Error Correction Enhancement with Interleavers for DC free codes

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***Abstract*—Spectral shaping codes are digital codes especially designed in such a manner that their alphabets give rise to predetermined frequency spectrum envelopes. This implies that the signal energy of these alphabets will be concentrated into a predetermined range of a given frequency spectrum. The design of digital codes in the above mentioned fashion is done in order to match specific channel constraints in the design of transmission, multiplexing or storage systems. In certain applications it is desirable that spectral shaping codes provide some error correction capability or that they possess an increased robustness to errors. These errors may be introduced during transmission, processing or storage of digital information. In this paper, an interleaving method is introduced to increase the robustness of DC free codes toward burst errors whilst maintaining the spectral shaping property of DC free codes. Interleaving when applied to a coded communication system results in a reduction in the mean bit error rate (BER) of coded bit streams.**

Keywords—DC free; Knuth; interleaver; error correction; power spectral density

I. INTRODUCTION

Spectral shaping codes include but are not limited to direct current (DC) free codes and spectral nulls codes. In certain applications it is desirable that spectral shaping codes provide some error correction capability or that they possess an increased robustness to errors. These errors may be introduced during transmission, processing or storage of digital information [1]. In general spectral shaping codes have limited robustness to errors and they possess limited error correcting capabilities. An interleaving method is introduced to increase the robustness of DC free codes toward burst errors whilst maintaining their spectral shaping properties.

DC free codes are digital sum constrained codes whose frequency spectrum vanishes at zero frequency [2]. The frequency region in which the spectrum is suppressed is known as the notch width and it is characterized by the cut off frequency, ω_0. The cut off frequency of a DC constrained sequence is formally defined by

$$H(\omega_0) = \frac{1}{2} \qquad (1)$$

The power spectral density (PSD) function of digital sum constrained codes is characterized by a parabolic shape in the low frequency range from DC to the cut off frequency [2]. The size of the notch width can be increased by designing higher order DC free codes. The order of a DC free codebook is denoted by K. The higher the K value, the higher the order of the DC free codebook.

Interleaving is a process in which code symbols from different code words are permutated according to a predefined mapping before transmission [3]. It is the most effective way for correcting single or multiple bursts of errors which may be encountered during transmission. The bursts of errors are spread across multiple code words when the code words are reconstructed at the receiver to prevent the corruption of large portions of successive bits. Interleaving is necessary in achieving time diversity because it effectively resists deep fading effects as well as noisy bursts of errors.

Interleaving reshuffles the original order of DC free sequences. When interleaving is applied to zero disparity DC free codes, it distorts the bit order and length of the DC free code words. Distortion in the bit order may result in a loss of the spectral shaping properties of a DC free codebook. This arises from the fact that zero disparity coding schemes require that the disparity $d(\boldsymbol{x})$ of code word $\boldsymbol{x}$ of length n, n being even, consisting of bipolar elements x_i, $1 \leq i \leq n$, $x_i \in \{-1,1,\}$ satisfies the following condition [2]:

$$d(\boldsymbol{x}) = \sum_{i=1}^{n} x_i = 0 \qquad (2)$$

Higher order zero disparity DC free codes require that the code word moments denoted by u_k meet the following conditions [4]:

$$u_k = \sum_{i=1}^{n} i^k x_i = 0 \quad ,k \in \{0,1,\dots\dots\dots K\}. \qquad (3)$$

Rearranging the order of DC free code words through interleaving may result in $d(\boldsymbol{x}) \neq 0$ or $u_k \neq 0$ for the new code book of code words formed at the interleaver output.

It is necessary to recover the DC free property of DC free codes after interleaving so that the interleaved sequences meet the DC free channel characteristics. While it is straightforward to observe that interleaving reshuffles and therefore distorts the order of a DC free codebook, it is interesting to investigate the effect of reshuffling the original pattern of a DC free codebook on the spectral shaping capabilities of the codebook. It would be expected that reshuffling the pattern of DC free sequences would automatically imply a loss in the spectral shaping capabilities of a DC free codebook. However, this is not necessarily the case depending on how the interleaver is designed and selection of interleaver parameters (J, w), J being the interleaver depth and w being the interleaver span.

In this work, an interleaving method which we will call a DC free interleaver is introduced which ensures a DC free interleaver output regardless of nature of input codebook. This interleaver is implemented using a predefined mapping which ensures a DC free interleaved output regardless of input.

The effects of reshuffling DC free codebooks through interleaving are investigated by analysing the evolution of the interleaver output PSD functions from the input DC free codebook PSD functions. It is determined for which interleaver depth J and span w the interleaver output gives a PSD function having a larger notch width than that of the input DC free codebook. The proposed DC free interleaver design is then introduced. The effect of reshuffling DC free codebooks using the DC free interleaver is investigated and a conclusion is made.

II. INVESTIGATION OF PARAMETERS AFFECTING NOTCH WIDTH SIZE OF DC FREE CODEBOOKS

A. Algorithm for generating DC free codebooks

DC free codebooks for the binary field are generated using the zero disparity coding scheme. The algorithm used to generate DC free codebooks is as follows,

i. Input $x \in \{-1, 1\}$
ii. Desired code word length n defined
iii. Using nested for loops combinations of x which satisfy,

$$\sum_{i=1}^{n} i^{K} x_i = 0 \quad , K \in \{0,1,2,..\} \tag{4}$$

where K is the order of the DC free codebook being generated are found. For $K > 0$, n is selected such that it is a multiple of four.

iv. Whenever (4) is satisfied the resulting x elements which satisfy the equality constituted a code word and the code word is stored in a matrix.
v. The matrix containing all possible combinations of x elements satisfying (4) constitute a complete DC free code book.

III. INTERLEAVING

A. Block interleaver for reshuffling DC free sequences

Reshuffling of DC free sequences is done through the use of an interleaver. The interleaver input is generated using the algorithm defined in Section II. A. The resulting codebook is used as the interleaver input.

Let J be the interleaver depth and w be the interleaver span. Let C be the input DC free codebook's cardinality. The interleaver is a J by w matrix designed in such a way that J and w are variable. J and w are chosen such that the product Jw equals the total number of bits in the input codebook, that is $Jw = Cn$. The interleaver is designed as follows:

i. Transmit input codebook serially and arrange the input bits column wise in the interleaver according to a predefined J value, J being the column length and interleaver depth
ii. When all input bits have been entered into interleaver according to the rule in i., transmit interleaver elements row wise. Each row is treated as a new code word.
iii. Transmit each new code word into a $J\ by\ w$ matrix which acts as the new codebook consisting of reshuffled or interleaved input codewords.

In order to determine the evolution of the resulting interleaved codebook's PSD function the PSD function of the interleaved codebook is plotted using the algorithm. The result obtained was then compared with that of the input codebook.

The effect of varying the interleaver parameters (J, w) on the spectral shaping capabilities of an interleaved codebook is also investigated. Investigations are carried out for (J, w) combinations in the regions $J < n$, $J = n$ and $J > n$.

B. Design of DC free interleaver

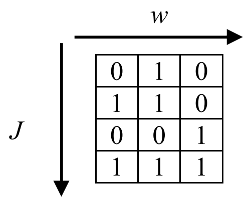
The interleaver design is as follows:

i. Interleaver makes use of the zero disparity coding scheme.
ii. Interleaver parameters (J, w) are variable therefore J and w can be defined at any particular instance.
iii. For even interleaver span w, the mapping (5)

$$\sum_{i=0}^{\frac{N}{J}-1} x_{Ji+1} = 0 \tag{5}$$

gives rise to the first balanced code word in the interleaver where J is the interleaver depth, N the total number of elements of input codebook and x an element from the input codebook. The mapping in (5) takes one element from each input code word contained in the codebook and arranges it row wise in a matrix. The new formed row constitutes a new balanced interleaved code word. To get the whole set of balanced interleaved words for a given input codebook, J has to decrement by one and the set $\{ x \}$ of elements from the input codebook has to decrement in the manner $\{x\} = \{x\} - \{previously\ formed\ interleaved\ code\ word\}$ for each subsequent row. The mapping given in (5) makes use of Knuth's algorithm [5] for balancing code words.In order for the mapping to work, J has to be selected such that it is a factor of the total number of elements in the codebook. Thus $\frac{N}{J} = w$, w being the interleaver span.

iv. For odd interleaver span w the mapping defined in (5) is not sufficient to result in a balanced output. As a result concatenation is used to result in even w. Say odd w is selected such that the resulting interleaver matrix is as in Fig. 1.



0	1	0
1	1	0
0	0	1
1	1	1

Fig. 1. Example of interleaver matrix with odd w

In Fig. 1, $J = 4$ and $w = 3$. In order for the mapping in (5) to be satisfied, it is required that w be even. Let the number of rows in the matrix in Fig. 1 be numbered i from the first row. Each subsequent row is therefore increment by 1. The previous row of elements i is concatenated with current row $i + 1$ in order to result with an even code word. The concatenation is as follows. Let $w(i), i \in \{1,2, \dots \dots . J\}$ denote the word in the i^{th} row of the J by w matrix. In order to create an even w a new word $v = w(i) + w(i + 1), i \in \{1, \dots \dots J - 1\}$ is generated. This even code word is then transmitted and stored in a new matrix. The complete matrix consisting of the new even concatenated code words is a $2w$ by $J - 1$ matrix D. The mapping described in (5) is then applied to matrix D resulting in a DC free output.

IV. RESULTS

Investigations are made on DC free codes of orders K = 0, 1 and 2 respectively. The investigations are carried out as highlighted below.

i. Effect of varying n on the spectral notch width size of a DC free codebook of a given order K.

ii. Effect of varying interleaver J and w on the spectral shaping properties of DC free codebooks.

iii. Effect of applying DC free interleaver on DC free codebooks

The results obtained are given below:

A. Effect of increasing n on spectral notch width size

***Example:* $\boldsymbol{K = 0}$**

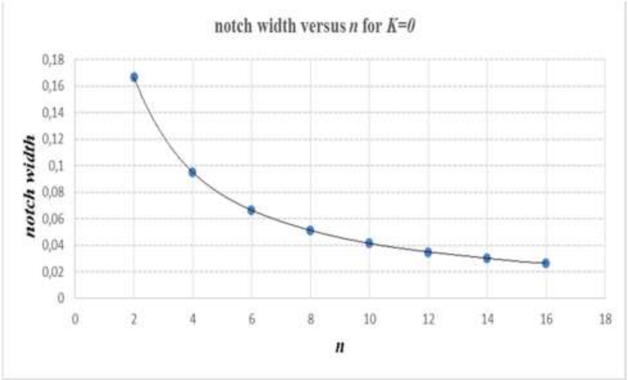


Fig. 2. Variation of notch width with n

Fig. 2 shows that the size of the spectral notch width is inversely proportional to n. In general, the relationship between notch width and code word length for DC free codes of orders K = 0, 1 and 2 respectively was found to be of polynomial regression form, the polynomials having a decreasing gradient.

B. Effect of interleaving on the spectral shaping properties of DC free codebooks

Example 1

Input : DC free codebook of order $K = 1$ *having code words of length* $n = 12$

$$\boldsymbol{J = 6, w = 116}$$

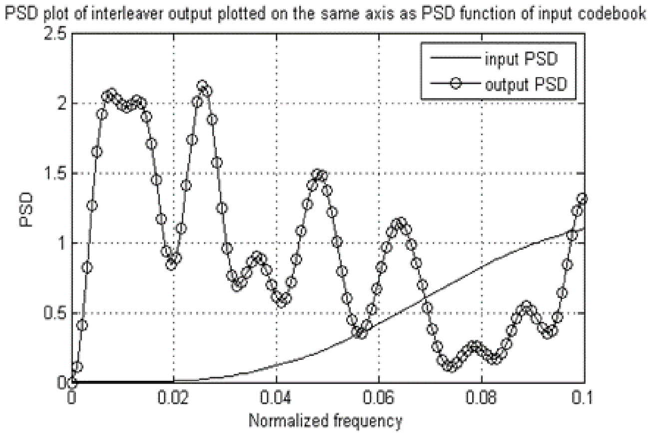


Fig. 3. Input codebook PSD versus interleaver output PSD for J=6

$J = 12, w = 58$

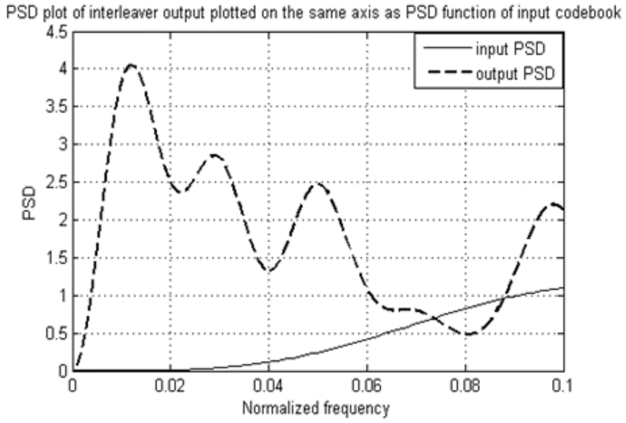


Fig. 4. Input codebook PSD versus interleaver output PSD for J=12

$J = 29, w = 24$

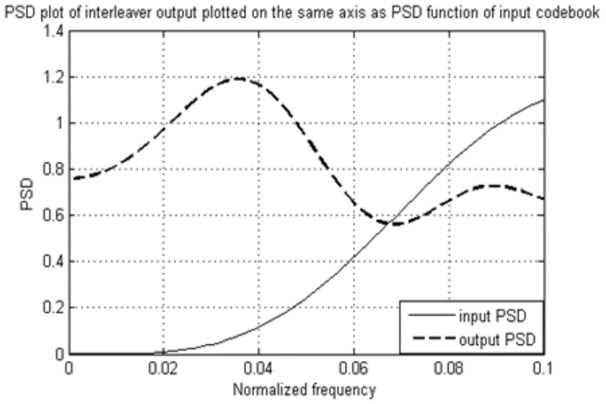


Fig. 5. Input codebook PSD versus interleaver output PSD for J=29

$J = 29, w = 24$ when DC free interleaver is used

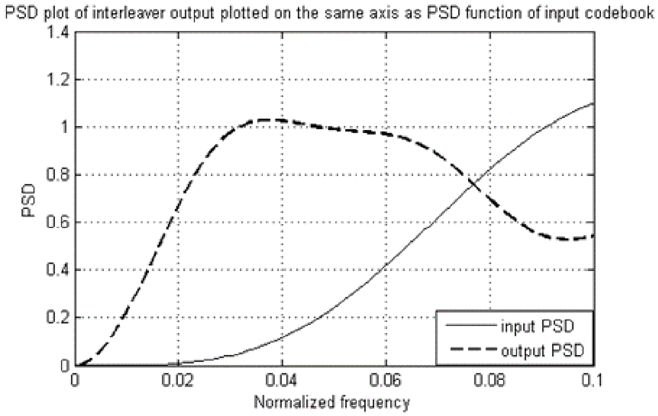


Fig. 6. Input codebook PSD versus interleaver output PSD for J=29

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$J = 348, w = 2$

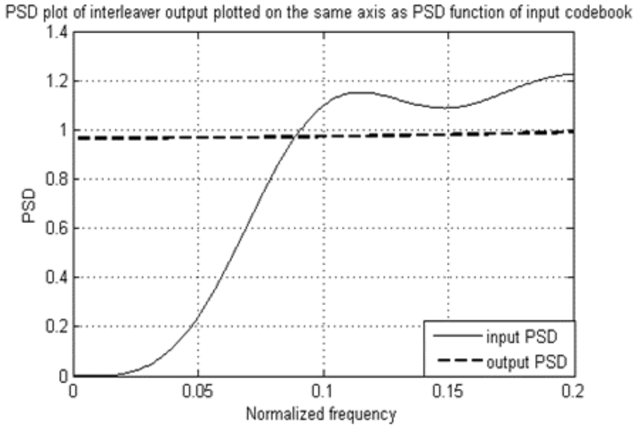

Fig. 7. Input codebook PSD versus interleaver output PSD for J=348

$J = 348, w = 2$ when DC free interleaver is used

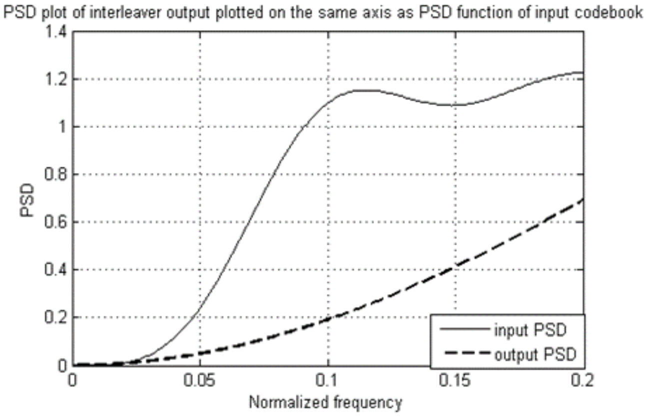

Fig. 8. Input codebook PSD versus interleaver output PSD for J=348

Figures 3 to 8 show the results that were obtained when normal interleaving as described in this paper as well as DC free interleaving were applied to a DC free codebook. From these Figures in addition to other investigations carried out for varying (J, w) combinations in the regions $J < n$, $J = n$ and $J > n$ the observations given below were made.

$J < n$

In the region $J < n$ certain selections of J and w give rise to a DC balanced interleaver output which has spectral shaping capabilities. In general odd J and odd w values give rise to an interleaver output which is not balanced and which therefore does not possess spectral shaping capabilities. J represents the cardinality of the new codebook from the interleaver output and w represents the code word length of the new code words formed at the interleaver output. It is straight forward to observe that odd w results in a code word with non-zero disparity which in the simplest scheme of generating DC free codes implies no balance. The results show that in most cases odd J results in an unbalanced interleaver output. Therefore eliminating odd J values in interleaver design specifications regardless of whether the corresponding w is even or not increases the probability of obtaining a balanced interleaver output. Similarly eliminating odd w values in interleaver design specifications regardless of whether the corresponding J is even or not increases the probability of obtaining a balanced interleaver output. For $K = 1$, the interleaver output is DC free for all values of J except for the cases in which the corresponding w for a given J value was odd.

Instead of eliminating J and w values in which either of the variables is odd, applying the DC free interleaver to unbalanced code words from interleaver output results in DC free interleaver output

The power spectral density function of the balanced interleaver output is DC free. However the notch width for the PSD function in the region $J < n$ is less than that of the input codebook. This is consistent with the results obtained from investigating the relationship between notch width and code word length for the psd functions of codebooks of a specified order K. In the region $J < n$, w is less than n of the input codebook. It follows that the corresponding PSD functions for the codebooks obtained from interleaver output in this region have notch widths less than that of the input codebook.

$$J = n$$

For $J = n$, the resulting codebook from the interleaver output is the transpose of the input codebook. The input DC free codebook is restricted in such a manner that n is always even. It follows that the cardinality C of the input codebook given by $C = \binom{n}{\frac{n}{2}}$ is always even. The plot of the psd function of the interleaver which in this case is the transpose of the input codebook is DC free. This is true for all input codebooks investigated of orders $K = 0,1,2$. From this result it can be concluded that in a DC free codebook, the codebook columns are always balanced, that is, their disparity is always zero. Therefore, a DC free codebook possesses balanced rows and columns, the rows being the code words.

$$J > n$$

In the region $J > n$ the PSD function of the interleaver output does not possess any spectral shaping capabilities and it is not DC free. This is true for all input codebooks of orders $K = 0,1,2$ which were investigated. The DC balance of the output codebook is disturbed by the interleaver when the condition $J > n$ is reached.

Application of the DC free interleaver in this region resulted in a DC free interleaver output. Interesting observations are made regarding the resultant notch width of the interleaver output's PSD function in the region $J > n$. In all cases for which w is not odd, application of Knuth's algorithm to the interleaver output results in DC free codebooks whose PSD functions have notch widths which increase as w decreases. According to the previous observations made in Section *IV-A*, as n increased, ω_0 decreased gradually. The converse is true. It follows therefore that decreasing n results in a gradual increase in ω_0 for a given order K. As a result it is expected that a decrease in w results in a gradual increase in ω_0 for interleaver output PSD function which is what the results showed. The notch width of the interleaver output codebook's PSD function exceeds that of the input codebook when $w < n$. The maximum notch width for the interleaver output's PSD function is found when $w = 2$.

V. CONCLUSION

In general, necessary and sufficient conditions for obtaining a DC free interleaver output using a simple block interleaver with variable interleaver parameters are $J \leq n$, $w \neq odd$ and $J \neq odd$. An exception occurs for $K = 1$ input DC free codebooks where balanced interleaver output is obtained for all J values in the region $J \leq n$ given that the corresponding w value is even. The notch width obtained for the PSD function of the balanced interleaver output in the region $J \leq n$ is much less than that of the input DC free codebook. $J = n$ is the cut off for obtaining a balanced interleaver output.

In the region $J > n$ all the J, w combinations give rise to a non dc free interleaver output. In both the regions $J \leq n$ and $J > n$ application of the DC free interleaver resulted in balanced interleaver output for (J, w) combinations which did not previously give rise to balanced interleaver output. Interleaver output PSD functions whose notch widths exceed that of the input are found in the region $J > n$ when $w < n$. Due to the way in which the DC free interleaver is designed, it results in a DC free output regardless of the input codebook.

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