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ON THE ELEMENTARY THEORY OF PAIRS OF REAL CLOSED FIELDS. II

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§0. Introduction. Let $\mathscr{L}$ be the first order language of field theory with an additional one place predicate symbol. In [B2] it was shown that the elementary theory T of the class $\mathscr{R}$ of all pairs of real closed fields, i.e., $\mathscr{L}$-structures $\langle K, L\rangle$, K a real closed field, L a real closed subfield of K, is undecidable.

The aim of this paper is to show that the elementary theory T_s of a nontrivial subclass of $\mathscr{R}$ containing many naturally occurring pairs of real closed fields is decidable (Theorem 3, §5). This result was announced in [B2]. An explicit axiom system for T_s will be given later. At this point let us just mention that any model of T_s is elementarily equivalent to a pair of power series fields $\langle R_0((T^A)), R_1((T^B))\rangle$ where R_0 is the field of real numbers, $R_1 = R_0$ or the field of real algebraic numbers, and $B \subseteq A$ are ordered divisible abelian groups. Conversely, all these pairs of power series fields are models of T_s.

Theorem 3 together with the undecidability result in [B2] answers some of the questions asked in Macintyre [M]. The proof of Theorem 3 uses the model theoretic techniques for valued fields introduced by Ax and Kochen [A-K] and Ershov [E] (see also [C-K]). The two main ingredients are

(i) the completeness of the elementary theory of real closed fields with a distinguished dense proper real closed subfield (due to Robinson [R]),

(ii) the decidability of the elementary theory of pairs of ordered divisible abelian groups (proved in §§1–4).

I would like to thank Angus Macintyre for fruitful discussions concerning the subject. The valuation theoretic method of classifying theories of pairs of real closed fields is taken from [M].

§1. Pairs of ordered groups. By a pair of ordered groups $\mathfrak{A} = \langle A, B\rangle$ we mean an ordered abelian group A together with a distinguished subgroup B. Our first goal is the following:

THEOREM 1. *The elementary theory P of pairs of divisible ordered abelian groups is decidable.*

REMARKS. 1. The language of P of course is the language of ordered groups with an additional predicate symbol for the distinguished subgroup.

2. The theory of pairs of ordered abelian groups (not necessarily divisible) is undecidable. A proof of this will be given at the end of §4.

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From now on "group" means "divisible torsionfree abelian group". Thus a group is just a vector space over the rationals.

Definition. A pair of ordered groups $\mathfrak{A} = \langle A, B\rangle$ is called simple if

(D1) $A \neq 0$ and $B = 0$, or

(D2) $A = B \neq 0$, or

(D3) B is dense in A and $B \neq A$.

It is well known (and easy to prove) that for each i, $1 \leq i \leq 3$, the theory $P \cup (Di)$ is complete and hence decidable. The theory P will be reduced to the theories $P \cup (Di)$ and a suitable theory of ordered sets with distinguished subsets. Since the latter theory is also decidable Theorem 1 will follow.

We recall some notions from the theory of ordered groups. For more details see Fuchs [F]. A subgroup C of an ordered group A is called convex if for all $\gamma \in C$ and all $\alpha \in A$, $|\alpha| < |\gamma|$ implies $\alpha \in C$ where $|\alpha| = \max\{\alpha, -\alpha\}$. C is principal, with generator γ say, if C is the smallest convex subgroup containing γ. If C is convex then the ordering of A induces an ordering of A/C making A/C into an ordered group. By a (convex) valuation of A we mean a function w from A onto an ordered set I with a maximal element ∞ such that (i) $w(\alpha) = \infty$ if and only if $\alpha = 0$, (ii) $w(\alpha + \beta) \geq \min\{w(\alpha), w(\beta)\}$, (iii) w is convex, i.e. for all $\alpha, \beta \in A$, $|\alpha| \leq |\beta|$ implies $w(\beta) \leq w(\alpha)$. For $u \in I$ put $A(u) = \{\alpha \in A \mid w(\alpha) \geq u\}$ and for $u \in I - \{\infty\}$ put $A_>(u) = \{\alpha \in A \mid w(\alpha) > u\}$. $A(u)$ and $A_>(u)$ are convex subgroups of A. Therefore the quotient groups $A(u)/A_>(u)$ are again ordered groups. If $w(\alpha) = u$ then $\alpha > 0$ if and only if $\alpha + A_>(u) > 0$ in $A(u)/A_>(u)$.

Now let $\mathfrak{A} = \langle A, B\rangle$ be a pair of ordered groups and $w: A \to I$ a valuation. For any subgroup $A' \subseteq A$ and any convex subgroup $C \subseteq A$ put $\mathfrak{A} \upharpoonright A' = \langle A', A' \cap B\rangle$ and $\mathfrak{A}/C = \langle A/C, (B + C)/C\rangle$. Clearly both structures are pairs of ordered groups. Finally, for $u \in I - \{\infty\}$ put $\mathscr{Q}(u) = (\mathfrak{A} \upharpoonright A(u))/A_>(u)$.

§**2. The natural valuation.** Let $\mathfrak{A} = \langle A, B\rangle$ be a pair of ordered groups and let I be the set of principal convex subgroups of A. I is ordered by inverse inclusion. Define $w: A \to I$ by $w(\alpha)$ = principal convex subgroup generated by α. w is a valuation of A, called the natural valuation (see [F]). All quotients $A(u)/A_>(u)$ are archimedean, i.e. isomorphic (as ordered groups) to subgroups of the reals. Therefore all quotients $\mathscr{Q}(u)$ are simple.

For $\alpha \in A$ put $U(\alpha) = \{u \in I \mid \alpha \in A(u) + B\}$ and let J be the smallest ordering extending I such that $\sup U(\alpha)$ exists in J for all $\alpha \in A$. Put $s(\alpha) = \sup U(\alpha)$. For $u \in J - I$ define $A(u)$ and $A_>(u)$ in the same way as for $u \in I$. Then $\mathscr{Q}(u) = 0$ for $u \notin I$. Finally put

$$L(u) = \Big(\bigcap_{u'<u} (A_>(u') + B)\Big)/(A(u) + B) \qquad (u \in J).$$

($L(u) = 0$ if $u = \min J$.) Note that $L(u) \neq 0$ for $u \in J - I$. $L(u)$ is just a group without ordering.

Now we are ready to associate with $\mathfrak{A}$ an ordering $\mathscr{T}(\mathfrak{A})$ with distinguished subsets as follows

$$\mathscr{T}(\mathfrak{A}) = \langle J; <, I, S, P_1, P_2, P_3\rangle$$

where J, $<$, I are defined as above and $S = \{u \in J \mid L(u) \neq 0\}$ and $P_i = \{u \in I \mid \mathscr{Q}(u)$ is simple of type $(Di)\}$.

We will show that the elementary theory of $\mathscr{T}(\mathfrak{A})$ determines the elementary theory of $\mathfrak{A}$.

§3. Elementary equivalence of pairs of ordered groups. We are going to axiomatize the situation of the last section. For this purpose we expand the language of pairs of ordered groups to a language appropriate for structures of the form $\mathfrak{A}^+ = \langle \mathfrak{A}, w, \mathscr{T}(\mathfrak{A})\rangle$ by adding predicate symbols for w, J, I, S and the P_i. Let P^+ be the theory (in the expanded language) with axioms expressing:

A1 $\mathfrak{A}$ is a pair of ordered groups;

A2 (i) J is an ordered set (disjoint from $\mathfrak{A}$) with largest element ∞;

(ii) $I - \{\infty\} = P_1 \dot{\cup} P_2 \dot{\cup} P_3$ (disjoint union);

(iii) $J = I \cup S$;

(iv) $\forall s \in S[\exists u \in J(u < s) \;\&\; \forall u \in J(u < s \rightarrow \exists u' \in P_2 \cup P_3(u < u' < s))]$;

A3 (i) w is a convex valuation of A onto I;

(ii) $\forall u \in I(u \in P_i \leftrightarrow \mathscr{Q}(u)$ is simple of type $(Di))$ $(i = 1, 2, 3)$;

(iii) $\forall u \in J(u \in S \leftrightarrow L(u) \neq 0)$;

(iv) $\forall \alpha \in A(s(\alpha) = \sup U(\alpha)$ exists in $J)$.

REMARKS. 1. The quotient structures $\mathscr{Q}(u)$, $L(u)$ and the set $U(\alpha)$ in A3 are defined with respect to the now arbitrary valuation w just as they were defined before with respect to the natural valuation. It should be clear that A3 is indeed an elementary statement.

2. A2(iv) is a consequence of the remaining axioms.

By the construction described in the last section any pair $\mathfrak{A}$ of ordered groups has an expansion $\mathfrak{A}^+$ to a model of P^+. Therefore Theorem 1 follows from

THEOREM 2. *P^+ is decidable.*

We need the following lemma which will be proved in the next section.

LEMMA 1. *Let $\bar{\mathfrak{A}} = \langle \mathfrak{A}, w, \mathscr{T}\rangle$, $\bar{\mathfrak{A}}' = \langle \mathfrak{A}' w', \mathscr{T}'\rangle$ be models of P^+. If $\mathscr{T}$ and $\mathscr{T}'$ are elementarily equivalent then $\bar{\mathfrak{A}}$ and $\bar{\mathfrak{A}}'$ are elementarily equivalent.*

Let $\mathscr{L}_5$ be the language of ordered sets with five distinguished subsets.

COROLLARY TO LEMMA 1. *For any model $\bar{\mathfrak{A}} = \langle \mathfrak{A}, w, \mathscr{T}\rangle$ of P^+ the set $P^+ \cup \mathrm{Th}_{\mathscr{L}_5}(\mathscr{T})$ is a complete axiom system for* $\mathrm{Th}(\bar{\mathfrak{A}})$.

LEMMA 2. *Let $\mathscr{T}$ be a countable $\mathscr{L}_5$-structure satisfying* A2. *Then there is a model $\bar{\mathfrak{A}}$ of P^+ of the form $\bar{\mathfrak{A}} = \langle \mathfrak{A}, w, \mathscr{T}\rangle$.*

PROOF. Let $\mathscr{T} = \langle J; <, I, S, P_1, P_2, P_3\rangle$ and let $C = \Gamma_{u \in I'} C_u$ be the lexicographic product over the ordered index set $I' = I - \{\infty\}$ where $C_u = \boldsymbol{R}$ for all $u \in I'$. Let A_0 be the subgroup of C of all elements of finite support and put

$$B = \{c_{u_1} + \cdots + c_{u_n} \in A_0 \mid n \in \boldsymbol{N}, u_i \in P_2 \cup P_3, c_{u_i} \in C_{u_i}, c_{u_i} \in \boldsymbol{Q} \text{ if } u_i \in P_3\}.$$

For each $u \in S$ choose $\alpha_u \in C$ such that $\sup\{w(\alpha_u - \beta) \mid \beta \in B\} = u \notin \{w(\alpha_u - \beta) \mid \beta \in B\}$ where w is the natural valuation of C. Finally put $A = A_0 + \sum_{u \in S} \boldsymbol{Q}\alpha_u$ and $\mathfrak{A} = \langle A, B\rangle$. Clearly $\bar{\mathfrak{A}} = \langle \mathfrak{A}, w, \mathscr{T}\rangle$ is the required model.

PROOF OF THEOREM 2. Since P^+ is r.e. it suffices to show that P^+ is co-r.e. Using compactness and Löwenheim-Skolem it follows from Lemma 2 and the Corollary to Lemma 1 that an arbitrary sentence φ in the language of P^+ is satisfiable in some model of P^+ if and only if there exists an $\mathscr{L}_5$-sentence ψ consistent with A2 such that $P^+ \vdash \psi \to \varphi$. Therefore Theorem 2 follows from the fact that the $\mathscr{L}_5$-theory of ordered sets with five distinguished subsets is decidable (see [L-L]).

§4. **Proof of Lemma 1.** Let $\mathfrak{A}$, $\mathfrak{A}'$ be $\aleph_0$-saturated models of P^+ satisfying the hypothesis of Lemma 1. We show that $\mathfrak{A}$, $\mathfrak{A}'$ are partially isomorphic and hence elementarily equivalent.

Let $\mathscr{P}$ be the set of all pairs $\langle f, h\rangle$ such that:

(1) there exist finite-dimensional subspaces A_0, A_0' of A, A' such that f is an isomorphism from $\mathfrak{A}\restriction A_0 = \langle A_0, B_0\rangle$ onto $\mathfrak{A}'\restriction A_0' = \langle A_0', B_0'\rangle$;

(2) h is a partial elementary map from $\mathscr{T}$ into $\mathscr{T}'$ (i.e. $\mathscr{T} \models \varphi(\mathbf{u}) \Leftrightarrow \mathscr{T}' \models \varphi(h(\mathbf{u}))$ for all formulas $\varphi(\mathbf{x})$ and all $\mathbf{u}$ from dom(h)) with finite domain containing $w(A_0) \cup s(A_0)$;

(3) for all $\alpha \in A_0$:

(i) $hw(\alpha) = w'f(\alpha)$,

(ii) $hs(\alpha) = s'f(\alpha)$,

(iii) $s(\alpha) \in U(\alpha) \Leftrightarrow s'f(\alpha) \in U'(f(\alpha))$,

(iv) if $s(\alpha) \in U(\alpha)$ then $w(\alpha - \beta) = s(\alpha)$ for some $\beta \in B_0$;

(4) for all $u \in \text{dom}(h)$: the partial map f_u from $\mathscr{Q}(u)$ into $\mathscr{Q}'(h(u))$ induced by f is elementary. (Note that f_u is well defined by (3)(i).)

Since $\langle 0, \{\langle\infty, \infty'\rangle\}\rangle$ is a member of $\mathscr{P}$, $\mathscr{P}$ is nonempty and it remains to prove the extension property. So let $\langle f, h\rangle \in \mathscr{P}$ and $n \in A \cup J - (\text{dom}(f) \cup \text{dom}(h))$. (The case $n' \in A' \cup J' - (\text{im}(f) \cup \text{im}(h))$ is symmetric.) If $n \in J$ then, by (2) and $\aleph_0$-saturation, there exists $n' \in J'$ such that $h_1 = h \cup \{\langle n, n'\rangle\}$ is elementary. Using A3(i), (ii) and completeness of the theories $P \cup (Di)$, $1 \le i \le 3$, it follows that $\langle f, h_1\rangle$ satisfies (4), hence $\langle f, h_1\rangle \in \mathscr{P}$. Now let $n = \alpha_1 \in A$. Put $A_1 = A_0 + \boldsymbol{Q}\alpha_1$, $B_1 = A_1 \cap B$.

Case 1. $B_1 \neq B_0$.

Choose $\beta \in B_1 - B_0$ such that $u = w(\beta)$ is maximal.

Case 1.1. $w(\beta - \alpha_0) \le u$ for all $\alpha_0 \in A_0$.

Using $\aleph_0$-saturation, first choose $u' \in J'$ such that $\bar{h} = h \cup \{\langle u, u'\rangle\}$ is elementary. (If $u \in \text{dom}(h)$ then $\bar{h} = h$ of course.) Then choose $\beta' \in B' \cap A'(u')$ such that $f_u \cup \{\langle \beta + A_>(u), \beta' + A_>'(u')\rangle\}$ is elementary. (If $u \notin \text{dom}(h)$ then f_u denotes the O-map $\mathscr{Q}(u) \to \mathscr{Q}'(u')$.) Define $\bar{f} : A_1 \to A'$ by $f \subseteq \bar{f}$ and $\bar{f}(\beta) = \beta'$. It is easily checked that $\langle \bar{f}, \bar{h}\rangle \in \mathscr{P}$. Since we will prove the analogous statement in the next case the proof is left to the reader.

Case 1.2. $w(\beta - \alpha_0) > u$ for some $\alpha_0 \in A_0$.

Choose $\alpha_0 \in A_0$ such that $v = w(\beta - \alpha_0)$ is maximal. Note that $v < s(\alpha_0) \notin U(\alpha_0)$, because otherwise $w(\beta_1) > u$ for some $\beta_1 \in B_1 - B_0$, by (3)(iv) (take $\beta_1 = \beta - \beta_0$ where $\beta_0 \in B_0$ such that $w(\alpha_0 - \beta_0) = s(\alpha_0)$). Using $\aleph_0$-saturation choose $v' \in J'$ such that $\bar{h} = h \cup \{\langle v, v'\rangle\}$ is elementary. Again using $\aleph_0$-saturation choose $\beta' \in B' \cap A'(v')$ such that $f_v \cup \{\langle(\beta - \alpha_0) + A_>(v), \beta' + A_>'(v')\rangle\}$ is elementary. Finally choose $\beta'' \in B$ such that $w'(\beta'' - f(\alpha_0)) > v'$ and define $\bar{f} : A_1 \to A'$ by $f \subseteq \bar{f}$, $\bar{f}(\beta) = \beta' + \beta''$. Choice of β'' is possible since $s(\alpha_0) \notin U(\alpha_0)$. Now we show that $\langle \bar{f}, \bar{h}\rangle \in \mathscr{P}$.

(3)(i): Let $\alpha_2 = q\beta - \alpha \in A_1$, $\alpha \in A_0$, $q \in \boldsymbol{Q}$. If $q = 0$ then (3)(i) holds by hypothesis. Therefore assume $q \neq 0$. Since $w(\alpha_2) \le v$ and $w(\beta - \alpha_0) = v$

by the choice of v and α_0 we obtain $w(\alpha_2) = w(q(\beta - \alpha_0) + (q\alpha_0 - \alpha)) = \min\{v, w(q\alpha_0 - \alpha)\}$. On the other hand

$$w'(\bar{f}(\alpha_2)) = w'(q(\beta'' - f(\alpha_0)) + q\beta' + f(q\alpha_0 - \alpha)) = \min\{v', w'(f(q\alpha_0-\alpha))\}$$

by the choice of β' and because $w'(\beta'' - f(\alpha_0)) > v'$. Therefore $\bar{h}w(\alpha_2) = w'\bar{f}(\alpha_2)$ and hence (3)(i). Next note that if $\alpha \in A$ and $\beta \in B$ then $U(\alpha + \beta) = U(\alpha)$ and $s(\alpha + \beta) = s(\alpha)$. This implies the remaining parts of (3) because A_1 is generated over A_0 by some element from B. (1), (2) and (4) are immediate consequences of (3) and the construction.

Case 2. $B_1 = B_0$.

Put $u = \max\{s(\alpha) \mid \alpha \in A_1 - A_0\}$.

Case 2.1. There exists $\alpha \in A_1 - A_0$ such that $s(\alpha) = u \in U(\alpha)$. By extending $\langle f, h\rangle$ according to Case 1 we may assume that

(a) $w(\alpha - \beta_0) = u$ for some $\beta_0 \in B_0$,

(b) $\forall \alpha_0 \in A_0 \cap (B + A_>(u)) \exists \beta_0 \in B_0\, w(\alpha_0 - \beta_0) > u$.

Replacing α by $\alpha - \beta_0$ as in (a) we may assume $w(\alpha) = u$. Choose $u' \in J'$ and $\alpha' \in A'(u')$ such that the two maps $\bar{h} = h \cup \{\langle u, u'\rangle\}$ and $f_u \cup \{\langle \alpha + A_>(u), \alpha' + A'_>(u')\rangle\}$ are elementary. Define $\bar{f}$ by $f \subseteq \bar{f}$ and $\bar{f}(\alpha) = \alpha'$.

Case 2.2. Not Case 2.1.

Choose $\alpha \in A_1 - A_0$ such that $s(\alpha) = u$. Choose $\beta \in B$ such that

(a) $\forall \alpha_0 \in A_0\, w(\alpha - \alpha_0) < w(\alpha - \beta) = v$,

(b) $\forall u_0 \in \mathrm{dom}(h)\, (v \le u_0 \to u \le u_0)$.

Again by extending $\langle f, h\rangle$ according to Case 1 we may assume $\beta \in A_0$ (and (b) still holds). Replacing α by $\alpha - \beta$ we may assume $w(\alpha) = v$. Using $\aleph_0$-saturation choose $u', v' \in J'$ and $\beta' \in B' \cap A'(v')$ such that $\bar{h} = h \cup \{\langle v, v'\rangle, \langle u, u'\rangle\}$ and $f_v \cup \{\langle \alpha + A_>(v), \beta' + A'_>(v')\rangle\}$ are elementary. Finally choose $\gamma' \in A'_>(v')$ such that $\gamma' \in \bigcap_{t<u'} (A'_>(t) + B')$ and $\gamma' \notin A'_0 + A'(u') + B$. This is possible because $u' \in S'$. Now define $\bar{f}$ by $f \subseteq \bar{f}, \bar{f}(\alpha) = \beta' + \gamma'$.

The verification of (1)–(4) for $\langle \bar{f}, \bar{h}\rangle$ in Cases 2.1 and 2.2 is left to the reader.

We close this section by proving Remark 2 after Theorem 1. Put $C = \bigoplus_{i\in\omega} C_i$ where each C_i is an ordered cyclic group with generator c_i. As a subgroup of the lexicographic product $P = \Gamma_{i\in\omega} C_i = \Pi_{i\in\omega} C_i$, C is an ordered group. For any subset $X \subseteq \omega$ define $a_X \in P$ by $a_X(i) = c_i$ if $i \in X$ and $a_X(i) = 0$ otherwise. Let $\mathcal{S}$ be an infinite set of pairwise disjoint infinite subsets of ω and let $I_1, \ldots, I_k \subseteq \mathcal{S}$ be pairwise disjoint. Let p be a prime number and let D be the subgroup of P generated by the elements $p^j a_X$, where $X \in I_j$, $1 \le j \le k$. Finally let B be an arbitrary subgroup of D and put $A = C + D$ and $\mathfrak{A} = \langle A, B\rangle$. It is easy to see that

$$\left(\bigcap_{i\in\omega}(A(i) + p^k A)\right)/p^k A \cong \bigoplus_{1\le j\le k} (Z/p^j Z)^{(\kappa_j)}$$

where $\kappa_j = \text{card } I_j$. Furthermore the quotient group on the left-hand side is definable in $\mathfrak{A}$ by means of

$$a \in \bigcap_{i\in\omega} (A(i) + p^k A) \Leftrightarrow \forall a' \in A - \{0\}\, \exists c \in p^k A(|a - c| < |a'|).$$

Therefore, for any given pair $\langle G, H\rangle$ of countable abelian groups such that $p^k G = 0$ we can find $I_1, \ldots, I_k \subseteq \mathscr{S}$ and B as above such that $\langle G, H\rangle$ is definable in $\mathfrak{A}$. By [B1] this implies undecidability of the theory of pairs of ordered abelian groups.

§5. Separated pairs of real closed fields.

DEFINITION. A pair of valued real closed fields is a structure $\mathscr{K} = \langle K, L, A, B, v\rangle$ such that $\langle K, L\rangle \in \mathscr{R}$ = class of all pairs of real closed fields, v is a valuation of K (now of course in the sense of field theory) with value group A and $B = v(L^\times)$. The residue class fields of K, L are denoted by K_v, L_v, or simply by $\underline{K}$, $\underline{L}$ if there is no danger of confusion. $\underline{\mathscr{K}}$ denotes the pair of fields $\langle \underline{K}, \underline{L}\rangle$. If v is given by its valuation ring V we also write K_V for K_v. As in the case of ordered groups v is called convex if for all $a, a' \in K$, $|a| \le |a'|$ implies $v(a') \le v(a)$, i.e. if the valuation ring associated to v is a convex subset of K. $\mathscr{R}^+$ is the class of all pairs $\mathscr{K}$ of valued real closed fields such that v is convex.

REMARKS. 1. The groups A, B occurring in $\mathscr{K}$ are divisible: If $a \in K$, $0 < a$, then $nv(a^{1/n}) = v(a)$. Therefore $\langle A, B\rangle$ is a pair of ordered divisible groups.

2. If v is convex then $\underline{\mathscr{K}} \in \mathscr{R}$.

3. Let $\langle A, B\rangle$ be an arbitrary pair of ordered divisible groups and $\langle R_0, R_1\rangle \in \mathscr{R}$. Then $\mathscr{K} = \langle R_0((T^A)), R_1((T^B)), A, B, v\rangle \in \mathscr{R}^+$ where $R_0((T^A))$ is the field of formal power series with coefficients in R_0 and exponents in A, and v is the natural valuation (see e.g. [P]). Also $\underline{\mathscr{K}} \cong \langle R_0, R_1\rangle$.

DEFINITION. Let $\varphi(x)$ be the $\mathscr{L}$-formula

$$\forall y(|x| \le y \le 2|x| \to \exists z \in L(y < z < y + 1)),$$

and for $\langle K, L\rangle \in \mathscr{R}$ let V_0 be the set of all $a \in K$ satisfying $\varphi(x)$.

LEMMA 3. *V_0 is the largest convex valuation ring of K such that L_{V_0} is dense in K_{V_0}.*

PROOF. First we show

(1) if $a_1 \in V_0$ and $0 \le a_2 \le 2a_1$ then $a_2 \in V_0$. Let $a_2 \le y \le 2a_2$. Since $a_1 \in V_0$ there exists $z_1 \in L$ such that $a_1 < z_1 < a_1 + 1$ and since $a_1 \le a_1 + y/4 \le 2a_1$ there exists $z_2 \in L$ such that $a_1 + y/4 < z_2 < a_1 + y/4 + 1$. Combining the inequations involving z_1, z_2 we obtain $y < 4(z_2 - z_1 + 1) < y + 8$ and hence $y < 4(z_2 - z_1 + 1) + q < y + 1$ for some $q \in \boldsymbol{Q}$. Since $z = 4(z_2 - z_1 + 1) + q \in L$ it follows that a_2 satisfies $\varphi(x)$, i.e. $a_2 \in V_0$.

Since $a \in V_0$ if and only if $|a| \in V_0$, (1) implies that V_0 is a convex subgroup of the additive group of K. Furthermore $1 \in V_0$.

Let $a, a' \in V_0$. We show $aa' \in V_0$. Assuming $0 < a' \le a$ it suffices to show $a^2 \in V_0$, by (1). If $a \le 1$ then $a^2 \in V_0$, again by (1). If $1 < a$ let $a^2 \le y \le 2a^2$. Then $a \le \sqrt{y} \le 2a$ so there exists $z_1 \in L$ such that $\sqrt{y} < z_1 < \sqrt{y} + 1$ hence $0 < z_1 - \sqrt{y} < 1$. Since $0 < z_1(z_1 - \sqrt{y}) < z_1 \le 2a + 1 \in V_0$, (1) yields $z_1(z_1 - \sqrt{y}) \in V_0$. Hence there exists $z_2 \in L$ such that $z_1(z_1 - \sqrt{y}) < z_2 < z_1(z_1 - \sqrt{y}) + 1$. So

$$\sqrt{y} < z_1 + \frac{1 - z_2}{z_1} < \sqrt{y} + \frac{1}{z_1}$$

and

$$y < \left(z_1 + \frac{1 - z_2}{z_1}\right)^2 < y + \frac{2\sqrt{y}}{z_1} + \frac{1}{z_1^2} < y + 3$$

because $1 < \sqrt{y} < z_1$. Adding a suitable $q \in Q$ to the expression between the $<$-signs we find $z \in L$ such that $y < z < y + 1$. Hence $a^2 \in V_0$.

It follows from what we have shown so far that V_0 is a convex valuation ring of K. In order to prove that L_{V_0} is dense in K_{V_0} let $a, a' \in V_0$ such that $0 < a < a'$ and $a \neq a'$ modulo the maximal ideal of V_0. Then $(a' - a)^{-1} \in V_0$ so there exists $z_1 \in L$ such that $(a' - a)^{-1} < z_1 < (a' - a)^{-1} + 1$. Now $z_1 \in V_0$ so $az_1 \in V_0$ hence there exists $z_2 \in L$ such that $az_1 < z_2 < az_1 + 1$, i.e. $a < z_2 z_1^{-1} < a + z_1^{-1} < a'$.

Finally let V_1 be an arbitrary convex valuation ring of K such that L_{V_1} is dense in K_{V_1}. Let $a \in V_1$ and $|a| \leq y \leq 2|a|$. Then $y \in V_1$, by convexity, hence there exists $z \in L \cap V_1$ such that $\underset{\sim}{y} < \underset{\sim}{z} < \underset{\sim}{y} + 1$ where $\underset{\sim}{y}$ is the residue class of y in K_{V_1}. Therefore $y < z < y + 1$ and so $a \in V_0$.

Definition (cf. [B2]). Let $\mathscr{K} = \langle K, L, A, B, v\rangle \in \mathscr{R}^+$. A sequence $\langle a_1, \ldots, a_n\rangle$ of elements from K is called ($\mathscr{K}$-) separated if for all $b_1, \ldots, b_n \in L$, $v(\sum_i a_i b_i) = \min_i v(a_i b_i)$. (As usual $v(0) = \infty > A$.) $\mathscr{K}$ is called separated if any finite-dimensional L-vectorspace $\subseteq K$ has a separated basis.

Now we are ready to introduce the theory T_s mentioned in the introduction: T_s is the $\mathscr{L}$-theory of all pairs $\langle K, L\rangle \in \mathscr{R}$ such that $\langle K, L, v_0(K^\times), v_0(L^\times), v_0\rangle$ is separated where v_0 is the valuation with valuation ring V_0.

Clearly T_s is axiomatizable. For each n there is an axiom expressing separatedness for n-dimensional L-subspaces of K. Our main result is

Theorem 3. 1. *Two models $\langle K, L\rangle$, $\langle K', L'\rangle$ of T_s are elementarily equivalent if and only if*

(i) $L_{V_0} = K_{V_0} \Leftrightarrow L'_{V'_0} = K'_{V'_0}$, *and*

(ii) *the associated pairs of value groups $\langle v_0(K^\times), v_0(L^\times)\rangle$ and $\langle v_0(K'^\times), v_0(L'^\times)\rangle$ are elementarily equivalent as pairs of ordered groups.*

2. *Any pair of power series fields $\langle R_0((T^A)), R_1((T^B))\rangle$ where $\langle A, B\rangle$ is a pair of ordered divisible abelian groups and R_0 = field of real numbers, $R_1 = R_0$ or = field of real algebraic numbers is a model of T_s. Any model of T_s is elementarily equivalent to such a pair of power series fields.*

3. *T_s is decidable.*

The proof follows the same pattern as the proof of Theorem 1. It is convenient to expand the language $\mathscr{L}$ to a language $\mathscr{L}^+$ appropriate for structures $\mathscr{K} = \langle K, L, A, B, v\rangle \in \mathscr{R}^+$ by adjoining symbols for the valuation and the value groups.

Let T^+ be the $\mathscr{L}^+$-theory with axioms expressing

(i) $\mathscr{K} = \langle K, L, A, B, v\rangle \in \mathscr{R}^+$,

(ii) $\mathscr{K}$ is separated,

(iii) L_v is dense in K_v.

The crucial step in the proof of Theorem 3 is the following lemma whose proof is postponed to the next section.

Lemma 4. *Let $\mathscr{K} = \langle K, L, A, B, v\rangle$, $\mathscr{K}' = \langle K', L', A', B', v'\rangle$ be models of T^+ such that*

(i) $\underset{\sim}{L} = \underset{\sim}{K} \Leftrightarrow \underset{\sim}{L}' = \underset{\sim}{K}'$, *and*

(ii) *the pairs of value groups $\langle A, B\rangle$ and $\langle A', B'\rangle$ are elementarily equivalent.*

Then $\mathscr{K}$ and $\mathscr{K}'$ are elementarily equivalent.

An immediate consequence of Lemma 4 is

Corollary 1. *Any model of T^+ is elementarily equivalent to a valued pair of the* *form* $\langle R_0((T^A)), R_1((T^B)), A, B, v\rangle$ *where* R_0, R_1, A, B *are as in Theorem* 3 *and* v is *the natural valuation of* $R_0((T^A))$. *Conversely any valued pair of power series fields of this special form is a model of* T^+.

PROOF. It follows from [B2, Lemma 3] that any valued pair of power series fields as in the corollary is a model of T^+.

Combining Lemma 4, Corollary 1 and Theorem 1 we get

COROLLARY 2. T^+ *is decidable.*

PROOF OF THEOREM 3. If $\langle K, L\rangle \models T_s$ then $\langle K, L, v_0(K^\times), v_0(L^\times), v_0\rangle \models T^+$. Conversely, if $\langle K, L, A, B, v\rangle \models T^+$ then $\langle K, L, v_0(K^\times), v_0(L^\times), v_0\rangle \models T^+$ because V_0 contains the valuation ring associated to v, by Lemma 3, and hence $v_0(K^\times)$ is a quotient of A. So $\langle K, L\rangle \models T_s$. Therefore the models of T_s are precisely the $\mathscr{L}$-reducts of models of T^+. Theorem 3 now follows from Lemma 4 and its corollaries.

§6. Proof of Lemma 4. The following simple properties of separated sequences will be used: Let $\mathscr{K} = \langle K, L, A, B, v\rangle$ and $\sigma = \langle a_1, \ldots, a_n\rangle$ be a sequence of elements from K.

(S1) If σ is separated then any subsequence of σ is separated.

(S2) If σ is separated and $b_1, \ldots, b_n \in L$ then $\langle a_1b_1, \ldots, a_nb_n\rangle$ is separated.

(S3) If σ is separated and $a \in K$ then $\langle aa_1, \ldots, aa_n\rangle$ is separated.

(S4) If $v(a_i) = 0$, $1 \le i \le n$, then σ is separated if and only if $\underline{a}_1, \ldots, \underline{a}_n$ are linearly independent over $\underline{L}$.

(S5) If σ is separated and $\langle a_{n+1}, \ldots, a_m\rangle$ is another separated sequence such that for all i, j, $1 \le i \le n$, and $n < j \le m$ implies $a_ia_j = 0$ or $v(a_i) \ne v(a_j) \bmod B$ then $\langle a_1, \ldots, a_m\rangle$ is separated.

(S1), (S2) and (S3) are immediate consequences of the definition.

PROOF OF (S4). "Only if": Obvious.

"If": Assume σ not separated and let $b_1, \ldots, b_n \in L$ such that $v(\sum_i a_ib_i) > \min_i v(a_ib_i)$. Multiplying by some $b \in L$ we may assume $\min_i v(b_i) = 0$. Then $\sum_i \underline{a}_i\underline{b}_i = 0$ hence $\underline{a}_1, \ldots, \underline{a}_n$ are linearly dependent over $\underline{L}$.

PROOF OF (S5).

$$v\Big(\sum_{1\le i\le m} a_ib_i\Big) = \min\Big(v\Big(\sum_{1\le i\le n} a_ib_i\Big),\ v\Big(\sum_{n+1\le i\le m} a_ib_i\Big)\Big) = \min_{1\le i\le m} v(a_ib_i).$$

DEFINITION. Let $\mathscr{K}_i = \langle K_i, L_i, A_i, B_i, v_i\rangle \in \mathscr{R}^+$, $i = 0, 1$, $\mathscr{K}_0$ a substructure of $\mathscr{K}_1$, i.e. $K_0 \subseteq K_1$, $v_0 = v_1 \restriction K_0$, $L_0 = K_0 \cap L_1$, $B_0 = A_0 \cap B_1$. $\mathscr{K}_0$ is called an admissible substructure of $\mathscr{K}_1$ if $\underline{K}_0$ and $\underline{L}_1$ are linearly disjoint over $\underline{L}_0$. (In particular $\underline{K}_0 \cap \underline{L}_1 = \underline{L}_0$.)

LEMMA 5. *Let* $\mathscr{K}_0$, $\mathscr{K}_1 \models T^+$, $\mathscr{K}_0$ *an admissible substructure of* $\mathscr{K}_1$.

(i) *Any* $\mathscr{K}_0$*-separated sequence* $\langle a_1, \ldots, a_n\rangle$ *from* K_0 *is* $\mathscr{K}_1$*-separated.*

(ii) K_0 *and* L_1 *are linearly disjoint over* L_0.

(i) is proved by induction on n. The cases $n = 1$ or $a_n = 0$ are trivial. Assume wlog that for some k, $1 \le k \le n$, $v(a_n) = v(a_i) \bmod B_1$ if and only if $i < k$. Since $B_0 = A_0 \cap B_1$ there exist $b_1, \ldots, b_{k-1} \in L_0$ such that $v(a_n) = v(a_ib_i)$, $1 \le i < k$. By (S1), (S2) the sequence $\langle a_1b_1, \ldots, a_{k-1}b_{k-1}, a_n\rangle$ is $\mathscr{K}_0$-separated. Therefore, by (S3), $\sigma = \langle a_1b_1/a_n, \ldots, a_{k-1}b_{k-1}/a_n, 1\rangle$ is $\mathscr{K}_0$-separated. σ satisfies the hypothesis of (S4). Using linear disjointness it follows from (S4) that σ is $\mathscr{K}_1$-separated.

Again using (S3), (S2) we conclude that $\langle a_1, \ldots, a_{k-1}, a_n \rangle$ is $\mathscr{K}_1$-separated. Applying the induction hypothesis to $\langle a_k, \ldots, a_{n-1} \rangle$ we obtain, by (S5), that $\langle a_1 \ldots, a_n \rangle$ is $\mathscr{K}_1$-separated.

(ii) Let $a_1, \ldots, a_n \in K_0$ be linearly independent over L_0. Since $\mathscr{K}_0 \models T^+$ the L_0-space $\sum_i L_0 a_i$ has a $\mathscr{K}_0$-separated basis $\langle a'_1, \ldots, a'_n \rangle$. By (i) this sequence is $\mathscr{K}_1$-separated and hence linearly independent over L_1. Therefore $a_1, \ldots, a_n$ are linearly independent over L_1.

LEMMA 6. *Let $\langle K, A, v \rangle$, $\langle K', A', v' \rangle$ be valued real closed fields, v, v' convex. Let $f : K \to K'$ and $F : A \to A'$ be isomorphisms such that for some subfield L of of K, $v(L^\times) = A$ and $v'f \restriction L = Fv \restriction L$. Then $v'f = Fv$.*

PROOF. Put $v_1 = Fv$, $v_2 = v'f$. Both v_1 and v_2 are convex valuations of K and $v_1 \restriction L^\times = v_2 \restriction L^\times$. Assume $v_1(a) < v_2(a)$ for some $a \in K$. Choose $b \in L$ such that $v_1(a) < v_1(b) = v_2(b) < v_2(a)$. Then $|b| < |a| < |b|$ by convexity, contradiction.

LEMMA 7. *Let $\langle K, A, v \rangle$, $\langle K', A', v' \rangle$ be $\aleph_1$-saturated valued real closed fields, v, v' convex. Let $K_0 \subseteq K_1 \subseteq K$ be countable real closed subfields and let $f_0 : K_0 \to K'$, $g : \underline{K}_1 \to \underline{K}'$, $F : v(K_1^\times) \to A'$ be isomorphic embeddings such that $v'f_0 = Fv \restriction K_0$ and $g \restriction \underline{K}_0 = \underline{f}_0$ where $\underline{f}_0$ is the induced map $\underline{K}_0 \to \underline{K}'$. Then there exists an isomorphic embedding $f_1 : K_1 \to K'$ extending f_0 such that $v'f_1 = Fv \restriction K_1$ and $g = \underline{f}_1$.*

This is a very special case of the main step in the usual model theoretic proofs of the Ax-Kochen-Ershov theorem on henselian fields (see e.g. [A], or [C-K, p. 271ff]). Details are left to the reader.

PROOF OF LEMMA 4. Let $\mathscr{K}$, $\mathscr{K}'$ be $\aleph_1$-saturated models of T^+ satisfying the hypothesis of Lemma 4. We show that they are partially isomorphic. Let I be the set of all pairs $\langle f, F \rangle$ such that:

(i) $\langle f, F \rangle$ is an isomorphism from a countable admissible substructure $\mathscr{K}_0 \models T^+$ of $\mathscr{K}$ onto an admissible substructure $\mathscr{K}'_0$ of $\mathscr{K}'$;

(ii) F is a partial elementary map from $\langle A, B \rangle$ to $\langle A', B' \rangle$;

(iii) the induced partial map $\underline{f}$ from $\underline{\mathscr{K}}$ to $\underline{\mathscr{K}}'$ is elementary.

REMARK. It follows from Robinson's proof [R] that (i) implies (iii). We will not use this fact.

$I \neq \varnothing$: Let K_0 (K'_0 resp.) be the algebraic closure of Q in K (in K' resp.) and let $f : K_0 \to K'_0$ be the unique isomorphism. Clearly $\langle f, 0 \rangle \in I$.

I has the extension property: Let $\langle f_0, F_0 \rangle \in I$ be an isomorphism from $\mathscr{K}_0$ onto $\mathscr{K}'_0$ and assume $a_1 \in K - K_0$. (The case $a'_1 \in K' - K'_0$ is symmetric.)

Choose a countable elementary substructure $\mathscr{K}_1 = \langle K_1, L_1, A_1, B_1, v \restriction K_1 \rangle$ of $\mathscr{K}$ such that $K_0 \cup \{a_1\} \subseteq K_1$. Obviously $\mathscr{K}_1 \models T^+$ is admissible. First choose an extension $F : A_1 \to A'$ of F_0 such that F is elementary as a partial map from $\langle A, B \rangle$ to $\langle A', B' \rangle$. Next choose an extension $g : \underline{K}_1 \to \underline{K}'$ of $\underline{f}_0$ such that g is elementary as a partial map from $\underline{\mathscr{K}}$ to $\underline{\mathscr{K}}'$. Now apply Lemma 7 to get an extension $f_1 : L_1 \to L'$ of $f_0 \restriction L_0$ such that $Fv \restriction L_1 = v'f_1$ and $\underline{f}_1 = g \restriction \underline{L}_1$. Put $L'_1 = \mathrm{im}(f_1)$.

By Lemma 5, K_0 and L_1 (K'_0 and L'_1 resp.) are linearly disjoint over L_0 (over L'_0 resp.). Therefore f_1 extends to a field isomorphism $f_2 : K_0 L_1 \to K'_0 L'_1$ such that $f_2 \restriction K_0 = f_0$.

Claim. (a) $v'f_2 = Fv \restriction K_0 L_1$ (and $v((K_0 L_1)^\times) = A_0 + B_1$),

(b) $\underline{f}_2 = g \restriction \underline{K_0 L_1}$,

(c) f_2 is order preserving.

(a) Since K_0L_1 is the quotient field of the set of all elements of the form

$$a = \sum_{1\le i\le n} a_ib_i, \qquad a_i \in K_0, \qquad b_i \in L_1, \tag{2}$$

it suffices to prove $v'f_2(a) = Fv(a)$ for elements a of this form. Furthermore, since $\mathscr{K}_0$ is separated, we may assume that the sequence $\langle a_1, \ldots, a_n\rangle$ is $\mathscr{K}_0$-separated and hence $\mathscr{K}$-separated, by admissibility. Since f_0 is an isomorphism the sequence $\langle f_0(a_1), \ldots, f_0(a_n)\rangle$ is $\mathscr{K}'_0$-separated, hence $\mathscr{K}'$-separated. Therefore

$$v(a) = \min v(a_ib_i) \in A_0 + B_1$$

and

$$\begin{aligned} v'(f_2(a)) &= v'\left(\textstyle\sum f_0(a_i)f_1(b_i)\right) = \min(v'f_0(a_i) + v'f_1(b_i)) \\ &= \min(Fv(a_i) + Fv(b_i)) = Fv(a). \end{aligned}$$

(b) Since f_2 coincides with g on $\underline{K}_0$ and $\underline{L}_1$ it suffices to show that $\underline{K_0L_1} \subseteq \underline{K}_0\underline{L}_1$. Let $c \in K_0L_1$, $v(c) = 0$. Write $c = a/a'$ where a, a' are of the form (2). By (a) there exist $a_0 \in K_0$, $b_0 \in L_1$ such that $v(a_0b_0) = v(a) = v(a')$. Dividing both a, a' by a_0b_0 we may assume $v(a) = v(a') = 0$ and a, a' are still of the form (2). Now it suffices to show that $\underline{a} \in \underline{K}_0\underline{L}_1$ (and similarly $\underline{a}' \in \underline{K}_0\underline{L}_1$). As above write $a = \sum_{1\le i\le n} a_ib_i$ where $\langle a_1, \ldots, a_n\rangle$ is $\mathscr{K}$-separated, $a_i \in K_0$, $b_i \in L_1$. Dropping the summands a_jb_j with $v(a_jb_j) > 0$ we may assume $v(a_ib_i) = 0$ for all i. Then $v(a_i) = -v(b_i) \in A_0 \cap B_1 = B_0$ so there exist $b'_i \in L_0$ such that $v(a_ib'_i) = 0$. Writing $a = \sum (a_ib'_i)(b_ib'^{-1}_i)$ we conclude $\underline{a} = \sum \underline{(a_ib'_i)}\,\underline{(b_ib'^{-1}_i)} \in \underline{K}_0\underline{L}_1$.

(c) Let $0 < a \in K_0L_1$. By (a) $v(K_0L_1)$ is divisible, hence there exists $c \in K_0L_1$ such that $v(ac^2) = 0$. Then $0 < ac^2$ and so $0 < \underline{ac^2}$. Since g is order preserving we obtain $0 < g(\underline{ac^2}) = \underline{f_2(ac^2)}$, by (b), hence $0 < f_2(ac^2)$ and finally $0 < f_2(a)$. The claim is proved.

From (c) it follows that f_2 extends to an embedding f_3 of the relative algebraic closure M of K_0L_1 into K'. M is just a real closure of K_0L_1. By Lemma 6 $v'f_3 = Fv\restriction M$, and $\underline{f_3} = g\restriction\underline{M}$ because $\underline{M}$ is a real closure of $\underline{K_0L_1}$ and therefore the extension of $g\restriction\underline{K_0L_1}$ to $\underline{M}$ is unique. By Lemma 7 f_3 extends to an embedding $f: K_1 \to K'$ such that $\underline{f} = g$ and $v'f = Fv$. Put $K'_1 = f(K_1)$. In order to show $\langle f, F\rangle \in I$ it remains to prove

(d) $K'_1 \cap L' = L'_1$,

(e) $\mathscr{K}'_1 = \mathscr{K}'\restriction K'_1$ is an admissible substructure of $\mathscr{K}'$.

(d) Assume there exists $a \in K_1 - L_1$ such that $f(a) \in L'$. Since $\mathscr{K}_1 \models T^+$ the L_1-vectorspace $L_1 + L_1a$ has a $\mathscr{K}_1$-separated basis $\langle a_1, a_2\rangle$. Since $f(a_1), f(a_2) \in L'$ we have $v(a_1), v(a_2) \in B_1$ hence there exist $b_1, b_2 \in L_1$ such that $v(a_1b_1) = v(a_2b_2) = 0$. $\langle a_1b_1, a_2b_2\rangle$ is $\mathscr{K}_1$-separated and so wlog $\underline{a_1b_1} \notin \underline{L}_1 = \underline{K}_1 \cap \underline{L}$ ((S2), (S4)). Therefore $\underline{f(a_1b_1)} = g(\underline{a_1b_1}) \notin \underline{L}'$ contradicting $f(a_1b_1) \in L'$.

(e) follows from the fact that $\mathscr{K}_1$ is admissible and g is partial elementary.

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