The Weibull-Inverted Exponential Distribution: A Generalization of the Inverse Exponential Distribution

Pelumi E. Oguntunde, *Member, IAENG*, Adebowale O. Adejumo, and Enahoro A. Owoloko

***Abstract*—In this paper, the Inverse Exponential distribution was extended using the weibull generalized family of distributions. The probability density function (pdf) and cumulative density function (cdf) of the resulting model were defined and some of its statistical properties were studied. The method of maximum likelihood estimation was proposed in estimating the model parameters. The model was applied to a real life data set in order to assess its flexibility over its parent distribution.**

***Index Terms*— Generalization, Estimation, Inverse Exponential, Properties, Weibull**

I. Introduction

The Inverse Exponential (IE) distribution was introduced as a modification of the Exponential distribution by [1] and thereby can be used in some situations where the Exponential distribution could not be applicable. For instance, the Exponential distribution is suitable for only constant hazard rates [2].

In recent years, generalized models have received greater attention as compared to standard theoretical models and in most cases, these generalized models tend to perform better than their baseline distributions when applied to real life phenomena.

The Weibull generalized family of distributions due to [3] is one of the several classes of generalized distributions available in the literature, its cdf was derived from;

$$F(x) = \int_0^{\frac{G(x)}{1-G(x)}} \alpha\beta t^{\beta-1} e^{-\alpha t^\beta} dt \qquad (1)$$

Hence, the cdf and pdf of the weibull generalized family of distribution is given by;

$$F(x) = 1 - \exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]^\beta\right\} \qquad (2)$$

and

$$f(x) = \alpha\beta g(x)\frac{G(x)^{\beta-1}}{\left[1-G(x)\right]^{\beta+1}}\exp\left\{-\alpha\left[\frac{G(x)}{1-G(x)}\right]^\beta\right\} \qquad (3)$$

respectively.

where $\alpha > 0$ and $\beta > 0$ are additional shape parameters

Details on other relevant generalized family of distributions are readily available in [4] and [5]

With this understanding, compound distributions like the Weibull-Weibull, Weibull-Burr XII, Weibull-Normal distributions [3], Weibull-Exponential distribution [6] and Weibull-Rayleigh distribution [7] have been studied.

In the same way, the Inverse Exponential (IE) distribution otherwise known as the Inverted Exponential distribution shall be extended in this paper to give the Weibull-Inverse Exponential (WIE) distribution and its properties shall be investigated.

The rest of this paper is organized as follows; in section 2, the proposed model is defined, some of its basic statistical properties are given in section 3 including estimation of model parameters, a real life application is provided in section 4 followed by a concluding remark. The R-code for the analysis can be made available on request.

II. The Weibull Inverted Exponential Distribution

To start with, the cdf and pdf of the Inverse Exponential distribution are given by;

$$G(x) = \exp\left(-\frac{\theta}{x}\right) \qquad (4)$$

and;

$$g(x) = \frac{\theta}{x^2}\exp\left(-\frac{\theta}{x}\right) \qquad (5)$$

respectively.

For $x > 0, \theta > 0$

where ; θ is a scale parameter

Then, the cdf of the WIE distribution is derived by inserting Equation (4) into Equation (2) to give;

Manuscript received: February 15, 2017; revised: March 12, 2017. This work was supported financially by Covenant University, Ota, Nigeria.

P. E. Oguntunde and E. A. Owoloko are with the Department of Mathematics, Covenant University, Ota, Nigeria. (corresponding author phone: +234 806 036 9637; e-mail: pelumi.oguntunde@covenantuniversity.edu.ng; peluemman@yahoo.com), alfred.owoloko@covenantuniversity.edu.ng

A. O. Adejumo works with the Department of Statistics, University of Ilorin, Nigeria and Department of Mathematics, Covenant University, Ota, Nigeria; email: adejumo.adebowale@covenantuniversity.edu.ng; aodejumo@gmail.com

$$F(x)=1-\exp\left\{-\alpha\left[\frac{\exp\left(-\frac{\theta}{x}\right)}{1-\exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\} \quad (6)$$

For $x>0,\alpha>0,\beta>0,\theta>0$

Its associated pdf can be simplified to give;

$$f(x)=\alpha\beta\frac{\theta}{x^2}\frac{\left\{\exp\left(-\frac{\theta}{x}\right)\right\}^{\beta}}{\left[1-\exp\left(-\frac{\theta}{x}\right)\right]^{\beta+1}}\exp\left\{-\alpha\left[\frac{\exp\left(-\frac{\theta}{x}\right)}{1-\exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\} \quad (7)$$

For $x>0,\alpha>0,\beta>0,\theta>0$

where ; θ is a scale parameter

α and β are shape parameters whose role are to vary tail weight

The shape of the WIE distribution could either be unimodal or decreasing (depending on the value of the parameters). The plots are not displayed for brevity purposes.

Some Properties of the WIE Distribution

In this section, some basic statistical properties of the WIE distribution are discussed.

Reliability Analysis

The survival function is given by;

$$S(x)=1-F(x)$$

Therefore, the survival function for the WIE distribution is given by;

$$S(x)=\exp\left\{-\alpha\left[\frac{\exp\left(-\frac{\theta}{x}\right)}{1-\exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\} \quad (8)$$

For $x>0,\alpha>0,\beta>0,\theta>0$

The hazard function is given by;

$$h(x)=\frac{f(x)}{1-F(x)}$$

Therefore, the hazard function for the WIE distribution is given by;

$$h(x)=\alpha\beta\frac{\theta}{x^2}\frac{\left\{\exp\left(-\frac{\theta}{x}\right)\right\}^{\beta}}{\left[1-\exp\left(-\frac{\theta}{x}\right)\right]^{\beta+1}} \quad (9)$$

For $x>0,\alpha>0,\beta>0,\theta>0$

It can be seen that the shape of the hazard function for the WIE distribution could be inverted bathtub, increasing or decreasing (depending on the value of the parameters).

Quantile Function and Median

Mathematically, quantile function can be obtained by:

$$Q(u)=F^{-1}(u)$$

Thus, the quantile function of the WIE distribution is obtained as:

$$Q(u)=\frac{\theta}{\log\left\{\left[-\alpha^{-1}\log(1-u)\right]^{-\frac{1}{\beta}}+1\right\}} \quad (10)$$

where; $u \sim Uniform(0,1)$

The median is obtained by substituting u=0.5 into Equation (10) as follow:

$$Median=\frac{\theta}{\log\left\{\left[-\alpha^{-1}\log(0.5)\right]^{-\frac{1}{\beta}}+1\right\}} \quad (11)$$

Parameter Estimation

Let $x_1,x_2,....,x_n$ be a random sample from the WIE distribution as defined in Equation (7), the likelihood function is given by;

$$f(x_1,x_2,...x_n;\alpha,\beta,\theta)=$$

$$\prod_{i=1}^{n}\left[\alpha\beta\frac{\theta}{x^2}\frac{\left\{\exp\left(-\frac{\theta}{x}\right)\right\}^{\beta}}{\left[1-\exp\left(-\frac{\theta}{x}\right)\right]^{\beta+1}}\exp\left\{-\alpha\left[\frac{\exp\left(-\frac{\theta}{x}\right)}{1-\exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}\right]$$

Let $l=\log\left[f(x_1,x_2,...,x_n;\alpha,\beta,\theta)\right]$

Then, the log-likelihood function denoted by l is given by;

$$l=n\log(\alpha)+n\log(\beta)+n\log(\theta)-2\sum_{i=1}^{n}\log(x_i)-$$

$$\beta\sum_{i=1}^{n}\left(\frac{\theta}{x_i}\right)-(\beta+1)\sum_{i=1}^{n}\log\left[1-\exp\left(-\frac{\theta}{x_i}\right)\right]-$$

$$\alpha\sum_{i=1}^{n}\left[\frac{\exp\left(-\frac{\theta}{x_i}\right)}{1-\exp\left(-\frac{\theta}{x_i}\right)}\right]^{\beta}$$

Solving the system of non-linear equations of $\frac{\partial l}{\partial\alpha}=0$, $\frac{\partial l}{\partial\beta}=0$ and $\frac{\partial l}{\partial\theta}=0$ gives the maximum likelihood estimates of parameters α, β and θ. Though, the solution cannot be obtained analytically but it can obtain numerically with the aid of statistical software.

III. APPLICATION

Here, the WIE distribution was applied to two real data sets and its flexibility over the baseline distribution (Inverse Exponential distribution) was assessed with the aid of R-software. Decisions in the analysis provided are based on Log-likelihood (LL) and Akaike Information Criteria (AIC). The higher the LL or the lower the AIC, the better the model and vice-versa.

The pdf of the competing models are as given in Table 1:

Table 1 : The pdf of competing models

Models	PDF
WIE	$f(x)=\alpha\beta\frac{\theta}{x^2}\frac{\left\{\exp\left(-\frac{\theta}{x}\right)\right\}^{\beta}}{\left[1-\exp\left(-\frac{\theta}{x}\right)\right]^{\beta+1}}\exp\left\{-\alpha\left[\frac{\exp\left(-\frac{\theta}{x}\right)}{1-\exp\left(-\frac{\theta}{x}\right)}\right]^{\beta}\right\}$
IE	$f(x)=\frac{\theta}{x^2}\exp\left(-\frac{\theta}{x}\right)$

DATA A: The first data used in this paper was given by [8] and it has also been used by [9]. It represents the survival times of a group of patients suffering from Head and Neck cancer diseases and treated using a combination of radiotherapy and chemotherapy (RT+CT). The data is as follows;

12.20, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36, 63.47, 68.46, 78.26, 74.47, 81.43, 84, 92, 94, 110, 112, 119, 127, 130, 133, 140, 146, 155, 159, 173, 179, 194, 195, 209, 249, 281, 319, 339, 432, 469, 519, 633, 725, 817, 1776

The summary of the data is given in Table 2:

Table 2: Data summary on survival times of patients suffering from head and neck cancer

Min	Max	Mean	Variance	Skewness	Kurtosis
12.2	1,776	223.5	93,286.41	3.3838	16.5596

The performance of the two competing distribution are shown in Table 3:

Table 3: Performance Rating

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	LL	AIC
WIE	0.7609	0.2488	0.0018	-200.40	406.81
IE	-	-	76.7000	-279.58	561.15

The result in Table 3 shows that the WIE distribution is indeed an improvement over the IE distribution as it has the lowest value of AIC and highest value of LL.

DATA B: The second data used in this research has been used before in [10] and [11]. It is the remission times of a random sample of 128 bladder cancer patients. Below is the data:
0.08, 2.09, 3.48,4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57,5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54,3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64,3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05,2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26,2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02,4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25,8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07,3.36, 6.93, 8.65, 12.63, 22.69.

The summary of the data is given in Table 4:

Table 4: Data summary on bladder cancer patients

Min.	Max.	Mean	Variance	Skewness	Kurtosis
0.08	79.05	9.366	110.425	3.2866	18.4831

The performance of the two competing distribution are shown in Table 5:

Table 5: Performance Rating

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	LL	AIC
WIE	0.3989	0.3105	0.0047	-227.98	461.95
IE	-	-	2.4847	-460.38	922.76

The result in Table 5 shows that the WIE distribution is indeed an improvement over the IE distribution as it has the lowest value of AIC and highest value of LL.

IV. CONCLUSION

The Weibull Inverted Exponential (WIE) distribution has been successfully defined. The shape of the model could either be unimodal or decreasing. Explicit expressions for the survival, hazard and quantile functions were given. The shapes of the hazard function indicate that the model would be useful for modeling data sets with decreasing, increasing or inverted bathtub failure rates. Real life application was provided and it was discovered that the WIE distribution is better than the IE distribution as the two additional shape parameters helped in withstanding data with strong asymmetry.

Further research would involve comparing the performance of the WIE distribution with other generalized models in order to assess its potentials.

ACKNOWLEGMENT

The authors are grateful to the anonymous reviewers who have made valuable comments on the draft of this paper. The financial support from Covenant University, Nigeria is also greatly appreciated.

REFERNCES

[1] A. Z. Keller & A. R. Kamath (1982). Reliability analysis of CNC Machine Tools, *Reliability Engineering*, 3, 449-473.

[2] G. Prakash (2012). Inverted Exponential distribution under a Bayesian viewpoint, *Journal of Modern Applied Statistical Methods*, 11(1), 190-202.

[3] M. Bourguignon, R. B. Silva, G. M. Cordeiro (2014). The Weibull-G Family of Probability Distributions, *Journal of Data Science*, 12, 53-68.

[4] M. Alizadeh, M. H. Tahir, G. M. Cordeiro, M. Mansoor, M. Zubair, G. G. Hamedani (2015). The Kumaraswamy Marshal-Olkin family of distributions, *Journal of the Egyptian Mathematical Society*, **23**, 546-557.

[5] E. A. Owoloko, P. E. Oguntunde & A. O. Adejumo (2015). Performance Rating of theTransmuted Exponential Distribution: An Analytical Approach, *SpringerPlus*, **4**(1), 1-15.

[6] P. E. Oguntunde, O. S. Balogun, H. I. Okagbue, S. A. Bishop (2015). The Weibull-Exponential Distribution: Its Properties and Applications, *Journal of Applied Sciences*, **15**(11), 1305-1311.

[7] F. Merovci & I. Elbatal (2015). Weibull Rayleigh Distribution: Theory and Applications, *Applied Mathematics & Information Sciences*, **9**(4), 2127-2137.

[8] B. Efron (1988). Logistic regression, survival analysis and the Kaplan-Meier curve, *Journal of the American Statistical Association*, **83**(402): 414-425.

[9] R. Shanker, H. Fesshaye, S. Selvaraj (2015). On Modeling Lifetimes Data Using Exponential and Lindley Distributions, *Biometric and Biostatistics International Journal*, **2**(5): 00042.

[10**]** E. T. Lee & J. W. Wang (2003). Statistical Methods for Survival Data Analysis (3rd Edition), John Wiley and Sons, New York, USA.

[11] P. E. Oguntunde, A. O. Adejumo & K. A. Adepoju (2016). Assessing the Flexibility of the Exponentiated Generalized Exponential Distribution, *Pacific Journal of Science and Technology*, 17(1), 49-57.

ISBN: 978-988-14047-4-9
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)