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Statistical Properties and Applications of A New Lindley Exponential Distribution

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ABSTRACT

In this study, a New Lindley Exponential distribution was studied using the Lindley generalized family of distributions. Expressions for its densities, survival function, hazard function, quantile function and distribution of order statistics were derived. Its sub-models were identified and the New Lindley Exponential distribution was applied to two real life datasets to assess its flexibility over the Lindley distribution and the Exponential distribution. The results indicate that the New Lindley Exponential distribution performed better than both the Lindley distribution and the Exponential distribution based on their log-likelihood and Akaike Information Criteria values.

Keywords: *Application, Distribution, Exponential, Lindley, Properties*

1. INTRODUCTION

Professionals in probability distribution theory and practitioners often make use of specific probability distributions based on either their mathematical simplicity or because of their flexibility. Some authors have worked on selecting standard theoretical distributions that best fits real life phenomena. For instance, the Lindley distribution has been compared with the Exponential distribution using fifteen data sets. The Lindley distribution performed better in fitting nine of the data sets while the Exponential distribution was better in fitting six out of the fifteen data sets considered. Meanwhile, the densities of the Exponential distribution appear to be simpler. Also, the performance of the Weibull distribution, Gamma distribution and the Log-normal distribution have been compared in modeling air pollutant data in Oguntunde et al., (2014).

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Recently, attention has been shifted to involve comparing the performance of compound distributions to that of standard theoretical distributions. This present study involves the formulation of a compound distribution; New Lindley Exponential distribution and aimed at investigating its statistical superiority over the Lindley distribution and the Exponential distribution.

The Lindley distribution is a continuous probability distribution that is qualitatively similar to the Exponential distribution; though, they are two distinct distributions. In practice, Lindley distribution has been used to model the waiting times of customers including lifetime data sets. Details about Lindley distribution can be found in several notable works like Cakmakyapan and Ozel (2016), Bhati et al., (2015), Shanker et al., (2015), Bakouch et al., (2011) and so on.

The cumulative density function (cdf) and the probability density function (pdf) of Lindley distribution are given by;

$$F(x)=1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x} \tag{1}$$

And;

$$f(x)=\frac{\theta^2}{1+\theta}(1+x)e^{-\theta x} \tag{2}$$

respectively.

For $x>0,\theta>0$

Where;

θ is a scale parameter

Some researchers have extended the Lindley distribution either by means of generalization or modification and the details can be found in Shanker (2016). In the same way, the Lindley distribution shall be extended in this study using the Lindley family of distributions. Meanwhile, two forms of the Lindley family of distributions are available in the literature. The first was introduced by Bhati et al., (2015) while the other was introduced by Cakmakyapan and Ozel (2016).

In Bhati et al., (2015), the survival function of the Lindley family of distributions was derived from;

$$S(x)=\frac{\theta^2}{1+\theta}\int_0^{\log[G(x)]}(1+t)e^{-\theta t}dt \tag{3}$$

For $x\in\Re,\theta>0$.

Where;

$G(x)$ is the cdf of the baseline or parent distribution

This form of generalization has been used to derive the Lindley Exponential distribution and it performed better than the Power Lindley distribution, Lindley distribution, Exponential distribution and the New Generalized Lindley distribution when applied to two real life data sets.

In Cakmakyapan and Ozel (2016), the cdf of the Lindley family of distributions was derived from the transformation;

$$F(x)=\int_0^{-\log[-G(x)]}\frac{\theta^2}{1+\theta}(1+t)e^{-\theta t}dt \tag{4}$$

Therefore, the cdf and the pdf of the Lindley family of distributions (or Lindley-G distribution) are given by;

$$F(x)=1-\left[1-\frac{\theta}{\theta+1}\left[\log\left(1-G(x)\right)\right]\right]\left[1-G(x)\right]^{\theta} \tag{5}$$

And;

$$f(x)=g(x)\left[1-\log\left(1-G(x)\right)\right]\left[1-G(x)\right]^{\theta-1}\frac{\theta^2}{\theta+1} \tag{6}$$

respectively.

Where;

$G(x)$ is the cdf of the baseline distribution

$g(x)$ is the pdf of the baseline distribution

θ is the additional shape parameter

This form of generalization has been used to study the Lindley Weibull distribution and it performed better than its competing distributions when applied to real life data sets. One very interesting thing that could be noticed about the Lindley-G distribution is the fact that it has only one additional shape parameter.

It should also be noted that several other families of generalized distributions exist in the literature. The Beta-G (Eugene et al., 2002), Kumaraswamy-G (Cordeiro and de Castro; 2011), Transmuted-G (Shaw and Buckley, 2007), Gamma-G (type 1) (Zografos and Balakrishnan, 2009), Gamma-G (type 2) (Ristic and Balakrishnan, 2012), Gamma-G (type 3) (Torabi and Montazari, 2012), McDonald-G (Alexander et al., 2012), Log-gamma-G (Amini et al., 2014), Exponentiated T-X (Alzaghal et al., 2013), Exponentiated-G (Cordeiro et al., 2013), Logistic-G (Torabi and Montazari, 2014), Gamma-X (Alzaatreh et al., 2014), Logistic-X (Tahir et al., 2015), Weibull-X (Alzaatreh et al., 2013), Weibull-G (Bourguignon et al., 2014), Marshall-Olkin-G (Marshall and Olkin, 1997), Beta Marshall-Olkin-G (Alizadeh et al., 2015a), Kumaraswamy Marshall-Olkin-G (Alizadeh et al., 2015b) and Kumaraswamy Transmuted-G family of distributions (Afifi et al., 2016) are known families of generalized distributions in the literature.

The aim of this study is to explore a New Lindley Exponential distribution using the Lindley-G distribution introduced by Cakmakyapan and Ozel (2016). Expressions for some of its basic statistical properties shall be established and its flexibility shall be assessed using real life data sets. The rest of this article is structured as follows; in section 2, the New Lindley Exponential distribution is defined and its properties investigated including estimation of model parameters, in section 3, real life applications of the New Lindley Exponential distribution are provided, followed by a concluding remark.

2. THE NEW LINDLEY EXPONENTIAL DISTRIBUTION

Consider the Exponential distribution with cdf and pdf given by;

$$G(x) = 1 - e^{-x/\lambda} \tag{7}$$

And;

$$g(x) = \frac{1}{\lambda} e^{-x/\lambda} \tag{8}$$

respectively.

Where;

λ is a scale parameter

Now, the cdf of the New Lindley Exponential distribution is derived by substituting Equation (7) into Equation (5) to give;

$$F(x) = 1 - \left[1 - \frac{\theta}{\theta+1}\left[\log\left(1 - \left\{1 - e^{-x/\lambda}\right\}\right)\right]\right]\left[1 - \left\{1 - e^{-x/\lambda}\right\}\right]^{\theta}$$

$$F(x) = 1 - \left[1 + \frac{\theta}{\theta+1}\left(\frac{x}{\lambda}\right)\right] e^{-(x/\lambda)\theta} \tag{9}$$

Its corresponding pdf is derived as;

$$f(x) = \frac{1}{\lambda} e^{-x/\lambda}\left[1 + \left(\frac{x}{\lambda}\right)\right]\left[e^{-x/\lambda}\right]^{\theta-1} \frac{\theta^2}{\theta+1}$$

$$f(x) = \frac{\theta^2}{\lambda(\theta+1)}\left[1 + \left(\frac{x}{\lambda}\right)\right] e^{-(x/\lambda)\theta} \tag{10}$$

For $x > 0, \theta > 0, \lambda > 0$.

Where;

θ is the shape parameter

λ is the scale parameter

Special Case:

If $\lambda = 1$, then the New Lindley Exponential distribution reduces to give the Lindley distribution.

Some possible plots for the pdf of the New Lindley Exponential distribution are shown in Figure 1;

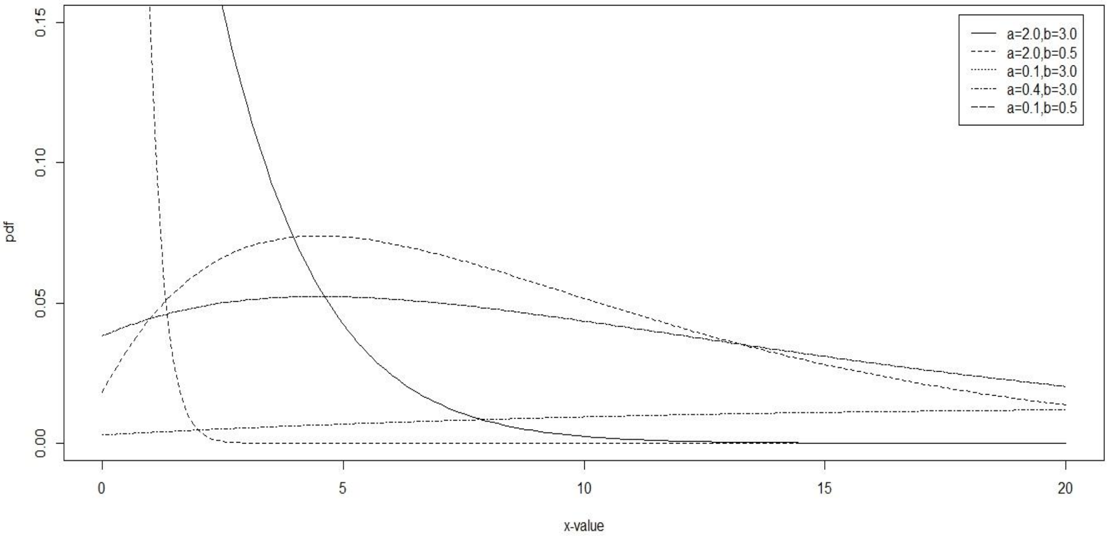

Fig. 1: PDF of the NLE Distribution

From Figure 1, it is obvious that the shape of the New Lindley Exponential could be "decreasing" just like that of the Exponential distribution, it could also be "increasing" and it could exhibit inverted bathtub shape; depending on the value of the parameters.

2.1 Reliability Analysis for the New Lindley Exponential distribution

In this sub-section, expressions for the survival function and the hazard function of the New Lindley Exponential distribution are provided.

The general mathematical expression for the survival function is given by;

$$S(x)=1-F(x)$$

Hence, the expression for the survival function of the New Lindley Exponential distribution is given by;

$$S(x)=1-\left\{1-\left[1+\frac{\theta}{\theta+1}\left(\frac{x}{\lambda}\right)\right]e^{-\left(x/\lambda\right)\theta}\right\}$$

$$S(x)=\left[1+\frac{\theta}{\theta+1}\left(\frac{x}{\lambda}\right)\right]e^{-\left(x/\lambda\right)\theta} \tag{11}$$

For $x>0,\theta>0,\lambda>0$.

Also, the hazard function is represented by;

$$h(x)=\frac{f(x)}{S(x)}$$

$$h(x)=\frac{\frac{\theta^2}{\lambda(\theta+1)}\left[1+\left(\frac{x}{\lambda}\right)\right]e^{-\left(x/\lambda\right)\theta}}{\left[1+\frac{\theta}{\theta+1}\left(\frac{x}{\lambda}\right)\right]e^{-\left(x/\lambda\right)\theta}}$$

This can be simplified to give;

$$h(x)=\frac{\theta^2\left[1+\left(\frac{x}{\lambda}\right)\right]}{\lambda(\theta+1)+\theta x} \tag{12}$$

For $x>0,\theta>0,\lambda>0$.

Some possible plots for the hazard function of the New Lindley Exponential distribution are shown in Figure 2.

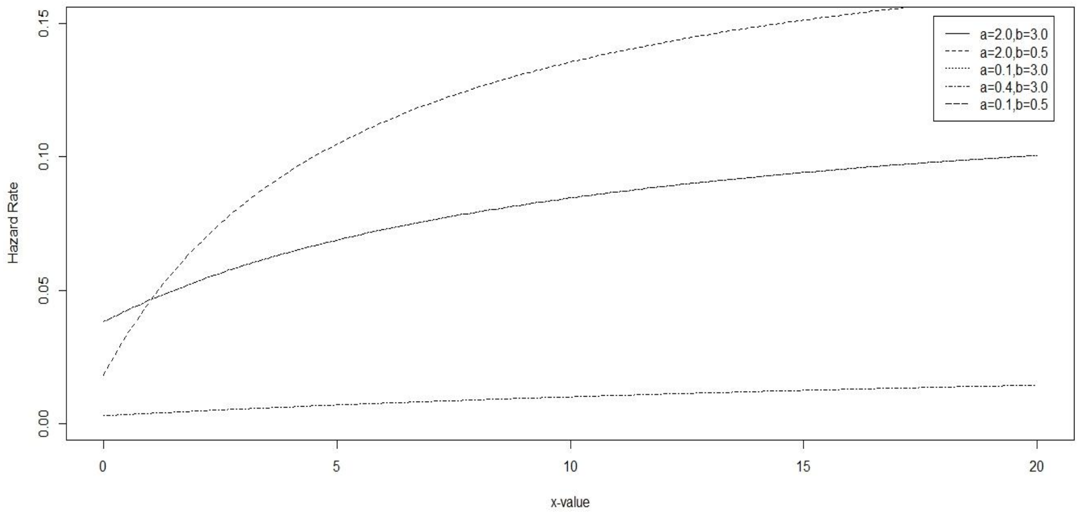

Fig. 2: Hazard Function of the NLE Distribution

Plots for the hazard function of the New Lindley distribution exhibit the same shape. This in turn means that the New Lindley Exponential distribution could be useful in modeling real life phenomena with increasing failure rates.

2.3 Distribution of Order Statistics

If $X_1, X_2, \ldots\ldots, X_n$ represent a random sample from a cdf $F(x)$ and an associated pdf $f(x)$ distributed according to the New Lindley Exponential distribution, then the pdf of jth order statistics of the New Lindley Exponential distribution is derived as;

$$f_{j:n}(x)=\frac{n!}{(j-1)!(n-j)!}f(x)F(x)^{j-1}\left[1-F(x)\right]^{n-j} \tag{13}$$

If Equations (9) and (10) are substituted into Equation (13) then;

$$f_{j:n}(x)=\frac{n!}{(j-1)!(n-j)!}\left\{\frac{\theta^2}{\lambda(\theta+1)}\left[1+\left(\frac{x}{\lambda}\right)\right]e^{-\left(x/\lambda\right)\theta}\right\}\left\{1-\left[1+\frac{\theta}{\theta+1}\left(\frac{x}{\lambda}\right)\right]e^{-\left(x/\lambda\right)\theta}\right\}^{j-1}\times$$

$$\left[\left\{\left[1+\frac{\theta}{\theta+1}\left(\frac{x}{\lambda}\right)\right]e^{-\left(x/\lambda\right)\theta}\right\}\right]^{n-j} \tag{14}$$

The pdf of the minimum order of statistics for the New Lindley Exponential distribution is given by;

$$f_{1:n}(x)=n\times\left\{\frac{\theta^2}{\lambda(\theta+1)}\left[1+\left(\frac{x}{\lambda}\right)\right]e^{-\left(x/\lambda\right)\theta}\right\}\left[\left\{\left[1+\frac{\theta}{\theta+1}\left(\frac{x}{\lambda}\right)\right]e^{-\left(x/\lambda\right)\theta}\right\}\right]^{n-1} \tag{15}$$

The pdf of the maximum order of statistics for the New Lindley Exponential distribution is given by;

$$f_{n:n}(x)=n\times\left\{\frac{\theta^2}{\lambda(\theta+1)}\left[1+\left(\frac{x}{\lambda}\right)\right]e^{-\left(x/\lambda\right)\theta}\right\}\left\{1-\left[1+\frac{\theta}{\theta+1}\left(\frac{x}{\lambda}\right)\right]e^{-\left(x/\lambda\right)\theta}\right\}^{n-1} \tag{14}$$

2.4 Estimation of Parameters

The method of maximum likelihood estimation (MLE) is used to estimate the unknown parameters of the New Lindley Exponential distribution as follows; Let $X_1, X_2, ..., X_n$ denote random samples from the New Lindley Exponential distribution with parameters θ and λ . The likelihood function is given by;

$$L(\theta;\lambda)=\prod_{i=1}^{n}\left\{\frac{\theta^2}{\lambda(\theta+1)}\left[1+\left(\frac{x}{\lambda}\right)\right]e^{-\left(x/\lambda\right)\theta}\right\} \tag{16}$$

The log-likelihood function denoted by l is given by;

$$l=2n\log\theta-n\log\lambda-n\log(\theta+1)+\sum_{i=1}^{n}\log\left[1+\left(\frac{x_i}{\lambda}\right)\right]-\theta\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right) \tag{17}$$

Solving the resulting simultaneous system of equations of $\frac{dl}{d\theta}=0$ and $\frac{dl}{d\lambda}=0$ gives the maximum likelihood estimates of parameters θ and λ . Meanwhile, the solution may not be obtainable analytically but it can be obtained numerically with the aid of statistical software like R, SAS, and so on when data sets are available.

3. APPLICATION TO REAL LIFE DATA

In this section, the New Lindley Exponential distribution is applied to two (2) real data sets. The aim is to assess its flexibility over the existing Lindley Exponential distribution, Lindley distribution and Exponential distribution.

The log-likelihood function for the Lindley distribution and the Exponential distribution used in this study are;

$$l=2n\log\theta-n\log(1+\theta)+\sum_{i=1}^{n}\log(1+x_i)-\theta\sum_{i=1}^{n}x_i$$

And

$$l=-n\log\lambda-\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)$$

respectively.

The results for the analysis in this present study are obtained with the aid of R software while the criteria used for selecting the best distribution are Log-likelihood and the Akaike Information Criteria (AIC). Meanwhile, the distribution with the highest log-likelihood value or the lowest AIC is considered the best.

DATA I: The first data was gotten from Nassar and Nada (2011) and it has also been used by Owoloko et al., (2016). It represents the monthly actual taxes revenue in Egypt (in 1,000 million Egyptian pounds) between January 2006 and November 2010. The data is as follows;

5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17, 8.6, 9.7,

39.2, 35.7, 15.7, 9.7, 10, 4.1, 36, 8.5, 8, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1,

20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11, 11.6, 11.9, 5.2, 6.8, 8.9,

7.1, 10.8

The data is summarized in Table 1.

Table 1: Summary of data on tax revenue

n	Min.	Max.	Mean	Median	Variance	Skewness	Kurtosis
59	4.10	39.20	13.49	10.60	64.8266	1.6083	5.2560

The performance of the competing distributions is given in Table 2:

Table 2: Performance Ratings using DATA I

Distributions	Estimates	Log-likelihood	AIC	Rank
New Lindley Exponential	$\theta = 0.00016, \lambda = 0.00111$	-198.1008	400.2015	1
Lindley	$\theta = 0.13922$	-200.6293	403.2586	2
Exponential	$\lambda = 13.488$	-212.5068	427.0136	3

From Table 2, the New Lindley Exponential distribution has the highest log-likelihood value and lowest AIC value of -198.1008 and 400.2015 respectively thus making it to perform better than the Lindley distribution and the Exponential distribution.

DATA II: The second data was gotten from Ghitany et al., (2008), Alqallaf et al., (2015) and Oguntunde et al., (2016). The data represents the waiting time (mins) of 100 bank customers before service is being rendered. The data is as follows;

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11, 11, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2,, 18.4, 18.9, 19, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27, 31.6, 33.1, 38.5

The data is summarized in Table 3.

Table 3: Summary of data on waiting time of bank customers

n	Min.	Max.	Mean	Median	Variance	Skewness	Kurtosis
100	0.800	38.500	9.877	8.100	52.3741	1.4728	5.5403

The performance of the competing distributions is given in Table 4:

Table 4: Performance Ratings using DATA II

Distributions	Estimates	Log-likelihood	AIC	Rank
New Lindley Exponential	$\theta = 0.00026, \lambda = 0.00127$	-317.3031	638.6062	1
Lindley	$\theta = 0.18657$	-319.0374	640.0748	2
Exponential	$\lambda = 9.8770$	-329.0209	660.0418	3

From Table 4, the New Lindley Exponential distribution has the highest log-likelihood value and lowest AIC value of -317.3031 and 638.6062 respectively thus making it to perform better than the Lindley distribution and the Exponential distribution.

4. CONCLUSION

A New Lindley Exponential distribution has been successfully studied. Explicit expressions for its probability density function, cumulative density function, survival function, reliability function and distribution of order statistics have been provided. The shape of the New Lindley Exponential distribution could be increasing, decreasing or inverted bathtub (depending on the parameter values). When applied to two real data sets, the New Lindley Exponential distribution performed better than the Lindley distribution and the Exponential distribution based on the log-likelihood and AIC values posed by the distributions.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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