On the Flexibility of the Transmuted Inverse Exponential Distribution

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***Abstract*—This research demonstrates that the Transmuted Inverse Exponential distribution is more robust than the Inverse Exponential distribution. The comparison was made using two real life data sets and the performance of the competing models were rated based on their loglikelihood value and Akaike Information Criteria (AIC) value. The analysis was performed using R-software. In addition, some further statistical properties of the Transmuted Inverse Exponential distribution were established.**

***Index Terms*— Distribution, Inverse Exponential, R software, Transmutation**

I INTRODUCTION

The Transmuted Inverse Exponential (TIE) distribution was obtained by [1] as one of the generalizations of the Inverse Exponential (IE) distribution. The TIE distribution was derived based on the contents of [2] who studied the quadratic rank transmutation map (QRTM).

Other generalized models that were proposed using the QRTM include; Transmuted Exponential distribution [3], Transmuted Weibull distribution [4], Transmuted Rayleigh distribution [5], Transmuted Lomax distribution [6], Transmuted Log-Logistic distribution [7], Transmuted Lindley distribution [8], Transmuted Exponentiated Frechet distribution [9], Transmuted Exponentiated Gamma distribution [10], Transmuted Inverse Rayleigh distribution [11], Transmuted Pareto distribution [12] and many more.

The probability density function (pdf) and the cumulative density function (cdf) of the TIE distribution with parameters θ and λ are given by;

$$f(x)=\frac{\theta}{x^2}\exp\left(-\frac{\theta}{x}\right)\left[1+\lambda-2\lambda\exp\left(-\frac{\theta}{x}\right)\right] \quad (1)$$

and;

$$F(x)=\left[\exp\left(-\frac{\theta}{x}\right)\right]\left\{1+\lambda-\lambda\exp\left(-\frac{\theta}{x}\right)\right\} \quad (2)$$

respectively

For $x>0$, $\theta>0$ and $|\lambda|\leq 1$

where θ is a scale parameter and λ is the transmuted parameter

Some possible plots for the pdf and hazard function of TIE distribution at various selected parameter values are shown in Figures 1 and 2:

It is obvious that the shape of the TIE distribution could be unimodal (depending on the value of the parameters).

This article intends to extend the work of [1] by providing some further properties of the TIE distribution and to demonstrate the usefulness of the model including its potential superiority over its sub-model.

II SOME FURTHER PROPERTIES OF THE TRANSMUTED INVERSE EXPONENTIAL DISTRIBUTION

Some properties like the moments, moment generating function, survival function, hazard function and quantile function can be found in [1] but further properties like the distribution of order statistics, odds function, reversed hazard function, including the estimation of model parameters shall be provided in this section.

Order Statistics

The pdf of the ith order statistics for random samples $X_1, X_2, ..., X_n$ from a pdf $f(x)$ and an associated cdf $F(x)$ is given by;

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}f(x)F(x)^{i-1}\left[1-F(x)\right]^{n-i} \quad (3)$$

Then, the pdf of the ith order statistics for the TIE distribution is as given in Equation (4) as;

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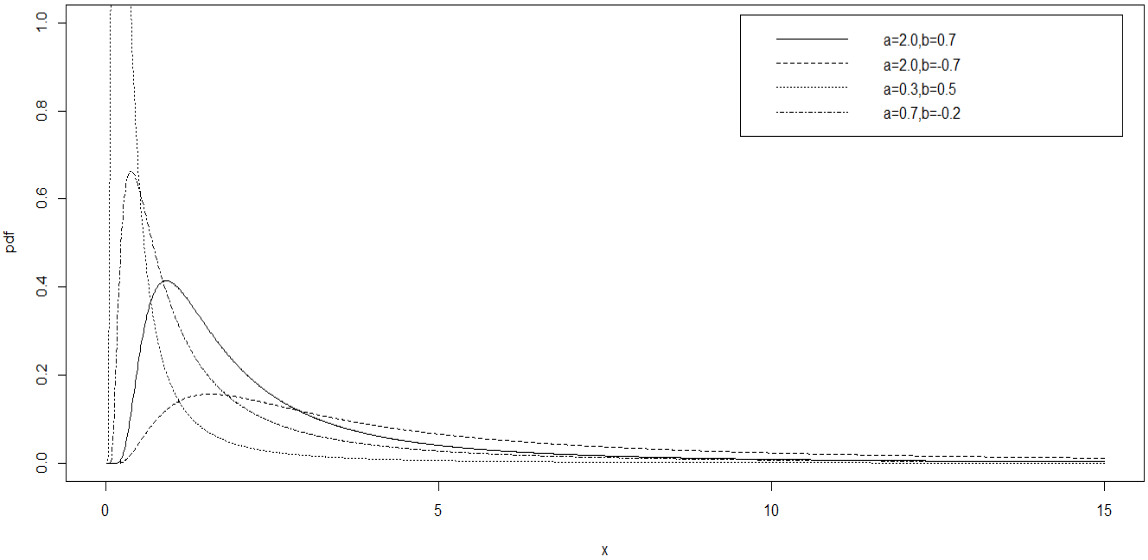

Fig. 1: Plot for the PDF of TIE distribution (where $a=\theta$ and $b=\lambda$)

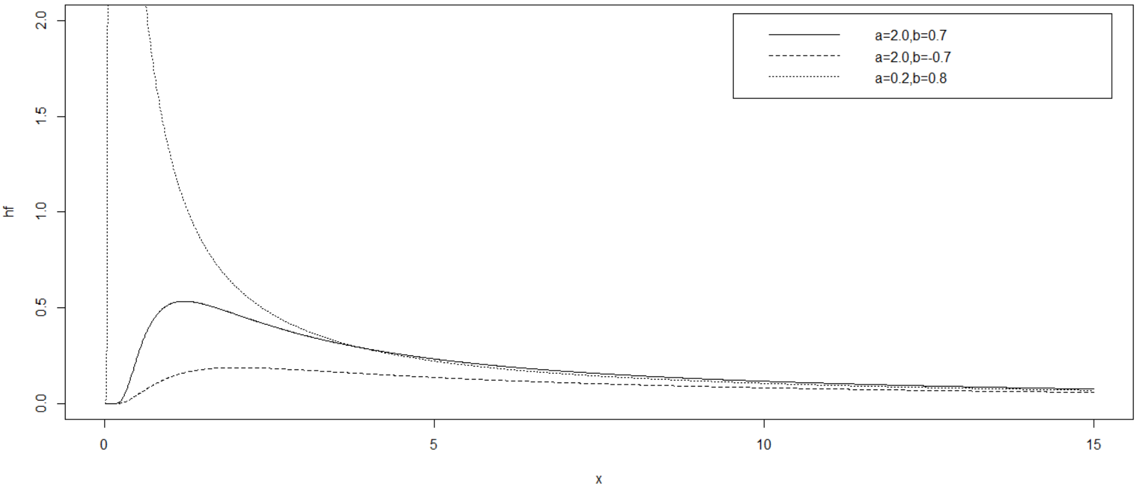

Fig. 2: Plot for the Hazard function of TIE distribution (where $a=\theta$ and $b=\lambda$)

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}\left\{\frac{\theta}{x^2}\exp\left(-\frac{\theta}{x}\right)\left[1+\lambda-2\lambda\exp\left(-\frac{\theta}{x}\right)\right]\right\}\times \left[\left[\exp\left(-\frac{\theta}{x}\right)\right]\left\{1+\lambda-\lambda\exp\left(-\frac{\theta}{x}\right)\right\}\right]^{i-1}\times \left[1-\left[\exp\left(-\frac{\theta}{x}\right)\right]\left\{1+\lambda-\lambda\exp\left(-\frac{\theta}{x}\right)\right\}\right]^{n-i} \quad (4)$$

The distributions of the minimum and maximum order statistics for the TIE distribution are thereby given by;

$$f_{X_{(1)}}(x)=n\times\left\{\frac{\theta}{x^2}\exp\left(-\frac{\theta}{x}\right)\left[1+\lambda-2\lambda\exp\left(-\frac{\theta}{x}\right)\right]\right\}\times \left[1-\left[\exp\left(-\frac{\theta}{x}\right)\right]\left\{1+\lambda-\lambda\exp\left(-\frac{\theta}{x}\right)\right\}\right]^{n-1} \quad (5)$$

and

$$f_{X_{(n)}}(x)=n\times\left\{\frac{\theta}{x^2}\exp\left(-\frac{\theta}{x}\right)\left[1+\lambda-2\lambda\exp\left(-\frac{\theta}{x}\right)\right]\right\}\times \left[\left[\exp\left(-\frac{\theta}{x}\right)\right]\left\{1+\lambda-\lambda\exp\left(-\frac{\theta}{x}\right)\right\}\right]^{n-1} \quad (6)$$

respectively

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Odds Function

Odds function can be defined as:

$$O(x) = \frac{F(x)}{S(x)} \quad (7)$$

Therefore, the odds function for the TIE distribution is:

$$O(x) = \frac{\left[\exp\left(-\frac{\theta}{x}\right)\right]\left\{1+\lambda-\lambda\exp\left(-\frac{\theta}{x}\right)\right\}}{1-\left[\exp\left(-\frac{\theta}{x}\right)\right]\left\{1+\lambda-\lambda\exp\left(-\frac{\theta}{x}\right)\right\}} \quad (8)$$

For $x > 0$, $\theta > 0$ and $|\lambda| \leq 1$

Reversed Hazard Function

Reversed hazard function is given by:

$$r(x) = \frac{f(x)}{F(x)} \quad (9)$$

Therefore, the expression for the reversed hazard function of the TIE distribution is given by:

$$r(x) = \frac{\frac{\theta}{x^2}\exp\left(-\frac{\theta}{x}\right)\left[1+\lambda-2\lambda\exp\left(-\frac{\theta}{x}\right)\right]}{\left[\exp\left(-\frac{\theta}{x}\right)\right]\left\{1+\lambda-\lambda\exp\left(-\frac{\theta}{x}\right)\right\}} \quad (10)$$

For $x > 0$, $\theta > 0$ and $|\lambda| \leq 1$

Estimation of Parameters

The parameters of the TIE distribution can be estimated using the method of maximum likelihood as follows ; let $X_1, X_2, ..., X_n$ be a random sample of size 'n' from Equation (1), then the likelihood function is given by;

$$f(x_1, x_2, ...x_n; \eta) = \prod_{i=1}^{n}\left[\frac{\theta}{x^2}\exp\left(-\frac{\theta}{x}\right)\left[1+\lambda-2\lambda\exp\left(-\frac{\theta}{x}\right)\right]\right] \quad (11)$$

The log-likelihood function is given by;

$$l = n\log(\theta) - 2\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right) + \sum_{i=1}^{n}\log\left[1+\lambda-2\lambda\exp\left(-\frac{\theta}{x_i}\right)\right] \quad (12)$$

The solution of $\frac{\partial l}{\partial \theta} = 0$ and $\frac{\partial l}{\partial \lambda} = 0$ gives the maximum likelihood estimates of the parameters.

The solution cannot be obtained analytically but it can be solved numerically using available statistical software.

III APPLICATIONS TO REAL LIFE DATA

In this section, the TIE distribution is applied to two real life data. The models under consideration are the TIE distribution and the IE distribution. R software shall be used to perform the analysis and the R-code would be made available on request. The pdfs of the competing models are given in Table 1.

Table 1: PDF of the competing models

Distributions	**PDF**
Transmuted Inverse Exponential	$f(x) = \frac{\theta}{x^2}\exp\left(-\frac{\theta}{x}\right)\left[1+\lambda-2\lambda\exp\left(-\frac{\theta}{x}\right)\right]$
Inverse Exponential	$f(x) = \frac{\theta}{x^2}\exp\left(-\frac{\theta}{x}\right)$

DATA I: The first data represents the waiting time (mins) of 100 bank customers before service is being rendered. It has previously been used by [13]. The data is as follows;

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11, 11, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2,, 18.4, 18.9, 19, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27, 31.6, 33.1, 38.5
The data is summarized in Table 2;

Table 2: Summary of the data on waiting times (in mins.) of bank customers

Min.	Max.	Mean	Var.	Skewness	Kurtosis
0.8	38.5	9.877	52.3741	1.4728	5.5403

The performances of the distributions under study are given in Table 3;

Table 3: Performance of the models with standard errors in parenthesis

Models	θ	λ	Log-Likelihood	AIC
Transmuted Inverse Exponential	10.7924 (0.9876)	1.8755 (0.1838)	-323.3	650.5
Inverse Exponential	5.3476 (0.5415)	-	-336.6	675.1

DATA II: The second data represents the vinyl chloride data (in mg/l) that was obtained from clean upgradient monitoring wells. It has been previously used by [14] and [15]. The data is as follows;

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5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8, 0.8, 0.4, 0.6, 0.9, 0.4, 2, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1, 0.2, 0.1, 0.1, 1.8, 0.9, 2, 4, 6.8, 1.2, 0.4, 0.2
The data is summarized in Table 4;

Table 4: Summary of the data on vinyl chloride

Min.	Max.	Mean	Variance	Skewness	Kurtosis
0.100	8.000	1.879	3.8126	1.6037	5.005

The performances of the distributions under study are given in Table 5;

Table 5: Performance of the models with standard errors in parenthesis

Models	θ	λ	Log-Likelihood	AIC
Transmuted Inverse Exponential	0.4138 (0.1089)	-0.6301 (0.3078)	-57.9202	119.8
Inverse Exponential	0.5725 (0.0982)	-	-59.1930	120.4

Remarks: The distribution that corresponds to the highest log-likelihood value or the lowest AIC value is considered to be the best.

IV CONCLUSION

The Transmuted Inverse Exponential distribution has been successfully extended to involve applications to real life data. The shape of the model could be unimodal (depending on the value of the parameters). In Tables 3 and 5, the TIE distribution has the highest log-likelihood value and it also has the lowest AIC value as compared to the Inverse Exponential distribution, this implies that the TIE distribution is more flexible than the IE distribution. In other words, one can confidently say the TIE distribution is an improvement over the IE distribution.

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REFERENCES

[1] P. E. Oguntunde & A. O. Adejumo (2015). The Transmuted Inverse Exponential Distribution, *International Journal of Advanced Statistics & Probability*, **3**(1), 1-7.
[2] W. Shaw & I. Buckley (2007). The alchemy of probability distributions: Beyond Gram-Charlier expansions and a skew-kurtotic-normal distribution from a rank transmutation map. *Research Report*.
[3] E. A. Owoloko, P. E. Oguntunde & A. O. Adejumo (2015). Performance Rating of the Transmuted Exponential Distribution: An Analytical Approach,, *SpringerPlus*, **4** (1), 1-15.
[4] G. R. Aryal & C. P. Tsokos (2011). Transmuted Weibull Distribution: A Genaralization of the Weibull Probability Distribution, *European Journal of Pure and Applied Mathematics*, 4(2), 89-102.
[5] F. Merovci (2013). Transmuted Rayleigh Distribution, *Austrian Journal of Statistics*, 42(1), 21-31.
[6] S. K. Ashour & M. A. Eltehiwy (2013). Transmuted Lomax Distribution, *American Journal of Applied Mathematics and Statistics*, 1(6), 121-127.
[7] G. R. Aryal (2013). Transmuted Log-Logistic Distribution, *Journal of Statistics Applications & Probability*, **2**(1), 11-20.
[8] F. Merovci (2013), Transmuted Lindley Distribution, *International. Journal of Open Problems in Computer and Mathematics*, **6**(2), 63-72.
[9] I. Elbatal, G. Asha & V. Raja (2014). Transmuted Exponentiated Fretchet Distribution : Properties and Applications, *Journal of Statistics Applications and Probability*, **3**(3), 379-394.
[10] M. A. Hussian (2014). Transmuted exponentiated gamma distribution; A generalization of the exponentiated gamma probability distribution, *Applied Mathematical Science*, 8(27):1297–1310.
[11] A. Ahmad, S. P. Ahmad & A. Ahmed (2014). Transmuted Inverse Rayleigh distribution: A generalization of the Inverse Rayleigh distribution, *Mathematical Theory & Modeling*, **4**(7), 90-98.
[12] F. Merovci & I. Puka (2014). Transmuted pareto distribution, *ProbStat Forum*, 7, 1-11.
[13] M. E. Ghitany, B. Atieh, S. Nadarajah (2008). Lindley Distribution and Its Applications, *Mathematics Computing and Simulation*, **78** 4), 493-506.
[14] D. K. Bhaumik, K. Kapur & R. D. Gibbons (2009). Testing Parameters of a Gamma Distribution for Small Samples, *Technometrics*, **51**(3), 326-334.
[15] R. Shanker, H. Fesshaye & S. Selvaraj (2015). On Modeling Lifetimes Data Using Exponential and Lindley Distributions, *Biometric and Biostatistics International Journal*, **2**(5), 00042