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Stability of the duality gap in linear optimization

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Abstract In this paper we consider the duality gap function g that measures the difference between the optimal values of the primal problem and of the dual problem in linear programming and in linear semi-infinite programming. We analyze its behavior when the data defining these problems may be perturbed, considering seven different scenarios. In particular we find some stability results by proving that, under mild conditions, either the duality gap of the perturbed problems is zero or $+\infty$ around the given data, or g has an infinite jump at it. We also give conditions guaranteeing that those data providing a finite duality gap are limits of sequences of data providing zero duality gap for sufficiently small perturbations, which is a generic result.

Keywords Linear programming · Linear semi-infinite programming · Duality gap function · Stability · Primal-dual partition

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1 Introduction

Linear optimization consists in the minimization of a linear objective function subject to linear constraints. Here the duality gap plays an important role both for theoretical and for

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computational purposes. It is defined as the difference of the optimal value of the primal problem and of the optimal value of its dual problem, and it depends on the data defining these optimization problems. We will analyze the behavior of this duality gap when these data are perturbed; in doing this we will consider seven different scenarios related to the parameters that present admissible perturbations and the ones that remain fixed. To be more precise we need to establish some definitions and notation.

The *primal problem* we consider in this paper is given by

$$P : \inf_{x \in \mathbb{R}^n} \{c'x : a'_t x \geq b_t \ \forall t \in T\}, \quad (1)$$

where T is an arbitrary set, $a_t \in \mathbb{R}^n$, $b_t \in \mathbb{R}$, for all $t \in T$, and $c \in \mathbb{R}^n$; c' denotes the transpose of c , while a'_t is the transpose of a_t . We say that P is a linear semi-infinite programming (LSIP) problem for arbitrary, possibly infinite, index set T ; P is said to be a linear programming (LP) problem when T is finite.

By defining $a : T \rightarrow \mathbb{R}^n$, $a(t) = a_t$, and $b : T \rightarrow \mathbb{R}$, $b(t) = b_t$, we identify the problem P with the triplet $(a, b, c) \in (\mathbb{R}^n)^T \times \mathbb{R}^T \times \mathbb{R}^n$.

We associate with P its Haar's dual problem D which is the ordinary dual problem in LP (See (3) for its definition in the next section). We denote by $v(P)$ the optimal value of P , with $v(P) = +\infty$ when P is an inconsistent problem, i.e. the feasible set of P is empty. Similarly, $v(D)$ stands for the optimal value of D , with $v(D) = -\infty$ when D is inconsistent. So, $v(P), v(D) \in \mathbb{R} := \mathbb{R} \cup \{\pm\infty\}$. The dual problem D has been defined in such a way that the *weak duality theorem* $v(D) \leq v(P)$ always holds. The *duality gap* of the pair (P, D) is the difference $v(P) - v(D)$, if $v(P), v(D) \in \mathbb{R}$, and $+\infty$, otherwise.

Recall that, in LSIP, the duality gap is related to the viability of solving P by discretization of the index set. More precisely, $v(P) = v(D)$ if and only if P is *discretizable*, i.e., there exists a sequence $\{T_r\}$ of finite subsets of T such that, denoting by $v(P_r)$ the optimal value of the LP subproblem

$$P_r : \inf_{x \in \mathbb{R}^n} \{c'x : a'_t x \geq b_t \ \forall t \in T_r\},$$

we have $v(P) = \lim_r v(P_r)$ (see [1] and [10, Table 8.1]). Hence, the importance of studying the behavior of this duality gap.

Obviously the duality gap depends on the data (a, b, c) , which sometimes are not known with certainty. In this paper we are interested in the behavior of this gap under small perturbations of all, or some, of the data by considering different scenarios relative to the admissible perturbations. To this aim, we embed the nominal problem P , corresponding to a given triplet (a, b, c) , into a suitable topological space of admissible perturbed triplets, the so-called *space of parameters* Θ . Denoting by $v^P(a, b, c)$ the optimal value of P and by $v^D(a, b, c)$ the optimal value of D , the duality gap of (a, b, c) is

$$g(a, b, c) := \begin{cases} v^P(a, b, c) - v^D(a, b, c), & \text{if } v^P(a, b, c), v^D(a, b, c) \in \mathbb{R}, \\ +\infty, & \text{otherwise.} \end{cases}$$

Observe that v^P and v^D are positively homogeneous, so that the *duality gap function* $g : \Theta \rightarrow \mathbb{R}_+ \cup \{+\infty\}$ is positively homogeneous too.

Some works, as [4] and [10], define the duality gap as 0 when the primal and dual problems have both the same infinite value (e.g. when one of them is inconsistent and the other is unbounded). Our present definition as $+\infty$ when $v^P(a, b, c)$ and $v^D(a, b, c)$ are not real numbers follows from the weak duality property and the usual criterion of defining $(\pm\infty) - (\pm\infty) = +\infty$ in the context of difference of convex (DC in short) functions when considering extended real-valued convex functions (see [17] and references therein).

Many authors have studied, from different perspectives, the effect of small changes in some elements of the triplet on the feasible set, on the optimal set, and on the optimal value. For instance, [21] considers only perturbations of b , [3] analyzes independent perturbations of b and c , as well as perturbations of the pair (a, b) , [8] considers simultaneous perturbations of the whole triplet (a, b, c) , [13] estimates the effect on the optimal value of perturbations of the *rim data* (b, c) in LP. On the other hand, [16], [5], and [6] deal with the generic uniqueness of optimal solutions in LP, the stability of the feasible set, and the stability of the optimal set, respectively, under perturbations of the *technological data* a . In particular, [9] studies the variation of the optimal value under directional perturbations of a in LP (other works dealing with arbitrary perturbations of a are mentioned there). It is worth noticing that, as observed by many authors (e.g. [11], [6] and [9]), the perturbations of a are the hardest to deal with for both the stability and the sensitivity analyses. Moreover, there is a broad literature on the relationship between the semicontinuity, or the subdifferentiability, of the optimal value under perturbations of b for non necessarily linear problems (e.g. [15]). To the authors' knowledge, this is the first work where a stability problem (in this case the stability of the duality gap) is simultaneously studied for perturbations of all nonempty subsets of the triplet (a, b, c) , which gives rise to seven different scenarios.

The term "gap function" has been first used in optimization by Auslender [2], who considered the variational inequality problem consisting of finding $x \in X$ such that

$$F(x)'(y-x) \geq 0 \quad \forall y \in X, \quad (2)$$

where X is a given nonempty closed convex subset of \mathbb{R}^n and $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a given continuously differentiable monotone mapping. In fact, introducing the so-called *gap function* $G(x) := \max_{y \in X} F(x)'(x-y)$, (2) turns out to be equivalent to the optimization problem $\inf_{x \in X} G(x)$. Different gap functions enjoying desirable properties have been proposed for the variational inequality problem (2) and for other related problems (e.g., vector variational, quasi-variational and pre-variational inequality problems, vector equilibrium problems, general equilibrium problems, etc.) in order to transform them into optimization problems to be solved with numerical optimization methods (typically, descent methods). Other gap functions have been introduced in convex optimization. For instance, [14] introduces a gap function expressing the duality gap of a given convex optimization problem in order to get a convex optimization algorithm.

In the framework of continuous infinite linear programming, with decision space X being a Banach space, [22] introduces the gap function $g(b, c)$ defined only for those pairs (b, c) providing primal and dual feasible problems (assumption guaranteeing that $g(b, c) \in \mathbb{R}$, by the weak duality theorem) and keeping the gradient vectors a_t fixed. It is worth observing that the LSIP problem P in (1) does not satisfy the assumptions of [22] except in the case of *continuous* LSIP (where T is a compact Hausdorff topological space and $Ax := (a'_t x)_{t \in T}$, for all $x \in \mathbb{R}^n$, defines a continuous linear map from \mathbb{R}^n to the Banach space $\mathcal{C}(T)$).

This paper is organized as follows: Section 2 introduces the dual problem and discusses known results about the basic primal-dual partition in the LSIP and LP frameworks. Section 3 provides characterizations of those parameters which maintain the consistency (or the inconsistency) of the corresponding primal (and dual) problem through sufficiently small perturbations of the data. Section 4 characterizes the parameters that maintain the simultaneous consistency of both the primal and the dual problems under small perturbations of the data in seven different scenarios. These results are applied to get conditions for the stability of the duality gap function and to give conditions guaranteeing that those data providing a finite duality gap are limits of sequences of data providing zero duality gap for sufficiently

small perturbations. Moreover, this section gives sufficient conditions for a given non simultaneously (primal and dual) consistent parameter to maintain this property under small perturbations. Finally, Section 5 is devoted to the particular case of LP problems.

2 The dual problem and the basic primal-dual partition

We begin this section by introducing some necessary notation. Given a topological real vector space X and $Y \subset X$, $\text{int} Y$, $\text{rint} Y$, $\text{cl} Y$, and $\text{bd} Y$ denote the interior, the relative interior, the closure, and the boundary of Y , respectively. Moreover, $\text{conv} Y$ stands for the convex hull of Y , whereas $\text{cone} Y := \mathbb{R}_+ \text{conv} Y$ means the convex conical hull of $Y \cup \{0_X\}$, where 0_X denotes the null of X . For $X = \mathbb{R}^n$, we represent by \geq the usual partial order, by $\mathbf{0}_n$ the vector of all 0's, and by $\|\cdot\|_\infty$ the Chebyshev norm. Finally, \lim_k stands for $\lim_{k \rightarrow \infty}$.

Now, we will recall known results from the LSIP and LP frameworks; all the primal problems P are as in (1), with $(a, b, c) \in (\mathbb{R}^n)^T \times \mathbb{R}^T \times \mathbb{R}^n$.

The Haar's dual problem of P is

$$D: \sup_{\lambda \in \mathbb{R}_+^{(T)}} \left\{ \sum_{t \in T} \lambda_t b_t : \sum_{t \in T} \lambda_t a_t = c \right\}, \quad (3)$$

where $\mathbb{R}_+^{(T)}$ is the positive cone in the space $\mathbb{R}^{(T)}$ of *generalized finite sequences* formed by all functions $\lambda \in \mathbb{R}^T$ such that $\lambda_t = 0$ for all $t \in T$ except maybe for a finite number of indices. Observe that D is a LSIP problem whenever T is infinite and it is a LP problem otherwise. If $T = \{1, \dots, m\}$, one has $\mathbb{R}^{(T)} = \mathbb{R}^T = \mathbb{R}^m$ and $\mathbb{R}_+^{(T)} = \mathbb{R}_+^m$, and P is the LP problem (in canonical format) given by

$$P: \inf_{x \in \mathbb{R}^n} \{c'x : Ax \geq b\}, \quad (4)$$

with $A (m \times n)$, while D is its ordinary dual problem (in standard format), i.e.,

$$D: \sup_{\lambda \in \mathbb{R}_+^m} \{b'\lambda : A'\lambda = c\}. \quad (5)$$

Other dual problems can be associated with P following general schemes. For instance, the Lagrangian dual problem, or the dual problem in Rockafellar's sense ([19]), whose feasible set is $\mathbb{R}^{(T)}$. Nevertheless most of them are equivalent to the Haar's dual problem, which is the one that we consider here.

The space of parameters Θ is determined by the admissible perturbations and it can be partitioned in different ways. The *basic primal (dual) partition* is formed by the cone Π_c (Δ_c) of parameters providing a consistent primal (dual) problem and the cone Π_i (Δ_i) of parameters providing an inconsistent primal (dual) problem. Related to the optimal values v^P and v^D , the basic primal partition $\{\Pi_c, \Pi_i\}$, is formed by the sets of parameters with $v^P < +\infty$ and $v^P = +\infty$, respectively, while the basic dual partition $\{\Delta_c, \Delta_i\}$, is determined by the parameters with $v^D > -\infty$ and $v^D = -\infty$, respectively.

We denote $\Theta_{\alpha\beta} := \Pi_\alpha \cap \Delta_\beta$, for $\alpha, \beta \in \{c, i\}$. The cells in Table 1 below, which are the intersections of the corresponding entries, indicate all the different states in the *basic primal-dual partition*.

	Δ_c	Δ_i
Π_c	Θ_{cc}	Θ_{ci}
Π_i	Θ_{ic}	Θ_{ii}

Table 1

Here the domain of the extended function g is $\Theta_{cc} = \Pi_c \cap \Delta_c$. Since the entries of Table 1 are cones, the sets of the basic primal-dual partition are cones too. In particular, the domain of g and its complement, $\Theta \setminus \Theta_{cc}$, are cones.

We associate with each triplet $(a, b, c) \in (\mathbb{R}^n)^T \times \mathbb{R}^T \times \mathbb{R}^n$ the *first moment cone*

$$\text{cone}(a) := \text{cone}\{a_t, t \in T\}$$

and the *second moment cone*

$$\text{cone}(a, b) := \text{cone}\{(a_t, b_t), t \in T\}.$$

We will also use the following notation for short:

$$\text{conv}(a) := \text{conv}\{a_t, t \in T\} \quad \text{and} \quad \text{conv}(a, b) := \text{conv}\{(a_t, b_t), t \in T\},$$

with $\text{cone}(a) = \mathbb{R}_+ \text{conv}(a)$ and $\text{cone}(a, b) = \mathbb{R}_+ \text{conv}(a, b)$. In LP, $\text{cone}(a)$ and $\text{cone}(a, b)$ are polyhedral convex cones while $\text{conv}(a)$ and $\text{conv}(a, b)$ are polytopes.

For any other parameter $(a^1, b^1, c^1) \in (\mathbb{R}^n)^T \times \mathbb{R}^T \times \mathbb{R}^n$ we will also use the brief notation $\text{cone}(a^1) := \text{cone}\{a_t^1, t \in T\}$ and so on.

It is known ([10, Chapter 4]) that the set Π_c of parameters providing primal consistent problems is characterized as

$$(a, b, c) \in \Pi_c \Leftrightarrow (\mathbf{0}_n, 1) \notin \text{cl cone}(a, b), \quad (6)$$

while the obvious characterization of Δ_c is

$$(a, b, c) \in \Delta_c \Leftrightarrow c \in \text{cone}(a). \quad (7)$$

From the previous equivalences,

$$(a, b, c) \in \Theta_{cc} \Leftrightarrow (\mathbf{0}_n, 1) \notin \text{cl cone}(a, b) \text{ and } c \in \text{cone}(a). \quad (8)$$

In LP, the simultaneous consistency of P and D implies a zero duality gap together with the solvability of both problems. Nonetheless, this is not the case in LSIP where it is possible to have a finite positive duality gap for consistent problems P and D , see [10, Example 2.1]. The classical LSIP duality theorems ([20], [4]) give conditions guaranteeing a zero duality gap together with the solvability of one of the coupled problems when P and D are both consistent. The first duality theorem establishes that $c \in \text{rint cone}(a)$ implies $g(a, b, c) = 0$ with P solvable while the second duality theorem asserts that the closedness of $\text{cone}(a, b) + \mathbb{R}_+\{(\mathbf{0}_n, -1)\}$ also entails $g(a, b, c) = 0$ but now together with the solvability of D .

As we have stated, the space of parameters Θ is determined by the admissible perturbations, i.e., the elements of the triplet (a, b, c) which can be perturbed. To avoid a somewhat cumbersome notation when dealing with different scenarios, we will always use the notation Θ for the parameter space which will have to be identified from the context; when all data are perturbable then $\Theta = (\mathbb{R}^n)^T \times \mathbb{R}^T \times \mathbb{R}^n$, when only a and b are perturbable while c remains fixed then $\Theta = (\mathbb{R}^n)^T \times \mathbb{R}^T \times \{c\}$ which is identified with $(\mathbb{R}^n)^T \times \mathbb{R}^T$, and so on. In this paper we consider the seven parameter spaces introduced in Table 2 below. Observe that all of them are real linear spaces.

Scenario	Perturbable data	Parameter space Θ
1	(a, b, c)	$(\mathbb{R}^n)^T \times \mathbb{R}^T \times \mathbb{R}^n$
2	(a, b)	$(\mathbb{R}^n)^T \times \mathbb{R}^T$
3	(a, c)	$(\mathbb{R}^n)^T \times \mathbb{R}^n$
4	(b, c)	$\mathbb{R}^T \times \mathbb{R}^n$
5	a	$(\mathbb{R}^n)^T$
6	b	\mathbb{R}^T
7	c	\mathbb{R}^n

Table 2

We choose topologies on the spaces of parameters corresponding to a reasonable measure of the size of the admissible perturbations. The box Chebyshev (pseudo) norm of $\theta = (a, b, c) \in (\mathbb{R}^n)^T \times \mathbb{R}^T \times \mathbb{R}^n$ is

$$\|(a, b, c)\|_\infty := \max \{ \|c\|_\infty, \sup_{t \in T} \|(a_t, b_t)\|_\infty \}.$$

The box Chebyshev (pseudo) distance between $\theta_1 \in (\mathbb{R}^n)^T \times \mathbb{R}^T \times \mathbb{R}^n$ and $\theta_2 \in (\mathbb{R}^n)^T \times \mathbb{R}^T \times \mathbb{R}^n$ is $d_\infty(\theta_1, \theta_2) := \|\theta_1 - \theta_2\|_\infty$. Obviously, d_∞ describes the topology of the uniform convergence. All the different spaces of parameters shall be seen as subspaces of $(\mathbb{R}^n)^T \times \mathbb{R}^T \times \mathbb{R}^n$ equipped with the corresponding restriction of d_∞ . In LP, where T is finite, all these parameter spaces are connected. However, it is not the case in LSIP when T is infinite because then it is well known that Θ is not a connected space with the box topology. The following example shows Θ_{cc} with different behaviors in different scenarios, for the same decision space \mathbb{R}^n and (finite) index set T .

Example 1 Let us consider the simplest case that $n = |T| = 1$, with $(a, b, c) = (0, 1, 0)$. It is easy to see that in

- scenario 1, $\Theta_{cc} = (\{0\} \times \mathbb{R}_- \times \{0\}) \cup (\mathbb{R}_{++} \times \mathbb{R} \times \mathbb{R}_+) \cup (\mathbb{R}_- \times \mathbb{R} \times \mathbb{R}_-)$ is connected but not convex,
- scenario 2, $\Theta_{cc} = \mathbb{R}^2 \setminus (\{0\} \times \mathbb{R}_{++})$ is also connected but not convex,
- scenario 3, $\Theta_{cc} = \mathbb{R}^2 \setminus (\{0\} \times \mathbb{R})$ is not even connected,
- scenario 4, $\Theta_{cc} = \mathbb{R}_- \times \{0\}$ is convex,
- scenario 5, $\Theta_{cc} = \mathbb{R} \setminus \{0\}$ is not connected,
- scenario 6, $\Theta_{cc} = \mathbb{R}_-$ is convex,
- scenario 7, $\Theta_{cc} = \emptyset$.

Concerning the function g , it is proper in scenarios 1-6, with $g|_{\Theta_{cc}}$ identically zero, and it is improper in scenario 7.

To the best of our knowledge, the stability of the duality gap function g has not been studied up to now in the LP and LSIP framework. Let us start by considering its algebraic behavior where Θ_{cc} is always convex, i.e. in scenarios 4, 6, and 7.

Scenario 6: Here (a, c) is fixed and we assume that $c \in \text{cone}(a)$ (otherwise $\Theta_{cc} = \emptyset$). Let $\alpha \in [0, 1]$ and $b^1, b^2 \in \Theta_{cc}$. Then there exist $x^1, x^2 \in \mathbb{R}^n$ such that $a'_t x^1 \geq b_t^1$ and $a'_t x^2 \geq b_t^2$ for all $t \in T$. Since $a'_t ((1 - \alpha)x^1 + \alpha x^2) \geq (1 - \alpha)b_t^1 + \alpha b_t^2$ for all $t \in T$, Θ_{cc} is convex. The convexity of v^P on Θ_{cc} can be easily obtained from this last property, while the convexity of v^D is obvious. So, g is a DC function.

Scenario 7: Assume that $(\mathbf{0}_n, 1) \notin \text{clcone}(a, b)$ (otherwise $\Theta_{cc} = \emptyset$). Then $\Theta_{cc} = \text{cone}(a)$ is a (non-empty) convex cone. It is known ([10, Theorem 8.1]) that v^D is a proper concave function while v^P coincides with its closure, so that it is concave too. Thus, g is DC too.

Consequently, in scenario 4, where Θ_{cc} is convex too (combine the arguments of scenarios 6 and 7), $g(\cdot, c)$ and $g(b, \cdot)$ are DC functions for any $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^T$, respectively.

3 Interiors of the sets of the basic primal and dual partitions

In order to address the stability of the duality gap we need to discuss first a characterization of the interiors of the sets of the basic primal (dual) partition. We do this mostly by gathering known results. The following simple property will be useful:

Lemma 1 ([18, Lemma 3.4]) *Let $z \in \mathbb{R}^n$ and $\{a_t, t \in T\} \subset \mathbb{R}^n$. If $z \in \text{intcone}(a)$, then there exists $\varepsilon > 0$ such that for all $z^1 \in \mathbb{R}^n$ and $\{a_t^1, t \in T\} \subset \mathbb{R}^n$ satisfying $\|z - z^1\|_\infty < \varepsilon$ and $\sup_{t \in T} \|a_t - a_t^1\|_\infty < \varepsilon$, we have $z^1 \in \text{intcone}(a^1)$.*

From now on we will consider a given data $(a, b, c) \in (\mathbb{R}^n)^T \times \mathbb{R}^T \times \mathbb{R}^n$, and $\theta \in \Theta$ will stand for the corresponding parameter in the different scenarios according to Table 2.

3.1 Interiors of the sets Π_c and Π_i

Lemma 2 (Interior of Π_c) *If $\theta \in \Pi_c$, then the following statements hold:*

(i)

$$\theta \in \text{int}\Pi_c \Leftrightarrow \mathbf{0}_{n+1} \notin \text{clconv}(a, b), \quad (9)$$

in scenarios 1, 2, 4, and 6.

(ii) *In scenarios 3 and 5, if $\sup_{t \in T} b_t \leq 0$, then $\text{int}\Pi_c = \Pi_c = \Theta$, otherwise (9) holds true.*

(iii) *$\text{int}\Pi_c = \Pi_c$ in scenario 7.*

Proof (i) follows from [10, Theorem 6.1] and by observing that the proof there only uses perturbations of b , so that this equivalence holds in scenarios 1, 2, 4, and 6.

(ii) If $\sup_{t \in T} b_t \leq 0$, then $\emptyset \neq \Pi_c = \Theta$ in scenarios 3 and 5 (where b remains fixed), and hence $\text{int}\Pi_c = \Pi_c$. Assume now that $\sup_{t \in T} b_t > 0$. Then an application of [5, Theorem 4.2

(iii) \Leftrightarrow (iv)] gives that $\theta \in \text{int}\Pi_c$ in scenarios 3 and 5 is equivalent to $\theta \in \text{int}\Pi_c$ in scenario 1, hence (9) follows.

(iii) In scenario 7, (a, b) remains fixed, so $\Pi_c = \Theta$ whenever $\Pi_c \neq \emptyset$. \square

Remark 1 It is obvious that in scenario 7, where (a, b) remains fixed, we always have $\Pi_c = \emptyset$ or $\Pi_c = \Theta$.

On the other hand, the inconsistency of some parameter, in any scenario, is maintained in some neighborhood when $(\mathbf{0}_n, 1) \in \text{intcone}(a, b)$. We will discuss this condition and other sufficient conditions for the interior of Π_i .

The next lemma provides an easy to check sufficient condition for θ to be an interior point of Π_i , in any scenario.

Lemma 3 *If $\sup_{t \in T} b_t = +\infty$ and $\sup_{t \in T} \|a_t\| < +\infty$, then $\theta \in \text{int}\Pi_i$ in scenarios 1 to 7.*

Proof Clearly in this case $(\mathbf{0}_n, 1) \in \text{clcone}(a^1, b^1)$ for all data at a finite distance of θ in any scenario. Thus, (6) yields $\theta \in \text{int}\Pi_i$. \square

Lemma 4 (*Interior of Π_i*) If $\theta \in \Pi_i$, then the following statements hold:

- (i) $(\mathbf{0}_n, 1) \in \text{int cone}(a, b)$ is a sufficient condition for $\theta \in \text{int } \Pi_i$ in any scenario.
(ii) If $\sup_{t \in T} b_t < +\infty$, then

$$\theta \in \text{int } \Pi_i \Leftrightarrow (\mathbf{0}_n, 1) \in \text{int cone}(a, b), \quad (10)$$

in scenarios 1, 2, 3, and 5.

- (iii) If $\inf_{t \in T} b_t > 0$, then $\theta \in \text{int } \Pi_i$, in scenarios 4 and 6.
(iv) $\text{int } \Pi_i = \Pi_i$ in scenario 7.

Proof (i) In any scenario, the inconsistency of some parameter is maintained in some neighborhood when $(\mathbf{0}_n, 1) \in \text{int cone}(a, b)$, on behalf of (6) and Lemma 1.

(ii) The proofs in [10, Theorem 6.3 and Lemma 6.5] only involve perturbations of a , so that (10) is valid in scenarios 1, 2, 3 and 5 provided that $\sup_{t \in T} b_t < +\infty$.

(iii) Let $0 < \varepsilon < \inf_{t \in T} b_t$. As $(\mathbf{0}_n, 1) \in \text{cl cone}(a, b)$, we can write

$$(\mathbf{0}_n, 1) = \lim_k \sum_{t \in T} \lambda_t^k (a_t, b_t),$$

for some sequence $\{\lambda^k\} \subset \mathbb{R}_+^{(T)}$. In particular, we have

$$\varepsilon \limsup_k \sum_{t \in T} \lambda_t^k \leq \lim_k \sum_{t \in T} \lambda_t^k b_t = 1,$$

Thus $\liminf_k \sum_{t \in T} \lambda_t^k (b_t - \frac{\varepsilon}{4}) > \frac{1}{2}$ and so $(\mathbf{0}_n, 1) \in \text{cl cone}(a, b - \frac{\varepsilon}{4})$. Now, for any $b^1 \in \mathbb{R}^{(T)}$ with $\|b^1 - b\|_\infty < \frac{\varepsilon}{4}$, it also holds that

$$(\mathbf{0}_n, 1) \in \text{cl cone}(a, b^1).$$

We conclude that $\theta \in \text{int } \Pi_i$ in scenarios 4 and 6, where a is fixed.

(iv) In scenario 7, (a, b) remains fixed, so $\Pi_i = \emptyset$ whenever $\Pi_i \neq \emptyset$. \square

Lemma 5 (*Interior of Π_i in LP*) Assume that T is finite. Let $\theta \in \Pi_i$, then (10) characterizes $\theta \in \text{int } \Pi_i$ in scenarios 1, 2, 3, and 5. Furthermore, Π_i is open in scenarios 4, 6, and 7.

Proof The cases of scenarios 1, 2, 3, and 5 follow from Lemma 4 (ii) because $\sup_{t \in T} b_t < +\infty$ is always true. Now, assume we are in scenarios 4 or 6 and take some $\theta \in \Pi_i$. Then (6) implies that $(\mathbf{0}_n, 1) \in \text{cone}(a, b)$ since this cone is closed because T is a finite set. If $T = \{1, \dots, m\}$, there exist nonnegative scalars $\lambda_1, \dots, \lambda_m$ such that

$$\sum_{t=1}^m \lambda_t (a_t, b_t) = (\mathbf{0}_n, 1).$$

Let $\sum_{t=1}^m \lambda_t = \delta > 0$. Then, for any $b^1 \in \mathbb{R}^m = \mathbb{R}^T$ such that $\|b^1 - b\|_\infty < \frac{1}{8}$ one has $\sum_{t=1}^m \lambda_t b_t^1 > 0$, so that $(\mathbf{0}_n, 1) \in \text{cone}(a, b^1)$ and hence the associated parameter $\theta^1 \in \Pi_i$. Thus θ is an interior point of Π_i . Finally, in scenario 7 we have only two possibilities $\Pi_i = \emptyset$ or $\Pi_i = \emptyset$, thus Π_i is always open in this case. \square

3.2 Interiors of Δ_c and Δ_i

Lemma 6 (*Interior of Δ_c*) If $\theta \in \Delta_c$, then

(i)

$$\theta \in \text{int}\Delta_c \Leftrightarrow c \in \text{intcone}(a). \quad (11)$$

in scenarios 1, 3, 4, and 7.

(ii) In scenarios 2 and 5, if $c = \mathbf{0}_n$, then $\theta \in \text{int}\Delta_c = \Delta_c = \Theta$, while for $c \neq \mathbf{0}_n$ (11) holds true.

(iii) $\text{int}\Delta_c = \Delta_c$ in scenario 6.

Proof First observe that $c \in \text{intcone}(a)$ is a sufficient condition for $\theta \in \text{int}\Delta_c$ in any case as a straightforward consequence of Lemma 1. So the implication (\Leftarrow) in (11) is always true in any scenario.

(i) (\Rightarrow) has been shown in [12, Theorem 5] by using only perturbations of c .

(ii) c remains fixed in scenarios 2 and 5. When $c = \mathbf{0}_n$, then it is clearly valid that $\Delta_c = \Theta$.

Let $c \neq \mathbf{0}_n$. Assume that $\theta \in \text{int}\Delta_c$ and that $c \notin \text{intcone}(a)$. Then $c \in \text{bdcone}(a)$ and so there exists an arbitrarily small vector $u \in \mathbb{R}^n \setminus \{\mathbf{0}_n\}$ such that $u'c \leq 0$ and $u'z \geq 0$ for all $z \in \text{cone}(a)$. We will show that $c \notin \text{cone}(a+u)$. Suppose by contradiction that $c \in \text{cone}(a+u)$, then we can write $c = \mu(z+u)$, with $\mu > 0$ as $c \neq \mathbf{0}_n$, and $z \in \text{conv}(a)$. Multiplying by u we get the following inconsistency:

$$0 \geq u'c = \mu u'(z+u) = \mu (u'z + \|u\|^2) > 0.$$

So we have $c \notin \text{cone}(a+u)$. Since $(a+u, b, c)$ defines a parameter $\theta'' \notin \Delta_c$ arbitrarily close to θ , we get the contradiction $\theta \notin \text{int}\Delta_c$.

(iii) In scenario 6 (a, c) is fixed, thus if $\Delta_c \neq \emptyset$, then $\Delta_c = \Theta$. \square

Lemma 7 (*Interior of Δ_i*) If $\theta \in \Delta_i$, then

(i)

$$\theta \in \text{int}\Delta_i \Leftrightarrow \mathbf{0}_n \notin \text{clconv}(a) \text{ and } c \notin \text{clcone}(a), \quad (12)$$

in scenarios 1 and 3. Moreover, it is also valid in scenarios 2 and 5 when $\{a_t\}_{t \in T}$ is bounded.

(ii)

$$\theta \in \text{int}\Delta_i \Leftrightarrow c \notin \text{clcone}(a),$$

in scenarios 4 and 7.

(iii) $\text{int}\Delta_i = \Delta_i$ in scenario 6.

Proof (i) The equivalence (12) has already been shown in [18, Theorem 3.5 (ii)] by using perturbations of a and c , which is valid only in scenarios 1 and 3. Nonetheless, that proof can be slightly modified in order to show the implication (\Leftarrow) using only perturbations of a , so it also holds in scenarios 2 and 5.

Now, we only need to show the implication (\Rightarrow) in scenarios 2 or 5. Here c is fixed in both scenarios. Assume that $\{a_t\}_{t \in T}$ is bounded. First, suppose that $\theta \in \text{int}\Delta_i$ and $\mathbf{0}_n \in \text{clconv}(a)$. For any $\varepsilon > 0$, let $z = \sum_{t \in T} \lambda_t a_t$ with $\lambda \in \mathbb{R}_+^{(T)}$ and $\sum_{t \in T} \lambda_t = 1$ be such that $\|z\|_\infty < \varepsilon$. Since $c \neq \mathbf{0}_n$, we can put

$$a_t^1 := a_t + \frac{\varepsilon}{\|c\|_\infty} c - z.$$

Then $\|a_t^1 - a_t\|_\infty < 2\varepsilon$ for all $t \in T$ and

$$\sum_{t \in T} \lambda_t a_t^1 = \frac{\varepsilon}{\|c\|_\infty} c,$$

which implies that $c \in \text{cone}(a^1)$, a contradiction with $\theta \in \text{int}\Delta_i$, thus $\mathbf{0}_n \notin \text{clconv}(a)$. Now, suppose that $\theta \in \text{int}\Delta_i$ and assume that $c \in \text{clcone}(a)$. Let $M > 0$ be such that $\text{conv}(a)$ is contained in the ball $B(\mathbf{0}_n; M)$ of radius M . As before, take any $\varepsilon > 0$, and let $z = \sum_{t \in T} \lambda_t a_t$ with $\lambda \in \mathbb{R}_+^{(T)}$ be such that

$$\|z\|_\infty > \frac{\|c\|_\infty}{2} \quad \text{and} \quad \|c - z\|_\infty < \frac{\varepsilon \|c\|_\infty}{2M}.$$

Put

$$\mu := \sum_{t \in T} \lambda_t > 0 \quad \text{and} \quad a_t^1 := a_t + \frac{c - z}{\mu}.$$

Since $\left\| \frac{z}{\mu} \right\|_\infty < M$, we obtain

$$\|a_t^1 - a_t\|_\infty < \frac{1}{\mu} \frac{\varepsilon \|c\|_\infty}{2M} < \frac{M}{\|z\|_\infty} \frac{\varepsilon \|c\|_\infty}{2M} < \frac{2}{\|c\|_\infty} \frac{\varepsilon \|c\|_\infty}{2} = \varepsilon$$

for all $t \in T$, and

$$\sum_{t \in T} \lambda_t a_t^1 = c,$$

which gives that $c \in \text{cone}(a^1)$, a contradiction with $\theta \in \text{int}\Delta_i$. Thus $c \notin \text{clcone}(a)$.

(ii) In scenarios 4 and 7 a remains fixed. Since $\theta \in \Delta_i$, it follows that $c \notin \text{cone}(a)$. If $c \notin \text{clcone}(a)$, clearly we can find a neighborhood V of c such that $c^1 \notin \text{clcone}(a)$ for all $c^1 \in V$; hence $\theta \in \text{int}\Delta_i$. On the other hand, if $c \in \text{clcone}(a)$, then there is a sequence $\{c^k\} \subset \text{cone}(a)$ that converges to c , which gives that $\theta \notin \text{int}\Delta_i$.

(iii) In scenario 6 (a, c) is fixed, thus $\Delta_i = \Theta$ whenever $\Delta_i \neq \emptyset$. \square

Remark 2 When the technological data a is not bounded the implication (\Rightarrow) in (i) in the previous lemma can be false in scenarios 2 and 5 as the following problem shows. Let $n = 2$, $T = \mathbf{N}$, and $P = \inf \{x_2 : kx_1 + k^2x_2 \geq 0, k \in \mathbf{N}\}$, then $c = (0, 1) \in \text{clcone}(\{(k, k^2), k \in \mathbf{N}\})$ and the associated parameter $\theta \in \text{int}\Delta_i$. Nevertheless, the implication (\Leftarrow) is always valid in these scenarios 2 and 5.

4 Stability of the duality gap

The partition $\{\text{int}\Theta_{cc}, \text{bd}\Theta_{cc}, \text{int}(\Theta \setminus \Theta_{cc})\}$ of Θ is related to our stability analysis of g . In fact, if $\theta \in \text{bd}\Theta_{cc}$, then g takes finite values and the value $+\infty$ on every neighborhood of θ , i.e., g has an infinite jump at θ , which is an extreme form of instability. If $\theta \in \text{int}(\Theta \setminus \Theta_{cc})$, then g is identically $+\infty$ in some neighborhood of θ , in which case we say that ∞ -stability holds. Finally, if $\theta \in \text{int}\Theta_{cc}$, we know that g has finite non-negative values in some neighborhood of θ . In this section we show that under mild assumptions g is identically 0 around θ , situation described with the term 0-stability. Here we also complete the list of known characterizations of $\text{int}\Theta_{cc}$, and analyze the relationship between the continuity of g and 0-stability, which turns out to be a generic property on Θ_{cc} .

4.1 Stability of the duality gap under primal-dual consistency

The next result, characterizing $\text{int}\Theta_{cc}$, involves the following two conditions:

$$(C1) \mathbf{0}_{n+1} \notin \text{cl conv}(a, b).$$

$$(C2) c \in \text{int cone}(a).$$

Observe that, by [10, Theorem 6.1], (C1) is equivalent to a primal Slater-type condition: there exists $\bar{x} \in \mathbb{R}^n$ (called *strong Slater point*) and $\varepsilon > 0$ such that $a'_t \bar{x} \geq b_t + \varepsilon$ for all $t \in T$. In turn, condition (C2) can also be seen as a dual Slater-type condition as it is guaranteed by the existence of $\lambda \in \mathbb{R}_+^T$ such that $c = \sum_{t \in T} \lambda_t a_t$ with $\{a_t : \lambda_t > 0\}$ being a set of n linearly independent vectors.

Theorem 1 (The interior of Θ_{cc}) *Table 3 characterizes the membership of an element of Θ_{cc} in its interior $\text{int}\Theta_{cc}$ for a given data $(a, b, c) \in (\mathbb{R}^n)^T \times \mathbb{R}^T \times \mathbb{R}^n$:*

Scenario	Parameter	Characterization of its membership in $\text{int}\Theta_{cc}$
1	(a, b, c)	(C1) and (C2)
2	(a, b)	(C1), if $c = \mathbf{0}_n$, (C1) and (C2), else.
3	(a, c)	(C2), if $\sup_{t \in T} b_t \leq 0$, (C1) and (C2), else.
4	(b, c)	(C1) and (C2)
5	a	(C1) and (C2), if $c \neq \mathbf{0}_n$ and $\sup_{t \in T} b_t > 0$, (C2), if $c \neq \mathbf{0}_n$ and $\sup_{t \in T} b_t \leq 0$, (C1), if $c = \mathbf{0}_n$ and $\sup_{t \in T} b_t > 0$, It holds for sure, if $c = \mathbf{0}_n$ and $\sup_{t \in T} b_t \leq 0$.
6	b	(C1)
7	c	(C2)

Table 3

Proof The fact that $\text{int}\Theta_{cc} = \text{int}\Pi_c \cap \text{int}\Delta_c$ together with Lemmas 2 and 6 provide the proof. \square

Notice that, in Example 1, (C1) holds while (C2) fails. So, the parameter belongs to $\text{int}\Theta_{cc}$ exclusively in scenarios 2, 5, and 6.

Remark 3 Concerning scenario 5, it could be the case of having $\text{int}\Theta_{cc} = \Theta_{cc}$ even though $\sup_{t \in T} b_t > 0$. As an example, let $b \in \mathbb{R}^T$ be a given function such that $\inf_{t \in T} b_t > 0$ and $c = \mathbf{0}_n$. We then have that (C1) holds for all $a \in (\mathbb{R}^n)^T$. In this case, given $a \in \Theta_{cc} \subset (\mathbb{R}^n)^T$, since $\sup_{t \in T} b_t \geq \inf_{t \in T} b_t > 0$, Theorem 1 yields $a \in \text{int}\Theta_{cc}$. This means that $\text{int}\Theta_{cc} = \Theta_{cc}$ not only when $c = \mathbf{0}_n$ and $\sup_{t \in T} b_t \leq 0$.

Theorem 2 (Characterization of 0-stability) *Let $\theta \in \Theta_{cc}$, with Θ being the parameter space for any scenario $k \in \{1, 3, 4, 5, 7\}$ or $k = 2$ with $c \neq \mathbf{0}_n$. Then the following statements are equivalent to each other:*

- (i) g is identically zero in some neighborhood of θ .
- (ii) g is continuous at θ .
- (iii) g is upper semi-continuous at θ .
- (iv) $\theta \in \text{int}\Theta_{cc}$.

If $|T| < \infty$, then the four statements (i) - (iv) are equivalent for any $k \in \{1, \dots, 7\}$.

Proof ((i) \Rightarrow (ii)) and [(ii) \Rightarrow (iii)] are trivial.

[(iii) \Rightarrow (iv)] Given $\gamma > g(\theta)$, there exists a neighborhood V_θ of θ such that $\gamma > g(\theta_1)$ for any $\theta_1 \in V_\theta$. Since g is finite-valued on V_θ , $V_\theta \subset \Theta_{cc}$ and the conclusion follows.

[(iv) \Rightarrow (i)] According to Theorem 1, if $\theta \in \text{int}\Theta_{cc}$, then θ satisfies (C2) in scenarios 1, 2 (with $c \neq \mathbf{0}_n$), 3, 4, 5 (with $c \neq \mathbf{0}_n$), and 7. By Lemma 1, there exists a neighborhood V_θ of θ (w.l.o.g. we assume that $V_\theta \subset \Theta_{cc}$) such that (C2) also holds for any $\theta_1 \in V_\theta$, in which case the first duality theorem yields $g(\theta_1) = 0$ for all $\theta_1 \in V_\theta$. The case of $c = \mathbf{0}_n$ in scenario 5 gives trivially $g = 0$ on Θ_{cc} .

Now we assume that $|T| < \infty$. Here [(iv) \Rightarrow (i)] in any scenario with no extra condition because of the strong duality property. \square

The next example shows that $\theta \in \text{int}\Theta_{cc}$ does not characterize the 0–stability of g in scenario 6 when T is infinite.

Example 2 Let $T = \mathbb{N}$ and let P be the problem

$$\begin{aligned} \inf_{x \in \mathbb{R}^2} \quad & x_2 \\ \text{s.t.} \quad & -x_1 \geq 0, \quad t = 1, \\ & x_2 \geq 0, \quad t = 2, \\ & -\frac{1}{t}x_1 + x_2 \geq 0, \quad t = 3, 4, \dots \end{aligned}$$

Here $a : \mathbb{N} \rightarrow \mathbb{R}^2$ is given by $a_1 = (-1, 0)'$, $a_2 = (0, 1)'$, $a_t = (-\frac{1}{t}, 1)'$, $t = 3, 4, \dots$; $b : \mathbb{N} \rightarrow \mathbb{R}^2$ is the null function, and $c = (0, 1)'$. The unique solution of $\sum_{t \in \mathbb{N}} \lambda_t a_t = c$ in $\mathbb{R}_+^{(\mathbb{N})}$ is $\lambda =$

$(0, 1, 0, \dots, 0, \dots)$. We will only allow for perturbations of b (scenario 6), thus $\Theta = \mathbb{R}^{\mathbb{N}}$. Clearly $\Delta_c = \Theta$. With respect to Π_c , observe that $b \in \Pi_c$ implies $\sup_{\mathbb{N}} b < +\infty$ by (6). On the other hand, $\mathbf{0}_2 \notin \text{conv}(a)$ provides $w \in \mathbb{R}^2 \setminus \{\mathbf{0}_2\}$ and $\delta > 0$ such that $a'_t w \geq \delta$ for all $t \in \mathbb{N}$. If $b \in \mathbb{R}^{\mathbb{N}}$ has an upper bound, then $\rho a'_t w \geq \sup_{\mathbb{N}} b$ for all $t \in \mathbb{N}$ for ρ sufficiently large, so that $b \in \Pi_c$. Hence $\Pi_c = \{b \in \mathbb{R}^{\mathbb{N}} : \sup_{\mathbb{N}} b < +\infty\}$, and therefore $\Theta_{cc} = \{b \in \mathbb{R}^{\mathbb{N}} : \sup_{\mathbb{N}} b < +\infty\}$. We associate with each $\varepsilon \geq 0$ an element $b^\varepsilon \in \Theta = \mathbb{R}^{\mathbb{N}}$ such that

$$b_t^\varepsilon = \begin{cases} -\varepsilon, & \text{if } t = 2, \\ 0, & \text{otherwise.} \end{cases}$$

From Theorem 1, we have $b^\varepsilon \in \text{int}\Theta_{cc}$ for all $\varepsilon \geq 0$. Moreover, it is easy to see that the primal and dual feasible set of b^ε are, respectively, $\mathbb{R}_- \times \mathbb{R}_+$ and the singleton set $\{\lambda\}$, where $\lambda_2 = 1$ and $\lambda_t = 0$ for all $t \in \mathbb{N} \setminus \{2\}$. Since $g(b^\varepsilon) = \varepsilon > 0$ for all $\varepsilon > 0$, g is not 0–stable at b^ε for all $\varepsilon \geq 0$.

Theorem 2 allows to assert that g is 0–stable at the elements of $\text{int}\Theta_{cc}$ in some scenarios, but not all of them. Now, observe that, in Example 1, $\text{int}\Theta_{cc}$ is dense in Θ_{cc} in all cases, except scenario 4 where $\text{int}\Theta_{cc}$ is empty. In fact, the next result shows that, under mild conditions, the 0–stability property is *generic* in topological sense, i.e. it is satisfied at an open dense subset of the domain of g (in fact, $\text{int}\Theta_{cc}$ is dense in Θ_{cc}). See [7] for other concepts of genericity, e.g., in the sense of the measure theory, where the condition that the complement of the given open set w.r.t. the set of parameters enjoying the aimed property is required to have zero measure instead of having an empty interior.

Lemma 8 Table 4 provides sufficient conditions guaranteeing the density of $\text{int } \Theta_{cc}$ in Θ_{cc} :

Scenario	Parameter	Suff. cond. for $\text{int } \Theta_{cc}$ to be dense in Θ_{cc}
1	(a, b, c)	$ T \geq n$
2	(a, b)	$ T \geq n$ or $c = \mathbf{0}_n$
3	(a, c)	$ T \geq n$ and $\sup_{t \in T} b_t \leq 0$
4	(b, c)	$\dim \text{cone}(a) = n$
5	a	$ T \geq n$ and $\sup_{t \in T} b_t \leq 0$
6	b	No condition is needed
7	c	$\dim \text{cone}(a) = n$

Table 4

Moreover, Θ_{cc} is open in scenario 5 whenever $c = \mathbf{0}_n$ and $[\sup_{t \in T} b_t \leq 0$ or $\inf_{t \in T} b_t > 0]$.

Proof We prove that the assumptions in Table 4 guarantee that $\Theta_{cc} \subset \text{clint } \Theta_{cc}$ by analyzing the scenarios in descending order. Assume that $\Theta_{cc} \neq \emptyset$ and that a parameter in Θ_{cc} is given. Scenario 7: First, observe that $\Theta_{cc} \neq \emptyset$ implies that $\Theta_{cc} = \text{cone}(a)$. The assumption entails $\text{intcone}(a) \neq \emptyset$, so that

$$\Theta_{cc} = \text{cone}(a) \subset \text{clcone}(a) = \text{clintcone}(a) = \text{clint } \Theta_{cc}.$$

Scenario 6: Let $b \in \Theta_{cc}$. Take any $\varepsilon > 0$, we will show that $b - \varepsilon \in \text{int } \Theta_{cc}$. Suppose, it is not the case, then from Theorem 1, $\mathbf{0}_{n+1} \in \text{clconv}(a, b - \varepsilon) = -(\mathbf{0}_n, \varepsilon) + \text{clconv}(a, b)$, which implies that $(\mathbf{0}_n, 1) \in \text{clcone}(a, b)$, i.e. $b \notin \Theta_{cc}$, a contradiction. Therefore $b - \varepsilon \in \text{int } \Theta_{cc}$ for all $\varepsilon > 0$, and so $b \in \text{clint } \Theta_{cc}$.

Scenario 5: Suppose $|T| \geq n$ and $\sup_{t \in T} b_t \leq 0$. If $c = \mathbf{0}_n$ then $\Theta_{cc} = \Theta$ and the result is trivial. Assume $c \neq \mathbf{0}_n$. Since $c \in \text{cone}(a)$, by Carathéodory's Theorem $c \in \text{cone}\{a_t, t \in S\}$, for some $S \subset T$, $|S| = n$, and $a_{s_1} \neq a_{s_2}$ if $s_1 \neq s_2$, $s_1, s_2 \in S$. Then there exist arbitrarily small vectors $p_t \in \mathbb{R}^n$, $t \in S$, such that $c \in \text{intcone}\{a_t + p_t, t \in S\}$, and $\{a_t + p_t, t \in S\}$ is a basis of \mathbb{R}^n . Defining $p_t = 0$ for all $t \in T \setminus S$, one gets $c \in \text{intcone}(a + p)$, with $p \in (\mathbb{R}^n)^{(T)} \subset (\mathbb{R}^n)^T$ such that $d_\infty(a, a + p) = \sup_{t \in T} \|p_t\|_\infty$ is arbitrarily small. Since $a + p \in \text{int } \Theta_{cc}$ by Theorem 1, $a \in \text{clint } \Theta_{cc}$. The last affirmation about the openness of Θ_{cc} in scenario 5 follows immediately from Theorem 1 and Remark 3.

Scenario 4: Let $(b, c) \in \Theta_{cc}$. On the one hand, by the argument of scenario 6, $\mathbf{0}_{n+1} \notin \text{clconv}(a, b - \frac{1}{r})$ for all $r \in \mathbb{N}$. On the other hand, by the argument of scenario 7, we can pick a sequence $\{c^r\} \subset \text{intcone}(a)$ such that $c^r \rightarrow c$. Since $(b - \frac{1}{r}, c^r) \in \text{int } \Theta_{cc}$ for all $r \in \mathbb{N}$ and $(b - \frac{1}{r}, c^r) \rightarrow (b, c)$, we have $(b, c) \in \text{clint } \Theta_{cc}$.

Scenario 3: If $c \neq \mathbf{0}_n$, then the proof is the same as the one of scenario 5; in case $c = \mathbf{0}_n$ take the perturbation p such that $\text{intcone}\{a_t + p_t, t \in S\} \neq \emptyset$ and choose $c^p \in \text{intcone}\{a_t + p_t, t \in S\}$ such that $d_\infty((a, c), (a + p, c^p))$ is arbitrarily small.

Scenario 2: First, as in scenario 6, observe that for any $\varepsilon > 0$ we have $\mathbf{0}_{n+1} \notin \text{clconv}(a, b - \varepsilon)$, which assures the existence of a neighborhood V of $(a, b - \varepsilon)$ in $\text{int } \Pi_c$. If $c = \mathbf{0}_n$, reasoning as in the proof of scenario 6 we get that $(a, b - \varepsilon) \in \text{int } \Theta_{cc}$ for all $\varepsilon > 0$, so $(a, b) \in \text{clint } \Theta_{cc}$. For $|T| \geq n$ and $c \neq \mathbf{0}_n$, we can copy the proof in scenario 5 to get an arbitrarily small perturbation p of a such that $(a + p, b - \varepsilon) \in \text{int } \Theta_{cc}$, concluding that $(a, b) \in \text{clint } \Theta_{cc}$.

Scenario 1: The proof can be found in [18, Theorem 4.1], where all the data are perturbed.

□

Theorem 3 (Generic 0–stability) *Table 4 provides sufficient conditions guaranteeing the genericity of the 0–stability of g in Θ_{cc} for scenarios 1, 2, 3, 4, 5 and 7. For $|T| < \infty$, it is also true in scenario 6.*

Proof It follows immediately from Theorem 2 and Lemma 8. \square

Concerning Example 1, we have seen that $\text{int } \Theta_{cc}$ is dense in Θ_{cc} for all the scenarios (except scenario 4), even though Theorem 3 does not apply to scenarios 3 and 7. So, the assumptions in Theorem 3 are sufficient, but not necessary, for the genericity of the 0–stability.

4.2 Stability of the duality gap under inconsistency

Now we analyze the ∞ –stability, i.e. conditions for a given $\theta \in \Theta$ to be in the exterior of Θ_{cc} which is the set $\text{int}(\Theta \setminus \Theta_{cc})$. First observe that $\text{int } \Pi_i \cup \text{int } \Delta_i \subset \text{int}(\Theta \setminus \Theta_{cc})$ always. The example below shows that this inclusion can be strict in scenarios 1, 2, 3, and 5. On the other hand the next proposition proves the equality of these two sets in scenarios 4, 6, and 7.

Proposition 1 *The equality $\text{int}(\Theta \setminus \Theta_{cc}) = \text{int } \Pi_i \cup \text{int } \Delta_i$ holds true in scenarios 4, 6 and 7.*

Proof The proof is immediate for scenarios 6 and 7. So, we only consider scenario 4. The inclusion \supset is clearly true. Assume that $\theta = (b, c) \in \text{int}(\Theta \setminus \Theta_{cc})$ and $\theta \notin \text{int } \Pi_i \cup \text{int } \Delta_i$. Then, for any $k \in \mathbb{N}$ there exist $\theta'_k = (b'_k, c'_k) \in \Pi_c$ and $\theta''_k = (b''_k, c''_k) \in \Delta_c$, with $\|\theta - \theta'_k\|_\infty < 1/k$ and $\|\theta - \theta''_k\|_\infty < 1/k$. Thus, $\theta_k = (b'_k, c''_k) \in \Theta_{cc}$, and $\|\theta - \theta_k\|_\infty < 1/k$, which contradicts the assumption $\theta = (b, c) \in \text{int}(\Theta \setminus \Theta_{cc})$. \square

The next example shows that the above proposition does not hold for the other scenarios.

Example 3 ($\text{int } \Pi_i \cup \text{int } \Delta_i \subsetneq \text{int}(\Theta \setminus \Theta_{cc})$) Let $n = 1$ and $T = \{0\} \cup \mathbb{N}$, and consider $a : T \rightarrow \mathbb{R}$ such that $a_0 = 1$, $a_k = 0$ for $k \in \mathbb{N}$, $b : T \rightarrow \mathbb{R}$ such that $b_0 = 0$, $b_k = \frac{1}{2} + \frac{1}{k}$ for $k \in \mathbb{N}$, and $c = -1$. Then the corresponding primal and dual problems are

$$\begin{array}{ll} P : \inf -x & D : \sup \sum_{k=1}^{\infty} \lambda_k \left(\frac{1}{2} + \frac{1}{k} \right) \\ \text{s.t. } x \geq 0, \text{ for } t = 0, & \text{s.t. } \lambda_0 + \sum_{k=1}^{\infty} \lambda_k \cdot 0 = -1, \\ 0x \geq \frac{1}{2} + \frac{1}{k}, k \in \mathbb{N}. & \lambda_0, \lambda_k \geq 0, k \in \mathbb{N}. \end{array}$$

Consider any of the scenarios 1, 2, 3, or 5, and let $\theta = (a, b, c)$ (making a slight abuse of notation in scenarios 2, 3, and 5 where b or c , or both, are fixed). Clearly $\theta \in \Theta \setminus \Theta_{cc}$. Since $0 \in \text{clconv}(a)$, we have that $\theta \notin \text{int } \Delta_i$. Moreover, $\sup b_t < +\infty$ and $(0, 1) \notin \text{int cone}(a, b)$ imply that $\theta \notin \text{int } \Pi_i$. However, we will show that $\theta \in \text{int}(\Theta \setminus \Theta_{cc})$. In fact, let $\hat{\theta} = (\hat{a}, \hat{b}, \hat{c}) \in \Theta$ with $\|\hat{\theta} - \theta\|_\infty < \frac{1}{4}$. If for some $k \in T$ the corresponding \hat{a}_k is negative, then we have that $(0, 1) \in \text{clcone}(\hat{a}, \hat{b})$ and thus $\hat{\theta} \in \Theta_{cc}$. Otherwise, if $\hat{a}_k \geq 0$ for all $k \in T$, then $\hat{c} \notin \text{cone}(\hat{a})$ and so $\hat{\theta} \in \Theta \setminus \Theta_{cc}$. Hence $\theta \in \text{int}(\Theta \setminus \Theta_{cc})$.

Therefore, we may have

$$\text{int } \Pi_i \cup \text{int } \Delta_i \subsetneq \text{int } (\Theta \setminus \Theta_{cc})$$

in the scenarios 1, 2, 3, and 5.

Now we provide sufficient conditions for the membership of $\theta \in \Theta \setminus \Theta_{cc}$ in $\text{int } (\Theta \setminus \Theta_{cc})$. The next result involves the following conditions:

$$\begin{aligned} \text{(S0)} & \begin{cases} (\mathbf{0}_n, 1) \in \text{int cone}(a, b), \\ \text{or} \\ \sup_{t \in T} \|a_t\|_\infty < +\infty \text{ and } \sup_{t \in T} b_t = +\infty, \end{cases} \\ \text{(S1)} & \mathbf{0}_n \notin \text{cl conv}(a) \text{ and } c \notin \text{cl cone}(a). \\ \text{(S2)} & \begin{cases} (\mathbf{0}_n, 1) \in \text{cl cone}(a, b) \text{ and } [\inf_{t \in T} b_t > 0 \text{ or } |T| < \infty], \\ \text{or} \\ c \notin \text{cl cone}(a). \end{cases} \\ \text{(S3)} & \begin{cases} (\mathbf{0}_n, 1) \in \text{cl cone}(a, b) \text{ and } [\inf_{t \in T} b_t > 0 \text{ or } |T| < \infty], \\ \text{or} \\ c \notin \text{cone}(a). \end{cases} \\ \text{(S4)} & \begin{cases} (\mathbf{0}_n, 1) \in \text{cl cone}(a, b), \\ \text{or} \\ c \notin \text{cl cone}(a). \end{cases} \end{aligned}$$

Notice that a parameter that satisfies any of the conditions (S0) to (S4) is in $\Theta \setminus \Theta_{cc}$ for the seven scenarios. The following theorem shows that the conditions stated in (S1) to (S4) imply $\theta \in \text{int } \Pi_i \cup \text{int } \Delta_i \subset \text{int } (\Theta \setminus \Theta_{cc})$ in the different scenarios 1-7.

Theorem 4 (The exterior of Θ_{cc}) *Let $(a, b, c) \in (\mathbb{R}^n)^T \times \mathbb{R}^T \times \mathbb{R}^n$ be given. The following statements hold:*

- (i) (S0) is a sufficient condition for the membership of an element of Θ in $\text{int } (\Theta \setminus \Theta_{cc})$ in any scenario.
- (ii) (S1) is a sufficient condition for the membership of an element of Θ in $\text{int } (\Theta \setminus \Theta_{cc})$ in scenarios 1, 2, 3, and 5.
- (iii) Table 5 provides characterizations of $\text{int } (\Theta \setminus \Theta_{cc})$ in scenarios 4, 6, and 7.

Scenario	Parameter	Characterization of its membership to $\text{int } (\Theta \setminus \Theta_{cc})$
4	(b, c)	(S2)
6	b	(S3)
7	c	(S4)

Table 5

Proof It follows immediately from Proposition 1 and the characterizations of $\text{int } \Pi_i$ and of $\text{int } \Delta_i$ given in Lemmas 4, 5 and 7 (taking into account Remark 2), because $\text{int } \Pi_i \cup \text{int } \Delta_i \subset \text{int } (\Theta \setminus \Theta_{cc})$ in any scenario. \square

5 Stable duality gap in linear programming

In this section we will consider the specific case of $|T| < \infty$, where P and D are both ordinary LP problems. We have seen that, when $|T| < \infty$, g is the indicator function of Θ_{cc} , so that it is 0-stable at θ if and only if $\theta \in \text{int}\Theta_{cc}$. It is clear that all the results valid in LSIP are also true for LP, nonetheless there are specific properties that are valid when all the parameter spaces are finite-dimensional. For instance, we know that Θ_{cc} in scenarios 4, 6, and 7 is a convex subset of Θ (which is now some Euclidean space \mathbb{R}^s), so it is always true that $\Theta_{cc} \subset \text{clrint}\Theta_{cc}$, where $\text{rint}\Theta_{cc}$ is the relative interior of Θ_{cc} , complementing in this way the genericity result provided by Theorem 3. Moreover, the (Lebesgue) measure of the set $\Theta_{cc} \setminus \text{int}\Theta_{cc}$ is always zero in these scenarios because of the convexity of Θ_{cc} , so the 0-stability property of g is also generic in the sense of [7] in scenarios 4, 6 and 7.

To be more precise, this section is about the reformulation of LP problems in order to apply the results of Section 3 allowing to classify a given $\theta \in \Theta_{cc}$ as an element of the partition $\{\text{int}\Theta_{cc}, \text{bd}\Theta_{cc}, \text{int}(\Theta \setminus \Theta_{cc})\}$, and the way to check the assumptions of such results, i.e. modeling and computational issues on the stability of the duality gap in LP.

5.1 Modeling issues

Any LP problem with uncertain data can be reformulated in canonical and standard formats (as P in (4) and D in (5)), with $X = \mathbb{R}^n$, $A (m \times n)$ a real matrix identified with $a \in \mathbb{R}^{nm}$, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$ by means of well-known algebraic transformations. In some cases (as in Example 4 below) the results in Section 4 can be directly applied to the given LP problem but, in general, these reformulations do not fit perfectly into one of the scenarios described in Table 2, so that the mentioned results only provide sufficient conditions for the classification of θ into one of the elements of the partition $\{\text{int}\Theta_{cc}, \text{bd}\Theta_{cc}, \text{int}(\Theta \setminus \Theta_{cc})\}$. Examples 5 and 6 below illustrate this reformulation task.

In this subsection we denote by Θ^k the space of parameters corresponding to scenario k . We also represent by $\mathbf{1}_p$ the $p \times 1$ vector of all 1's.

Example 4 In this linear assignment problem the variable $y_{ij} \in \{0, 1\}$ represents the assignment of agent i to task j , $i, j = 1, \dots, p$, taking value 1 (with cost $\gamma_j \geq 0$) if the assignment is done and 0 otherwise. The problem can be formulated as

$$\begin{aligned} D_L : \inf \sum_{i,j=1,\dots,p} \gamma_j y_{ij} \\ \text{s.t. } \sum_{i=1,\dots,p} y_{ij} = 1, \forall j = 1, \dots, p, \\ \sum_{j=1,\dots,p} y_{ij} = 1, \forall i = 1, \dots, p, \\ y_{ij} \geq 0, i, j = 1, \dots, p, \end{aligned}$$

where we have replaced the Boolean condition $y_{ij} \in \{0, 1\}$ by $y_{ij} \geq 0$ because the constraint matrix $A (p^2 \times 2p)$ is totally unimodular. Obviously, D_L is a particular instance of the problem D in (5), with $m = p^2$, $n = 2p$, $c = \mathbf{1}_{2p}$, and $b_{ij} = -\gamma_j$, $i, j = 1, \dots, p$. Observe that the unique perturbable data in D_L are the costs γ_j , i.e., we are in scenario 6, with $\Theta = \Theta^6 = \mathbb{R}^{p^2}$. If we order the p^2 indices $t = (i, j)$ with the lexicographical order as:

$(1, 1), (1, 2), \dots, (1, p), (2, 1), \dots, (p, p)$, then the transpose A' of A can be written in the form

$$A' = \begin{bmatrix} I_p & I_p & \cdots & \cdots & I_p \\ \mathbf{1}'_p & \mathbf{0} & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{1}'_p & \cdots & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & \mathbf{1}'_p \end{bmatrix},$$

where I_p is the unit $p \times p$ matrix. Observe that the (i, i) -column of A' is the $2p$ -vector $(e_i, e_i)'$, where e_i is the canonical p -vector. Then $b \in \Delta_c$ because c is the sum of all the (i, i) -columns of A' . The fact that $\sup b_{ij} \leq 0$, gives that P_L is consistent, and hence $b \in \Theta_{cc}$. Thus, Theorem 1 applies directly to obtain that

$$g \text{ is 0-stable at } \theta \Leftrightarrow (C1).$$

Now, it is clear that no convex combination of the a_{ij} 's is null, so (C1) is valid and we obtain the 0-stability of g at θ . (We could also have obtained this result from an application of the theory of totally unimodular matrices which assures us that D_L is solvable).

Example 5 Allocating q scarce resources among n production lines can be formulated as

$$D_R : \sup_{y \in \mathbb{R}_+^n} \left\{ \sum_{j=1, \dots, n} p_j y_j : W'y \leq c \right\},$$

where y_j represents the production level of the j -th good (a decision variable), p_j the price of the j -th good, $c_i \in \mathbb{R}_+$ the available amount of the i -th resource, and $W (n \times q)$ the technological matrix (with non-negative entries). Introducing a slack variable $z \in \mathbb{R}_+^q$, and defining $\lambda' := (y', z')$, $b' := (p_1, \dots, p_n, 0, \dots, 0) \in \mathbb{R}^{n+q}$, and $A' := [W' \mid I_q]$, where I_q denotes the unit $q \times q$ matrix, D_R can be expressed as D in (5), with $m = n + q$. The corresponding primal problem is consistent because of the assumption of non-negative entries in W . Since D_R is also consistent, $\theta \in \Theta_{cc}$.

(i) If only the resource vector c is uncertain, then $\Theta = \Theta^7 = \mathbb{R}^q$ and, by Theorem 1,

$$g \text{ is 0-stable at } \theta \Leftrightarrow (C2). \quad (13)$$

(ii) If, additionally, the price vector b is uncertain too, then $\{b\} \times \Theta^7 \subsetneq \Theta \subsetneq \Theta^4 = \mathbb{R}^{n+2q}$ (observe that only the first n components of b can be perturbed) and Theorem 1 yields

$$g \text{ is 0-stable at } \theta \Leftrightarrow (C1) \text{ and } (C2). \quad (14)$$

Now, it is easy to observe that (C1) is always true in this framework when W has non-null rows or all the prices are positive, hence in this case we also obtain the equivalence (13) between the 0-stability of g at θ and (C2).

(iii) If all the data in D_R , c , p , and W , are perturbable, then $\{a\} \times \{b\} \times \Theta^7 \subsetneq \Theta \subsetneq \Theta^1 = \mathbb{R}^{(n+q)q+n+2q}$ and the conclusion is the same as in (ii).

Example 6 Example 1.1 in [6], dealing with two person, finite, zero-sum games, involves a particular LP problem P_G of the form

$$P_G : \inf \mathbf{1}'_n x \\ \text{s.t. } U'x \geq \mathbf{1}_m, \quad x \geq \mathbf{0}_n,$$

where $U(n \times m)$ denotes the pay-off matrix, whose entries can be assumed w.l.o.g. to be positive but subject to small perturbations while the remaining data are deterministic. Obviously, P_G can be formulated as P in (4), with $A' = [U \mid I_n]$, $b' = (\mathbf{1}'_m, \mathbf{0}'_n)$ and $c = \mathbf{1}_n$. Since P_G is solvable, $\theta \in \Theta_{cc}$. Moreover, the unique perturbable data are the entries of U , so that $\Theta \subsetneq \Theta^5 = \mathbb{R}^{(m+n)n}$ and Theorem 1 yields the sufficient condition:

$$(C1) \text{ and } (C2) \Rightarrow g \text{ is 0-stable at } \theta.$$

It is clear that (C2) is satisfied because $c = \mathbf{1}_n$ and $\text{cone}(a) = \mathbb{R}_+^{(m+n)n}$. Furthermore, (C1) is also valid by a simple inspection of the form of A and b . Therefore g is 0-stable at θ .

5.2 Computational issues

Denote by M a $p \times n$ matrix without zero rows such that $\text{cone}(a) = \{x \in \mathbb{R}^n : Mx \geq \mathbf{0}_p\}$ and by N a $q \times (n+1)$ matrix without zero rows such that $\text{cone}(a, b) = \{x \in \mathbb{R}^{n+1} : Nx \geq \mathbf{0}_q\}$ (p and q are natural numbers). According to (8),

$$(a, b, c) \in \Theta_{cc} \Leftrightarrow N \begin{pmatrix} \mathbf{0}_n \\ 1 \end{pmatrix} \not\geq \mathbf{0}_q \text{ and } Mc \geq \mathbf{0}_p.$$

(Example 7 below illustrates the effective computation of M and N .) Recall that (C1) is the Slater condition in LP.

Theorems 1 and 4 are still valid if we replace the geometric conditions (C1), (C2), (S0), ..., (S4) by the following vector inequalities and/or conditions expressed in terms of optimal values of LP problems:

$$(C1) \sup \{x_{n+1} : Ax - x_{n+1} \mathbf{1}_m \geq b\} > 0.$$

$$(C2) Mc > \mathbf{0}_p.$$

$$(S0) N \begin{pmatrix} \mathbf{0}_n \\ 1 \end{pmatrix} > \mathbf{0}_q.$$

$$(S1) \sup \{x_{n+1} : Ax - x_{n+1} \mathbf{1}_m \geq \mathbf{0}_m\} = +\infty \text{ and } Mc \not\geq \mathbf{0}_p.$$

$$(S2) \equiv (S3) \equiv (S4) \begin{cases} N \begin{pmatrix} \mathbf{0}_n \\ 1 \end{pmatrix} \geq \mathbf{0}_q, \\ \text{or} \\ Mc \not\geq \mathbf{0}_p. \end{cases}$$

All matrices and vectors in the above conditions, except M and N , are expressed in terms of the data defining P and D . Concerning the matrices M and N , they can also be obtained from the data by eliminating $\lambda_1, \dots, \lambda_m$ from the linear systems

$$\left\{ x = \sum_{t=1, \dots, m} \lambda_t a_t; \lambda_t \geq 0, t = 1, \dots, m \right\}$$

and

$$\left\{ (x, x_{n+1}) = \sum_{t=1, \dots, m} \lambda_t (a_t, b_t); \lambda_t \geq 0, t = 1, \dots, m \right\},$$

respectively. Thus, all the above conditions are checkable.

Example 7 Let

$$A = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

We start by computing the auxiliary matrices M and N . Gauss elimination of λ_1 , λ_3 , and λ_4 in (5.2) yields the inequality system

$$\{2x_1 - x_3 + \lambda_2 - \lambda_5 \geq 0; x_2 - \lambda_2 - \lambda_5 \geq 0; -x_3 + \lambda_2 - \lambda_5 \geq 0; \lambda_2 \geq 0; \lambda_5 \geq 0\}, \quad (15)$$

and Fourier elimination of λ_2, λ_5 from (15) provides the aimed linear representation of the first moment cone

$$\text{cone}(a) = \{x \in \mathbb{R}^3 : 2x_1 + x_2 - x_3 \geq 0; x_2 - x_3 \geq 0; x_2 \geq 0\},$$

so that

$$M = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

In a similar way one gets

$$N = \begin{bmatrix} 3 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 2 & 1 & 2 \end{bmatrix},$$

with $N \begin{pmatrix} \mathbf{0}_3 \\ 1 \end{pmatrix} = (-1, -1, 2, -2)' \not\geq \mathbf{0}$, which gives the consistency of the primal problem for any c . Now we consider different c 's in the triplet (a, b, c) :

(i) $c = (0, 3, -2)'$. As $Mc = (5, 5, 3) \geq \mathbf{0}_3$, we have $\theta \in \Theta_{cc}$. Moreover, since $\sup\{x_4 : Ax - x_4 \mathbf{1}_5 \geq b\} = +\infty$, (C1) and (C2) hold, so that $\theta \in \text{int}\Theta_{cc}$ and hence g is 0-stable at θ in all scenarios.

(ii) $c = (0, 1, 1)'$. We still have $\theta \in \Theta_{cc}$, but now $Mc = (0, 0, 1)' \not\geq \mathbf{0}_3$ has null components. Thus, since (C1) holds while (C2) fails, Theorem 1 allows to conclude that $\theta \in \text{int}\Theta_{cc}$ only in scenario 6 (under perturbations of b). Then g is 0-stable at θ in scenario 6 and it is not 0-stable at θ in the other scenarios.

(iii) $c = (-1, 0, 0)'$. Now $Mc = (-2, 0, 0)' \not\geq \mathbf{0}_3$, so that $\theta \in \Theta \setminus \Theta_{cc}$. Since

$$\sup\{x_4 : Ax - x_4 \mathbf{1}_5 \geq \mathbf{0}_5\} = +\infty,$$

(S1)-(S4) hold. Hence, by Theorem 4, $\theta \in \text{int}(\Theta \setminus \Theta_{cc})$ for any scenario.

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