \tag{1.1}$$

and often called the Faddeev–Marchenko or Bargman–Jost–Kohn potentials [34], Chapter 2.2.1. The Bessel potential $\kappa(\kappa+1)/x^2$ considered here does not belong to this class as the integral (1.1) diverges both at the origin and at infinity.

The direct and inverse scattering theory on the line has also successfully been developed for potentials in $L_1^1(\mathbb{R})$ [13], [16], [28], [31]. Recently it has been extended to a larger class of Schrödinger operators with Miura distributional potentials in H_{loc}^{-1} [17], [23], [24]. The Miura potentials that were considered in these works are of the form $q = u' + u^2$, with $u \in L_{2,\text{loc}}(\mathbb{R}) \cap L_1(\mathbb{R})$. We notice that the function u is related to the modified Korteweg–de Vries (mKdV) equation in the same manner as q is related to the Korteweg–de Vries (KdV) equation, see [32]. The corresponding differential expression giving the Hamiltonian can then be factorized as

$$-\frac{d^2}{dx^2} + q = -\left(\frac{d}{dx} + u\right)\left(\frac{d}{dx} - u\right),$$

and the class of Miura potentials treated in these works include the Faddeev–Marchenko class and allow potentials with, e.g., local singularities of Coulomb $1/x$-type or Dirac delta-functions. The formal identity

$$\ell_\kappa = -\left(\frac{d}{dx} - \frac{\kappa}{x}\right)\left(\frac{d}{dx} + \frac{\kappa}{x}\right)$$

might suggest that the Bessel potential could be viewed as a Miura potential; however, since the function κ/x is neither integrable at infinity nor at the origin, the approach based on Miura potentials is not applicable.

We observe that the inverse scattering problem for Schrödinger operators $H_\kappa + q$ with $\kappa \in \mathbb{N}$ and q belonging to the Faddeev–Marchenko class was also considered in the context of the corresponding three-dimensional problem for the operator $-\Delta + Q$ with spherically-symmetric potential $Q(\boldsymbol{x}) := q(|\boldsymbol{x}|)$, cf. [7], [33]. The essential difference is, however, that the unperturbed (or reference) Hamiltonian is then H_κ and not H_0. Moreover, in this problem there exists an efficient "double commutation" (or multiple Darboux) procedure that reduces the inverse scattering problem to the case $\kappa = 0$, albeit with some modified potential q_κ that can explicitly be calculated from q and κ, see [16]. In fact, the same double commutation can be applied to $H_\kappa + q$ for the general case $\kappa \in \mathbb{R}$, reducing it to the basic case $\kappa \in \left[-\frac{1}{2}, \frac{1}{2}\right)$, which thus explains the importance of studying operators of the form H_κ with $\kappa \in \left[-\frac{1}{2}, \frac{1}{2}\right)$.

Scattering with some other singular reference potentials has also been discussed in the literature. For instance, scattering on Coulombic potentials was treated in e.g. [9], [10], and singular potentials that describe "point" interactions were thoroughly investigated in the books [4], [5], where additional references might be found. Inverse scattering for long-range oscillating potentials leading to scattering functions with finite phase shifts was considered in [27]. The methods developed for such potentials do not apply, however, to the case of Bessel potentials.

In this paper, we show that despite the fact that the Bessel potential $\kappa(\kappa+1)/x^2$ is too singular for applying the methods of the classical scattering theory, a meaningful stationary scattering theory between H_0 and H_κ exists when $\kappa \in \left[-\frac{1}{2}, \frac{1}{2}\right)$. We remark that ℓ_κ is invariant under the change $\kappa \mapsto -1-\kappa$, so that only $\kappa \geqq -1/2$ need to be considered. On the other hand, the minimal operator generated by ℓ_κ is essentially self-adjoint on $C_0^\infty(\mathbb{R}_+)$ when $\kappa \geqq 1/2$, so that no scattering is possible between H_0 and H_κ in that case. Thus $\kappa \in \left[-\frac{1}{2}, \frac{1}{2}\right)$ is a natural limitation for a scattering theory between H_0 and H_κ to exist at all. And indeed the effect of "scattering ambiguities", where different potentials generate the same scattering data, has been observed for potentials exhibiting a c/x^2-type behaviour at infinity with $c \geqq 3/4$, see [3], [12].

Let us remark that a non-stationary scattering theory between H_0 and H_κ can also be developed, and the scattering operator is the operator of multiplication by the scattering function S constructed in the present paper, just as it is the case in the above-mentioned "classical scattering theory". This will be discussed elsewhere.

We show that all the classical objects of the potential scattering theory have their counterparts in our setting, albeit with a special interpretation. For instance, the scattering function S turns out to take the values $e^{-\pi i\kappa}$ and $e^{\pi i\kappa}$ on the positive and negative semi-axes respectively. Thus S is discontinuous at the origin and at infinity, and the function $1-S$ does not vanish at infinity, in contrast to all situations treated so far (see [15] and [31], Chapter 3.3, for the classical setting and [3], [12], [27] for some cases of singular potentials). We then derive the Marchenko equation and show that the kernel f of the corresponding integral operator F is the Fourier transform of S taken in the sense of distributions. The operator F is not compact but rather a multiple of the classical Carleman (also called Stieltjes) operator [6]. Thus one cannot follow the standard arguments in solving the Marchenko equation for the kernel k of the transformation operators. We show, however, that the Marchenko equation when interpreted as a relation between operators in some operator algebra is indeed soluble and the solution gives the transformation operator sending $e^{i\omega x}$ into the special solutions of the equation $\ell_\kappa y = \omega^2 y$. Finally, the kernel of the transformation operator reconstructs the potential we have started with via the same formula as in the case of regular potentials.

Although we consider here a concrete operator problem which allows for explicit calculation of all quantities of interest, our treatment is not confined to this special case. In fact, it can also be extended to Schrödinger operators

$$H_\kappa(v) = -\left(\frac{d}{dx} - \frac{\kappa}{x} + v\right)\left(\frac{d}{dx} + \frac{\kappa}{x} - v\right) \tag{1.2}$$

with $v \in L_1(\mathbb{R}_+)$. For smooth enough v this operator might be written as $H_\kappa + q$ with $q = v' + v^2 - 2\kappa v/x$. The objects we constructed here give the first approximation, or the "leading singularity", of their counterparts for operators (1.2); thus the precise knowledge of these objects is important for a subsequent analysis of operators of the form (1.2).

The paper is organized as follows. In the next section we define rigorously the operator H_κ as the Friedrichs extension of the minimal operator generated by ℓ_κ. The transformation operator $I + K$ and its inverse $I + L$ are constructed in Section 3. The Jost solutions and the scattering function are constructed in Section 4 first using the explicit formulae and then by means of the transformation operators. The Marchenko equation relating the scattering function S and the transformation operator $I + K$ is derived in Section 5 and its solution in the special operator algebra and the reconstruction of the potential are discussed in Section 6. The final Section 7 discusses two examples demonstrating that discontinuity of S at the origin is caused by the singularity of the potential at infinity and, conversely, that the behaviour of S at infinity is determined by the singularity of the potential at the origin. Finally, in two appendices we collect some information about Bessel special functions and the Hankel and Mellin transforms which we extensively use in the present work.

Notation. Throughout the paper, we shall write $\mathbb{C}^+$ for the open complex upper-half plane, $\mathscr{B}$ for the algebra of all bounded linear operators acting in the Hilbert space $L_2(\mathbb{R}_+)$, and L.i.m. for the limit in the topology of the space $L_2(\mathbb{R}_+)$. As usual, $\Gamma(\cdot)$ stands for the Euler Gamma function.

2. Differential operators

2.1. Minimal and maximal operators. For $\kappa \in \left[-\frac{1}{2}, \frac{1}{2}\right)$, we consider the differential expression ℓ_κ on its natural domain

$$\operatorname{dom} \ell_\kappa = \{y, y' \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+)\}$$

and denote by T_κ the symmetric operator in $L_2(\mathbb{R}_+)$ acting on the set $C_0^\infty(\mathbb{R}_+)$ of test functions on $\mathbb{R}_+$ by $T_\kappa y := \ell_\kappa y$. By definition, the minimal operator $T_{\kappa,\min}$ is the closure of T_κ and the maximal operator $T_{\kappa,\max}$ is the adjoint of the latter, i.e., $T_{\kappa,\max} = (T_\kappa)^* = (T_{\kappa,\min})^*$.

Lemma 2.1. *The maximal operator $T_{\kappa,\max}$ is given by $T_{\kappa,\max} f = \ell_\kappa f$ on the set of functions*

$$\operatorname{dom} T_{\kappa,\max} = \{y \in L_2(\mathbb{R}_+) \cap \operatorname{dom} \ell_\kappa \mid \ell_\kappa y \in L_2(\mathbb{R}_+)\}.$$

Proof. In order that $g \in L_2(\mathbb{R}_+)$ belongs to the domain of the maximal operator, it is necessary and sufficient that the functional

$$G(\phi) := \int_0^\infty (\ell_\kappa \phi)(x)\overline{g(x)}\,dx$$

defined on $C_0^\infty(\mathbb{R}_+)$ should be continuous in $L_2(\mathbb{R}_+)$.

Assume that $g \in \operatorname{dom} T_{\kappa,\max}$ and fix an arbitrary $\varepsilon > 0$. Then the functional

$$\phi \mapsto -\int_\varepsilon^\infty \phi''(x)\overline{g(x)}\,dx, \quad \phi \in C_0^\infty(\varepsilon, \infty),$$

is continuous in $L_2(\varepsilon, \infty)$. It follows that the distribution g'' is in $L_2(\varepsilon, \infty)$ and thus g belongs to $W_2^2(\varepsilon, \infty)$. Since ε was arbitrary, we see that $g \in \operatorname{dom} \ell_\kappa$ and that for every $\phi \in C_0^\infty(\mathbb{R}_+)$ we can integrate by parts in the expression for $G(\phi)$ to get

$$G(\phi) = \int_0^\infty \phi(x)\overline{(\ell_\kappa g)(x)}\,dx. \tag{2.1}$$

Since G is continuous in $L_2(\mathbb{R}_+)$, it follows that $\ell_\kappa g \in L_2(\mathbb{R}_+)$.

Conversely, if $g \in L_2(\mathbb{R}_+) \cap \operatorname{dom} \ell_\kappa$ is such that the function $\ell_\kappa g$ belongs to $L_2(\mathbb{R}_+)$, then, for every $\varepsilon > 0$, the function g'' is in $L_2(\varepsilon, \infty)$ and thus $g \in W_2^2(\varepsilon, \infty)$. It follows that equality (2.1) holds for every $\phi \in C_0^\infty(\mathbb{R}_+)$ and thus the functional G is continuous in $L_2(\mathbb{R}_+)$. This shows that g belongs to the domain of the maximal operator $T_{\kappa,\max}$. □

Remark 2.2. It follows from the above proof that every function $y \in \operatorname{dom} T_{\kappa,\max}$ belongs to $W_2^2(\varepsilon, \infty)$, for every $\varepsilon > 0$.

Remark 2.3. In the paper [14] the operator $T_{\kappa,\max}$ was *defined* as acting by $T_{\kappa,\max} y = \ell_\kappa y$ on the set $\operatorname{dom} T_{\kappa,\max}$ of the above lemma.

2.2. The operator H_κ. The differential expression ℓ_κ for the κ considered is in the limit circle case at the origin and in the limit point case at infinity in the Weyl classification. Indeed, two linearly independent solutions of the equation

$$-y'' + \frac{\kappa(\kappa+1)}{x^2}y = \omega^2 y \tag{2.2}$$

are, e.g., $\phi_\kappa(x,\omega) := \sqrt{\omega x}J_{\kappa+1/2}(\omega x)$ and $\psi_\kappa(x,\omega) := \sqrt{\omega x}J_{-\kappa-1/2}(\omega x)$ with J_ν being the Bessel function of first kind and order ν (for $\kappa = -1/2$, we take $\psi_{-1/2}(x,\omega) := \sqrt{\omega x}\,Y_0(\omega x)$, with Y_0 being the Bessel function of second kind and order 0). Thus the Weyl limit circle/limit point classification of ℓ_κ follows from the asymptotic behaviour of the Bessel functions at the origin and at infinity, see Appendix A.

Therefore the minimal operator $T_{\kappa,\min}$ is symmetric but not self-adjoint. Since

$$\int_0^\infty (\ell_\kappa f)(x)\overline{f(x)}\,dx = \int_0^\infty \left|f'(x) + \frac{\kappa}{x} f(x)\right|^2 dx \geqq 0$$

for all $f \in C_0^\infty(\mathbb{R}_+)$, the operator $T_{\kappa,\min}$ is nonnegative. It follows from the results of [14] that the Friedrichs extension H_κ of $T_{\kappa,\min}$ is the restriction of $T_{\kappa,\max}$ by the boundary condition at the origin

$$\lim_{x\to+0} x^\kappa y(x) = 0 \tag{2.3}$$

for $\kappa \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and by the boundary condition

$$y(x) = O(\sqrt{x}), \quad x \to +0,$$

for $\kappa = -1/2$. Clearly, the operator H_κ is nonnegative; moreover, it has no eigenvalues and its continuous spectrum coincides with the positive half-line $\mathbb{R}_+$ and is absolutely continuous there (see [14]). Some other spectral properties of operators of the form H_κ (in particular, definition and properties of the related m-function) were investigated in [19].

3. Transformation operators

Both direct and inverse scattering theories for Schrödinger operators heavily rely on the existence of the Jost solutions $e(\cdot,\omega)$. These are solutions of the equations $\ell_\kappa y = \omega^2 y$ of the form $e^{i\omega x}(1 + o(1))$ as $x \to \infty$. For our model case, the Jost solutions can explicitly be constructed as linear combinations of the special solutions $\phi_\kappa(\cdot,\omega)$ and $\psi_\kappa(\cdot,\omega)$ (see Section 4); the latter, in turn, are expressed via the Bessel functions $J_{\kappa+1/2}$ and $J_{-\kappa-1/2}$ (or Y_0 for $\kappa = -1/2$, see Subsection A.2).

However, if one adds to H_κ a nontrivial potential q belonging to the Faddeev–Marchenko class, then no explicit formulae for solutions are available and one could try to follow the classical approach via the transformation operators. In this section we show that in the unperturbed case $q = 0$ the transformation operators indeed exist and study some of their properties. In Sections 5 and 6 below, these transformation operators will be related to the scattering data via the Marchenko equation and will be used to reconstruct the potential of H_κ.

3.1. Direct construction of the transformation operators. We look for the transformation operator $I + K$ with K an integral operator of the form

$$(Ky)(x) = \int_x^\infty k(x,t)y(t)\,dt$$

that satisfies the relation

$$T_{\kappa,\max}(I + K) = (I + K)T_{0,\max}.$$

Assume that there exists such a K with kernel k that is bounded in the domain $c \leqq x < t < \infty$ for every $c > 0$. Then for every ω in the open upper half-plane $\mathbb{C}^+$ the function $y(\cdot,\omega) := (I+K)e^{i\omega x}$ solves the equation $\ell_\kappa y = \omega^2 y$ and is of the form $e^{i\omega x}\big(1+o(1)\big)$ as $x \to +\infty$. Therefore $y(\cdot,\omega)$ gives then the Jost solution $e(\cdot,\omega)$, i.e., the following integral representation holds:

$$e(x,\omega) = (I+K)e^{i\omega x} = e^{i\omega x} + \int_x^\infty k(x,t)e^{i\omega t}\,dt.$$

Along with $I+K$ we consider its (formally) inverse operator $I+L$ satisfying

$$(I+L)T_{\kappa,\max} = T_{0,\max}(I+L). \tag{3.1}$$

As in the classical situation of Schrödinger operators with potentials belonging to the Faddeev–Marchenko class, we expect that L is also an integral operator with an upper-triangular kernel, i.e., that

$$(Ly)(x) = \int_x^\infty l(x,t)y(t)\,dt. \tag{3.2}$$

It turns out that transformation operators $I+K$ and $I+L$ of the above form indeed exist, are bounded and boundedly invertible, and $(I+K)^{-1} = I+L$. Both operators can be constructed explicitly; we start with $I+L$ since its kernel has a simpler form. By analogy with the classical theory (see [31], Section 3.1, [28], Section 1.1.3), we expect that the kernel l should satisfy the wave equation

$$-\frac{\partial^2 l}{\partial x^2} = -\frac{\partial^2 l}{\partial t^2} + \frac{\kappa(\kappa+1)}{t^2}l \tag{3.3}$$

and the boundary conditions

$$\begin{gathered} \frac{d}{dx}l(x,x) = \frac{1}{2}\frac{\kappa(\kappa+1)}{x^2}, \\ \lim_{x+t\to\infty}\frac{\partial}{\partial x}l(x,t) = \lim_{x+t\to\infty}\frac{\partial}{\partial t}l(x,t) = 0. \end{gathered} \tag{3.4}$$

The crucial observation is that the system (3.3)–(3.4) is homogeneous in the sense that, for every $\lambda > 0$, the function $\lambda l(\lambda x, \lambda t)$ is a solution of (3.3)–(3.4) along with $l(x,t)$. This suggests that we can look for homogeneous solutions of that system satisfying the relation

$$l(x,t) = \frac{1}{t}l\left(\frac{x}{t},1\right).$$

Set $u(\xi) := l(\xi,1)$; then the function u must satisfy the ordinary differential equation

$$\big((1-\xi^2)u\big)'' + \kappa(\kappa+1)u = 0$$

and the boundary condition

$$u(1) = -\frac{\kappa(\kappa+1)}{2}.$$

Recalling that a solution of the Legendre equation

$$((1-\xi^2)y')' + \kappa(\kappa+1)y = 0 \tag{3.5}$$

satisfying the terminal conditions

$$y(1) = 1, \quad y'(1) = \frac{\kappa(\kappa+1)}{2}$$

is given by the Legendre function P_κ of first kind and order κ (see [35], Chapter 15, [1], Chapter 8), one immediately recognizes that $u = -P'_\kappa$.

Set therefore

$$l(x,t) := -\frac{1}{t}P'_\kappa\left(\frac{x}{t}\right), \quad x < t,$$

and denote by L the integral operator of (3.2). Since P_κ is an analytic function in a complex neighbourhood of $[0,1]$, P'_κ is bounded on $[0,1]$ by some constant c. Hence $|l(x,t)| \leqq c/t$ for all $t > 0$, and the Hardy inequality [22], Section 9.9 shows that L is a bounded operator in $L_2(\mathbb{R}_+)$. By Corollary 3.3 below, the operator $I + L$ is boundedly invertible in $L_2(\mathbb{R}_+)$.

Theorem 3.1. *The operator $I + L$ is the transformation operator, i.e., it performs similarity of the operators $T_{0,\max}$ and $T_{\kappa,\max}$ via* (3.1).

Proof. Take an arbitrary $y \in \operatorname{dom} T_{\kappa,\max}$, set $f := \ell_\kappa y$ and $g := (I+L)y$, and fix an $\varepsilon > 0$. By Remark 2.2, the function y belongs to $W_2^2(\varepsilon,\infty)$, so that $y'' \in L_2(\varepsilon,\infty)$. Integrating by parts twice in the integral $\int_x^\infty l(x,t)y''(t)\,dt$ for $x > \varepsilon$ and using the relations (3.3)–(3.4), we arrive at the equality

$$f(x) + \int_x^\infty l(x,t)f(t)\,dt = -g''(x) \tag{3.6}$$

in the sense of distributions over (ε,∞).

Observe that the function f belongs to $L_2(\mathbb{R}_+)$ by the definition of $\operatorname{dom} T_{\kappa,\max}$ and that $g = (I+L)y$ and $(I+L)f$ are in $L_2(\mathbb{R}_+)$. Since $\varepsilon > 0$ was arbitrary, we conclude that the distribution g'' belongs to $L_2(\mathbb{R}_+)$. Therefore $g \in \operatorname{dom} T_{0,\max}$ and $(I+L)f = \ell_0 g$, which establishes the inclusion $(I+L)T_{\kappa,\max} \subset T_{0,\max}(I+L)$.

To prove the reverse inclusion, we take an arbitrary $y \in L_2(\mathbb{R}_+)$ for which the function $g := (I+L)y$ belongs to the domain of $T_{0,\max}$, i.e., to $W_2^2(\mathbb{R}_+)$. We fix an arbitrary

$\varepsilon > 0$ and observe that Ly is absolutely continuous on the interval (ε, ∞) and that the derivative

$$(Ly)'(x) = P_\kappa'(1)\frac{y(x)}{x} - \int_x^\infty P_\kappa''\left(\frac{x}{t}\right)\frac{1}{t^2}f(t)\,dt$$

belongs to $L_2(\varepsilon, \infty)$; in particular, $Ly \in W_2^1(\varepsilon, \infty)$. Since $g \in W_2^2(\mathbb{R}_+)$, we conclude that $y = g - Ly \in W_2^1(\varepsilon, \infty)$ and then, by replicating the arguments, that $y \in W_2^2(\varepsilon, \infty)$.

As a result, the distribution $f := \ell_\kappa y$ is in $L_2(\varepsilon, \infty)$, and integration by parts again leads to the equality (3.6) for $x > \varepsilon$. As $\varepsilon > 0$ was arbitrary and $-g'' \in L_2(\mathbb{R}_+)$, we see that $(I + L)f$ belongs to $L_2(\mathbb{R}_+)$ and equals $-g''$. By Corollary 3.3 below, the operator $I + L$ is boundedly invertible in $L_2(\mathbb{R}_+)$, so that $f \in L_2(\mathbb{R}_+)$. It follows that y belongs to $\operatorname{dom} T_{\kappa,\max}$ and that $(I + L)T_{\kappa,\max} \supset T_{0,\max}(I + L)$, thus completing the proof. □

3.2. Some symbol calculus. The operator $I + L$ has an upper triangular kernel, which suggests that it might belong to the subalgebra $\mathscr{A}^+$ introduced in Subsection 6.2 below. To verify this, we have to calculate the symbol ζ_κ of $I + L$ and to show that it belongs to the Hardy space H^∞.

By definition, we have

$$\zeta_\kappa(z) := 1 - \mathcal{M}(P_\kappa'\chi_{[0,1]})(z) = 1 - \int_0^1 t^{-iz-1/2}P_\kappa'(t)\,dt, \tag{3.7}$$

where $\mathcal{M}$ denotes the Mellin transform (see Appendix B), P_κ is the Legendre function of first kind and order κ, see [1], Chapter 8, [35], Chapter 15, and $\chi_{[0,1]}$ is the indicator function of the interval $[0, 1]$.

Lemma 3.2. *For $z \in \overline{\mathbb{C}^+}$, the following identity holds*:

$$\zeta_\kappa(z) = \frac{\Gamma\left(\frac{1}{4} - \frac{i}{2}z\right)\Gamma\left(\frac{3}{4} - \frac{i}{2}z\right)}{\Gamma\left(\frac{1}{4} - \frac{i}{2}z - \frac{\kappa}{2}\right)\Gamma\left(\frac{3}{4} - \frac{i}{2}z + \frac{\kappa}{2}\right)}. \tag{3.8}$$

Proof. We multiply the Legendre equation (3.5) by t and then take its Mellin transform to get the relation

$$\int_0^1 \big((1 - t^2)P_\kappa'(t)\big)' t^{-iz+\frac{1}{2}}\,dt + \kappa(\kappa + 1)\int_0^1 P_\kappa(t)t^{-iz+\frac{1}{2}}\,dt = 0.$$

Integrating by parts yields

$$\zeta_\kappa(z + 2i) = 1 - \int_0^1 P_\kappa'(t)t^{-iz+\frac{3}{2}}\,dt = \left(-iz + \frac{3}{2}\right)\int_0^1 P_\kappa(t)t^{-iz+\frac{1}{2}}\,dt$$

and

$$\int_0^1 \big((1-t^2)P_\kappa'(t)\big)' t^{-iz+\frac{1}{2}}\,dt = \left(-iz+\frac{1}{2}\right)\big(\zeta_\kappa(z)-\zeta_\kappa(z+2i)\big);$$

therefore the above relation takes the form

$$\left(-iz+\frac{1}{2}\right)\big(\zeta_\kappa(z)-\zeta_\kappa(z+2i)\big)+\frac{\kappa(\kappa+1)}{-iz+\dfrac{3}{2}}\,\zeta_\kappa(z+2i)=0.$$

Setting $a(z) := \left(z^2+\frac{1}{4}\right)^{-1}$, we get

$$\frac{\zeta_\kappa(z)}{\zeta_\kappa(z+2i)} = 1+\kappa(\kappa+1)a(z+i),$$

or, by iteration,

$$\zeta_\kappa(z) = \zeta_\kappa(z+2ni)\prod_{k=0}^{n-1}\big(1+\kappa(\kappa+1)a(z+i+2ki)\big). \tag{3.9}$$

The Riemann–Lebesgue lemma applied to the integral in (3.7) yields the equality

$$\lim_{y\to+\infty}\zeta_\kappa(z+iy)=1$$

for every $z\in\mathbb{C}^+$. Passing to the limit in (3.9), we derive the relation

$$\zeta_\kappa(z) = \prod_{k=0}^{\infty}\big(1+\kappa(\kappa+1)a(z+i+2ki)\big). \tag{3.10}$$

Observing that

$$1+\kappa(\kappa+1)a(z) = 1+\frac{\kappa(\kappa+1)}{z^2+1/4} = \left(1+\frac{i\kappa}{z+i/2}\right)\left(1-\frac{i\kappa}{z-i/2}\right),$$

we can recast (3.10) as

$$\zeta_\kappa(z) = \prod_{k=0}^{\infty}\left(1+\frac{\kappa/2}{k+3/4-iz/2}\right)\left(1-\frac{\kappa/2}{k+1/4-iz/2}\right), \tag{3.11}$$

which gives (3.8) by [21], Equation 8.325(1). □

Corollary 3.3. *The operator $I+L$ belongs to the algebra $\mathscr{A}^+$, is boundedly invertible in $\mathscr{A}^+$, and the inverse $I+K$ has symbol $1/\zeta_\kappa(z)$.*

Proof. We have to show that the symbols ζ_κ and $1/\zeta_\kappa$ belong to the Hardy class H^∞ in the upper complex half-plane. Regrouping the factors in (3.11), we see that

$$\zeta_\kappa(z) = \prod_{n=0}^{\infty}\left(1 - \frac{\kappa(\kappa+1)}{(2n+1-iz)^2 - 1/4}\right).$$

Since for $z \in \overline{\mathbb{C}^+}$ the estimate

$$\left|\frac{\kappa(\kappa+1)}{(2n+1-iz)^2 - 1/4}\right| \leqq \frac{|\kappa(\kappa+1)|}{4n^2 + 3/4}$$

holds, the above product converges uniformly in $\overline{\mathbb{C}^+}$ to a bounded analytic function. The claim about $1/\zeta_\kappa$ is justified in the same manner by using the representation

$$1/\zeta_\kappa(z) = \prod_{n=0}^{\infty}\left(1 - \frac{\kappa(\kappa+1)}{(2n+1-iz)^2 - (\kappa+1/2)^2}\right),$$

and the proof is complete. □

3.3. The transformation operator $I + K$. According to Corollary 3.3, the transformation operator $I + K$ is bounded in $L_2(\mathbb{R}_+)$ and belongs to the algebra $\mathscr{A}^+$; namely,

$$(I+K)y(x) = y(x) + \int_x^\infty v\left(\frac{x}{t}\right)\frac{1}{t}y(t)\,dt$$

with a function v that is related to the symbol $\eta_\kappa := 1/\zeta_\kappa$ via the Mellin transform $\mathcal{M}$, viz.

$$\eta_\kappa = 1 + \mathcal{M}(v\chi_{[0,1]}), \tag{3.12}$$

see Subsection 6.2 and Appendix B. Here we shall use the explicit formula for η_κ in order to derive some properties of v and $I + K$ that will be useful for studying the scattering function in Subsection 4.2.

By virtue of (3.8), the symbol η_κ has the form

$$\eta_\kappa(z) = \frac{\Gamma\left(\frac{1}{4} - \frac{i}{2}z - \frac{\kappa}{2}\right)\Gamma\left(\frac{3}{4} - \frac{i}{2}z + \frac{\kappa}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{i}{2}z\right)\Gamma\left(\frac{3}{4} - \frac{i}{2}z\right)} \tag{3.13}$$

and is a meromorphic function with simple poles at the points

$$z_n := -\left(n + \frac{1}{2} + (-1)^{n+1}\kappa\right)i, \quad n \in \mathbb{Z}_+.$$

Denote by a_n the residue of η_κ at the pole z_n; then we have the following result.

Lemma 3.4. *As $n \to \infty$, the residues a_n admit the representation*

$$a_n = i(-1)^n \frac{\tan \pi\kappa}{\pi}\left(1 - \frac{\kappa(\kappa+1)}{4n}\right) + O(n^{-2}).$$

Proof. We only treat the poles z_{2n} since the formulae for z_{2n+1} are obtained from those for z_{2n} by replacing κ by $-1-\kappa$.

We recall that the Gamma function Γ has simple poles at all non-positive integers and that its residue at the point $-n$ is equal to $(-1)^n/n!$. Therefore,

$$\lim_{z \to z_{2n}} \left(\frac{1}{4} - \frac{\kappa}{2} - \frac{i}{2}z + n\right)\Gamma\left(\frac{1}{4} - \frac{\kappa}{2} - \frac{i}{2}z\right) = \frac{(-1)^n}{n!},$$

which yields

$$a_{2n} = 2i\frac{(-1)^n}{n!} \frac{\Gamma\left(\frac{1}{2} + \kappa - n\right)}{\Gamma\left(\frac{\kappa}{2} - n\right)\Gamma\left(\frac{1}{2} + \frac{\kappa}{2} - n\right)}. \tag{3.14}$$

Using the relation $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$, we find that

$$\begin{aligned} a_{2n} &= 2i\frac{\sin\left(\pi\frac{\kappa}{2}\right)\sin\pi\left(\frac{1}{2}+\frac{\kappa}{2}\right)}{n!\pi\sin\pi\left(\frac{1}{2}-\kappa\right)} \frac{\Gamma\left(n+1-\frac{\kappa}{2}\right)\Gamma\left(n+\frac{1}{2}-\frac{\kappa}{2}\right)}{\Gamma\left(n+\frac{1}{2}-\kappa\right)} \\ &= i\frac{\tan\pi\kappa}{\pi} \frac{\Gamma\left(n+1-\frac{\kappa}{2}\right)\Gamma\left(n+\frac{1}{2}-\frac{\kappa}{2}\right)}{\Gamma(n+1)\Gamma\left(n+\frac{1}{2}-\kappa\right)}. \end{aligned}$$

By virtue of [21], Equation 8.325(1), we get

$$\frac{\Gamma\left(n+1-\frac{\kappa}{2}\right)\Gamma\left(n+\frac{1}{2}-\frac{\kappa}{2}\right)}{\Gamma(n+1)\Gamma\left(n+\frac{1}{2}-\kappa\right)} = \prod_{k=0}^{\infty}\left(1 + \frac{\frac{\kappa}{2}}{n+k+1-\frac{\kappa}{2}}\right)\left(1 - \frac{\frac{\kappa}{2}}{n+k+\frac{1}{2}-\frac{\kappa}{2}}\right).$$

Since

$$\left(1 + \frac{\frac{\kappa}{2}}{n+k+1-\frac{\kappa}{2}}\right)\left(1 - \frac{\frac{\kappa}{2}}{n+k+\frac{1}{2}-\frac{\kappa}{2}}\right) = 1 - \frac{\kappa(\kappa+1)}{4}\frac{1}{(n+k)^2} + O\big((n+k)^{-3}\big)$$

and

$$\sum_{k=0}^{\infty} \frac{1}{(n+k)^2} = \frac{1}{n} + O(n^{-2}),$$

the required representation of a_n follows. □

Lemma 3.5. *The symbol η_κ has the form*

$$\eta_\kappa(z) = 1 + \sum_{n=0}^{\infty} \left(\frac{a_{2n}}{z - z_{2n}} + \frac{a_{2n+1}}{z - z_{2n+1}} \right), \tag{3.15}$$

where the series converges uniformly on every compact subset of $\mathbb{C}$ not containing the poles.

Proof. We denote by ξ the function given by the right-hand side of (3.15). By virtue of Lemma 3.4 the series for ξ converges uniformly on every compact subset of $\mathbb{C}$ not containing the numbers z_n, $n \in \mathbb{Z}_+$, whence ξ is a meromorphic function with simple poles at z_n. It follows that the function $\eta_\kappa - \xi$ is entire.

Fix $\varepsilon > 0$ and denote by D_ε the complement of the ε-neighbourhood of the set $\{z_n\}_{n \in \mathbb{Z}_+}$. Then the function η_κ is uniformly bounded on D_ε due to the product representation (3.10). By Lemma 3.4, the function ξ is also uniformly bounded on D_ε. Hence $\eta_\kappa - \xi$ is a constant function by the Liouville theorem, and this constant is zero in view of the relations

$$\lim_{x \to +\infty} \eta_\kappa(x) = \lim_{x \to +\infty} \xi(x) = 1.$$

The lemma is proved. □

Corollary 3.6. *The function v has the representation*

$$v(s) = -ia_0 s^{-\kappa} - ia_1 s^{1+\kappa} - i \sum_{n=1}^{\infty} (a_{2n} s^{2n-\kappa} + a_{2n+1} s^{2n+1+\kappa}),$$

in which the series converges uniformly on $[0, 1]$.

Indeed, the above formula follows from (3.12), (3.15), and the relation

$$\mathcal{M}(x^\alpha \chi_{[0,1]}) = i\left[z + i\left(\alpha + \frac{1}{2}\right)\right]^{-1}.$$

The uniform convergence of the series is guaranteed by the asymptotics of the a_n established in Lemma 3.4.

Corollary 3.6 implies that $v(s) = -ia_0 s^{-\kappa} + s^{1+\kappa} \tilde{v}(s)$ with a function $\tilde{v}$ that is continuous on $[0, 1]$, which yields the following representation of the transformation operator $I + K$.

Corollary 3.7. *The transformation operator $I + K$ has the form*

$$I + K = I - ia_0 B_{-\kappa} + \tilde{K},$$

where $B_{-\kappa}$ is the Hardy operator of Example 6.2(i) *and $\tilde{K}$ acts via*

$$(\tilde{K}y)(x) = \int_x^\infty \left(\frac{x}{t}\right)^{1+\kappa} \tilde{v}\left(\frac{x}{t}\right) \frac{1}{t} y(t)\,dt$$

with some function $\tilde{v}$ that is continuous on $[0, 1]$.

4. The scattering function

4.1. Direct construction. By definition, the scattering function S is the coefficient in the linear combination $e(x, -\omega) - S(\omega)e(x, \omega)$ that produces a solution of (2.2) satisfying the initial condition (2.3) at the origin.

Recall that the Jost solution $e(\cdot, \omega)$ is a solution of equation (2.2) that for ω in the closed upper half-plane has the asymptotics

$$e(x, \omega) = e^{i\omega x}\bigl(1 + o(1)\bigr)$$

as $x \to +\infty$. For $\kappa \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ the asymptotic behaviour of the solutions ϕ_κ and ψ_κ at infinity (see Subsection A.2) yields

$$e(x, \omega) = \sqrt{\frac{\pi}{2}} \frac{1}{\cos \pi\kappa} \left[ie^{-i\frac{\pi}{2}\kappa}\phi_\kappa(x, \omega) + e^{i\frac{\pi}{2}\kappa}\psi_\kappa(x, \omega)\right]. \tag{4.1}$$

Using the relation $\Gamma\left(\frac{1}{2} - \kappa\right)\Gamma\left(\frac{1}{2} + \kappa\right) = \dfrac{\pi}{\cos \pi\kappa}$ and the asymptotics of the ϕ_κ and ψ_κ at the origin, we find that

$$\begin{aligned} e(x, \omega) &= \frac{\sqrt{\pi}e^{i\frac{\pi}{2}\kappa}}{\cos \pi\kappa} \frac{2^\kappa}{\Gamma\left(\frac{1}{2} - \kappa\right)} (\omega x)^{-\kappa}\bigl(1 + o(1)\bigr) \\ &= \frac{e^{i\frac{\pi}{2}\kappa}}{\sqrt{\pi}} 2^\kappa \Gamma\left(\frac{1}{2} + \kappa\right)(\omega x)^{-\kappa}\bigl(1 + o(1)\bigr) \end{aligned} \tag{4.2}$$

as $x \to +0$. For positive ω we thus find that

$$S(\omega) = \lim_{x \to +0} \frac{e(x, -\omega)}{e(x, \omega)} = \frac{e^{-\kappa(\log \omega + \pi i)}}{e^{-\kappa \log \omega}} = e^{-\pi i\kappa}.$$

Since we only consider ω in the upper half-plane, for negative ω we should interpret $(-\omega)^{-\kappa}$ as $e^{-\kappa\log(-\omega)}$ and thus similarly get

$$S(\omega) = \lim_{x\to+0}\frac{e(x,-\omega)}{e(x,\omega)} = \frac{e^{-\kappa\log(-\omega)}}{e^{-\kappa(\log(-\omega)+\pi i)}} = e^{\pi i\kappa}.$$

For $\kappa = -1/2$ the asymptotics (4.2) of $e(x,\omega)$ as $x\to+0$ gets an extra factor $\log(\omega x)$, which, however, does not influence the value of $S(\omega)$. Therefore the scattering function is piecewise constant for all $\kappa \in \left[-\frac{1}{2},\frac{1}{2}\right)$ and equals

$$S(\omega) = \begin{cases} e^{-\pi i\kappa}, & \omega > 0,\\ e^{\pi i\kappa}, & \omega < 0.\end{cases} \tag{4.3}$$

We observe that in the case of Schrödinger operators with Faddeev–Marchenko potentials the scattering function S is continuous and is close to 1 in the sense that the difference $1 - S$ belongs to $L_2(\mathbb{R})$, see [31], Chapter 3.3. Here this is not the case; nevertheless one can use S to uniquely reconstruct the operator H_κ, see Section 6 below.

4.2. Construction via the transformation operators. Here we show that the same result can be derived without knowing the explicit formulae expressing the Jost solution $e(\cdot,\omega)$ via the Bessel functions but rather using the representation of $e(\cdot,\omega)$ via the transformation operators.

Consider first the case $\kappa > 0$. By Corollary 3.7, for all ω in the open upper half-plane we get

$$e(x,\omega) = e^{i\omega x} - ia_0x^{-\kappa}\int_x^\infty \frac{e^{i\omega t}}{t^{1-\kappa}}\,dt + \int_x^\infty \left(\frac{x}{t}\right)^{1+\kappa}\tilde{v}\left(\frac{x}{t}\right)\frac{e^{i\omega t}}{t}\,dt \tag{4.4}$$

with a function $\tilde{v}$ that is continuous over $[0,1]$. Observe that for every fixed $x > 0$ the function $e(x,\cdot)$ is analytic in the whole complex plane. Since the integrals above converge uniformly in the domain

$$\{\omega\in\mathbb{C} \mid \operatorname{Im}\omega \geqq 0, |\omega| \geqq \varepsilon\}$$

for every $\varepsilon > 0$ (the first one by the Abel–Dirichlet test and the second one by the dominated convergence test), we conclude that the representation (4.4) holds also for all real nonzero ω.

Further, the last integral remains bounded as $x\to+0$, so that, by (3.14), we get

$$\begin{aligned}\lim_{x\to+0} x^\kappa e(x,\omega) &= -ia_0\int_0^\infty \frac{e^{i\omega t}}{t^{1-\kappa}}\,dt = -ia_0\Gamma(\kappa)e^{i\frac{\pi}{2}\kappa}\omega^{-\kappa}\\ &= 2e^{i\frac{\pi}{2}\kappa}\frac{\Gamma(\kappa)\Gamma\left(\frac{1}{2}+\kappa\right)}{\Gamma\left(\frac{\kappa}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\kappa}{2}\right)}\omega^{-\kappa} = \frac{e^{i\frac{\pi}{2}\kappa}}{\sqrt{\pi}}2^\kappa\Gamma\left(\frac{1}{2}+\kappa\right)\omega^{-\kappa},\end{aligned} \tag{4.5}$$

which should be compared with (4.2). In the last equality above, we have used the double argument formula [21], Formula 8.335:

$$\Gamma(x)\Gamma\left(\frac{1}{2}+x\right) = \sqrt{\pi}2^{1-2x}\Gamma(2x)$$

for the Gamma functions. Therefore,

$$S(\omega) = \lim_{x\to+0}\frac{e(x,-\omega)}{e(x,\omega)} = \lim_{x\to+0}\frac{x^\kappa e(x,-\omega)}{x^\kappa e(x,\omega)} = \frac{(-\omega)^{-\kappa}}{\omega^{-\kappa}},$$

which results in (4.3).

The study of the behaviour of the Jost solution $e(\cdot,\omega)$ at the origin for negative κ will be based on a different integral representation. Firstly, by Corollary 3.6 the function v is then continuous on $[0,1]$ and

$$\lim_{s\to+0} s^\kappa v(s) = -ia_0. \tag{4.6}$$

Therefore the formula

$$e(x,\omega) = e^{i\omega x} + \int_x^\infty v\left(\frac{x}{t}\right)\frac{1}{t}\,e^{i\omega t}\,dt$$

established in Section 3 for $x>0$ and $\omega\in\mathbb{C}^+$ remains true for real ω, due to the uniform convergence of the integral for $\omega\in\overline{\mathbb{C}^+}$.

We set $V(s) := \int_0^s v(\xi)/\xi\,d\xi$ and notice that, by l'Hôpital's rule,

$$\lim_{s\to+0} s^\kappa V(s) = -\frac{1}{\kappa}\lim_{s\to+0} s^\kappa v(s) = \frac{ia_0}{\kappa}. \tag{4.7}$$

Now for $x\in(0,1]$ the integration by parts gives

$$e(x,\omega) = e^{i\omega x}\bigl(1+V(1)\bigr) - e^{i\omega}V(x) + i\omega\int_x^1 V\left(\frac{x}{t}\right)e^{i\omega t}\,dt + \int_1^\infty v\left(\frac{x}{t}\right)\frac{1}{t}\,e^{i\omega t}\,dt.$$

Next we show that $1+V(1)=0$. Indeed, both the symbol η_κ and the Mellin transform of $v\chi_{[0,1]}$ can be continued analytically into the half-plane $\operatorname{Im} z > \kappa - \frac{1}{2}$. Since $z = -\frac{i}{2}$ is a zero of η_κ in view of (3.13), we derive from (3.12) that

$$0 = \eta_\kappa\left(\frac{i}{2}\right) = 1 + \int_0^1 t^{-1}v(t)\,dt = 1 + V(1)$$

as required. It follows that

$$\lim_{x\to+0} x^\kappa e(x,\omega) = -e^{i\omega} \lim_{x\to+0} x^\kappa V(x) + i\omega \lim_{x\to+0} \int_x^1 \left(\frac{x}{t}\right)^\kappa V\left(\frac{x}{t}\right) t^\kappa e^{i\omega t}\, dt$$
$$+ \lim_{x\to+0} \int_1^\infty \left(\frac{x}{t}\right)^\kappa v\left(\frac{x}{t}\right) \frac{1}{t^{1-\kappa}} e^{i\omega t}\, dt.$$

Using (4.6), (4.7), and applying the Lebesgue dominated convergence theorem to the above integrals, we get

$$\lim_{x\to+0} x^\kappa e(x,\omega) = -ia_0 \left[\frac{e^{i\omega}}{\kappa} - \frac{i\omega}{\kappa}\int_0^1 t^\kappa e^{i\omega t}\, dt + \int_1^\infty t^{\kappa-1} e^{i\omega t}\, dt\right]$$
$$= -\frac{a_0\omega}{\kappa}\int_0^\infty t^\kappa e^{i\omega t}\, dt.$$

[21], Formula 3.381.5 finally yields the relation

$$\lim_{x\to+0} x^\kappa e(x,\omega) = -\frac{a_0\omega^{-\kappa}}{\kappa}\Gamma(1+\kappa) e^{\frac{i\pi}{2}(1+\kappa)} = -ia_0\Gamma(\kappa) e^{\frac{i\pi}{2}\kappa}\omega^{-\kappa}$$

as for the case $\kappa > 0$, cf. (4.5), which results in the expression (4.3) for the scattering function S.

5. Derivation of the Marchenko equation

In Section 3, we constructed the transformation operator $I + K$ that maps solutions of the unperturbed equation $-y'' = \omega^2 y$ into the solutions of the equation $\ell_\kappa y = \omega y$ and preserves their behaviour at infinity. In particular, the asymptotics of the solution ϕ_κ as $x \to +\infty$ yields the relation

$$\phi_\kappa(x,\omega) = \frac{ie^{\frac{i\pi}{2}\kappa}}{\sqrt{2\pi}}[e(x,-\omega) - S(\omega)e(x,\omega)]$$
$$= \frac{ie^{\frac{i\pi}{2}\kappa}}{\sqrt{2\pi}}(I+K)[e^{-i\omega t} - S(\omega)e^{i\omega t}](x).$$

We recall (see Subsection A.3) that $\mathcal{J}_\kappa$ denotes the Hankel transform in $L_2(\mathbb{R}_+)$ given by

$$(\mathcal{J}_\kappa f)(\omega) := \int_0^\infty \phi_\kappa(x,\omega) f(x)\, dx,$$

which is a unitary operator in $L_2(\mathbb{R}_+)$. We shall write $\mathcal{F}_\pm$ for the truncated Fourier transforms, viz.

$$(\mathcal{F}_\pm f)(\omega) := \int_0^\infty e^{\pm i\omega x} f(x)\, dx.$$

These are bounded operators in $L_2(\mathbb{R}_+)$ defined by

$$(\mathcal{F}_\pm f)(\omega) := \operatorname*{L.i.m.}_{N\to\infty} \int_0^N e^{\pm i\omega x} f(x)\,dx.$$

Denoting the operator of multiplication by the scattering function S by the same letter S and substituting for ϕ_κ in the transform $\mathcal{J}_\kappa$, we find that

$$\mathcal{J}_\kappa = \frac{ie^{i\frac{\pi}{2}\kappa}}{\sqrt{2\pi}}(\mathcal{F}_- - S\mathcal{F}_+)(I+K)^*,$$

which yields the relation

$$I = \mathcal{J}_\kappa^* \mathcal{J}_\kappa = \frac{1}{2\pi}(I+K)(\mathcal{F}_- - S\mathcal{F}_+)^*(\mathcal{F}_- - S\mathcal{F}_+)(I+K)^*.$$

Recalling that $|S(\omega)| = 1$ and $S(-\omega) = \overline{S(\omega)}$ for all real ω and observing that $(\mathcal{F}_\pm)^* = \mathcal{F}_\mp$, we find that

$$(I+K)^{-1}(I+K^*)^{-1} = \frac{1}{2\pi}[\mathcal{F}_+\mathcal{F}_- + \mathcal{F}_-\mathcal{F}_+ - \mathcal{F}_+S\mathcal{F}_+ - \mathcal{F}_-S\mathcal{F}_-].$$

For every function $\phi \in L_2(\mathbb{R}_+)$ of compact support we find that

$$(\mathcal{F}_+\mathcal{F}_-\phi)(x) + (\mathcal{F}_-\mathcal{F}_+\phi)(x) = \operatorname*{L.i.m.}_{n\to\infty} \int_0^\infty f(t)\frac{\sin n(x-t)}{x-t}\,dt.$$

Since $\sin nx/x$ is the Fourier transform of the characteristic function $\chi_{[-n,n]}$ of the interval $[-n,n]$, the operator of convolution with $\sin nx/x$ converges as $n\to\infty$ to $2\pi I$ in the strong operator topology of $L_2(\mathbb{R}_+)$. Therefore $\mathcal{F}_+\mathcal{F}_- + \mathcal{F}_-\mathcal{F}_+ = 2\pi I$. Further, straightforward calculations give

$$(\mathcal{F}_+S\mathcal{F}_+\phi)(x) + (\mathcal{F}_-S\mathcal{F}_-\phi)(x) = -2\pi\sin(\pi\kappa)(\mathcal{C}\phi)(x) + 2\operatorname*{L.i.m.}_{n\to\infty}\int_0^\infty \phi(t)\frac{\sin[n(x+t)-\pi\kappa]}{x+t}\,dt,$$

where $\mathcal{C}$ is the Carleman (or Stieltjes) transform defined by

$$(\mathcal{C}\phi)(x) := \frac{1}{\pi}\int_0^\infty \frac{\phi(t)}{x+t}\,dt.$$

The Carleman operator $\mathcal{C}$ belongs to the operator algebra $\mathscr{A}$ introduced in Section 6 below; in particular, it is bounded in $L_2(\mathbb{R}_+)$ (see Example 6.2(b)). Therefore the integral operators I_n given by

$$(I_n\phi)(x) := \int_0^\infty \phi(t)\frac{\sin[n(x+t)-\pi\kappa]}{x+t}\,dt$$

are uniformly bounded in $L_2(\mathbb{R}_+)$. If ϕ belongs to $C_0^\infty(\mathbb{R}_+)$, so that $\varepsilon := \inf \operatorname{supp} \phi > 0$, then integration by parts gives the pointwise estimate

$$|(I_n\phi)(x)| \leqq \frac{C}{n}\frac{1}{\varepsilon + x}$$

for some constant C depending only on ϕ. It follows that $I_n\phi \to 0$ in $L_2(\mathbb{R}_+)$ for every such ϕ, which implies that I_n converge to zero in the strong operator topology of $L_2(\mathbb{R}_+)$.

Combining the above formulae, we finally arrive at the relation

$$(I+K)^{-1}(I+K^*)^{-1} = I - \sin(\pi\kappa)\mathcal{C}, \tag{5.1}$$

which states that the operator $I - \sin(\pi\kappa)\mathcal{C}$ is factorized in the operator algebra $\mathscr{A}$ (see Section 6 below) by means of the operator $I + L = (I+K)^{-1}$ and its adjoint.

Applying the operator $I + K$ to both sides of (5.1) and rewriting the resulting equation in terms of kernels, we derive the Marchenko equation,

$$k(x,t) + f(x+t) + \int_x^\infty k(x,s) f(s+t)\, ds = 0, \quad x < t, \tag{5.2}$$

with $f(s) := -\sin(\pi\kappa)/s$. Conversely, it is known that if some functions k and f are related by the Marchenko equation, then the corresponding integral operators K and F,

$$(Fy)(x) := \int_0^\infty f(x+t)y(t)\, dt$$

are related via the factorization relation (5.1) with $I - \sin(\pi\kappa)\mathcal{C}$ there replaced by $I + F$.

Remark 5.1. In the classical situation of Schrödinger operators with potentials in the Faddeev–Marchenko class the function S satisfies the relation $S(-\omega) = \overline{S(\omega)}$ and the inclusion $1 - S \in L_2(\mathbb{R})$. Observing that $\mathcal{F}_+^2 + \mathcal{F}_-^2 = 0$, we find that, for every $\phi \in L_2(\mathbb{R}_+)$ of compact support,

$$\begin{aligned}
&\big(\mathcal{F}_+ S(\omega)\mathcal{F}_+\phi\big)(x) + \big(\mathcal{F}_- S(-\omega)\mathcal{F}_-\phi\big)(x)\\
&\quad = \big(\mathcal{F}_+[S(\omega)-1]\mathcal{F}_+\phi\big)(x) + \big(\mathcal{F}_-[S(-\omega)-1]\mathcal{F}_-\phi\big)(x)\\
&\quad = \int_0^\infty \phi(t)\, \underset{n\to\infty}{\text{L.i.m.}} \int_{-n}^{n} e^{i\omega(x+t)}[S(\omega)-1]\, d\omega\\
&\quad = 2\pi\int_0^\infty f(x+t)\phi(t)\, dt,
\end{aligned}$$

with f being the Fourier transform of $S - 1$. It is known that the function f is also integrable and thus the integral operator F is compact.

In our situation with Bessel potentials the scattering function S is given by (4.3). Its distributional Fourier transform equals the distribution $-\sin(\pi\kappa)/x$ taken in the principal value sense, which is in complete agreement with the above. The integral operator F under the Mellin transform becomes the operator of multiplication by the symbol $-\sin(\pi\kappa)/\cosh(\pi z)$ in the space $L_2(\mathbb{R})$ (cf. Example 6.2(b)) and thus it is no longer compact.

6. Solution of the Marchenko equation and factorization of the operator $I + F$

As explained at the end of the previous section, the problem of solving the Marchenko equation (5.2) is equivalent to that of factorizing the operator $I + F$. For the operator H_κ, it is easier to solve the latter one, and we treat it in this section.

6.1. Canonical factorization of operators. For every $t \in [0, \infty)$, we denote by χ_t the characteristic function of the interval $[0, t]$ and introduce the orthoprojector P_t in $L_2(\mathbb{R}_+)$ by

$$(P_t f)(x) = \chi_t(x) f(x).$$

An operator $A \in \mathscr{B}$ is called *upper triangular* (resp. *lower triangular*) if

$$(I - P_t)AP(t) = 0 \quad (\text{resp. } P_t A(I - P_t) = 0)$$

for every $t \in \mathbb{R}_+$. The subset $\mathscr{B}^+$ (resp. $\mathscr{B}^-$) of all upper triangular (resp. lower triangular) operators in $\mathscr{B}$ forms a closed subalgebra of $\mathscr{B}$.

Definition 6.1. Assume that $\mathscr{B}_0$ is a subalgebra of $\mathscr{B}$ and set $\mathscr{B}_0^\pm := \mathscr{B}^\pm \cap \mathscr{B}_0$. We say that an operator $A \in \mathscr{B}_0$ admits a *canonical factorization in* $\mathscr{B}_0$ if there are operators $A^+ \in \mathscr{B}_0^+$ and $A^- \in \mathscr{B}_0^-$ that are invertible respectively in $\mathscr{B}_0^+$ and $\mathscr{B}_0^-$ and such that $A = A^+ A^-$.

We refer to [20], Chapter IV, for general results on factorization in operator algebras.

6.2. The algebra $\mathscr{A}$. Now we introduce a special commutative subalgebra $\mathscr{A}$ of $\mathscr{B}$. Given an arbitrary function $\theta \in L_\infty(\mathbb{R})$, we denote by M_θ the operator of multiplication by θ, $M_\theta f = \theta f$, and set

$$\hat{M}_\theta := \mathcal{M}^{-1} M_\theta \mathcal{M},$$

with $\mathcal{M}$ denoting the Mellin transform, see Appendix B. We call θ the *symbol* of the operator $\hat{M}_\theta$. The family

$$\mathscr{A} := \{\hat{M}_\theta \mid \theta \in L_\infty(\mathbb{R})\}$$

forms a closed self-adjoint commutative subalgebra of the Banach algebra $\mathscr{B}$ with unity. The mapping

$$L_\infty(\mathbb{R}) \ni \theta \mapsto \hat{M}_\theta \in \mathscr{A}$$

is an algebra isomorphism and $(\hat{M}_\theta)^* = \hat{M}_{\bar{\theta}}$.

Let $\phi \in L_2(0,\infty)$ be such that $\theta := \mathcal{M}\phi \in L_\infty(\mathbb{R})$; then the operator $\hat{M}_\theta$ is an integral operator K_ϕ in $L_2(\mathbb{R}_+)$ given by

$$(K_\phi f)(x) := \int_0^\infty \phi\left(\frac{x}{t}\right)\frac{1}{t}f(t)\,dt.$$

Indeed, using the property (B.2), we find that

$$\hat{M}_\theta f = \mathcal{M}^{-1}[(\mathcal{M}\phi)\cdot(\mathcal{M}f)] = \mathcal{M}^{-1}\mathcal{M}(\phi \star f) = \phi \star f$$

for all $f \in C_0^\infty(\mathbb{R}_+)$, and the result follows.

The above construction gives many classical integral operators in the algebra $\mathscr{A}$.

Example 6.2. The algebra $\mathscr{A}$ contains the following operators:

(a) The Hardy operator B_α, $\operatorname{Re}\alpha > -1/2$, given by

$$(B_\alpha f)(x) := \int_x^\infty \left(\frac{x}{t}\right)^\alpha \frac{1}{t}f(t)\,dt;$$

the corresponding symbol is $\theta := \mathcal{M}(x^\alpha \chi_{[0,1]}) = \left(-iz + \alpha + \frac{1}{2}\right)^{-1}$.

(b) The Carleman operator $\mathcal{C}$ given by

$$(\mathcal{C}f)(x) := \frac{1}{\pi}\int_0^\infty \frac{f(t)}{x+t}\,dt;$$

the corresponding symbol is $\theta := \pi^{-1}\mathcal{M}\big(1/(1+x)\big)$. Recalling the formula [21], Equation 3.241(2),

$$\frac{1}{\pi}\int_0^\infty \frac{x^{\mu-1}}{1+x}\,dx = \frac{1}{\sin \pi\mu}, \qquad 0 < \operatorname{Re} < 1,$$

we get $\theta(z) = 1/\cosh(\pi z)$.

6.3. Factorization in the algebra $\mathscr{A}$. We set $\mathscr{A}^+ := \mathscr{A} \cap \mathscr{B}^+$ and $\mathscr{A}^- := \mathscr{A} \cap \mathscr{B}^-$. The operators in $\mathscr{A}^\pm$ are characterized by the following property.

Lemma 6.3. *For $\theta \in L_\infty(\mathbb{R})$, the operator $\hat{M}_\theta$ is in $\mathscr{A}^+$ if and only if θ belongs to the Hardy space H^∞ of functions that are bounded and analytic in the upper complex half-plane $\mathbb{C}^+$. Analogously, $\hat{M}_\theta \in \mathscr{A}^-$ if and only if θ belongs to the Hardy space $H^{-,\infty}$ of functions that are bounded and analytic in the lower complex half-plane $\mathbb{C}^-$.*

Proof. We only give the proof for $\mathscr{A}^+$. Assume that $\hat{M}_\theta \in \mathscr{A}^+$. Since the Hardy operator B_0 of Example 6.2(a) belongs to $\mathscr{A}^+$, we get $B_0\hat{M}_\theta \in \mathscr{A}^+$. Clearly, $B_0\hat{M}_\theta = \hat{M}_{\theta_1}$

with

$$\theta_1(z) = \frac{i\theta(z)}{z + i/2} \in L_2(\mathbb{R});$$

therefore the function $\phi := \mathcal{M}^{-1}\theta_1$ is in $L_2(\mathbb{R}_+)$. Fix an arbitrary $a > 0$ and take $f \in L_2(\mathbb{R}_+)$ with support in the interval $(0, a)$. By the definition of $\mathscr{A}^+$, we get

$$\int_0^a \phi\left(\frac{x}{t}\right)\frac{1}{t} f(t)\,dt = 0$$

for all $x > a$. This implies that $\phi(\xi) = 0$ for a.e. $\xi > 1$, whence $\theta_1 := \mathcal{M}\phi$ belongs to the space H^2. As a result, the function θ is analytic in the upper half-plane $\mathbb{C}^+$, whence $\theta \in H^\infty$.

Assume now that $\theta \in H^\infty$. If in addition $\theta \in L_2(\mathbb{R})$, then the function $\phi := \mathcal{M}^{-1}\theta$ belongs to $L_2(\mathbb{R}_+)$ and has its support in the interval $[0, 1]$. Therefore

$$(\hat{M}_\theta f)(x) = (K_\phi f)(x) = \int_x^\infty \phi\left(\frac{x}{t}\right)\frac{1}{t} f(t)\,dt$$

for all $f \in L_2(\mathbb{R}_+)$, so that $\hat{M}_\theta \in \mathscr{A}^+$. A generic $\theta \in H^\infty$ is the limit in H^∞ of the sequence $(\theta_n) \in H^\infty \cap L_2(\mathbb{R})$ with

$$\theta_n(z) = \frac{in\theta(z)}{z + in}.$$

Since $\hat{M}_{\theta_n}$ belong to $\mathscr{A}^+$ for all $n \in \mathbb{N}$ and converge in $\mathscr{A}$ to $\hat{M}_\theta$ as $n \to \infty$, the closedness of $\mathscr{A}^+$ yields that $\hat{M}_\theta$ belongs to $\mathscr{A}^+$ as claimed. □

Assume that $\theta \in L_\infty$ is such that $1/\theta$ also belongs to L_∞. Assume also that the Riemann–Hilbert factorization problem for θ is soluble, i.e., that

$$\theta = \theta_1\overline{\theta_2} \tag{6.1}$$

for some functions θ_1 and θ_2 in H^∞. Since θ_1 and θ_2 must be essentially bounded away from zero on the real line, we conclude that $1/\theta_1$ and $1/\theta_2$ also belong to H^∞. Therefore

$$\hat{M}_\theta = \hat{M}_{\theta_1}\left(\hat{M}_{\theta_2}\right)^*, \tag{6.2}$$

and the operators $\hat{M}_{\theta_1}$ and $\left(\hat{M}_{\theta_2}\right)^*$ belong respectively to $\mathscr{A}_2^+$ and $\mathscr{A}^-$ and are boundedly invertible there. Conversely, equation (6.2) clearly implies (6.1). Thus the two problems, the Riemann–Hilbert factorization problem for the function θ and the canonical factorization problem in $\mathscr{A}$ for the operator $\hat{M}_\theta$, are equivalent.

6.4. Factorization of the operator $I + F$. Consider now the problem of canonical factorization in the algebra $\mathscr{A}$ of the operator $A_\kappa := I + F$, i.e.,

$$A_\kappa = I - \sin(\pi\kappa)\mathcal{C},$$

where $\mathcal{C}$ is the Carleman operator of Example 6.2(b). By the above, $A_\kappa = \hat{M}_{\theta_\kappa}$, where

$$\theta_\kappa(z) = 1 - \frac{\sin \pi\kappa}{\cosh \pi z}.$$

Set

$$\eta_\kappa^+(z) := \frac{\Gamma\left(\frac{1}{4} - \frac{i}{2}z\right)\Gamma\left(\frac{3}{4} - \frac{i}{2}z\right)}{\Gamma\left(\frac{1}{4} - \frac{i}{2}z - \frac{\kappa}{2}\right)\Gamma\left(\frac{3}{4} - \frac{i}{2}z + \frac{\kappa}{2}\right)}$$

and $\eta_\kappa^-(z) = \eta_\kappa^+(-z) = \overline{\eta_\kappa^+(\bar{z})}$. By Corollary 3.3, the function η_κ^+ belongs to the Hardy space H^∞ in the upper half-plane and thus $\overline{\eta_\kappa^-} \in H^\infty$. Moreover, using the identity

$$\Gamma\left(\frac{1}{2} + z\right)\Gamma\left(\frac{1}{2} - z\right) = \frac{\pi}{\cos \pi z},$$

we find that

$$\begin{aligned}\eta_\kappa^+(z)\eta_\kappa^-(z) &= \frac{\cos\pi\left(\frac{i}{2}z + \frac{1}{4} + \frac{\kappa}{2}\right)\cos\pi\left(\frac{i}{2}z - \frac{1}{4} - \frac{\kappa}{2}\right)}{\cos\pi\left(\frac{i}{2}z + \frac{1}{4}\right)\cos\pi\left(\frac{i}{2}z - \frac{1}{4}\right)}\\ &= \frac{\cosh \pi z - \sin \pi\kappa}{\cosh \pi z}\\ &= \theta_\kappa(z).\end{aligned}$$

As a result, we arrive at the canonical factorization of A_κ in the form

$$A_\kappa = \hat{M}_{\eta_\kappa^+}\hat{M}_{\eta_\kappa^-} = \hat{M}_{\eta_\kappa^+}(\hat{M}_{\eta_\kappa^+})^*.$$

Recalling (3.8), we see that η_κ^+ is the symbol of the transformation operator $I + L$; whence $\hat{M}_{\eta_\kappa^+} = I + L$, and we have the equality

$$I + F_\kappa = (I + L)(I + L^*),$$

which is equivalent to the Marchenko equation

$$k(x,t) + f(x+t) + \int_x^\infty k(x,s)f(s+t)\,ds = 0, \quad x < t,$$

for the kernel k of the transformation operator $I + K = (I + L)^{-1}$.

6.5. Reconstruction of the potential. The inverse scattering problem consists in reconstructing the Schrödinger operator from its scattering function S. Given S as in (4.3), we form its distributional Fourier transform $f(x) = -\sin(\pi\kappa)/x$, denote by F the integral operator in $L_2(\mathbb{R}_+)$ with kernel $f(x+t)$, and then factorize the operator $I + F$ as explained in Subsection 6.4. This way we get kernels k and l of the corresponding transformation operators discussed in Section 3, and at the final stage of the reconstruction algorithm, we set

$$q(x) := -2\frac{d}{dx}k(x,x) = 2\frac{d}{dx}l(x,x).$$

By the above, we have

$$l(x,x) = -P'_\kappa(1)\frac{1}{x};$$

recalling the normalization of the Legendre function P_κ, we see that

$$q(x) = 2\frac{\kappa(\kappa+1)}{2}\frac{1}{x^2} = \frac{\kappa(\kappa+1)}{x^2}$$

as it should be. This completes the solution of the inverse scattering problem.

7. Approximations by half-regular potentials

In the scattering theory for Schrödinger operators with potentials in the Faddeev–Marchenko class the corresponding scattering functions S are continuous on the whole line and $1 - S$ are square integrable [15]. Both these properties do not hold for the operator H_κ. Indeed, as we have seen in Section 4, the scattering function S of H_κ is piecewise constant; in particular, it has a jump discontinuity at $\omega = 0$ and $1 - S$ takes nonzero constant values for positive and negative ω.

The Bessel potential $\kappa(\kappa+1)/x^2$ is singular both at the origin and at infinity (in the sense that it does not decay sufficiently fast there). The purpose of this section is to demonstrate that the discontinuity of the scattering function S at the origin is caused by the behaviour of the potential at infinity and, conversely, the behaviour of S at infinity is influenced mainly by the singularity of the potential at the origin. To do this, we consider two model examples of Schrödinger operators $H_{\kappa,0,n}$ and $H_{\kappa,1,n}$ with potentials

$$q_{0,n}(x) := \chi_{[1/n,\infty)}(x)\kappa(\kappa+1)/x^2$$

and

$$q_{1,n}(x) := \chi_{(0,n]}(x)\kappa(\kappa+1)/x^2,$$

which are regular respectively at the origin and at infinity. We shall show that the scattering functions $S_{0,n}$ and $S_{1,n}$ of $H_{\kappa,0,n}$ and $H_{\kappa,1,n}$ have the following properties:

(a) $S_{0,n}$ has the limit values $S_{0,n}(\pm 0) = e^{\mp i\pi\kappa}$ and $1 - S_{0,n}$ belongs to $L_2(\mathbb{R})$.

(b) $S_{1,n}$ is continuous on the whole line and the limits at $\pm\infty$ exist and are equal to $e^{\mp i\pi\kappa}$.

We remark that the standard scaling arguments will also enable us to show that, as $n \to \infty$, the scattering functions $S_{j,n}$, $j = 0, 1$, converge pointwise to S. Indeed, if y is a solution of the equation

$$-y'' + q_{0,n} y = \omega^2 y,$$

then $y_1(x) := y(xn)$ solves the equation

$$-y'' + q_{0,1} y = (\omega/n)^2 y.$$

Therefore $S_{0,n}(\omega) = S_{0,1}(\omega/n)$ and

$$S_{0,n}(\omega) \to \begin{cases} S_{0,1}(+0) = e^{-i\pi\kappa}, & \omega > 0, \\ S_{0,1}(-0) = e^{i\pi\kappa}, & \omega < 0, \end{cases}$$

pointwise as $n \to \infty$. Similarly, the pointwise convergence of $S_{1,n}$ to S follows from the relation $S_{1,n}(\omega) = S_{1,1}(n\omega)$ and the behaviour of $S_{1,1}$ at infinity.

7.1. Approximation by potentials regular at the origin. By the above scaling arguments, it only suffices to study the operator $H_{\kappa,0,1}$. The Jost solution $e_{0,1}(x,\omega)$ coincides with the Jost solution $e(x,\omega)$ of the operator H_κ for $x > 1$ and equals

$$A(\omega) \sin \omega(x-1) + B(\omega) \cos \omega(x-1)$$

for $x \in [0,1]$. Equating the limit values at 1 from both sides for the function $e_{0,1}$ and its derivative, we conclude that

$$e_{0,1}(x,\omega) = \begin{cases} e'(1,\omega) \dfrac{\sin \omega(x-1)}{\omega} + e(1,\omega) \cos \omega(x-1), & x \in [0,1], \\ e(x,\omega), & x > 1. \end{cases}$$

Thus

$$e_{0,1}(0,\omega) = -e'(1,\omega) \sin \omega/\omega + e(1,\omega) \cos \omega$$

is a continuous function outside the origin. Since the functions $e_{0,1}(0,\omega)$ and $e_{0,1}(0,-\omega)$ are linearly independent if $\omega \neq 0$ and $e_{0,1}(0,-\omega) = \overline{e_{0,1}(0,\omega)}$, the function $e_{0,1}(0,\omega)$ never vanishes for real nonzero ω. Therefore the scattering function

$$S_{0,1}(\omega) = e_{0,1}(0,-\omega)/e_{0,1}(0,\omega)$$

is continuous for $\omega \neq 0$.

Using the asymptotics of the Bessel functions and their derivatives at the origin and recalling formulae (A.3) and (4.1), we conclude that $e_{0,1}(0,\omega) = C\omega^{-\kappa}\big(1 + o(1)\big)$ as $\omega \to +0$ for a constant C independent of ω. Therefore we find that $S_{0,1}(\omega) \to e^{-i\pi\kappa}$ as $\omega \to +0$ and $S_{0,1}(\omega) \to e^{i\pi\kappa}$ as $\omega \to -0$, cf. Section 4.

To derive the behaviour of $S_{0,1}$ at infinity, we use the representation

$$e(x,\omega) = e^{i\omega x} + \int_x^\infty v\left(\frac{x}{t}\right)\frac{e^{i\omega t}}{t}\,dt$$
$$= e^{i\omega x} - ia_0\int_x^\infty \left(\frac{x}{t}\right)^{-\kappa}\frac{e^{i\omega t}}{t}\,dt + \int_x^\infty \left(\frac{x}{t}\right)^{1+\kappa}\tilde v\left(\frac{x}{t}\right)\frac{e^{i\omega t}}{t}\,dt$$

of the Jost function of the operator H via the transformation operator $I+K$, see Section 3. Since the function v belongs to $L_2(0,1)$, the above integral exists as a Fourier transform of the L_2-function $v(x/t)/t$ of the variable t and thus $e(1,\omega) = e^{i\omega} + g_1(\omega)$ for some $g_1 \in L_2(\mathbb{R})$. Moreover, since $\tilde v$ is continuous over $[0,1]$, we see that g_1 is a continuous function that tends to zero as $\omega \to \infty$.

Next, differentiation and integration by parts yields

$$e'(1,\omega) = i\omega e^{i\omega} - v(1)e^{i\omega} + \int_1^\infty v'\left(\frac{1}{t}\right)\frac{e^{i\omega t}}{t^2}\,dt$$
$$= i\omega e^{i\omega} - \int_1^\infty v\left(\frac{1}{t}\right)\left[i\omega - \frac{2}{t}\right]\frac{e^{i\omega t}}{t^2}\,dt;$$

since the function $t^{-2}v(1/t)$ belongs to $L_1(1,\infty)$, we find that

$$e'(1,\omega) = i\omega[e^{i\omega} + g_2(\omega) + o(\omega^{-1})]$$

as $\omega \to \pm\infty$ for some $g_2 \in L_2(\mathbb{R})$ that is continuous and vanishes at infinity by the Riemann–Lebesgue lemma. Therefore,

$$e_{0,1}(0,\omega) = 1 + g(\omega) + o(\omega^{-1})$$

for $g(\omega) := \cos\omega g_1(\omega) + i\sin\omega g_2(\omega) \in L_2(\mathbb{R})$ that vanishes at infinity, and it follows that $1 - S_{0,1}$ belongs to $L_2(\mathbb{R})$.

The above considerations are summarized in the following proposition.

Proposition 7.1. *The scattering function $S_{0,1}$ is continuous outside the origin, assumes the limit values $\lim_{\omega\to\pm 0} S_{0,1}(\omega) = e^{\mp i\pi\kappa}$, and $1 - S_{0,1}$ belongs to $L_2(\mathbb{R})$.*

7.2. Approximation by potentials regular at infinity. As explained earlier, it suffices to only consider the potential $q_{1,1}$. Since it vanishes for $x > 1$, the Jost solution $e_{1,1}(x,\omega)$ coincides there with $e^{i\omega x}$. For $x \in (0,1)$, $e_{1,1}(\cdot,\omega)$ is a linear combination $A(\omega)\phi(x,\omega) + B(\omega)\psi(\cdot,\omega)$ of the special solutions ϕ and ψ of the Bessel equation (A.2). Equating the values of the two expressions for $e_{1,1}$ and its derivative at $x = 1$, we find that

$$A(\omega) = \frac{\pi}{\cos\pi\kappa}\frac{e^{i\omega}}{2\omega}[i\omega\psi(1,\omega) - \psi'(1,\omega)],$$
$$B(\omega) = \frac{\pi}{\cos\pi\kappa}\frac{e^{i\omega}}{2\omega}[\phi'(1,\omega) - i\omega\phi(1,\omega)].$$

By definition, the scattering function $S_{1,1}(\omega)$ is defined by the requirement that the linear combination $e_{1,1}(x,-\omega) + S_{1,1}(\omega)e_{1,1}(x,\omega)$ should satisfy the boundary condition at $x = 0$. We thus find that

$$S_{1,1}(\omega) = \lim_{x\to+0} \frac{x^\kappa e_{1,1}(x,-\omega)}{x^\kappa e_{1,1}(x,\omega)}$$

$$= \frac{B(-\omega)}{B(\omega)} \lim_{x\to+0} \frac{x^\kappa \psi(x,-\omega)}{x^\kappa \psi(x,\omega)} = \frac{B(-\omega)}{B(\omega)} S(\omega).$$

Recalling the asymptotics of the special solutions ϕ and ψ of the Bessel equation (A.2) at the origin and at infinity (see Appendix A.2), we arrive at the following conclusion.

Proposition 7.2. *The scattering function $S_{1,1}$ is continuous on the whole line, assumes the value -1 at the origin, and tends to $e^{\mp i\pi\kappa}$ at $\pm\infty$.*

8. Conclusions

We showed that the classical direct and inverse scattering theory for Schrödinger operators on the semi-axis can be successfully extended to operators H_κ, $\kappa \in \left[-\frac{1}{2}, \frac{1}{2}\right)$, with Bessel-type potentials $\kappa(\kappa+1)/x^2$. In particular, we constructed transformation operators, Jost solutions, scattering function S, derived the Marchenko equation and demonstrated that its solution reconstructs the potential we have started with. Here we have come across a new phenomenon that the scattering function S is no longer continuous but rather has two jump discontinuities, one at the origin and the other at infinity. The jump at the origin is in some sense caused by the behaviour of the potential at infinity, while the behaviour of S at infinity is determined by the singularity of the potential at the origin.

In the problem considered here all the objects have an explicit form in terms of special functions and "classical" operators of analysis. It gives, however, an insight into more general situations; in particular, it suggests that both the direct and inverse scattering theory can be further developed for perturbations of the Bessel-type potentials we have considered, e.g., for operators generated by the differential expressions

$$-\left(\frac{d}{dx} - \frac{\kappa}{x} + v\right)\left(\frac{d}{dx} + \frac{\kappa}{x} - v\right)$$

with suitable v. Our model gives a hint of what should be expected in such a more general case, which will be discussed elsewhere.

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Appendix A. Bessel functions and Hankel transform

A.1. Asymptotics of Bessel functions. The Bessel function J_ν of the first kind and order ν is a particular solution of the Bessel equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\nu^2)y=0 \tag{A.1}$$

and is given by the convergent series

$$J_\nu(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+\nu+1)}\left(\frac{z}{2}\right)^{2n+\nu},$$

where Γ is the Euler Gamma function. It is an entire function of z and obeys the following asymptotics:

$$J_\nu(x)\asymp\begin{cases}\sqrt{\dfrac{2}{\pi x}}\cos\left(x-\dfrac{\pi}{2}\nu-\dfrac{\pi}{4}\right), & x\to+\infty,\\ \left(\dfrac{x}{2}\right)^\nu\dfrac{1}{\Gamma(\nu+1)}, & x\to 0.\end{cases}$$

For non-integer ν, $J_{-\nu}$ is a solution of (A.1) that is linearly independent of J_ν.

A.2. Special solutions of (2.2). If y is a solution of (A.1), then $u(x):=\sqrt{\omega x}\,y(\omega x)$ solves the equation

$$-u''+\frac{\nu^2-\frac{1}{4}}{x^2}u=\omega^2u. \tag{A.2}$$

Therefore for $\kappa\in\left(-\frac{1}{2},\frac{1}{2}\right)$ the functions

$$\phi_\kappa(x,\omega):=\sqrt{\omega x}J_{\kappa+1/2}(\omega x),\quad \psi_\kappa(x,\omega):=\sqrt{\omega x}J_{-\kappa-1/2}(\omega x) \tag{A.3}$$

form a basis of solutions for the equation (2.2). They obey the following asymptotics:

$$\phi_\kappa(x,\omega)\asymp\begin{cases}\sqrt{\dfrac{2}{\pi}}\sin\left(\omega x-\dfrac{\pi}{2}\kappa\right), & x\to+\infty,\\ (\omega x)^{\kappa+1}\dfrac{1}{2^{\kappa+\frac12}\Gamma\left(\kappa+\dfrac{3}{2}\right)}, & x\to+0;\end{cases}$$

$$\psi_\kappa(x,\omega)\asymp\begin{cases}\sqrt{\dfrac{2}{\pi}}\cos\left(\omega x+\dfrac{\pi}{2}\kappa\right), & x\to+\infty,\\ (\omega x)^{-\kappa}\dfrac{2^{\kappa+\frac12}}{\Gamma\left(-\kappa+\dfrac{1}{2}\right)}, & x\to+0.\end{cases}$$

For $\kappa = -1/2$, the singular solution $\psi_{-1/2}$ is given by $\psi_{-1/2}(x,\omega) = \sqrt{\omega x}\, Y_0(\omega x)$; the asymptotics at infinity remains the same, while at the origin we get

$$\psi_{-1/2}(x,\omega) \asymp \sqrt{\omega x}\log(\omega x/2), \quad x \to +0.$$

A.3. The Hankel transform. For positive ω, $\phi_\kappa(\cdot,\omega)$ are generalized eigenfunctions of the operator H_κ corresponding to the point ω^2 in the continuous spectrum. They generate the integral transform $\mathcal{J}_\kappa$ in $L_2(\mathbb{R}^+)$ called the Hankel transform, given by

$$(\mathcal{J}_\kappa f)(\omega) := \int_0^\infty \phi_\kappa(x,\omega) f(x)\, dx,$$

where the integral here is understood as

$$\operatorname*{L.i.m.}_{N\to\infty} \int_0^N \phi_\kappa(x,\omega) f(x)\, dx.$$

It is well known [11], [14] that $\mathcal{J}_\kappa$ is a unitary operator in $L_2(\mathbb{R}_+)$. This follows from the so-called distributional "closure relation":

$$\int_0^\infty \phi_\kappa(x,\omega_1)\phi_\kappa(x,\omega_2) = \delta(\omega_1 - \omega_2),$$

where δ denotes the Dirac delta function and ω_1 and ω_2 are arbitrary positive numbers.

We also notice that $\mathcal{J}_\kappa$ diagonalizes the operator H_κ, i.e., that

$$(\mathcal{J}_\kappa H_\kappa f)(\omega) = \omega^2 (\mathcal{J}_\kappa f)(\omega),$$

see [14].

Appendix B. The Mellin transform

The *Mellin transform* $\mathcal{M}$ is a linear mapping from $L_2(\mathbb{R}_+)$ into $L_2(\mathbb{R})$ given by the formula

$$(\mathcal{M}f)(z) = \int_0^\infty t^{-iz-1/2} f(t)\, dt. \tag{B.1}$$

The operator $(2\pi)^{-1/2}\mathcal{M}$ is unitary and the inverse Mellin transform is given by

$$(\mathcal{M}^{-1}g)(t) = \frac{1}{2\pi}\int_{\mathbb{R}} t^{iz-1/2} g(z)\, dz.$$

Clearly, we have $(\mathcal{M}f')(z) = -\left(iz + \frac{1}{2}\right)(\mathcal{M}f)(z-i)$. If we set

$$(f \star g)(x) := \int_0^\infty f\left(\frac{x}{t}\right)\frac{1}{t} g(t)\, dt$$

for $f, g \in C_0^\infty(\mathbb{R}_+)$, then one can verify directly that

$$\mathcal{M}(f \star g) = (\mathcal{M}f) \cdot (\mathcal{M}g), \tag{B.2}$$

so that the operation $\star$ plays the same role for the Mellin transform as the usual convolution does for the Fourier transform.

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