On the Solution of the Cahn-Hilliard Equation via the Perturbation Iteration Transform Method

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***Abstract*— Recently, a new approach called the Perturbation Iteration Transform Method has been introduced. This approach is based on the fusion of the Perturbation Iteration Algorithm and the Laplace Transform Method. In this paper, the solution of the nonlinear partial differential equation: Cahn-Hilliard equation is presented by using this new scheme. Some numerical tests are presented to make apparent the potential of this new approach. The results show that the approximate solutions of these equations are very close to their exact solutions even with less computational stress.**

***Index Terms*— perturbation iteration algorithm, Cahn-Hilliard equation, Laplace transform method, nonlinear PDEs, perturbation iteration transform method**

I. Introduction

Nonlinear Partial Differential Equations continues to be an active field of study in Physics, Engineering and Applied Mathematics. Among such PDEs, we have the Cahn-Hilliard equation, which was introduced by Cahn and Hilliard in [1] in order to illustrate the phase separation phenomenon in a solid. Several authors have also worked on Nonlinear PDEs [7-10].

In this work, we are concerned with the solution of the Cahn-Hilliard Equation, which is of the form:

$$u_t(x,t) = ru_{xx}(x,t) + su(x,t) - tu^n(x,t) \qquad (1.1)$$

with the initial condition:

$$u(x,0) = f(x) \qquad (1.2)$$

where s, t are both real numbers, r and n are positive integers.

Many authors have investigated the Cahn-Hilliard equation both analytically and numerically.

Shehata in [2] obtained and compared the numerical solution of a Cahn-Hilliard equation by using both the Homotopy Perturbation Method (HPM) and the Adomian Decomposition Method (ADM). Furihata in [3] also obtained the numerical solution of Cahn-Hilliard equation via the finite difference method. [4] solved these equations with the Differential Transform Method (DTM) and [5] - [6] used the Exp-function method to obtain the exact solutions of the Cahn-Hilliard. Many articles have investigated the analytical and numerical solution of the Cahn-Hilliard equation [16-21].

The Perturbation Iteration Transform Method (PITM) is the combined form of the Perturbation Iteration Algorithm and the Laplace Transform. The idea of using the PITM was proposed by in [13]. They used the method to solve both the linear and nonlinear Klein-Gordon equations. For more articles on PITM, see [12-15].

The remaining part of this study thus arranged: section II gives a review of the PIA and section III deals with the illustration of the PITM. Numerical examples are presented in section IV, where PITM is applied to some Cahn-Hilliard equations to prove its effectiveness. Section V presents the graphs of the solutions in section IV. The final section, VI gives the concluding remark.

II. Perturbation Iteration Algorithm [11], [13]

Here, we illustrate how the Perturbation Iteration Algorithm works. Suppose a perturbation algorithm is been developed by taking the correction terms of the first derivatives in the Taylor series expansion and also one correction term in the perturbation expansion. This algorithm will be named: PIA(1,1).

We now consider a partial differential equation of the form:

$$F(\dot{u}, u'', u, \varepsilon) = 0 \qquad (2.1)$$

where $u = u(x,t)$, $\dot{u} = \frac{\partial u}{\partial t}$, $u'' = \frac{\partial^2 u}{\partial x^2}$ and ε is the introduced perturbation parameter.

And if we use just one correction term in the perturbation expansion, we have:

$$u_{n+1} = u_n + \varepsilon(u_c)_n \qquad (2.2)$$

Putting (2.2) into (2.1) and expanding in a Taylor series with first derivatives will yield.

$$\begin{cases} F(\dot{u},u'',u,0) + F_{\dot{u}}(\dot{u},u'',u,0)\varepsilon(\dot{u}_c)_n + F_{u''}(\dot{u},u'',u,0)\varepsilon(u''_c)_n \\ \qquad + F_u(\dot{u},u'',u,0)\varepsilon(u_c)_n + F_\varepsilon(\dot{u},u'',u,0)\varepsilon = 0 \end{cases} \qquad (2.3)$$

where $u = u(x,t)$, $F_{\dot{u}} = \frac{\partial F}{\partial \dot{u}}$, $F_{u''} = \frac{\partial F}{\partial u''}$, $F_u = \frac{\partial F}{\partial u}$, $F_\varepsilon = \frac{\partial F}{\partial \varepsilon}$ and ε is the perturbation parameter to be evaluated at zero.

Reorganizing (2.3), we have

$$(\dot{u}_c)_n + \frac{F_{u''}}{F_{\dot{u}}}(u''_c)_n = -\frac{F_\varepsilon + \frac{F}{\varepsilon}}{F_{\dot{u}}} - \frac{F_u}{F_{\dot{u}}}(u_c)_n \qquad (2.4)$$

Starting with an initial guess, u_0, evaluate the term, $(u_c)_0$ from (2.4) and then substitute the result into (2.2) for u_1. We continue this iteration procedure by using Equations (2.4) and (2.2) until a satisfactory result is obtained.

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III. PERTURBATION ITERATION TRANSFORM METHOD [13]

In this section, we demonstrate the basic idea of the PITM. Consider the general nonlinear PDE of the form:

$$Lu(x,t)+Mu(x,t)+Nu(x,t)=g(x,t) \tag{3.1}$$

with the associated initial condition:

$$u(x,0)=f(x) \tag{3.2}$$

where $L=\frac{\partial}{\partial t}$ is the first order linear differential operator, $M=\frac{\partial^2}{\partial x^2}$ is the second order linear differential operator, $Nu(x,t)$ represents both the linear and the nonlinear terms, and $g(x,t)$ is the source term.

We take the Laplace transform of both sides of (3.1) to have

$$L\left[Lu(x,t)\right]+L\left[Mu(x,t)\right]+L\left[Nu(x,t)\right]=L\left[g(x,t)\right] \tag{3.3}$$

On using the differential property of Laplace transform in (3.3), we get

$$\begin{aligned}L\left[Lu(x,t)\right]=&\frac{f(x)}{s}+\frac{1}{s}L\left[h(x,t)\right]-\frac{1}{s}L\left[Mu(x,t)\right]\\&-\frac{1}{s}L\left[Nu(x,t)\right]\end{aligned} \tag{3.4}$$

Applying the Inverse Laplace Transform to both sides of (3.4) gives

$$u(x,t)=E(x,t)-L^{-1}\left[\frac{1}{s}L\left[Mu(x,t)+Nu(x,t)\right]\right] \tag{3.5}$$

where $E(x,t)$ is the term obtained from the source term and the associated initial condition.

Now, by using the PITM, (3.5) becomes:

$$\begin{aligned}&u(x,t)-E(x,t)+u_c(x,t)\varepsilon\\&\quad-L^{-1}\left[\frac{1}{s}L\left[Mu(x,t)+Nu(x,t)\right]\right]\varepsilon=0\end{aligned} \tag{3.6}$$

Hence,

$$u_c(x,t)=\frac{E(x,t)-u(x,t)}{\varepsilon}-L^{-1}\left[\frac{1}{s}L\left[Mu(x,t)+Nu(x,t)\right]\right] \tag{3.7}$$

Equation (3.7) is the combined form of the Laplace transform method and the perturbation iteration method. From (3.7), the term, $(u_c)_0$ is then calculated and substituted into (2.2) to obtain u_1. This iteration procedure is repeated for u_2, u_3 and so on. The approximate solution is thus obtained by the formula:

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t) \tag{3.8}$$

That is,

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots \tag{3.9}$$

IV. NUMERICAL EXAMPLES

In this section, we apply the proposed method to the Cahn-Hilliard Equations.

Case I:

Consider the Cahn-Hilliard Equation:

$$u_t(x,t)=u_{xx}(x,t)-u^3(x,t)+u(x,t) \tag{4.1}$$

subject to:

$$u(x,0)=\frac{1}{1+e^{\frac{x}{\sqrt{2}}}} \tag{4.2}$$

with the exact solution:

$$u(x,t)=\frac{1}{1+e^{\frac{x}{\sqrt{2}}-\frac{3t}{2}}} \tag{4.3}$$

Solution to Case I:

Taking the Laplace Transform of both sides of (4.1) with the initial condition (4.2), we get

$$L\left[u(x,t)\right]=\frac{1}{1+e^{\frac{x}{\sqrt{2}}}}\cdot\frac{1}{s}+\frac{1}{s}L\left[u_{xx}(x,t)-u^3(x,t)+u(x,t)\right] \tag{4.4}$$

Applying the Inverse Laplace Transform to both sides of (4.4) gives

$$u(x,t)=\frac{1}{1+e^{\frac{x}{\sqrt{2}}}}+L^{-1}\left[\frac{1}{s}L\left[u_{xx}(x,t)-u^3(x,t)+u(x,t)\right]\right] \tag{4.5}$$

Now, by PITM, (4.5) becomes:

$$\begin{aligned}&u(x,t)-\frac{1}{1+e^{\frac{x}{\sqrt{2}}}}+u_c(x,t)\varepsilon\\&\quad-L^{-1}\left[\frac{1}{s}L\left[u_{xx}(x,t)-u^3(x,t)+u(x,t)\right]\right]\varepsilon=0\end{aligned} \tag{4.6}$$

Thus,

$$\begin{aligned}u_c(x,t)=&\varepsilon^{-1}\left(-u(x,t)+\left(1+e^{\frac{x}{\sqrt{2}}}\right)^{-1}\right)\\&+L^{-1}\left[\frac{1}{s}L\left[u_{xx}(x,t)-u^3(x,t)+u(x,t)\right]\right]\end{aligned} \tag{4.7}$$

This implies that:

$$u_0(x,t)=\frac{1}{1+e^{\frac{x}{\sqrt{2}}}}$$

$$u_1(x,t)=\left[\frac{e^{\sqrt{2}x}}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}-\frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}-\frac{e^{\frac{x}{\sqrt{2}}}}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2}+\frac{1}{1+e^{\frac{x}{\sqrt{2}}}}\right]t$$

$$u_2(x,t)=\frac{9e^{\frac{x}{\sqrt{2}}}\left(1+e^{\sqrt{2}x}\right)t^2}{8\left(1+e^{\frac{x}{\sqrt{2}}}\right)^4}-\frac{27e^{\frac{3x}{\sqrt{2}}}t^4}{32\left(1+e^{\frac{x}{\sqrt{2}}}\right)^6}$$

$$\begin{aligned}u_3(x,t)=&\frac{3e^{\frac{x}{\sqrt{2}}}\left(3-6e^{\frac{x}{\sqrt{2}}}-6e^{\frac{3x}{\sqrt{2}}}+22e^{\sqrt{2}x}+3e^{2\sqrt{2}x}\right)t^3}{16\left(1+e^{\frac{x}{\sqrt{2}}}\right)^6}\\&-\frac{27e^{\frac{3x}{\sqrt{2}}}\left(11-20e^{\frac{x}{\sqrt{2}}}+11e^{\sqrt{2}x}\right)t^5}{320\left(1+e^{\frac{x}{\sqrt{2}}}\right)^8}-\frac{729e^{\frac{3x}{\sqrt{2}}}\left(1+e^{\sqrt{2}x}\right)^3t^7}{3584\left(1+e^{\frac{x}{\sqrt{2}}}\right)^{12}}\\&+\frac{729e^{\frac{5x}{\sqrt{2}}}\left(1+e^{\sqrt{2}x}\right)^2t^9}{2048\left(1+e^{\frac{x}{\sqrt{2}}}\right)^{14}}-\frac{19683e^{\frac{7x}{\sqrt{2}}}\left(1+e^{\sqrt{2}x}\right)t^{11}}{90112\left(1+e^{\frac{x}{\sqrt{2}}}\right)^{16}}\\&+\frac{19683e^{\frac{9x}{\sqrt{2}}}t^{13}}{425984\left(1+e^{\frac{x}{\sqrt{2}}}\right)^{18}}\end{aligned}$$

$$\vdots$$

Therefore, the solution $u(x,t)$ is given by:

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots$$

$$=\frac{1}{1+e^{\frac{x}{\sqrt{2}}}}+\left[\frac{e^{\sqrt{2}x}}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}-\frac{1}{\left(1+e^{\frac{x}{\sqrt{2}}}\right)^3}-\frac{e^{\frac{x}{\sqrt{2}}}}{2\left(1+e^{\frac{x}{\sqrt{2}}}\right)^2}+\frac{1}{1+e^{\frac{x}{\sqrt{2}}}}\right]t$$

$$+\frac{9e^{\frac{x}{\sqrt{2}}}\left(1+e^{\sqrt{2}x}\right)t^2}{8\left(1+e^{\frac{x}{\sqrt{2}}}\right)^4}-\frac{27e^{\frac{3x}{\sqrt{2}}}t^4}{32\left(1+e^{\frac{x}{\sqrt{2}}}\right)^6}$$

$$+\frac{3e^{\frac{x}{\sqrt{2}}}\left(3-6e^{\frac{x}{\sqrt{2}}}-6e^{\frac{3x}{\sqrt{2}}}+22e^{\sqrt{2}x}+3e^{2\sqrt{2}x}\right)t^3}{16\left(1+e^{\frac{x}{\sqrt{2}}}\right)^6}$$

$$-\frac{27e^{\frac{3x}{\sqrt{2}}}\left(11-20e^{\frac{x}{\sqrt{2}}}+11e^{\sqrt{2}x}\right)t^5}{320\left(1+e^{\frac{x}{\sqrt{2}}}\right)^8}-\frac{729e^{\frac{3x}{\sqrt{2}}}\left(1+e^{\sqrt{2}x}\right)^3t^7}{3584\left(1+e^{\frac{x}{\sqrt{2}}}\right)^{12}}$$

$$+\frac{729e^{\frac{5x}{\sqrt{2}}}\left(1+e^{\sqrt{2}x}\right)^2t^9}{2048\left(1+e^{\frac{x}{\sqrt{2}}}\right)^{14}}-\frac{19683e^{\frac{7x}{\sqrt{2}}}\left(1+e^{\sqrt{2}x}\right)t^{11}}{90112\left(1+e^{\frac{x}{\sqrt{2}}}\right)^{16}}$$

$$+\frac{19683e^{\frac{9x}{\sqrt{2}}}t^{13}}{425984\left(1+e^{\frac{x}{\sqrt{2}}}\right)^{18}}+\cdots \qquad (4.8)$$

Equation (4.8) is an approximate solution of case I.

Case II:

Consider the Cahn-Hilliard Equation:

$$u_t(x,t)=u_{xx}(x,t)-u^3(x,t)+u(x,t) \qquad (4.9)$$

subject to:

$$u(x,0)=e^x \qquad (4.10)$$

Solution to Case II:

Taking the Laplace Transform of both sides of (4.9) with the initial condition (4.10), we get

$$L\left[u(x,t)\right]=\frac{e^x}{s}+\frac{1}{s}L\left[u_{xx}(x,t)-u^3(x,t)+u(x,t)\right] \qquad (4.11)$$

Applying the Inverse Laplace Transform to both sides of (4.11) gives

$$u(x,t)=e^x+L^{-1}\left[\frac{1}{s}L\left[u_{xx}(x,t)-u^3(x,t)+u(x,t)\right]\right] \qquad (4.12)$$

Now, by PITM, (4.12) becomes:

$$u(x,t)-e^x+u_c(x,t)\varepsilon$$
$$-L^{-1}\left[\frac{1}{s}L\left[u_{xx}(x,t)-u^3(x,t)+u(x,t)\right]\right]\varepsilon=0 \qquad (4.13)$$

Thus,

$$u_c(x,t)=\frac{-u(x,t)+e^x}{\varepsilon}+L^{-1}\left[\frac{1}{s}L\left[u_{xx}(x,t)-u^3(x,t)+u(x,t)\right]\right] \qquad (4.14)$$

This implies that:

$$u_0(x,t)=e^x$$

$$u_1(x,t)=\left[2e^x-e^{3x}\right]t$$

$$u_2(x,t)=\frac{1}{4}e^{3x}\left(-2+e^{2x}\right)^3t^4-e^x\left(-2+5e^{2x}\right)t^2$$

$$u_3(x,t)=-\frac{2}{3}e^x\left(-2+25e^{2x}\right)t^3$$
$$+\frac{1}{10}e^{3x}\left(-40+156e^{2x}-150e^{4x}+41e^{6x}\right)t^5$$
$$+\frac{1}{7}e^{3x}\left(-2+5e^{2x}\right)^3t^7-\frac{1}{12}e^{5x}\left(-2+e^{2x}\right)^3\left(-2+5e^{2x}\right)^2t^9$$
$$+\frac{3}{176}(-128e^{7x}+704e^{9x}-1440e^{11x}+1520e^{13x}$$
$$-920e^{15x}+324e^{17x}-62e^{19x}+5e^{21x})t^{11}$$
$$-\frac{1}{832}(-512e^{9x}+2304e^{11x}-4608e^{13x}+5376e^{15x}$$
$$-4032e^{17x}+2016e^{19x}-672e^{21x}+144e^{23x}-18e^{25x}+e^{27x})t^{13}$$

$$\vdots$$

Therefore, the solution $u(x,t)$ is given by:

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+\cdots$$
$$=e^x+\left[2e^x-e^{3x}\right]t+\frac{1}{4}e^{3x}\left(-2+e^{2x}\right)^3t^4-e^x\left(-2+5e^{2x}\right)t^2$$
$$-\frac{2}{3}e^x\left(-2+25e^{2x}\right)t^3+\frac{1}{10}e^{3x}\left(-40+156e^{2x}-150e^{4x}+41e^{6x}\right)t^5$$
$$+\frac{1}{7}e^{3x}\left(-2+5e^{2x}\right)^3t^7-\frac{1}{12}e^{5x}\left(-2+e^{2x}\right)^3\left(-2+5e^{2x}\right)^2t^9$$
$$+\frac{3}{176}(-128e^{7x}+704e^{9x}-1440e^{11x}+1520e^{13x}$$
$$-920e^{15x}+324e^{17x}-62e^{19x}+5e^{21x})t^{11}$$
$$-\frac{1}{832}(-512e^{9x}+2304e^{11x}-4608e^{13x}+5376e^{15x}-4032e^{17x}$$
$$+2016e^{19x}-672e^{21x}+144e^{23x}-18e^{25x}+e^{27x})t^{13}+\ldots \qquad (4.15)$$

Equation (4.15) is an approximate solution of case II.

Equation (4.15) is the approximate solution of Case II.

V. Discussion of Results

In this section, we present the graphs for both the approximate and exact solutions to the Cahn-Hilliard in Case I. The approximate solutions contain terms up to the fourth power. In Table I, the solution of case 1 is analysed.

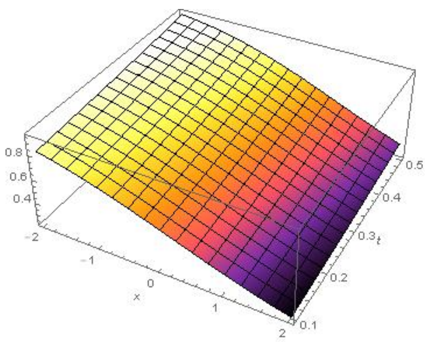

Fig. 1: Approximate solution of Cahn-Hilliard Equation in Case I

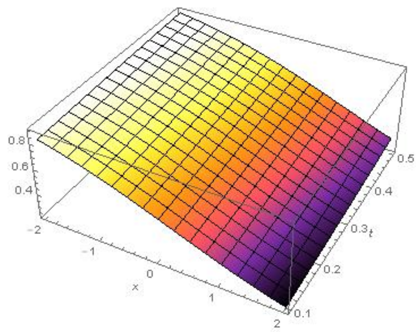

Fig. 2: Exact solution of Cahn-Hilliard Equation in Case I

Table I: Solution comparison of case 1

(x,t)	$u_{3term-apprx}$	$u_{5term-apprx}$	$Abs-error$
$(0,\ 0)$	0.500000	0.500000	0.000000
$(0.1,0.9)$	0.958248	0.962758	0.004510
$(0.2,0.8)$	0.871402	0.874352	0.002951
$(0.3,0.7)$	0.788224	0.790056	0.001832
$(0.4,0.6)$	0.709070	0.710123	0.001052
$(0.5,0.5)$	0.634227	0.634767	0.000540
$(0.6,0.4)$	0.563914	0.564149	0.000235
$(0.7,0.3)$	0.498290	0.498369	0.000078
$(0.8,0.2)$	0.437453	0.437469	0.000016
$(0.9,0.1)$	0.381441	0.381442	0.000002

VI. Conclusion

In this paper, we presented the solution of the nonlinear partial differential equation: Cahn-Hilliard Equation by using a new solution technique referred to as Perturbation Iteration Transform Method. The results obtained revealed a fast convergent rate to their exact solutions without any form of discretization or linearization. We therefore, recommend this solution technique for solving any nonlinear partial differential equations in other aspects of pure and applied sciences.

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