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On the Lax–Phillips scattering theory

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Synopsis

We give a simple description of the wave operators appearing in the Lax–Phillips scattering theory. This is used to derive a relation between the scattering matrix and a kind of time delay operator and to characterize all scattering systems having the same scattering operator.

1. Introduction

Let $\{U(t)\}_{t\in\mathbb{R}}$ be a strongly continuous one-parameter unitary group on a separable Hilbert space $\mathfrak{H}$. Assume that there exist two closed subspaces $\mathfrak{D}_+$ and $\mathfrak{D}_-$–called outgoing and incoming subspace, respectively–having the following properties:

$$U(t)\mathfrak{D}_+\subseteq\mathfrak{D}_+ \quad \text{for} \quad t\geqq 0, \tag{1}$$

$$U(t)\mathfrak{D}_-\subseteq\mathfrak{D}_- \quad \text{for} \quad t\leqq 0, \tag{2}$$

$$\bigcap_t U(t)\mathfrak{D}_+=\{0\}=\bigcap_t U(t)\mathfrak{D}_-, \tag{3}$$

$$\overline{\bigcup_t U(t)\mathfrak{D}_+}=\mathfrak{H}=\overline{\bigcup_t U(t)\mathfrak{D}_-}. \tag{4}$$

Then we say that $\{\mathfrak{H}, U(t), \mathfrak{D}_+, \mathfrak{D}_-\}$ form a *general Lax–Phillips scattering system* (GLPS system). If in addition the relation

$$\mathfrak{D}_+\perp\mathfrak{D}_- \tag{5}$$

is satisfied, we speak of a *Lax–Phillips scattering system* (LPS system).

Given a GLPS system, there is an auxiliary Hilbert space $\mathfrak{K}$ and two unitary maps R_+ and R_- of $\mathfrak{H}$ onto $L^2(\mathbb{R}, dx;\mathfrak{K})$ such that [**3, 5**]

$$R_+\mathfrak{D}_+=L^2(\mathbb{R}_+, dx;\mathfrak{K}), \tag{6}$$

$$R_-\mathfrak{D}_-=L^2(\mathbb{R}_-, dx;\mathfrak{K}), \tag{7}$$

$$R_\pm U(t)R_\pm^{-1}=T(t), \tag{8}$$

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where $T(t)$ is the translation group on $L^2(\mathbb{R}, dx; \mathfrak{K})$, i.e.

$$[T(t)f](x) = f(x-t) \quad \text{for} \quad f \in L^2(\mathbb{R}, dx; \mathfrak{K}). \tag{9}$$

The scattering operator S is defined by

$$S := R_+ R_-^{-1}. \tag{10}$$

S is a unitary operator on $L^2(\mathbb{R}, dx; \mathfrak{K})$ and commutes with $T(t)$. For LPS systems one also has [**3**, p. 52]

$$SL^2(\mathbb{R}_-, dx; \mathfrak{K}) \subseteq L^2(\mathbb{R}_-, dx; \mathfrak{K}). \tag{11}$$

If the assumption (5) is satisfied, then the subspace

$$\mathfrak{D} := \mathfrak{H} \ominus (\mathfrak{D}_+ \oplus \mathfrak{D}_-) \tag{12}$$

may be called the interacting subspace. This is to be seen in the formulation of the Lax–Phillips scattering theory where one has a free dynamics $U_0(t)$ besides the interacting dynamics $U(t)$ [**3**, p. 53–54; **5**]. We shall denote by D_+, D_- and D the orthogonal projection with range $\mathfrak{D}_+$, $\mathfrak{D}_-$ and $\mathfrak{D}$ respectively.

Recently Lax and Phillips [**4**] introduced and studied an operator T, called the time delay operator, by

$$T = s - \lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} dt e^{-\varepsilon|t|} U(-t) D U(t), \tag{13}$$

where the integral is to be taken in the strong sense. They obtained various interesting properties of T. In particular they showed that, under additional assumptions on the LPS system, T is connected with the scattering matrix $\mathcal{S}(\lambda)$ by the following relation

$$\mathcal{T}(\lambda) = -i\mathcal{S}(\lambda)\, \partial_\lambda \mathcal{S}(\lambda)^* \tag{14}$$

in the outgoing spectral representation (T commutes with $U(t)$, hence it is decomposable). The expectation values of T are interpreted as the total energy of interaction of the wave with the scatterer.

In quantum mechanics the time delay is defined by a somewhat different expression involving also the free dynamics. In this paper we transcribe this quantum mechanical definition into the framework of LPS systems by using the associated free dynamics $U_0(t)$. We relate this new time delay operator to that defined in (13) and deduce equation (14) under weaker assumptions than those in [**4**] by a method which avoids the use of analytic continuation.

Our treatment of the "time delay operator" is based on the description of the structure of the wave operators for LPS systems with a fixed free dynamics $U_0(t)$. This again is connected with the problem of characterizing the class of all possible scattering operators for LPS systems. In contrast to the corresponding problem in the abstract time-dependent (e.g. quantum mechanical) scattering theory (see [**7**]) the answer is here quite simple because of the additional restrictions which one has in the Lax–Phillips scattering theory.

2. Characterization of the wave and scattering operators

We begin with some remarks on the class of scattering operators appearing in GLPS and LPS systems. This is useful for our later characterization of the wave operators and furthermore of interest for comparison with the corresponding situation in abstract time-dependent scattering theory (see [**7**]).

For a GLPS system, the scattering operator S defined on the space $L^2(\mathbb{R}, dx; \mathfrak{K})$ is unitary and commutes with the translation group $T(t)$ in this space, whereas for a LPS system it satisfies in addition the relation (11). One may formulate the following converse problem: given $\mathfrak{H}$, a strongly continuous unitary group $\{U(t)\}_{t\in\mathbb{R}}$ on $\mathfrak{H}$, a unitary map $R: \mathfrak{H} \to L^2(\mathbb{R}, dx; \mathfrak{K})$ such that $RU(t)R^{-1} = T(t)$ and a unitary operator S' on $L^2(\mathbb{R}, dx; \mathfrak{K})$ that commutes with $T(t)$, can one find two subspaces $\mathfrak{D}_\pm$ of $\mathfrak{H}$ which are outgoing and incoming for $U(t)$ such that S' is the scattering operator associated with $\{\mathfrak{H}, U(t), \mathfrak{D}_+, \mathfrak{D}_-\}$?

The solution is simple. Define $\mathfrak{D}_+ = R^{-1}L^2(\mathbb{R}_+, dx; \mathfrak{K})$ and $\mathfrak{D}_- = R^{-1}S'L^2(\mathbb{R}_-, dx; \mathfrak{K})$. Then it is easy to verify that $\mathfrak{D}_+$ and $\mathfrak{D}_-$ satisfy the assumptions (1)–(4). The operators $R_\pm$ are defined by $R_+ = R$ and $R_- = S'^*R$. We see that $R_\pm$ satisfy the assumptions (6)–(8) and therefore are the right unitary maps for $\{\mathfrak{H}, U(t), \mathfrak{D}_+, \mathfrak{D}_-\}$. The scattering operator is $S \equiv R_+R_-^{-1} = RR^{-1}S' = S'$. If in addition S' satisfies (11), i.e. if S' leaves $L^2(\mathbb{R}_-, dx; \mathfrak{K})$ invariant, it follows from the unitarity of R^{-1} that $\mathfrak{D}_+ \perp \mathfrak{D}_-$.

These considerations show that the class of scattering operators S for GLPS systems is completely characterized by the two properties: unitarity and $[S, T(t)] = 0$, and that the class of scattering operators for LPS systems is completely characterized by the three conditions: unitarity, $[S, T(t)] = 0$ and $SL^2(\mathbb{R}_-, dx; \mathfrak{K}) \subseteq L^2(\mathbb{R}_-, dx; \mathfrak{K})$.

Our next point is to embed the LPS systems in the framework of two-space scattering theory. The essential steps for this were already described in [**3**, p. 53–54].. Assume we have a LPS system $\{\mathfrak{H}, U(t), \mathfrak{D}_+, \mathfrak{D}_-\}$. We define an operator J from the representation space $L^2(\mathbb{R}, dx; \mathfrak{K})$ to $\mathfrak{H}$ as follows. We fix a number $a \geqq 0$ and set

$$Jf := R_+^{-1}T(-a)f \quad \text{for} \quad f \in L^2([a, \infty), dx; \mathfrak{K}),$$
$$Jf := R_-^{-1}T(a)f \quad \text{for} \quad f \in L^2((-\infty, -a], dx; \mathfrak{K}),$$

whereas on $L^2((-a, a), dx; \mathfrak{K})$, J is an arbitrary bounded operator mapping $L^2((-a, a), dx; \mathfrak{K})$ into $\mathfrak{H} \ominus (\mathfrak{D}_+ \oplus \mathfrak{D}_-)$. Clearly

$$JL^2([a, \infty), dx; \mathfrak{K}) = D_+,$$
$$JL^2((-\infty, -a], dx; \mathfrak{K}) = \mathfrak{D}_-,$$

and $U(t)Jf = JT(t)f$ for $f \in L^2([a, \infty), dx; \mathfrak{K})$ and $t \geqq 0$ as well as for $f \in L^2((-\infty, -a]\, dx; \mathfrak{K})$ and $t \leqq 0$.

It is now easily seen that the LPS systems may be included in the following variant, which we denote by (LP), of the Lax–Phillips scattering theory involving a free dynamics $U_0(t)$ (see e.g. [**5**, p. 219]; it suffices to identify below $\mathfrak{H}_0$ with $L^2(\mathbb{R}, dx; \mathfrak{K})$, R_0 with the identity operator on $L^2(\mathbb{R}, dx; \mathfrak{K})$ and $U_0(t)$ with $T(t)$):

(LP) There are two strongly continuous unitary groups $U(t)$ and $U_0(t)$ acting on the Hilbert space $\mathfrak{H}$ and $\mathfrak{H}_0$, respectively, and a bounded operator J from $\mathfrak{H}_0$ to

$\mathfrak{H}$ satisfying:

(i) There exist two orthogonal subspaces $\mathfrak{E}_+$ and $\mathfrak{E}_-$ of $\mathfrak{H}_0$ such that J is isometric on $\mathfrak{E}_+ \oplus \mathfrak{E}_-$ and maps $\mathfrak{H}_0 \ominus (\mathfrak{E}_+ \oplus \mathfrak{E}_-)$ into $\mathfrak{H} \ominus J(\mathfrak{E}_+ \oplus \mathfrak{E}_-)$,

(ii) $\mathfrak{E}_+$ and $\mathfrak{E}_-$ are outgoing and incoming subspaces for $U_0(t)$, $\mathfrak{D}_+ := J\mathfrak{E}_+$ and $\mathfrak{D}_- := J\mathfrak{E}_-$ are outgoing and incoming subspaces for $U(t)$,

(iii) $U(t)J = JU_0(t)$ on $\mathfrak{E}_+$ for $t \geqq 0$ and $U(t)J = JU_0(t)$ on $\mathfrak{E}_-$ for $t \leqq 0$,

(iv) There is a Hilbert space $\mathfrak{K}$, a non-negative number a and a unitary map R_0 from $\mathfrak{H}_0$ onto $L^2(\mathbb{R}, dx; \mathfrak{K})$ such that

$$R_0\mathfrak{E}_+ = L^2([a, \infty), dx; \mathfrak{K})$$
$$R_0\mathfrak{E}_- = L^2((-\infty, -a], dx; \mathfrak{K})$$

and

$$R_0 U_0(t) R_0^{-1} = T(t) \quad \text{on} \quad L^2(\mathbb{R}, dx; \mathfrak{K}).$$

At this point we remark that it is, in general, not possible to define the free dynamics $U_0(t)$ on the same Hilbert space as $U(t)$ without modifying the subspaces $\mathfrak{D}_+$ and $\mathfrak{D}_-$. The reason is that the subspace $\mathfrak{D}$ defined in (12) can have arbitrary dimension (see e.g. [**3**, p. 24]), and therefore a unitary map J from $L^2((-a, a), dx; \mathfrak{K})$ onto $\mathfrak{D}$ is in general not realizable. In most applications, however, $\dim \mathfrak{D} = \infty$ and then we may choose $\mathfrak{H} = \mathfrak{H}_0$ and $\mathfrak{D}_\pm = \mathfrak{E}_\pm$. On the other hand one could also assume without loss of generality that $\mathfrak{H}_0 = \mathfrak{E}_+ \oplus \mathfrak{E}_-$, i.e. $\mathfrak{H}_0 = \mathfrak{D}_+ \oplus \mathfrak{D}_-$ up to an isomorphism. In this form $\mathfrak{H}$ is larger than $\mathfrak{H}_0$, the difference being the interacting subspace $\mathfrak{D}$. In order to preserve the symmetry in the roles of $\mathfrak{H}$ and $\mathfrak{H}_0$, we shall continue to work with the general formulation given by (LP).

Assuming one has a system satisfying (LP), one can easily prove the existence and unitarity of the wave operators

$$W_\pm := s - \lim_{t \to \pm\infty} U(-t) J U_0(t) \tag{15}$$

and show that [**5**, Th. XI.86]

$$W_+ f = Jf \quad \text{if} \quad f \in \mathfrak{E}_+, \tag{16}$$
$$W_- f = Jf \quad \text{if} \quad f \in \mathfrak{E}_-. \tag{17}$$

Furthermore, the scattering operator defined by

$$S_0 := W_+^{-1} W_- \tag{18}$$

is connected with the corresponding scattering operator S in the LPS system by the relation

$$S = R_0 U_0(-2a) S_0 R_0^{-1}. \tag{19}$$

From (19) and (11) (or from (16) and (17)) one gets that

$$S_0 \mathfrak{E}_- \subseteq \mathfrak{E}_+^\perp, \tag{20}$$

or equivalently

$$\hat{S}_0 L^2((-\infty, -a], dx; \mathfrak{K}) \subseteq L^2((-\infty, a], dx; \mathfrak{K}) \tag{21}$$

if one defines $\hat{S}_0 := R_0 S_0 R_0^{-1}$.

Let us introduce the notation E_+, E_- and E for the orthogonal projections in $\mathfrak{H}_0$ with range $\mathfrak{E}_+, \mathfrak{E}_-$ and $\mathfrak{E} := \mathfrak{H}_0 \ominus (\mathfrak{E}_+ \oplus \mathfrak{E}_-)$ respectively. We are now prepared to prove the following results.

PROPOSITION 1. *Let $U_0(t)$ be a strongly continuous unitary group on a Hilbert space $\mathfrak{H}_0$ and J a bounded operator from $\mathfrak{H}_0$ to another Hilbert space $\mathfrak{H}$. Assume that the conditions* (i) *and* (iv) *of* (LP) *are satisfied.*

(a) *Let $U(t)$ be a strongly continuous unitary group on $\mathfrak{H}$ such that the other conditions of* (LP) *are also satisfied for the system $\{\mathfrak{H}, \mathfrak{H}_0, U(t), U_0(t), J, \mathfrak{E}_\pm, R_0\}$. Then the wave operator W_- defined by* (15) *for this* (LP) *system is a unitary operator of the form*

$$W_- = JE_- + JE_+S_0 + V = D_-J + D_+JS_0 + V, \tag{22}$$

where V is a partial isometry from $\mathfrak{H}_0$ to $\mathfrak{H}$ with

$$VV^* = D, \quad V^*V := \tilde{E} = I - E_- - S_0^*E_+S_0. \tag{23}$$

(b) *Let W be a unitary operator from $\mathfrak{H}_0$ to $\mathfrak{H}$ of the form*

$$W = JE_- + JE_+S' + DV', \tag{24}$$

where S' is a unitary operator on $\mathfrak{H}_0$ commuting with $U_0(t)$, D is the orthogonal projection in $\mathfrak{H}$ with range $\mathfrak{H} \ominus J(\mathfrak{E}_+ \oplus \mathfrak{E}_-)$ and $V' \in B(\mathfrak{H}_0, \mathfrak{H})$. Define $U(t) := WU_0(t)W^{-1}$. Then the system $\{\mathfrak{H}, \mathfrak{H}_0, U(t), U_0(t), J, \mathfrak{E}_\pm, R_0\}$ satisfies the assumptions of (LP); *the wave and scattering operators for this* (LP) *system are given by $W_- = W$, $W_+ = WS'^*$ and $S_0 = S'$. (In particular $V := DV'$ is a partial isometry from $\mathfrak{H}_0$ to $\mathfrak{H}$ satisfying* (23)).

Proof. (a) It follows from (LP, i) that

$$D_\pm J = JE_\pm \tag{25}$$

and

$$D_\pm = JE_\pm J^*. \tag{26}$$

By (26), (16), (17) and (LP, i) we have $D_\pm W_\pm = JE_\pm J^* W_\pm = JE_\pm$, i.e.

$$D_\pm = JE_\pm W_\pm^{-1}. \tag{27}$$

Hence

$$\begin{aligned} W_- &= (D_- + D_+ + D)W_- = JE_-W_-^{-1}W_- + JE_+W_+^{-1}W_- + DW_- \\ &= JE_- + JE_+S_0 + DW_-. \end{aligned}$$

Setting $V = DW_-$, we obtain the first identity in (22). The second one is then immediate by virtue of (25). To check (23), we notice that $(DW_-)(DW_-)^* = DW_-W_-^{-1}D = D$ and

$$\begin{aligned} (DW_-)^*(DW_-) &= W_-^{-1}DW_- = I - W_-^{-1}D_-W_- - W_-^{-1}D_+W_- \\ &= I - E_- - W_-^{-1}W_+E_+W_+^{-1}W_- = I - E_- - S_0^*E_+S_0, \end{aligned}$$

where we have used the relation

$$D_\pm = W_\pm E_\pm W_\pm^{-1} \tag{28}$$

which follows from (27), (16) and (17).

(b) Define $D_\pm$ to be the orthogonal projection with range $J\mathfrak{E}_\pm$. By (24), (25) and the unitarity of W, we have for $f\in\mathfrak{H}_0$

$$\|f\|^2=\|Wf\|^2=\|D_-Jf\|^2+\|D_+JS'f\|^2+\|DV'f\|^2. \tag{29}$$

Now if $f\in\mathfrak{E}_-$, $\|D_-Jf\|^2=\|f\|^2$. Hence (29) implies that

$$WE_-=D_-JE_-=JE_-. \tag{30}$$

Similarly, if $S'f\in\mathfrak{E}_+$, i.e. $f\in S'^*\mathfrak{E}_+$, then $\|D_+JS'f\|^2=\|S'f\|^2=\|f\|^2$, hence

$$WS'^*E_+=D_+JE_+=JE_+. \tag{31}$$

We now prove condition (iii) of (LP). By (30), $U(t)JE_-=WU_0(t)W^{-1}WE_-=WU_0(t)E_-$. Now let $t\leqq 0$. Then $U_0(t)\mathfrak{E}_-\subseteq\mathfrak{E}_-$ by (LP, iv), so that by (30) $WU_0(t)E_-=JU_0(t)E_-$. This proves that $U(t)J=JU_0(t)$ on $\mathfrak{E}_-$ for $t\leqq 0$. Similarly, for $t\geqq 0$, $U_0(t)\mathfrak{E}_+\subseteq\mathfrak{E}_+$, and one obtains by using (31) that

$$\begin{aligned}U(t)JE_+&=WU_0(t)W^{-1}WS'^*E_+=WS'^*U_0(t)E_+\\&=JU_0(t)E_+.\end{aligned}$$

We next prove (LP, ii). The assertions on $\mathfrak{E}_+$ and $\mathfrak{E}_-$ follow from (LP, iv). For $\mathfrak{D}_\pm$, the properties (1), (2) and (3) follow immediately from the corresponding properties of $\mathfrak{E}_\pm$ and (LP, iii). To verify (4) for the subspace $\mathfrak{D}_-$, we use (30):

$$\begin{aligned}\overline{\bigcup_t U(t)J\mathfrak{E}_-}&=\overline{\bigcup_t WU_0(t)W^{-1}J\mathfrak{E}_-}=W\overline{\bigcup_t U_0(t)\mathfrak{E}_-}\\&=W\mathfrak{H}_0=\mathfrak{H}.\end{aligned}$$

Similarly, by using (31):

$$\begin{aligned}\overline{\bigcup_t U(t)J\mathfrak{E}_+}&=\overline{\bigcup_t WU_0(t)W^{-1}J\mathfrak{E}_+}\\&=WS'^*\overline{\bigcup_t U_0(t)\mathfrak{E}_+}=WS'^*\mathfrak{H}_0=\mathfrak{H}.\end{aligned}$$

It remains to calculate the wave operators $W_\pm$. Let $\mathfrak{M}$ be the dense subset of $\mathfrak{H}_0$ formed by all $f\in\mathfrak{H}_0$ such that the vector-valued function $(R_0f)(\cdot)$ in $L^2(\mathbb{R},dx;\mathfrak{K})$ has compact support. Given $f\in\mathfrak{M}$, there is a positive number s such that $U_0(\pm s)f\in\mathfrak{E}_\pm$. For $t<-s$ we then have $U_0(t)f\in\mathfrak{E}_-$, hence by (30)

$$U(-t)JU_0(t)f=WU_0(-t)W^{-1}WU_0(t)f=Wf.$$

Since $\mathfrak{M}$ is dense in $\mathfrak{H}_0$, this shows that $W_-=W$. Similarly, by (31), we get for $t>s$

$$U(-t)JU_0(t)f=WU_0(-t)S'^*U_0(t)f=WS'^*f,$$

i.e. $W_+=WS'^*$. ■

At the beginning of this section we characterized the class of scattering operators for LPS systems. In terms of S_0, these conditions read: (α) S_0 is a unitary operator on $\mathfrak{H}_0$, (β) S_0 commutes with $U_0(t)$, (γ) $S_0\mathfrak{E}_-\subseteq\mathfrak{E}_+^\perp$. Incidentally, we notice that the operator S' in Proposition 1(b) will also satisfy (γ), although this condition was not explicitly postulated. Indeed we have shown that S' is identical with the operator S_0 of a LPS system.

We now use Proposition 1 to study the question of determining all possible (LP) systems having a given operator S_0 as their scattering operator. We assume given $\mathfrak{H}_0$, $U_0(t)$, $\mathfrak{E}_+$, $\mathfrak{E}_-$, R_0 and an operator S_0 satisfying (α), (β) and (γ). For fixed $\mathfrak{H}$ and J, there is a solution $U(t)$ if and only if

$$\dim \mathfrak{H} \ominus J(\mathfrak{E}_+ \oplus \mathfrak{E}_-) = \dim (I - E_- - S_0^* E_+ S_0)\mathfrak{H}_0.$$

(Note that $I - E_- - S_0^* E_+ S_0$ is a projection, since $S_0^* E_+ S_0 E_- = 0$ by (γ) and hence the range of the projection $S_0^* E_+ S_0$ is a subspace of $(I - E_-)\mathfrak{H}_0$.)

If the above dimensions are the same, one obtains all possible solutions $U(t)$ by choosing in (22) all possible partial isometries V from $\tilde{E}\mathfrak{H}_0$ to $\mathfrak{H} \ominus J(\mathfrak{E}_+ \oplus \mathfrak{E}_-)$ and setting $U_t = WU_0(t)W^{-1}$. If these dimensions are different, one may find solutions $U(t)$ either by changing $\mathfrak{H} \ominus J(\mathfrak{E}_+ \oplus \mathfrak{E}_-)$ so that the dimensions become equal, or by modifying the subspaces $\mathfrak{E}_\pm$. In fact, if a is the number appearing in (LP, iv) and $b > a$, then $\mathfrak{E}_+^b := R_0^{-1} L^2([b, \infty), dx; \mathfrak{K})$ and $\mathfrak{E}_-^b := R_0^{-1} L^2((-\infty, -b], dx; \mathfrak{K})$ also verify (i) and (iv) of (LP). Furthermore, if $E_\pm^b$ denote the projections with range $\mathfrak{E}_\pm^b$, we have

$$S_0^* E_+^b S_0 \mathfrak{H}_0 \subseteq S_0^* E_+^a S_0 \mathfrak{H}_0 \subseteq (I - E_-^a)\mathfrak{H}_0 \subseteq (I - E_-^b)\mathfrak{H}_0,$$

so that $(I - E_-^b) - S_0^* E_+^b S_0$ is a projection. One clearly has $\dim \mathfrak{H} \ominus J(\mathfrak{E}_+^b \oplus \mathfrak{E}_-^b) = \dim (I - E_-^b - S_0^* E_+^b S_0)\mathfrak{H}_0 = \infty$, so that solutions $U(t)$ exist by our preceding remarks. To get all solutions in this case, we may vary the parameter b over (a, ∞), and for fixed b take all partial isometries V from $(I - E_-^b - S_0^* E_+^b S_0)\mathfrak{H}_0$ to $\mathfrak{H} \ominus J(\mathfrak{E}_+^b \oplus \mathfrak{E}_-^b)$.

Finally, if $\mathfrak{H}$ and J are not given, one has more freedom in the choice of $U(t)$. It is then possible to find a "maximal" group $U(t)$ as follows: Let c be the smallest non-negative number such that $S_0 E_-^c \mathfrak{H}_0 \subseteq (I - E_+^c)\mathfrak{H}_0$. (Note that if Γ denotes the set of all $a \geqq 0$ such that $S_0 E_-^a = (I - E_+^a) S_0 E_-^a$ and if $c = \inf \Gamma$, then $S_0 E_-^c = (I - E_+^c) S_0 E_-^c$ by the strong continuity in b of the projections $E_\pm^b$; thus $\inf \Gamma \in \Gamma$.) Define $\mathfrak{H}$ by $\mathfrak{H} := \mathfrak{E}_-^c \oplus \mathfrak{E}_+^c \oplus \mathfrak{N}$, where $\mathfrak{N}$ is a Hilbert space of the same dimension as $(I - E_-^c - S_0^* E_+^c S_0)\mathfrak{H}_0$. The corresponding solutions $U(t)$ are maximal in the sense that (iii) of (LP) holds on maximal subspaces $\mathfrak{E}_\pm$.

3. The time delay operator

In this section we use the formulation of the LPS systems in the framework where we have a free dynamics $U_0(t)$. To simplify the notation, we restrict ourselves to the case where J is unitary; we shall thus assume $\mathfrak{H} = \mathfrak{H}_0$, $J = I$ (the identity operator), $D_\pm = E_\pm$ and $D = E$.

Let χ_Δ be the characteristic function of the interval Δ in $\mathbb{R}$ and, for $a \in \mathbb{R}$ and $\varepsilon > 0$, set

$$\varphi_{a,\varepsilon}(x) := \int_{-\infty}^{x-a} dt \exp(-\varepsilon |t|).$$

We shall need the projections F_Δ and the operators $\Phi_{a,\varepsilon}$ defined by

$$\begin{aligned}(R_0F_\Delta f)(x) &= \chi_\Delta(x)(R_0f)(x)\\ (R_0\Phi_{a,\varepsilon}f)(x) &= \varphi_{a,\varepsilon}(x)(R_0f)(x).\end{aligned} \tag{32}$$

These operators are related as follows (see the Appendix for a proof):

LEMMA.

(a)
$$\int_{-\infty}^{\infty} dt e^{-\varepsilon|t|}U_0(-t)F_{(a,\infty)}U_0(t) = \Phi_{a,\varepsilon}. \tag{33}$$

(b) *If* $-\infty < a \leqq b < \infty$, *then*

$$s-\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty} dt e^{-\varepsilon|t|}U_0(-t)F_{(a,b)}U_0(t) = (b-a)I. \tag{34}$$

In quantum mechanics the time delay operator is defined by

$$T_0 = \lim_{r\to\infty}\int_{-\infty}^{\infty} dt U_0(-t)[W_-^*P_rW - P_r]U_0(t),$$

where P_r is the orthogonal projection onto the subspace $\mathfrak{D}_r$ of $L^2(\mathbb{R}^3)$ formed by all functions $f(\cdot)$ with support in the ball of radius r centered at the origin, and where the limit is defined on some dense subset of $L^2(\mathbb{R}^3)$ in a suitable sense (see [**2**]).

Here we use a formally similar expression to define our "time delay operator" for the Lax–Phillips scattering theory. The projection P_r is replaced by the projection $E^r = I - E_+^r - E_-^r$, where $E_\pm^r$ are the projections defined at the end of Section 2. Thus we define T_0 by

$$T_0 = s-\lim_{r\to\infty} s-\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty} dt e^{-\varepsilon|t|}U_0(-t)[W_-^*E^rW_- - E^r]U_0(t), \tag{35}$$

provided the limit exists (on some linear subset $D(T_0)$ of $\mathfrak{H}$). Some properties of T_0 are collected in the following theorem.

PROPOSITION 2. *T_0 commutes with $U_0(t)$ and is given by*

$$T_0 = s-\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty} dt e^{-\varepsilon|t|}U_0(-t)[W_-^*EW_- - E]U_0(t) \tag{36}$$

$$= s-\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty} dt e^{-\varepsilon|t|}U_0(-t)[E_+^0 - S_0^*E_+^0S_0]U_0(t) \tag{37}$$

$$= s-\lim_{\varepsilon\to+0}(\Phi_{0,\varepsilon} - S_0^*\Phi_{0,\varepsilon}S_0). \tag{38}$$

Here $E \equiv E^a$ is the projection onto the interacting subspace, where a is the number appearing in (iv) *of* (LP). (*The statement means that, if any of the limits* (35), (36), (37) *or* (38) *exists, then they all exist and are equal*).

Proof. (i) The commutativity of T_0 and $U_0(t)$ is obtained in the same way as the corresponding result in [**4**].

(ii) Let $r \geqq a$. Using $E^r = E + F_{(-r,-a)} + F_{(a,r)}$ and (22), we get

$$
\begin{aligned}
W_-^* E^r W_- - E^r &= W_-^* E W_- - E - F_{(-r,-a)} - F_{(a,r)} \\
&\quad + (E_- + S_0^* E_+ + V^* E)(F_{(-r,-a)} + F_{(a,r)}) \\
&\quad \times (E_- + E_+ S_0 + EV) \\
&= (W_-^* E W_- - E) - (F_{(-r,-a)} + F_{(a,r)}) + F_{(-r,-a)} \\
&\quad + S_0^* F_{(a,r)} S_0 \\
&= (W_-^* E W_- - E) + S_0^* F_{(a,r)} S_0 - F_{(a,r)}.
\end{aligned}
$$

This identity implies the equivalence of (35) and (36) since, by virtue of (34),

$$
s - \lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} dt e^{-\varepsilon |t|} U_0(-t)[S_0^* F_{(a,r)} S_0 - F_{(a,r)}] U_0(t) = 0.
$$

(iii) By (22) and (23), we have

$$
\begin{aligned}
W_-^* E W_- &= (E_- + S_0^* E_+ + V^* E) E (E_- + E_+ S_0 + EV) \\
&= V^* E V = V^* V = E + E_+ - S_0^* E_+ S_0.
\end{aligned}
$$

Thus

$$
W_-^* E W_- - E = E_+ - S_0^* E_+ S_0 = U(a)[E_+^0 - S_0^* E_+^0 S_0] U(-a).
$$

Insertion into (36) leads to

$$
T_0 = U_0(a)\, s - \lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} dt e^{-\varepsilon |t|} U_0(-t/[E_+^0 - S_0^* E_+^0 S_0] U_0(t) U_0(-a),
$$

which gives (37) since T_0 commutes with $U_0(a)$. Equation (38) follows immediately from (37) and (33). ■

Equation (38) gives a convenient expression for the time delay operator T_0. It can be used to relate T_0 to the scattering operator S_0. For this one introduces the spectral representation of H_0, the infinitesimal generator of the group $U_0(t)$ which is such that $U_0(t) = \exp(-iH_0 t)$. If $\mathscr{F}$ denotes the Fourier transformation in $L^2(\mathbb{R}, dx; \mathfrak{K})$, then the operator $K_0 := \mathscr{F} R_0$ diagonalizes H_0. Since S_0 and T_0 commute with $U_0(t)$, $K_0 S_0 K_0^{-1}$ and $K_0 T_0 K_0^{-1}$ are decomposable in $L^2(\mathbb{R}, d\lambda; \mathfrak{K})$, i.e. $K_0 S_0 K_0^{-1} = \{S_0(\lambda)\}$ and $K_0 T_0 K_0^{-1} = \{T_0(\lambda)\}$, where $S_0(\lambda)$ and $T_0(\lambda)$ are operators acting in $\mathfrak{K}$. The relation between these operators is given in the following proposition.

PROPOSITION 3. *Assume* $\lambda \mapsto S_0(\lambda)$ [*or* $\lambda \mapsto S_0(\lambda)^*$] *is strongly continuously differentiable. Then* $D(T_0)$ *contains the dense set* $\mathfrak{M}$ *of all f such that* $\lambda \mapsto (K_0 f)(\lambda)$ *has compact support.* T_0 *is essentially self-adjoint and satisfies*

$$
T_0(\lambda) = -iS_0(\lambda)^* \, \partial_\lambda S_0(\lambda) \qquad \text{or} \qquad T_0(\lambda) = i[\partial_\lambda S_0(\lambda)^*] S_0(\lambda). \tag{39}
$$

Proof. By (38)

$$
T_0 f = s - \lim_{\varepsilon \to +0} T_{0,\varepsilon} f := s - \lim_{\varepsilon \to +0} S_0^*[S_0, \Phi_{0,\varepsilon} - \varepsilon^{-1} I] f, \tag{40}
$$

where $[A, B]$ denotes the commutator of A and B. The Fourier transform of

$\phi_{0,\varepsilon}-\varepsilon^{-1}$, as a distribution, is readily computed to be $(-i\varepsilon)[\pi\lambda(\varepsilon^2+\lambda^2)]^{-1}$. Hence, in $L^2(\mathbb{R}, d\lambda; \mathfrak{K})$, the operator $S_0^*[S_0, \Phi_{0,\varepsilon}-\varepsilon^{-1}I]$ is an integral operator with kernel

$$q_\varepsilon(\lambda, \lambda') = -\frac{i\varepsilon}{\varepsilon^2+(\lambda-\lambda')^2} S_0(\lambda)^* \frac{S_0(\lambda)-S_0(\lambda')}{\pi(\lambda-\lambda')}$$

$$:= -\frac{i\varepsilon}{\varepsilon^2+(\lambda-\lambda')^2} g(\lambda, \lambda'), \tag{41}$$

where $g(\lambda, \lambda')$ maps $\mathfrak{K}$ into $\mathfrak{K}$. By our assumptions, $g(\lambda, \lambda')$ is continuous in λ and λ'. Also, for each compact set Δ, there is a constant c_Δ such that

$$\sup_{\lambda'\in\Delta} \|g(\lambda, \lambda')\| \leqq c_\Delta(1+|\lambda|)^{-1}. \tag{42}$$

Let $\mathfrak{M}_0 := \{f \in \mathfrak{M} \mid \lambda \mapsto (K_0 f)(\lambda)$ is strongly continuous$\}$. Using standard estimates of Poisson integrals (e.g. [**6**, Th. II2.1], one finds from the above properties of $g(\lambda, \lambda')$ that, for $f \in \mathfrak{M}_0$,

$$\int d\lambda' \frac{\varepsilon}{\varepsilon^2+(\lambda-\lambda')^2} g(\lambda, \lambda')(K_0 f)(\lambda') \to \pi g(\lambda, \lambda)(K_0 f)(\lambda) \tag{43}$$

pointwise, as $\varepsilon \to +0$, and that the norm of the l.h.s. of (43) is majorized, uniformly in $0<\varepsilon\leqq 1$, by the square-integrable function $c(f)(1+|\lambda|)^{-1}$. Hence, for $f \in \mathfrak{M}_0$,

$$\begin{aligned} s-\lim_{\varepsilon\to+0} S_0^*[S_0, \Phi_{0,\varepsilon}-\varepsilon^{-1}I]f &= \{\pi g(\lambda, \lambda)(K_0 f)(\lambda)\} \\ &= \{-iS_0(\lambda)^*[\partial_\lambda S_0(\lambda)](K_0 f)(\lambda)\}. \end{aligned} \tag{44}$$

(44) shows that $\mathfrak{M}_0 \subseteq D(T_0)$. To prove that $\mathfrak{M} \subseteq D(T_0)$, it now suffices to show that for each $0<a<\infty$, $\|T_{0,\varepsilon}F_{(-a,a)}\|$ is bounded by a constant $C(a)$ which is independent of ε. For this, we use the triangle inequality, (42) and [**6**, Th. II2.1, Eqn (2.2)]:

$$\begin{aligned} \int d\lambda\, &\|(K_0 T_{0,\varepsilon} F_{(-a,a)} f)(\lambda)\|^2 \\ &\leqq \int d\lambda \left| \int_{-a}^{a} d\lambda' \frac{\varepsilon}{\varepsilon^2+(\lambda-\lambda')^2} \|g(\lambda, \lambda')\|\, \|(K_0 f)(\lambda')\| \right|^2 \\ &\leqq c_{[-a,a]}^2 \|K_0 f\|^2 = c_{[-a,a]}^2 \|f\|^2, \end{aligned} \tag{45}$$

so that $\|T_{0,\varepsilon}F_{(-a,a)}\| \leqq c_{[-a,a]}$. Hence T_0 is a symmetric extension of the essentially self-adjoint operator A given by the r.h.s. of (44), with $D(A)=\mathfrak{M}$. This shows that T_0 is essentially self-adjoint. ■

Remark. If one assumes in adddition that $\|\partial_\lambda S_0(\lambda)\| \leqq c < \infty$ for all $\lambda \in \mathbb{R}$, one has $\|g(\lambda, \lambda')\| \leqq \pi^{-1}c$, so that by the argument in (45), $\|T_{0,\varepsilon}\| \leqq \pi^{-1}c$ for all $\varepsilon>0$. Hence in this case $D(T_0)=\mathfrak{H}$.

The relation between T_0 and the operator T introduced in [**4**] is very simple.

Using the fact that here $D=E=F_{(-a,a)}$, we obtain from (13), (36) and (34) that

$$W_-^*TW_- = s-\lim_{\varepsilon\to+0}\int_{-\infty}^{\infty} dte^{-\varepsilon|t|}U_0(-t)W_-^*EW_-U_0(t)$$
$$= T_0+(2a)I. \tag{46}$$

Thus the total energy of interaction of a wave in the state W_-f is equal to the expectation value of T_0 in the state f plus the product of the energy of the state f and the size 2a of the "interaction region".

Since $R_+=R_0U_0(-a)W_+^{-1}$ [**5**], (46) implies that

$$R_+TR_+^{-1}=R_0S_0T_0S_0^*R_0^{-1}+(2a)I,$$

i.e. in the outgoing spectral representation of $U(t)$:

$$T(\lambda)=S_0(\lambda)T_0(\lambda)S_0(\lambda)^*+2a=iS_0(\lambda)\ \partial_\lambda S_0(\lambda)^*+2a,$$

where we have used the second expression in (39) and the unitarity of $S_0(\lambda)$. Now by (19), $S_0(\lambda)=\exp(-2ia\lambda)S(\lambda)$, so that (under the hypothesis of Proposition 3)

$$T(\lambda)=iS(\lambda)\ \partial_\lambda S(\lambda)^*. \tag{47}$$

This is nothing but the relation (14) due to Lax and Phillips [**4**]. In fact, these authors introduce the spectral representation of $U(t)$ such that $U(t)$ is multiplication by $\exp(i\lambda t)$, whereas here $U(t)$ is multiplication by $\exp(-i\lambda t)$. Hence (14) follows from (47) upon replacing λ by $-\lambda$ and taking into account that $\mathscr{S}(\lambda)=S(-\lambda)$ and $\mathscr{T}(\lambda)=T(-\lambda)$.

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Appendix: Proof of the lemma

We have

$$U_0(-s)F_{(a,c)}U_0(s)=F_{(a-s,c-s)}. \tag{48}$$

For $\varepsilon>0$ and any a, c in $[-\infty,\infty]$, $\int_{-\infty}^{\infty} dte^{-\varepsilon|t|}U_0(-t)F_{(a,c)}U_0(t)$ exists as an improper strong Riemann integral. We may therefore calculate the integral over $[-M, M]$ as a limit of approximating Riemann sums for a particular sequence $\{\Pi_n\}$ of partitions. We take Π_n to be a partition of $[-M, M]$ into n intervals Δ_k of length $\eta=2M/n$, and we denote by t_k the left endpoint of Δ_k. Then, for $\varepsilon>0$ (see [**1**, Ch. 6]),

$$\int_{-M}^{M} dte^{-\varepsilon|t|}F_{(-t,\infty)}f = s-\lim_{n\to\infty}\eta\sum_{k=1}^{n}e^{-\varepsilon|t_k|}F_{(-t_k,\infty)}f$$
$$= s-\lim_{n\to\infty}\left(\eta\sum_{k=1}^{n}\sum_{i=1}^{k}e^{-\varepsilon|t_i|}F_{\Delta_k}f+\eta\sum_{i=1}^{n}e^{-\varepsilon|t_i|}F_{(M,\infty)}f\right)$$
$$= \int_{-M}^{M}\left(\int_{-M}^{x}dte^{-\varepsilon|t|}\right)F_{dx}f+\int_{-M}^{M}dte^{-\varepsilon|t|}F_{(M,\infty)}f.$$

Letting $M \to \infty$, we obtain

$$\int_{-\infty}^{\infty} dte^{-\varepsilon|t|}F_{(-t,\infty)}f = \int_{-\infty}^{\infty} \varphi_{0,\varepsilon}(x)F_{dx}f = \Phi_{0,\varepsilon}f,$$

which, together with (48), implies (33).

By (33), the l.h.s. of (34) equals $\Phi_{a,\varepsilon} - \Phi_{b,\varepsilon}$. Now, since $|\phi_{a,\varepsilon}(x)| \leqq 2$, we have for each $f \in \mathfrak{H}$

$$\|\Phi_{a,\varepsilon}f - \Phi_{b,\varepsilon}f - (b-a)f\|^2 \leqq 3[8+(b-a)^2]\,\|[I - F_{(-M,M)}]f\|^2 + \sup_{-M \leqq x \leqq M} |\varphi_{a,\varepsilon}(x) - \varphi_{b,\varepsilon}(x) - (b-a)|^2\,\|F_{(-M,M)}f\|^2.$$

Since for each fixed $M < \infty$, $\phi_{a,\varepsilon} - \phi_{b,\varepsilon} - (b-a)$ converges to zero in $L^\infty([-M, M])$, this implies that

$$\|\Phi_{a,\varepsilon}f - \Phi_{b,\varepsilon}f - (b-a)f\| \to 0 \quad \text{as} \quad \varepsilon \to +0,$$

which proves (34). ■

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