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A priori bounds for a class of nonlinear elliptic equations and applications to physical problems

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Synopsis

Upper and lower bounds for the solutions of a nonlinear Dirichlet problem are given and isoperimetric inequalities for the maximal pressure of an ideal charged gas are constructed. The method used here is based on a geometrical result for two-dimensional abstract surfaces.

1. Introduction

Let $D\subset\mathbb{R}^2$ be a *simply connected* domain and $x=(x_1, x_2)$ be a generic point. The starting-point for our investigations is problems of the type

$$\Delta w(x)+\lambda(x)e^{w(x)}=0 \text{ in } D, \quad w(x)=\varphi(x) \text{ on } \partial D. \tag{1.1}$$

If v is the harmonic function in D coinciding with φ on the boundary, then (1.1) can be written as

$$\Delta u(x)+\lambda(x)e^{v(x)+u(x)}=0 \text{ in } D, \quad u(x)=0 \text{ on } \partial D, \tag{1.2}$$

where $u=w-v$.

In this paper we shall give bounds for the solutions of (1.2) requiring $\lambda(x)$ and $v(x)$ to satisfy the conditions

(C1) $$\lambda(x)\leqq\lambda_0$$

(C2) $$\Delta v(x)\geqq 0 \text{ in } D.$$

We shall transform (1.2) into an integral equation and use a method already developed in [**3**] for the particular case $\lambda(x)\equiv\lambda_0>0$ and $v\equiv 0$. It is based on the introduction of a special coordinate system defined by the level lines of the Green's function. The results are related to a theorem of Pólya and Szegö [**10**] concerning the warping function. It states that the solution of the problem $\Delta u+1=0$ in D, $u=0$ on ∂D satisfies $\max u\geqq \dot{R}^2/4$, where $\dot{R}$ is the maximal conformal radius. An extension to higher dimensions has been given by Payne [**8**] and a generalization is found in [**4**]. We then use our results to derive bounds for the pressure of a charged gas in a container [**7**]. It turns out that there exists a sharp estimate for the maximum value of the pressure in terms of the total mass and the geometrical quantity $\dot{R}$. As a further application we study a nonlinear Dirichlet problem arising in combustion theory [**6**]. In particular we are interested in the behaviour of the solutions at interior points. We also add a little discussion on the loci where the solutions attain their maxima. It shows some typical difficulties of this class of problems.

2. General inequalities

The first step is to transform the partial differential equation (1.2) into an integral equation. Let $g(x, y)$ be the Green's function of the Laplace operator. It is of the form

$$g(x, y)=\tfrac{1}{2}\pi \log (R_x/|x-y|)+H(x, y), \tag{2.1}$$

where H is determined such that for fixed $y \in D$

(i) $g(., y)$ vanishes on ∂D

(ii) $H(., y)$ is continuous in $\bar{D}$ and harmonic in D

(iii) $\lim_{x\to y} H(x, y)=0$.

R_x is called the *conformal radius* of x with respect to D.

With the help of the Green's function (1.2) can be written as

$$u(x)=\int_D g(x, y)\lambda(y)e^{v(y)+u(y)}\,dy \tag{2.2}$$

where $dy = dy_1\, dy_2$. Let us introduce the following notation:

$$D(t):=\{y\in D\colon g(x,.)\geqq t\},\quad a(t):=\int_{D(t)} e^{u(y)+v(y)}\,dy.$$

$a(t)$ can be interpreted as the area of $D(t)$ in the Riemannian metric $d\mathfrak{S}^2 = e^{u+v}\,ds^2$, where ds denotes Euclidean arc length.

The next lemma will be the key for all our investigations. It is based on a geometrical result and is already contained in [**3**]. For the sake of completeness we shall repeat it here.

LEMMA 2.1. *Under the conditions* (C1) *and* (C2), *the function* $m(t):= e^{-4\pi t}(1/a(t)-\lambda_0/(8\pi))$ *is non-decreasing.*

Proof. We observe that $D(t)$ is homeomorphic to a circle and that by the strong maximum principle $|\text{grad}\, g(x,.)|$ does not vanish on $\partial D(t)$. Hence the following formula holds [**5**]

$$\frac{da}{dt}=-\oint_{\partial D(t)} e^{u+v}\,|\text{grad}\, g(x, y)|^{-1}\,ds_y. \tag{2.3}$$

By the Schwarz inequality

$$\oint_{\partial D(t)} e^{u+v}\,|\text{grad}\, g|^{-1}\,ds_y \oint_{\partial D(t)} |\text{grad}\, g|\,ds_y \geqq \left\{\oint_{\partial D(t)} e^{(u+v)/2}\,ds_y\right\}^2. \tag{2.4}$$

Since $\oint_{\partial D(t)} |\text{grad}\, g|\,ds_y = 1$, (2.3) and (2.4) imply

$$-\frac{da}{dt}\geqq\left\{\oint_{\partial D(t)} e^{(u+v)/2}\,ds_y\right\}^2. \tag{2.5}$$

In view of the conditions (C1) and (C2) the Gaussian curvature $K = -\Delta(u+v)/2\exp(u+v)$ of the metric $d\mathfrak{S}^2 := \exp(u+v)\,ds^2$ is bounded from above by $\lambda_0/2$. Alexandrov showed that under these assumptions the following inequality holds between the length $L(t) := \oint_{\partial D(t)} d\mathfrak{S}$ of $\partial D(t)$ and the area $a(t)$ of $D(t)$ **[1]**

$$L^2(t) \geqq 4\pi a(t) - \frac{\lambda_0}{2} a^2(t).$$

This inequality together with (2.5) yields

$$-\frac{da}{dt} \geqq 4\pi a(t) - \frac{\lambda_0}{2} a^2(t).$$

The assertion is now obvious.

From this lemma we obtain immediately the

COROLLARY 2.1. *Under the assumptions stated above we have, setting* $\beta(x) := \frac{\lambda_0}{8} R_x^2 e^{v(x)}$,

(i) $$\frac{1}{a(0)} - \frac{\lambda_0}{8\pi} \leqq m(t) \leqq \frac{\lambda_0}{8\pi\beta(x)e^{u(x)}}$$

(ii) $$a(t) \geqq \frac{8\pi}{\lambda_0}\{e^{4\pi t}/(\beta(x)e^{u(x)}) + 1\}^{-1}$$

(iii) $$\int_0^\infty a(t)\,dt \geqq \frac{2}{\lambda_0}\log(1 + \beta(x)e^{u(x)}).$$

Proof. The first statement expresses the fact that $m(0) \leqq m(t) \leqq m(\infty)$. The second assertion follows from the inequality $m(t) \leqq m(\infty)$ and the third assertion follows from the second by integration.

3. Problems with $\lambda_0 \leqq 0$

Consider the problem (1.2) subject to the conditions (C1), (C2) and

(C3) $$\lambda(x) \leqq \lambda_0 \leqq 0.$$

From (2.2) and (C1) we conclude that

$$u(x) \leqq \lambda_0 \int_D g(x, y)e^{v(y)+u(y)}\,dy. \tag{3.1}$$

If we integrate along the level curves of $g(x, .)$, we get

$$\int_D g(x, y)e^{v(y)+u(y)}\,dy = -\int_0^\infty t\,da(t) = \int_0^\infty a(t)\,dt. \tag{3.2}$$

Equation (3.2) holds irrespective of the sign of λ. Inequality (iii) of Corollary 2.1 together with (3.1) and (3.2) yields

$$u(x) \leqq 2\log(1 + \beta(x)e^{u(x)}). \tag{3.3}$$

Whence

$$e^{u(x)/2} \leqq \frac{1}{2\beta(x)} + \left\{\frac{1}{4\beta^2(x)} - \frac{1}{\beta(x)}\right\}^{\frac{1}{2}}. \tag{3.4}$$

Remarks

1. The estimate (3.4) remains valid, if we replace R_x by any lower bound, for example we could take as a lower bound $\delta(x) := \text{dist}(x, \partial D)$ [**5**].

2. Since $R_x \to 0$ as $x \to \partial D$, the right-hand side of (3.4) tends to infinity if x approaches the boundary. The estimate (3.4) is therefore very bad for points near the boundary.

4. The problem of a charged gas

In the study of the equilibrium of a uniformly charged gas in a perfectly conducting container, we are led to the following problem which we shall describe briefly. For a more detailed discussion, especially on the physical model, we refer to [**7**]. Let p be the pressure and ρ be the mass density of the gas. For an ideal gas the equation of state is of the form $p = \gamma\rho$. In this case equilibrium occurs when $w := \log p$ satisfies the differential equation $\Delta w = ce^w$, where c is a constant depending on some physical properties of the system. In equilibrium the pressure attains its maximum at the surface of the container, and it is constant there. The problem consists in determining the pressure for a given total mass of the gas. We shall restrict ourselves to containers of the form of an infinitely long cylinder with cross-section D. Mathematically we have to solve

$$\left.\begin{aligned} \Delta w &= ce^w \quad \text{in } D, \quad c > 0 \\ w &= \varphi_0 \quad \text{on } \partial D, \quad \varphi_0 \text{ being an unknown constant} \\ \int_D e^w\,dx &= M \quad \text{given.} \end{aligned}\right\} \tag{4.1}$$

M/γ corresponds to the mass of the gas per unit length. Keller [**7**] showed that for any given M there exists a unique solution w, which by the maximum principle is bounded from above by φ_0. The problem (4.1) is of form (1.2) with $\lambda(x) = \lambda_0 = -c$, $v = \varphi_0$. Thus $\beta(x) = -\frac{c}{8} R_x^2 e^{\varphi_0}$ and from Corollary 2.1(i) we then deduce

COROLLARY 4.1. *For the solution of* (4.1) *we have*

$$R_x^2 e^{w(x)} \leqq \frac{8M}{8\pi + Mc}.$$

Equality holds if D is a circle and x is taken at its centre.

This result expresses the fact already observed by Keller [**7**] that the pressure at inner points cannot be made arbitrarily large by putting more gas into the container.

A further consequence of Corollary 2.1 is the

THEOREM 4.1. *Under the same assumptions as above we have*

$$M(4\pi+\tfrac{1}{2}cM)\left\{\oint_{\partial D} ds\right\}^{-2} \leqq e^{\varphi_0} \leqq \frac{M(8\pi+Mc)}{8\pi^2\dot{R}^2}.$$

Equality holds on both sides for the circle.

Proof. The lower bound follows from Alexandrov's inequality (cf. Section 2). Since $a(0)=M$, the lower bound for $m(t)$ in Corollary 2.1(i) now gives

$$a(t) \leqq \left\{e^{4\pi t}\left(\frac{1}{M}+\frac{c}{8\pi}\right)-\frac{c}{8\pi}\right\}^{-1}. \tag{4.2}$$

By inserting this expression into (3.1) and (3.2) we get

$$e^{u(x)/2} \geqq \frac{8\pi}{8\pi+Mc} \tag{4.3}$$

which together with Corollary 4.1 yields

$$e^{\varphi_0} \leqq \frac{M(8\pi+Mc)}{8\pi^2 R_x^2}.$$

Since this inequality holds for all x, the optimal choice is to take for R_x the maximal conformal radius $\dot{R}$. This completes the proof.

For some simple regions $\dot{R}$ can be computed numerically. A table of such values is contained in [**10**].

5. Problems with $\lambda_0 \geqq 0$

Let us consider Problem (1.2) and suppose that in addition to (C1) and (C2) it satisfies

(C4) $$\lambda(x) \equiv \lambda_0 \geqq 0.$$

From (2.2) and (3.2) it then follows that

$$u(x) = \lambda_0 \int_0^\infty a(t)\, dt$$

and from Corollary 2.1(ii)

$$e^{u(x)/2} \geqq 1+\beta(x)e^{u(x)}. \tag{5.1}$$

This relation is trivial for $x \to \partial D$. Inequality (5.1) can be written in the form

$$\left\{e^{u(x)/2}-\frac{1}{2\beta(x)}\right\}^2+\frac{1}{\beta(x)}-\frac{1}{4\beta^2(x)} \leqq 0. \tag{5.2}$$

From this inequality we deduce the

THEOREM 5.1. *Let the assumptions of Section 5 hold. Then*
(i) *a necessary condition for* (1.2) *to have a solution is*

$$\lambda_0 \leqq \inf_x 2(R_x^2 e^{v(x)})^{-1},$$

(ii) *for any solution* $u(x)$

$$\sqrt{(4\beta^2(x))^{-1} - \beta^{-1}(x)} - (2\beta(x))^{-1} \leqq e^{u(x)/2} \leqq \sqrt{(4\beta^2(x))^{-1} - \beta^{-1}(x)} + (2\beta(x))^{-1}.$$

For the special case $v \equiv 0$ these results have already been proved in [**3**]. In this case all inequalities are isoperimetric in the sense that the equality sign is attained for the circle.

Remarks
1. All estimates remain valid, if R_x is replaced by any lower bound.
2. If $x \to \partial D$, then the lower bound tends to 1, and the upper bound tends to ∞.

6. Remarks on the Gelfand problem

The Gelfand equation arises in combustion and in the theory of a chemically active gas. It is of the form [**6**]

$$\Delta u + \lambda_0 e^u = 0 \text{ in } D, \quad u = 0 \text{ on } \partial D. \tag{6.1}$$

We assume $D \subset \mathbb{R}^2$ to be simply connected in order to apply the results of the last section.

From Theorem 5.1(i) with $v = 0$ it follows that a necessary condition for (6.1) to have a solution is $\lambda_0 \leqq 2\dot{R}^{-2}$. In [**2**] it has been shown that a solution always exists for $\lambda_0 \in [0, 2\pi/A]$ where A is the total area of D.

Since we have $R_x \geqq \delta(x)$, we conclude from (5.1) that

$$\delta^2(x) \leqq 8\lambda_0^{-1}\{e^{-u(x)/2} - e^{-u(x)}\}. \tag{6.2}$$

Hence, if for fixed λ_0 there exists a solution such that $\max u$ is large, then it attains its maximum at a point close to the boundary. Let us denote this point by ξ. (It is possible that there are several of them.) For constructing a lower bound for $\delta(\xi)$ we use a technique of Payne and Stakgold [**9**], which is based on the maximum principle.

LEMMA 6.1. (Payne and Stakgold). *If D is convex and u is any solution of Problem* (6.1), *then the expression* $\operatorname{grad}^2 u + 2\lambda_0 e^u$ *assumes its maximum at a critical point of u.*

Let us abbreviate $\max u$ by u_0. Then by the previous lemma

$$\max_{x \in D} \{\operatorname{grad}^2 u + 2\lambda_0 e^u\} = 2\lambda_o e^{u_0}.$$

Consequently

$$|\operatorname{grad} u(x)| \leqq \sqrt{2\lambda_0(e^{u_0} - e^{u(x)})}. \tag{6.3}$$

Suppose that the nearest point from ξ to ∂D is x_1. With du/dr we denote the derivative along the ray joining ξ and x_1. In view of (6.3)

$$\left|\frac{du}{dr}\right| \leqq |\text{grad } u| \leqq \sqrt{2\lambda_0(e^{u_0}-e^u)};$$

and

$$\int_0^{u_0} \frac{du}{\{2\lambda_0(e^{u_0}-e^u)\}^{1/2}} \leqq \delta(\xi).$$

An explicit computation yields

$$\delta(\xi) \geqq \frac{1}{\sqrt{2\lambda_0 e^{u_0/2}}} \log \frac{e^{u_0/2}+\sqrt{e^{u_0}-1}}{e^{u_0/2}-\sqrt{e^{u_0}-1}}.$$

Remark. The right-hand side tends to 0 as $u_0 \to \infty$. Therefore this estimate cannot be combined with (6.2) to yield an upper bound for u_0. The question remains open whether there exists a branch of solutions $(u(x:\lambda), \lambda)$ whose maximum norm tends to infinity for a positive value of λ.

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