Finite Difference Method and Laplace Transform for Boundary Value Problems

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***Abstract*—This article presents the solution of boundary value problems using finite difference scheme and Laplace transform method. Some examples are solved to illustrate the methods; Laplace transforms gives a closed form solution while in finite difference scheme the extended interval enhances the convergence of the solution.**

***Index Terms*— Finite difference method, Laplace transforms, boundary value problems**

I. Introduction

Two-point boundary value problems have received a considerable attention due to its importance in many areas of sciences and engineering. These types of differential equations arise very frequently in fluid mechanics, quantum mechanics, optimal control, chemical-reactor theory, aerodynamics, reaction-diffusion process and geophysics.

Various analytical and numerical techniques proposed for the solution of differential equations are available in literature; some of these are Differential Transform Method [1-6], Rung-Kutta 4th Order Method [7], Bernoulli Polynomials [8], Cubic Spline Method [9], Sinc Collocation Method [10], Modified Picard Technique [11], Block Method [12-14], Adomian Decomposition Method [15-20], Homotopy Perturbation Method [21-23].

In this work, finite difference method proposed for the solution of two-point boundary value problems has been widely applied [24-26]. However, in this article the step length is extended and it is observed that the approach enhances the convergence of the result when compared with the exact from Laplace transforms (which gives a close form of solution), See Tables 1 and 2.

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II. ANALYSIS OF FINITE DIFFERECE SCHEME

Consider the second order boundary value problem below

$$\psi'' + p(\eta)\psi' + q(\eta)\psi = r(\eta), \eta \in [\alpha, \beta] \tag{1}$$

with the boundary conditions

$$\psi(\alpha) = A \text{ and } \psi(\beta) = B \tag{2}$$

The intervals $[a,b]$ is subdivided into n equal subintervals. The subintervals length is referred to as h , given that

$$h = \frac{\beta - \alpha}{n} \tag{3}$$

We consider the following points

$$\alpha = \eta_0, \eta_1 = \eta_0 + h, \eta_2 = \eta_0 + 2h, \ldots, \eta_m = \eta_0 + mh, \ldots, \eta_n = \eta_0 + nh \tag{4}$$

The numerical solution at any point η_m is denoted by ψ_m and the theoretical solution is written as $\psi(\eta_m)$

We shall consider the central difference approximation for the approximation of the differential equation. The approximation is shown below

$$\psi'_m = \frac{1}{2h}[\psi_{m+1} - \psi_m]; \qquad \psi''_m = \frac{1}{h^2}[\psi_{m+1} - 2\psi_m + \psi_{m-1}] \tag{5}$$

using (5) in (1),we obtain

$$\frac{1}{2}[\psi_{m+1} - 2\psi_m + \psi_{m-1}] + \frac{p(\eta_m)}{2h}[\psi_{m+1} - \psi_{m-1}] + q(\eta_m) \tag{6}$$

simplifying gives

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$$2\left[\psi_{m+1}-2\psi_m+\psi_{m-1}\right]+hp(\eta_m)\left[\psi_{m+1}-\psi_{m-1}\right]+ 2h^2q(\eta_m) \tag{7}$$

Equation (7) can be written as

$$a_m\psi_{m-1}+b_m\psi_m+c_m\psi_{m+1}=d_m, m=1,2,3,\ldots \tag{8}$$

where

$$\begin{aligned} a_m &= 2-hp(\eta_m),\\ b_m &= -4+2h^2q(\eta_m),\\ c_m &= 2+hp(\eta_m),\\ d_m &= 2h^2r(\eta_m) \end{aligned} \tag{9}$$

The following equations are obtained from (8)

$$a_1\psi_0+b_1\psi_1+c_1\psi_2=d_1 \tag{10}$$

$$a_2\psi_0+b_2\psi_1+c_2\psi_2=d_2 \text{ ,etc.} \tag{11}$$

The equations above result to a system of equations of the form $A\psi=d$ for the unknowns $\psi_1,\psi_2,\psi_3,\ldots,\psi_{n-1}$, where A is the coefficient matrix. Solving the system of equations above gives the solution of the boundary value problems

III. NUMERICAL EXAMPLES

Example 1: Consider the two-point boundary value problem below

$$\psi''(\eta)-\psi(\eta)=1, \psi(0)=0, \psi(1)=e-1 \tag{12}$$

The theoretical solution of (12) is

$$\psi(\eta)=e^t-1 \tag{13}$$

Solution by Laplace Transform

The Laplace transform of equation (12) gives

$$L\{\psi''\}-L\{\psi\}=L\{1\} \tag{14}$$

$$s^2\psi-s\psi(0)-\psi'(0)-\psi=\frac{1}{s} \tag{15}$$

Let $L\{\psi'(0)\}=m$

Equation (15) becomes

$$s^2\psi-s\psi(0)-m-\psi=\frac{1}{s} \tag{16}$$

and simplifying, we obtain

$$\psi=\frac{1}{s\left(s^2-1\right)}+\frac{m}{\left(s^2-1\right)} \tag{17}$$

Resolving into partial fraction, we get

$$\psi=-\frac{1}{s}+\frac{1}{s(s+1)}+\frac{1}{2(s-1)}+\frac{m}{2(s-1)}-\frac{m}{2(s+1)} \tag{18}$$

The inverse Laplace gives

$$\psi=-1+\frac{1}{2}e^{-\eta}+\frac{1}{2}e^{\eta}+\frac{m}{2}e^{\eta}-\frac{m}{2}e^{-\eta} \tag{19}$$

Using $y(1)=e-1$, we obtain

$$e-1=-1+\frac{1}{2}e^{-1}+\frac{1}{2}e^{1}+\frac{m}{2}e^{1}-\frac{m}{2}e^{-1} \tag{20}$$

which gives m = 1, then

$$\psi(\eta)=-1+\frac{1}{2}e^{-\eta}+\frac{1}{2}e^{\eta}+\frac{1}{2}e^{\eta}-\frac{1}{2}e^{-\eta} \tag{21}$$

Then

$$\psi(\eta)=e^{\eta}-1 \tag{22}$$

which is the exact solution

Solution by Finite Difference Method

Equation (12) is written with the following step lengths

$$h=\frac{1}{10}, n=\frac{\beta-\alpha}{h}=\frac{1-0}{\frac{1}{10}}=10 \tag{23}$$

From the above we have

$$\begin{aligned} &\psi(0)=0, \quad \psi(0.1)=?, \quad \psi(0.2)=?,\\ &\psi(0.3),\ldots, \quad \psi(1)=e-1 \end{aligned} \tag{24}$$

Using the central difference approximations for equation (12), we have

$$100\left[\psi_{m+1}-2\psi_m+\psi_{m-1}\right]-\psi_m=1 \tag{25}$$

For

$$m=1, \psi_0=1: \quad -201\psi_1+100\psi_2=1 \tag{26}$$

$$m=2: \quad 100\psi_1-201\psi_2+100\psi_3=1 \tag{27}$$

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$$m=3: \quad 100\psi_2-201\psi_3+100\psi_4=1$$
$$\vdots \tag{28}$$

$$m=9,\psi_{10}=e-1: \quad 100\psi_8-201\psi_9+ 100e-100=1 \tag{29}$$

Solving the system of equations (26-29) gives the solution of the boundary value problems; and the comparism with the close form solution of Laplace transform is presented in table1.

Table I: NUMERICAL SOLUTION FOR EXAMPLE 1

n	LAPLACE TRANSFORM	FDM	ERROR
0	0	0	0
0.1	0.105170918	0.105221343	5.0425E-05
0.2	0.221402758	0.221494899	9.2141E-05
0.3	0.349858808	0.349983405	0.0001246
0.4	0.491824698	0.491971744	0.00014705
0.5	0.648721271	0.648879802	0.00015853
0.6	0.8221188	0.822276656	0.00015786
0.7	1.013752707	1.013896278	0.00014357
0.8	1.225540928	1.225654862	0.00011393
0.9	1.459603111	1.459669995	6.6884E-05
1	1.718281828	1.718281828	4.5905E-12

Example 2: Consider the boundary value problems below

$$\psi''-\psi'=1,\psi(0)=2,\psi(1)=2(e-1) \tag{30}$$

The theoretical solution is

$$\psi(\eta)=2e^{\eta}-\eta-1 \tag{31}$$

Solution by Laplace Transform
The Laplace transform of equation (30) is

$$L\{\psi''\}-L\{\psi'\}=L\{1\} \tag{32}$$

$$\left(s^2\psi-s\psi(0)-\psi'(0)\right)-\left(s\psi-\psi(0)\right)=\frac{1}{s} \tag{33}$$

Let $L\{\psi'(0)\}=m$, equation (33) becomes

$$s^2\psi-s-m-s\psi+1=\frac{1}{s} \tag{34}$$

Simplifying, we obtain

$$\psi=\frac{1}{s\left(s^2-s\right)}+\frac{1}{s-1}+\frac{m}{s^2-s}-\frac{1}{s^2-s} \tag{35}$$

Resolving into partial fraction, we obtain

$$\psi=-\frac{1}{s^2}+\frac{1}{s-1}-\frac{1}{s}+\frac{1}{s-1}+\frac{m}{s-1}-\frac{m}{s} \tag{36}$$

The inverse Laplace transform of (36) is given as

$$\psi=-\eta+e^{\eta}+me^{\eta}-m \tag{37}$$

Applying $\psi(1)=2e-2$

$$2e-2=-\eta+e^{\eta}+me^{\eta}-m \tag{38}$$

Simplifying, we obtain $m=1$. Then equation (37) becomes

$$y=-\eta+e^{\eta}+e^{\eta}-1 \quad \Rightarrow \quad \psi(\eta)=2e^{\eta}-\eta-1 \tag{39}$$

Equation (39) is the closed form solution of the boundary value problems (30)

Solution by Finite Difference Method
Equation (30) is written with the following step lengths

$$\psi''-\psi'=1,\psi(0)=0,\psi(1)=2e-2$$

$$h=\frac{1}{10},n=\frac{\beta-\alpha}{h}=\frac{1-0}{\frac{1}{10}}=10 \tag{40}$$

With the nodal points above, we have

$$\psi(0)=0, \quad \psi(0.1)=?, \quad \psi(0.2)=?, \quad \psi(0.3),\ldots, \quad \psi(1)=2e-2 \tag{41}$$

Applying central difference approximations for equation (30), we obtain

$$100\left[\psi_{m+1}-2\psi_m+\psi_{m-1}\right]- 5\left[\psi_{m+1}-\psi_{m-1}\right]=1 \tag{42}$$

For

$$m=1,\psi_0=1: \quad -200\psi_1+95\psi_2=104 \tag{43}$$

$$m=2: \quad 105\psi_1-200\psi_2+95\psi_3=1 \tag{44}$$

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$$m=3:\quad 105\psi_2-200\psi_3+95\psi_4=1 \tag{45}$$
$$\vdots$$

$$m=9,\psi_{10}=2e-2:\quad 105\psi_8-200\psi_9+ 190e-190=1 \tag{46}$$

The system of equations (43-46) are solved and compared with the closed form solution of the Laplace transforms in Table 2.

Table I NUMERICAL SOLUTION FOR EXAMPLE 2

n	LAPLACE TRANSFORM	FDM	ERROR
0	1	1	0
0.1	1.110341836	1.11024861	9.3226E-05
0.2	1.242805516	1.242628652	0.00017686
0.3	1.399717615	1.399469751	0.00024786
0.4	1.583649395	1.583346756	0.00030264
0.5	1.797442541	1.79710555	0.00033699
0.6	2.044237601	2.043891587	0.00034601
0.7	2.327505415	2.327181416	0.000324
0.8	2.651081857	2.650817543	0.00026431
0.9	3.019206222	3.019046947	0.00015928
1	3.436563657	3.436563656	9.1809E-10

IV. CONCLUSION

In this article, Finite Difference Technique and Laplace transform are employed to solve two point boundary value problems. The step length is extended in finite difference method to enhance the convergence of the method; the results are compared with the close form solution of Laplace transform in Tables 1 and 2.

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