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Displacements by successive rotations for vehicles subject to given constraints

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SUMMARY

The displacements of a vehicle on a plane can be subject to constraints depending on the nature of the vehicle. One can, for instance, think of the existence of a smallest turning circle for a car.

In this paper our purpose is to show, on a simple example, how such constraints can be handled. We, in fact, consider the case of a vehicle the motions of which consist of a finite sequence of rotations, each rotation being subject to the following constraints.

1) The radius of the circles along which the displacements of the vehicle take place are larger than a critical radius.
2) The centers of the successive rotations are located along a straight line defined by the geometry of the vehicle.

The mathematical analysis of this problem relies on a suitable choice of frames of reference in which the expression of the constraints is particularly simple. It is then shown that, under the above constraints, an arbitrary displacement can always be achieved by three approriate rotations.

KEYWORDS: Successive rotations; Vehicles; Constraints; Arbitrary displacements.

1. INTRODUCTION

In robotics the displacements of objects, vehicles or, more generally, systems are defined by a number of controlled axes. For a given position of the system the controlled axes allow, in general, to define a restricted family of possible trajectories in such a way that it will frequently be impossible to pass from an initial position to an arbitrary one without performing displacements along successive trajectories. For instance, an aeroplane will not be able to reach the same location in space, but with the reverse direction, with a single displacement. In this paper we will be mainly interested in the motions of vehicles on a plane.

The displacements of vehicles as, for example, mobile robots on a plane are subject to a set of constraints restricting the types of motions which can effectively be performed. These constraints can be of different natures. Some of them are imposed by the mechanical system allowing the robot to move on the plane. We can, for instance, think of a car the motions of which are naturally limited by its smallest turning circle.

Some other restrictions may result from obstacles located on the plane. In other words, there may exist domains of the plane through which the robot is not allowed to run.

It may be of interest to know to which extent a robot can, under some specified constraints, move, from a given position of the plane to an arbitrary one, with *the smallest number of displacements*. In this study we have been interested in the displacements of a vehicle on a plane which can be expressed with the help of a finite number p or rotations (performed successively). We assume that each rotation satisfies the following requirements:

1) The radius of the circles along which the displacements of the vehicle take place is larger than a critical radius R_0.

2) The centers of the successive rotations are located along straight lines defined by the geometry of the vehicle.

In the example of the displacements of a car, given above, the first requirement corresponds to the restriction imposed by the smallest turning circle, whereas the second expresses the fact that the center of rotation must be located along the straight line defined by the rear wheels.

In summary, our problem is thus to determine the least number p of rotations, satisfying conditions 1) and 2) and allowing to pass from a given position of the plane to another one, given but arbitrary.

It is easy to imagine that the above restrictions are more easily expressed in a frame of reference which occupies a special position with respect to the vehicle. We consequently devote the second section to a derivation of the general transformations allowing to pass from a fixed frame of reference to another one in which the rotation automatically fulfills the second requirement.

The effects of these constraints on these transformations are studied in the third section and a general expression, corresponding to p rotations, successively performed, is derived. Making use of this result, we show, in the fourth section, that the problem can always be solved for $p \leq 3$. Special situations corresponding to $p = 2$ are studied.

The existence of such transformations for $p = 3$ being established, we give in the fifth section a procedure allowing one to geometrically build a solution. In the last

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section, we finally consider some possible generalizations of the present work.

2. GENERAL TRANSFORMATIONS BETWEEN FRAMES OF REFERENCE

As already pointed out in the introduction our purpose in this section is to derive the form of a general transformation from a fixed frame of reference to a "moving one".

Let us recall that in many cases constraints depending on the nature of the vehicle are expressed in a simpler way in some special frames of reference directly related to the displacements and to the geometry of the vehicle.

Having in mind the derivation of such transformations we introduce the following notations:

- E^2: classical Euclidian plane (dimension 2),
- Σ: fixed frame of reference with an orthonormal basis $(\underline{e}_1, \underline{e}_2)$,
- Σ^j: frames of reference with origin $\underline{a}^{(j)}$ with respect to Σ and orthonormal basis $(\underline{e}_1^{(j)}, \underline{e}_2^{(j)})$, $j = 1, 2, \ldots, p$.

Let Q be a point of E^2 described by the vectors $\underline{x}$ with respect to Σ and $\underline{x}^{(j)}$ with respect to Σ^j. From the above definitions we have that:

$$\underline{x} = \underline{x}^{(j)} + \underline{a}^{(j)}, \quad j = 1, 2, \ldots, p. \tag{2.1}$$

We denote by D_j a rotation around the origin of Σ^j and look for the mapping P_j, of E^2 into E^2, which corresponds to D_j when considered from the fixed frame of reference Σ. We have:

$$P_j\underline{x} = D_j(\underline{x} - \underline{a}^{(j)}) + \underline{a}^{(j)}. \tag{2.2}$$

The general mapping we are interested in is obtained by successively performing the rotations $D_1, D_2, \ldots, D_p$ in the corresponding frames $\Sigma^1, \ldots, \Sigma^p$. In Σ this leads to the mapping P defined by:

$$P = P_p \cdots P_1. \tag{2.3}$$

We thus get

$$P\underline{x} = D_p \cdots D_1\underline{x} + D_p \cdots D_2(I - D_1)\underline{a}^{(1)} + \cdots + (I - D_p)\underline{a}^{(p)}. \tag{2.4}$$

As already mentioned in the introduction the frames of reference Σ^1 to Σ^p are not arbitrary but depend on the successive positions of the vehicle. In order to take this fact into account, we introduce some additional notations. We set:

- Ξ^j, frame of reference bound to the vehicle, with orthonormal bases $(\underline{\varepsilon}_1^{(j)}, \underline{\varepsilon}_2^{(j)})$, before performing the rotation by D_j,
- $\underline{b}^{(j)}$, origin of Ξ^j from Σ^j,
- $\underline{c}^{(j)}$, origin of Ξ^j from Σ.

Recalling that $\underline{a}^{(j)}$ is the origin of Σ^j from Σ, we see (Figure 1) that

$$\underline{c}^{(j)} = \underline{a}^{(j)} + \underline{b}^{(j)}. \tag{2.5}$$

In summary, the Σ^j are the frames of reference from which the displacements of the vehicle will be easily described; there origins are given by the $\underline{a}^{(j)}$.

The positions of the vehicle are described by a frame of reference bound to it. Its locations will successively correspond to the Ξ^j, the origins of which are given, from Σ, by the $\underline{c}^{(j)}$.

We note that the rotation D_j, performed in Σ^j, sends Ξ^j into Ξ^{j+1}. This implies that

$$\underline{c}^{(j+1)} = \underline{a}^{(j)} + D_j\underline{b}^{(j)}, \tag{2.6}$$

or, taking (2.5) into account, that

$$\underline{c}^{(j+1)} = \underline{c}^{(j)} - (I - D_j)\underline{b}^{(j)}. \tag{2.7}$$

One easily checks that this recursive relation allows us to derive an explicit expression for $\underline{c}^{(j)}$. We have:

$$\underline{c}^{(j)} = \underline{c}^{(1)} + (D_1 - I)\underline{b}^{(1)} + (D_2 - I)\underline{b}^{(2)} + \cdots + (D_{j-1} - I)\underline{b}^{(j-1)}. \tag{2.8}$$

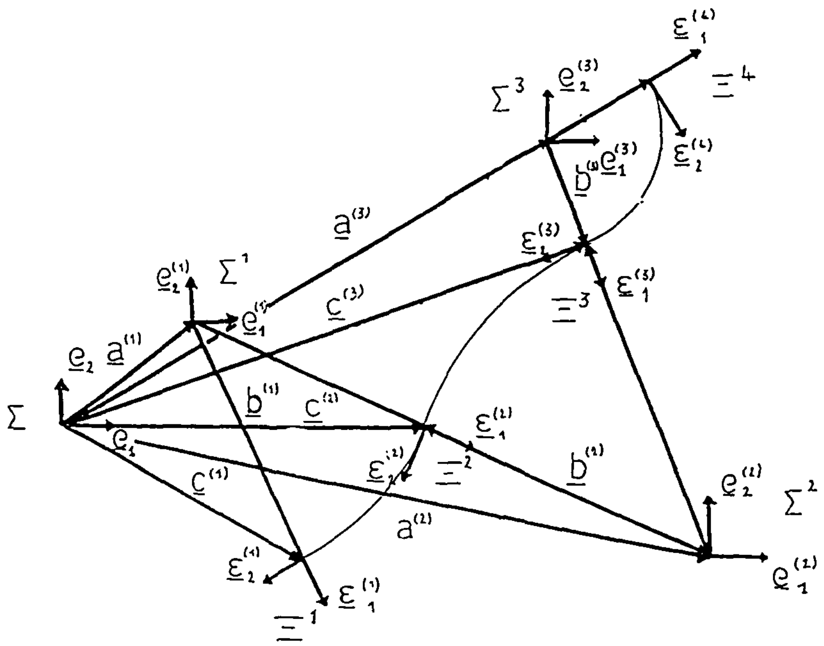

Fig. 1. Displacement corresponding to three successive rotations.

With this result we also draw from (2.5) that

$$\underset{\sim}{a}^{(j)} = \underset{\sim}{c}^{(1)} - [\underset{\sim}{b}^{(j)} + (I - D_{j-1})\underset{\sim}{b}^{(j-1)} + \cdots + (I - D_1)\underset{\sim}{b}^{(1)}]. \tag{2.9}$$

Without loss of generality we identify Σ with Ξ^1 that is we consider that, in its initial position, the frame of reference bound to the vehicle is the fixed frame of reference Σ.

From now on we will thus assume that

$$\underset{\sim}{c}^{(1)} = 0. \tag{2.10}$$

Taking (2.9) into account we obtain from (2.4) with an induction procedure the following expression for P.

$$P\underset{\sim}{x} = D_p D_{p-1} \cdots D_1 \underset{\sim}{x} - \sum_{j=1}^{p} (I - D_j)\underset{\sim}{b}^{(j)}, \tag{2.11}$$

or, with (2.8)

$$P\underset{\sim}{x} = D_p D_{p-1} \cdots D_1 \underset{\sim}{x} + \underset{\sim}{c}^{(p+1)}. \tag{2.12}$$

In fact, for $p = 1$, (2.4) reads

$$P\underset{\sim}{x} = D_1 \underset{\sim}{x} + (I - D_1)\underset{\sim}{a}^{(1)} \tag{2.13}$$

and, with (2.10), (2.9) becomes

$$\underset{\sim}{a}^{(1)} = -\underset{\sim}{b}_j^{(1)}. \tag{2.14}$$

(2.11) is thus satisfied for $p = 1$.

We assume that the result is valid for $p - 1$ and prove it for p. In order to show this we put (2.4) in the form

$$P\underset{\sim}{x} = D_p[D_{p-1} \cdots D_1 \underset{\sim}{x} + D_{p-1} \cdots D_2(I - D_1)\underset{\sim}{a}^{(1)} + \cdots + (I - D_{p-1})\underset{\sim}{a}^{(p-1)}] + (I - D_p)\underset{\sim}{a}^{(p)}, \tag{2.15}$$

or, with the induction hypothesis and (2.11),

$$P\underset{\sim}{x} = D_p\left[D_{p-1} \cdots D_1 \underset{\sim}{x} - \sum_{j=1}^{p-1} (I - D_j)\underset{\sim}{b}^{(j)}\right] - (I - D_p)\left(\underset{\sim}{b}^{(p)} + \sum_{j=1}^{p-1} (I - D_j)\underset{\sim}{b}^{(j)}\right). \tag{2.16}$$

This expression immediately leads to (2.11).

From a geometrical point of view the mapping P performs a rotation by $D_p \cdots D_1$ of the position vector $\underset{\sim}{x}$, followed by a translation which corresponds to the evolution of the origin of the frame of reference bound to the vehicle through the different rotations.

Remarks

1. Let us note the expression

$$(I - D_j)\underset{\sim}{b}^{(j)} \tag{2.17}$$

is precisely the translation allowing to pass from Ξ^j to Ξ^{j+1}.

2. The general expression for P, given by (2.11), is important for it directly depends on the $\underset{\sim}{b}^{(j)}$, that is on the relative position of Ξ^j with respect to Σ^j. We will see that the restrictions imposed on the rotations will take a particularly simple form when expressed in terms of the $\underset{\sim}{b}^{(j)}$.
3. All the results derived up to here remain valid for motions in E^3.

3. THE CONSTRAINTS

Up to here no restrictions have been enforced upon the transformations P_j defined above. Our next task will be to give a precise definition of the constraints imposed on the mapping. We require that:

$$|\underset{\sim}{b}^{(j)}| \geq R_0, \quad j = 1, 2, \ldots, p, \tag{3.1}$$

$$(\underset{\sim}{b}^{(j)}, \underset{\sim}{\varepsilon}_2^{(j)}) = 0, \; j = 1, 2, \ldots, p, \tag{3.2}$$

where R_0 is a positive number, known as the *critical radius*.

$\underset{\sim}{b}^{(j)}$ being a vector of Σ^j, denoting the origin of Ξ^j, we see that (3.1) means that the radius of the circle described by the vehicle, when moving through the rotation D_j, must be larger than the critical radius R_0. The second requirement corresponds to the fact that the direction defined by the first basis vector of the frame of reference bound to the vehicle must be identical to the direction defined by the tangent to the circle on which the displacement takes place.

In the example of the car moving on a plane, the first condition means that the turning circle must have a radius larger than R_0 whereas the second tells us that this circle must be orthogonal to the straight line defined by the axle of the rear wheel.

Condition (3.2) is not very convenient for it directly depends on the bases $(\underset{\sim}{\varepsilon}_j^{(1)}, \underset{\sim}{\varepsilon}_j^{(2)})$ of Ξ^j. In order to remove this dependence we note that, with the identification of Σ and Ξ^1, previously done, $(\underset{\sim}{\varepsilon}_j^{(1)} = \underset{\sim}{e}_j, j = 1, 2)$, by (2.12), for $p = j$,

$$P\underset{\sim}{e}_k = D_j \cdots D_1 \underset{\sim}{e}_k + \underset{\sim}{c}^{j+1} \tag{3.3}$$

or $\underset{\sim}{c}^{j+1}$ being the origin of Ξ^{j+1}.

$$\underset{\sim}{\varepsilon}_k^{(j)} = D_{j-1} D_{j-2} \cdots D_1 \underset{\sim}{e}_k, \quad j = 1, \ldots, p, \; k = 1, 2. \tag{3.4}$$

Taking account of (3.4) we thus find from (3.2) that

$$\underset{\sim}{b}^{(j)} = b_j \underset{\sim}{\varepsilon}_1^{(j)} = D_{j-1} D_{j-2} \cdots D_1 b_j \underset{\sim}{e}_1, \tag{3.5}$$

where the first equality can be considered as a definition of b_j.

Defining the vectors

$$\underset{\sim}{d}^{(j)} = -b_j \underset{\sim}{e}_1, \quad j = 1, \ldots, p. \tag{3.6}$$

We can finally write (2.11) in the form

$$P\underset{\sim}{x} = D_p \cdots D_1 \underset{\sim}{x} + \sum_{j=1}^{p} (I - D_j) D_{j-1} \cdots D_1 \underset{\sim}{d}^{(j)}. \tag{3.7}$$

By (3.5) (and (3.6)), (3.7) represents a transformation for which condition (3.2) is satisfied for each rotation; it moreover depends on vectors which are all defined in the frame of reference Σ.

The only requirement we still have to fulfill is (3.1). With (3.5) and (3.6) again it reads

$$|b_j| = |\underset{\sim}{d}^{(j)}| \geq R_0. \tag{3.8}$$

Gathering the obtained results we see that we are led to find solutions of the following problem:

Problem (P)

Let Σ and Σ' be two arbitrary frames of reference of E^2 with orthonormal bases respectively given by $(\underset{\sim}{e}_1, \underset{\sim}{e}_2)$ and $(\underset{\sim}{e}'_1, \underset{\sim}{e}'_2)$.

Find the smallest integer p for which the mapping P of E^2 into E^2 defined by

$$P\underset{\sim}{x} = D_p \cdots D_1\underset{\sim}{x} + \sum_{j=1}^{p} (I - D_j)D_{j-1} \cdots D_1\underset{\sim}{d}^{(j)}, \quad (3.9)$$

sends $\underset{\sim}{e}_k$ into $\underset{\sim}{e}'_k$, for $k = 1, 2$, under the following conditions:

- $$\underset{\sim}{d}^{(j)} = -b_j\underset{\sim}{e}_1, \quad b_j \in \mathbb{R}, \quad (3.10)$$
  where b_j satisfies
  $$|b_j| \geq R_0. \quad (3.11)$$
- $$D_j \in SO(2), \quad j = 1, \ldots, p, \quad (3.12)$$
  where $SO(2)$ is the group of orthogonal transformations with unit determinant (rotations without inversion).

4. SOLUTIONS OR PROBLEM (*P*)

In the search for solutions of the above problem, to which this section is devoted, we will use the following classical result.

Theorem 1. Every displacement of a rigid body in the Euclidian plane E^2 can be expressed by a transformation P, of E^2 into E^2 of the form

$$D\underset{\sim}{x} + \underset{\sim}{t}, \quad (4.1)$$

where $D \in SO(2)$ and $\underset{\sim}{t} \in \mathbb{R}^2$.

Proof: See for example Whittaker.[1]

We recall that a rigid body is geometrically represented by a finite domain of E^2 the shape of which is not changed by the displacements; in other words, in a frame of reference bound to the rigid body the coordinates of its points, which constitute the body, are independent of the displacements.

With this result problem (P) reduces to

Problem (P')

$D \in SO(2)$ and $\underset{\sim}{t} \in \mathbb{R}^2$ being given, find the smallest integer p for which there exists a mapping P of E^2 into F^2 defined by

$$D_p \cdots D_1\underset{\sim}{x} + \sum_{k=1}^{p} (I - D_j)D_{j-1} \cdots D_1\underset{\sim}{d}^{(j)} \quad (4.2)$$

satisfying (3.10) to (3.12),

$$D = D_p \cdots D_1 \quad (4.3)$$

and

$$\underset{\sim}{t} = \sum_{j=1}^{p} (I - D_j)D_{j-1} \cdots D_1\underset{\sim}{d}^{(j)}. \quad (4.4)$$

In the following the transformations D and D_j, $j = 1, \ldots, p$ will be identified with their corresponding matrices computed in Σ. That is

$$D = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}, \quad D_j = \begin{pmatrix} c_j & s_j \\ -s_j & c_j \end{pmatrix}, \quad j = 1, 2, \ldots, p \quad (4.5)$$

where

$$c^2 + s^2 = 1 \quad \text{and} \quad c_j^2 + s_j^2 = 1. \quad (4.6)$$

In order to solve problem (P') we will consider separately the cases $p = 2$ and $p = 3$.

I) *Case* $p = 2$

When D and $\underset{\sim}{t}$ are such that the smallest p is equal to 2, we find from (4.3) and (4.4) that

$$D = D_1D_2, \quad (4.7)$$

$$\underset{\sim}{t} = \underset{\sim}{d}^{(1)} + D_1(\underset{\sim}{d}^{(2)} - \underset{\sim}{d}^{(1)}) - D\underset{\sim}{d}^{(2)}, \quad (4.8)$$

or in components, with (3.10) ($b_j = -d_j\, j = 1, 2$) that

$$t_1 = d_1 + c_1(d_2 - d_1) - cd_2, \quad (4.9)$$

$$t_2 = -s_1(d_2 - d_1) + sd_2. \quad (4.10)$$

Taking the square of (4.9) and (4.10) we get, with (4.6),

$$(t_1 - d_1 + cd_2)^2 + (t_2 - sd_2)^2 = (d_2 - d_1)^2. \quad (4.11)$$

Two cases must be considered:

a) $D = I$, i.e. $c = 1$ and $s = 0$.

(4.11) thus becomes

$$d_1t_1 - d_2t_1 = \tfrac{1}{2}\,|\underset{\sim}{t}|^2. \quad (4.12)$$

b) $D \neq I$, i.e. $c \neq 1$.

In this case (4.11) can be written in the form

$$\left(d_1 + \frac{ct_1 - st_2}{1 - c}\right)\left(d_2 - \frac{t_1}{1 - c}\right) = -\frac{(t_1\sqrt{1 + c} - t_2\sqrt{1 - c})^2}{2(1 - c)^2}. \quad (4.13)$$

Several results can be drawn from (4.12) and (4.13) they are contained in the next theorem.

Theorem 2

i) When a displacement is reduced to a non-vanishing translation without rotation, the direction of which is different of the one given by $\underset{\sim}{e}_2$, the smallest p is equal to 2 and the equation

$$d_1 = d_2 + \frac{1}{2}\frac{|\underset{\sim}{t}|^2}{t_1} \quad (4.14)$$

always admits solutions satisfying the conditions

$$|d_j| \geq R_0, \quad j = 1, 2. \quad (4.15)$$

ii) When the displacement is not reduced to a translation and when

$$k = \frac{(t_1\sqrt{1 + c} - t_2\sqrt{1 - c})^2}{2(1 - c)^2} \quad (4.16)$$

is not vanishing, conditions

$$\left|\frac{ct_1 - st_2}{1 - c}\right| > R_0 \quad (4.17)$$

or

$$\left|\frac{t_1}{1 - c}\right| > R_0. \quad (4.18)$$

are sufficient for the existence of solutions satisfying (4.15).

iii) When the displacement is reduced to a rotation without translation (4.15) can never be fulfilled (for $p = 2$).

Proof: i) $\underset{\sim}{t}$ being non-trivial, with $t_1 \neq 0$, (4.12) yields the relation

$$d_1 = d_2 + \frac{1}{2}\frac{|\underset{\sim}{t}|^2}{t_1}. \quad (4.19)$$

We choose d_2 such that $|d_2| = R_0$ and $\operatorname{sgn} d_2 = \operatorname{sgn} t_1$. This choice implies that $|d_1| > R_0$ whence (4.15) is satisfied.

ii) Let

$$a = -\frac{ct_1 - st_2}{1-c}, \quad (4.20)$$

$$b = \frac{t_1}{1-c}, \quad (4.21)$$

With these notations (4.13) becomes

$$(d_1 - a)(d_2 - b) = -k. \quad (4.22)$$

Assuming that (4.17) is satisfied, we see from (4.22) that, k being different of naught, we can choose d_1 arbitrarily close to a and such that $|d_1| > R_0$. This implies that $|d_2| > R_0$ for an appropriate choice of d_1.

This proves the existence of solutions for which (4.15) is satisfied. The case, where (4.18) is fulfilled, is handled in the same way.

iii) When the displacement consists of a non-trivial rotation (without translation) (4.11) reduces to

$$d_1 d_2 = 0 \quad (4.23)$$

so that (4.15) can never be satisfied.

II) *Case* $p = 3$

When $p = 3$, (4.3) and (4.4) become,

$$D = D_3 D_2 D_1, \quad (4.24)$$

$$\underset{\sim}{t} = (I - D_1)\underset{\sim}{d}^{(1)} + (I - D_2)D_1\underset{\sim}{d}^{(2)} + (I - D_3)D_2D_1\underset{\sim}{d}^{(3)}. \quad (4.25)$$

Let us set:

$$\tilde{D} = D_2 D_1, \quad (4.26)$$

$$\tilde{\underset{\sim}{t}} = \underset{\sim}{t} - \tilde{D}\underset{\sim}{d}^{(3)} + D\underset{\sim}{d}^{(3)}. \quad (4.27)$$

With these notations (4.25) takes the form

$$\tilde{\underset{\sim}{t}} = \underset{\sim}{d}^{(1)} + D_1(\underset{\sim}{d}^{(2)} - \underset{\sim}{d}^{(1)}) - \tilde{D}\underset{\sim}{d}^{(2)}. \quad (4.28)$$

Comparing (4.26) and (4.28) with (4.7) and (4.8) we see that the problem for $p = 3$ is reduced to the problem for $p = 2$ but for the translation $\tilde{\underset{\sim}{t}}$ which is defined by (4.27) and consequently depends on d_3 and D_3. Our task is thus to show that it is always possible to choose d_3 and D_3 in such a way that

1) $|d_3| > R_0$.
2) The resulting problem for $\tilde{D} = D_1 D_2$ and $\tilde{\underset{\sim}{t}}$, which is a problem for $p = 2$, has a solution.

We obtain this result in two steps:

- Similarly to (4.5), we denote by $\tilde{s}$ and $\tilde{c}$ the coefficients of the matrix $\tilde{D}$ and show that the condition (4.18) can always be satisfied. In fact, with (4.27)

$$\frac{\tilde{t}_1}{1-\tilde{c}} = \frac{t_1}{1-\tilde{c}} + \frac{c-\tilde{c}}{1-\tilde{c}}d_3, \quad (4.29)$$

from which we find

$$\left|\frac{\tilde{t}_1}{1-\tilde{c}}\right| \geq \left|\frac{c-\tilde{c}}{1-\tilde{c}}\right| |d_3| - \left|\frac{t_1}{1-\tilde{c}}\right|. \quad (4.30)$$

For any value of $\tilde{c}$, such that

$$c \neq \tilde{c}, \quad \tilde{c} \neq 1. \quad (4.31)$$

Since we can thus find d_3 such that

1. $$\left|\frac{\tilde{t}_1}{1-\tilde{c}}\right| > R_0, \quad (4.32)$$

2. $$|d_3| > R_0. \quad (4.33)$$

This shows that, if $\tilde{k}$ given by (4.16), with $\tilde{t}_1$, $\tilde{t}_2$ and $\tilde{c}$ instead of t_1, t_2 and c, is non-vanishing, Theorem 2 applies.

- We now have to see that the $\tilde{k}$ just defined can effectively be chosen different from zero. From (4.16) we have that

$$\tilde{k} = \frac{(\tilde{t}_1\sqrt{1+\tilde{c}} - \tilde{t}_2\sqrt{1-\tilde{c}})^2}{2(1-\tilde{c})^2}, \quad (4.34)$$

and from (4.27) that

$$\tilde{t}_1 = t_1 - \tilde{c}d_3 + cd_3, \quad (4.35)$$

$$\tilde{t}_2 = t_2 + \tilde{s}d_3 + sd_3. \quad (4.36)$$

From (4.34) $\tilde{k} \neq 0$ means that

$$\tilde{t}_1(1+\tilde{c}) \neq \tilde{t}_2\tilde{s}, \quad (4.37)$$

or with (4.35) and (4.36) that

$$(1+\tilde{c})t_1 - t_2\tilde{s} + d_3(c - \tilde{c} - c\tilde{c} + s\tilde{s}) \neq d_3. \quad (4.38)$$

Since the only conditions on d_3 are (4.32), which is derived from (4.30), and (4.33), we see that it will always be possible to choose d_3 in such a way that (4.38) be effectively different from zero, as long as

$$(c - \tilde{c} - c\tilde{c} + s\tilde{s}) \neq 1. \quad (4.39)$$

The only conditions on $\tilde{c}$ (and $\tilde{s} = \sqrt{1-\tilde{c}^2}$) are given by (4.31) so that there always exists a possible choice of $\tilde{c}$ for which (4.39) is satisfied (if $c = 0$, then $t \neq 0$ and (4.38) is easily satisfied). With these different results we see that the solution of our problem can be obtained in the following way:

- Choose $\tilde{c}$, and thus $\tilde{s}$, in such a way that (4.31) and (4.39) are fulfilled.
- Define D_3 by
$$D_3 = \tilde{D}^{-1}D. \quad (4.40)$$
- Choose d_3 such that (4.32), (4.33) and (4.38) are satisfied.
- Compute $\tilde{\underset{\sim}{t}}$ given by (4.28).
- The condition (4.32) being verified Theorem 2 applies and possible d_1, d_2, D_1 and D_2 can be computed through formula (4.13) where t_1, t_2, s and c are replaced by $\tilde{t}_1, \tilde{t}_2, \tilde{s}$ and $\tilde{c}$.

We summarize our results in the next theorem.

Theorem 3

i) The value of the integer p defining the number of necessary rotations of problem (P) (or (P')) is always smaller or equal to 3.

ii) In the absence of translation p is always equal to 3.

Proof: i) results from the above derivation whereas ii) is a consequence of Theorem 2 and of i).

5. SOLUTIONS DERIVED BY GEOMETRICAL MEANS

Having shown that a displacement of a vehicle subject to the constraints defined by (3.1) and (3.2) can always be achieved by three rotations we show in this section that for some special choices of the rotations D_1, D_2 and D_3 geometrical solutions can be easily obtained.

We consider separately again the $p = 2$ and $p = 3$ cases.

I) *The* $p = 2$ *case*

We recall that, according to (2.10), we have, for $p = 2$, a general transformation of the form

$$P\underset{\sim}{x} = D_2 D_1 \underset{\sim}{x} - (I - D_1)\underset{\sim}{b}^{(1)} - (I - D_2)\underset{\sim}{b}^{(2)}. \quad (5.1)$$

Pure translations

We have shown that translations of a vehicle in a direction different of the one given by $\underset{\sim}{e}_2$ can always be achieved by two rotations. In our model the different positions of a vehicle are entirely defined by the unit vector bound to it which is, before the jth rotation, perpendicular to $\underset{\sim}{b}^{(j)}$.

Let $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_2'$ be two unit vectors of E^2 corresponding to two positions of a vehicle. We now show how it is possible to geometrically derive a mapping P (of the form (5.1)) sending $\underset{\sim}{e}_2$ into $\underset{\sim}{e}_2'$ and satisfying (3.1) and (3.2).

Let $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_2'$ be parallel unit vectors (not linearly dependent) which are sent into one another by a translation (not in the direction defined by $\underset{\sim}{e}_2$).

In order to build the transformation sending $\underset{\sim}{e}_2$ into $\underset{\sim}{e}_2'$ we draw a circle S_1 of radius $b_1 > R_0$, tangential to $\underset{\sim}{e}_2$ and not involving $\underset{\sim}{e}_2'$ (Figure 2). The straight line passing through the origins of $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_2'$ crosses S_1 at a point m. We now define the center C_2 of the circle S_2 as the intersection of the straight line joining m with the center C_1 of S_1 and the straight line orthogonal to $\underset{\sim}{e}_2'$; the radius b_2 of S_2 being equal to the segment from C_2 to m. This construction is the well known homothetic transformation of center m and ratio b_2/b_1.

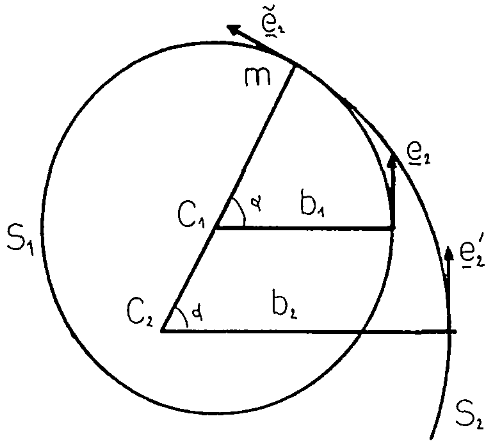

Fig. 2. $p = 2$ case. Translation in a direction which is not parallel to $\underset{\sim}{e}_2$.

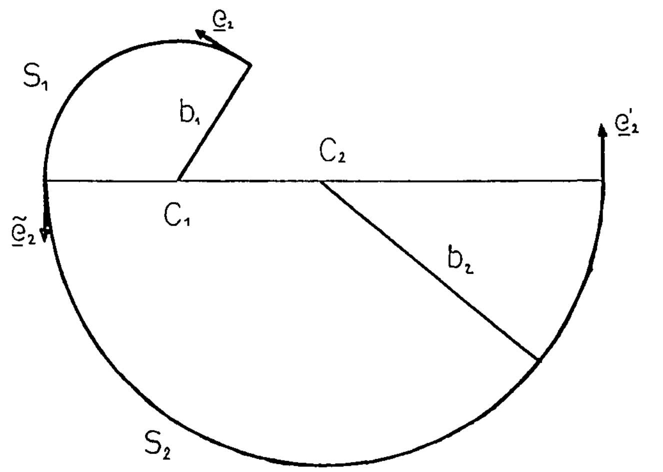

Fig. 3. $p = 2$ case. Arbitrary displacement but for two positions which are "sufficiently far away".

It is clear, by the properties of homothetic transformations, that the rotation angles corresponding to the passage from $\underset{\sim}{e}_2$ to $\tilde{\underset{\sim}{e}}_2$ and from $\tilde{\underset{\sim}{e}}_2$ to $\underset{\sim}{e}_2'$ are, but for the sign, equal. We have thus built a transformation for which b_1 and b_2 are larger than R_0 and where $D_1 = D_1^{-1}$ as it should.

It is now an easy matter to see that the transformation we have just built is the same as (5.1), for $D_2 = D_1^{-1}$.

General transformation

We have seen that, under some circumstances, when the distance between $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_2'$ is sufficiently large, $p = 2$ solutions can exist.

$\underset{\sim}{e}_2$ and $\underset{\sim}{e}_2'$ being given as in the Figure 3, we take the center C_1 of the circle S_1 as the intersection of the normals to $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_2'$; the radius of S_1 being given by the segment joining C_1 to the origin of $\underset{\sim}{e}_2$.

We define $\tilde{\underset{\sim}{e}}_2$ as the unit vector tangent to S_1 and located on the normal to $\underset{\sim}{e}_2' \cdot \tilde{\underset{\sim}{e}}_2$ and $\underset{\sim}{e}_2'$ are then prallel but of opposite directions so that there exists a circle S_2 which is tangent to these two vectors.

The process will be possible when b_1 and b_2 are larger than R_0. If this is the case the rotations D_1 and D_2 are defined by the motions of respectively $\underset{\sim}{e}_2$ to $\tilde{\underset{\sim}{e}}_2$ on S_1 and $\tilde{\underset{\sim}{e}}_2$ to $\underset{\sim}{e}_2'$ on S_2.

The correspondence with (5.1) is now immediate.

II) *The* $p = 3$ *case. Translation*

As shown in section 3 the translation of $\underset{\sim}{e}_2$, along a straight line carrying $\underset{\sim}{e}_2$ itself, has no solution for $p = 2$. We first consider this special case.

We assume that the two vectors $\underset{\sim}{e}_2$ and $\underset{\sim}{e}_2'$ are carried by a straight line, as in Figure 4. We now carry $\underset{\sim}{e}_2$ along a circle S_1, tangent to $\underset{\sim}{e}_2$, with radius $b_1 > R_0$ through a rotation of $\pi/2$. The image $\tilde{\underset{\sim}{e}}_2$ of $\underset{\sim}{e}_2$ is then rotated along a circle S_2 the center of which is defined by the intersection of the normal to $\underset{\sim}{e}_2'$, with the normal to $\tilde{\underset{\sim}{e}}_2$. The rotation performed is again of $\pi/2$ and the image $\tilde{\underset{\sim}{e}}_2'$ of $\tilde{\underset{\sim}{e}}_2$ is a unit vector, located on the normal to $\underset{\sim}{e}_2'$ and parallel to $\underset{\sim}{e}_2'$ but with opposite direction. It is then clear that $\tilde{\underset{\sim}{e}}_2$ can be carried into $\underset{\sim}{e}_2'$ by a motion along the circle S_3 which is tangent to these two vectors.

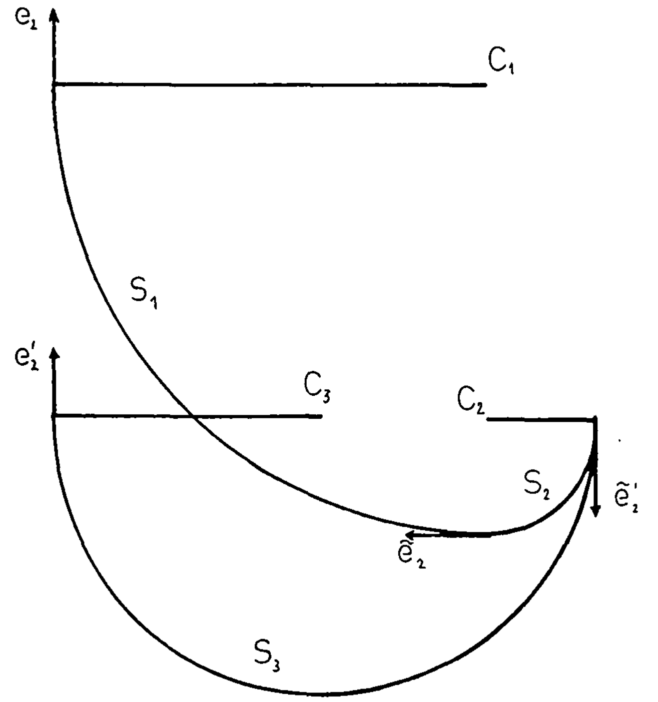

Fig. 4. $p = 3$ case. Translation in a direction parallel to $\underset{\sim}{e}_2$.

We point out that the radius b_2 and b_3 of the circles S_2 and S_3 can always be chosen larger than R_0 for an appropriate choice of b_1.

It is now an easy matter to see that this construction corresponds to the relation (2.11) for $p = 3$, $D_1 = D_2$, rotations of $\pi/2$, and D_3, rotation of $-\pi$.

General transformation

We finally consider the case of a general motion for which $D \neq I$, the translation being arbitrary. Let $\underset{\sim}{e}_2$ and

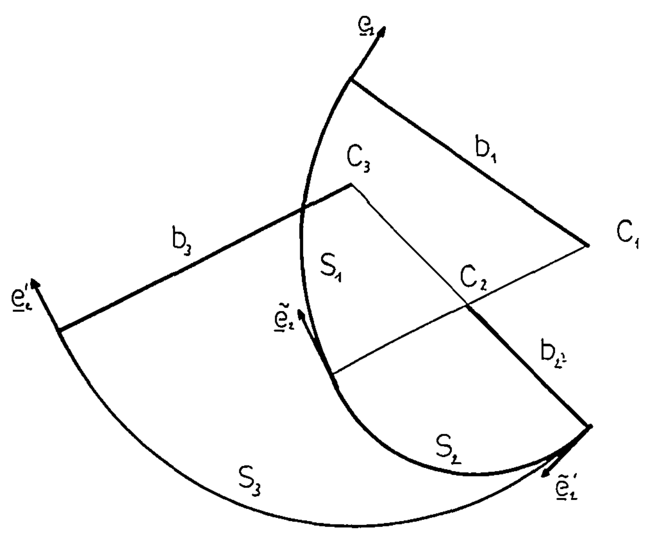

Fig. 5. Arbitrary displacement.

$\underset{\sim}{e}_2'$ be two unit vectors of E^2, not carried by parallel straight lines, as in Figure 5. By a displacement on a circle of radius $b_1 > R_0$, tangent to $\underset{\sim}{e}_2$, we send $\underset{\sim}{e}_2$ into a vector $\tilde{\underset{\sim}{e}}_2$ parallel to $\underset{\sim}{e}_2'$. If $\tilde{\underset{\sim}{e}}_2$ is not carried by the same straight line we are back to one of the $p = 2$ cases handled above.

Finally we remark that by an appropriate choice of b_1 we can always avoid the case where $\tilde{\underset{\sim}{e}}_2$ and $\underset{\sim}{e}_2'$ are carried on the same straight line.

The correspondence with (2.11) for $p = 3$ is again immediate, the rotations being such that $D = D_1$ and $D_3 = D_2^{-1}$.

6. CONCLUSIONS

Our purpose in this paper is to show, on the basis of a simple example, how motions of vehicles subject to some constraints can be handled.

It may, however, be worthwhile pointing out that the first two sections have been written in a form which can immediately be generalised in the Euclidian space E^n. From the above results it is apparent that a certain amount of freedom still remains on the choices of the possible motions. This freedom could be used to impose some additional conditions on the motions suggested by the operations a given robot is supposed to achieve. We think for example, of the presence on the plane of circular obstacles, that is of circular domains, the vehicle cannot cross, or of a choice of a path for which the covered distance is minimal.

We, however, think that such studies should not be pursued any further on general situations but must be handled in the case of some specific problems.

In this paper we have been considering operations of rotation only. It is easy to see that translations could be included without difficulties, as a limit of a rotation the angle of which is then a function of this radius tending to zero.

In fact, when the rotation angle is small enough an increase of the radius by a factor $\lambda > 1$ leads to a decrease of this angle by a factor λ^{-1}, so that the expression

$$\lim_{\infty} \left(I - D\left(\frac{\alpha}{\lambda}\right)\right)\lambda \underset{\sim}{d}, \tag{6.1}$$

where $\underset{\sim}{d}$ is a given vector and α a given rotation angle, is well defined and corresponds to a translation.

Such limits could be introduced in (3.7), allowing one to combine rotations and translations.

Reference

1. E.T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge University Press, Cambridge, UK, 1970) p. 4.