About the development of problem solving skills via change of representation

Summary of Ph.D. thesis

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1 Introduction

1.1 Motivating the choice of topic

One of the main goals of mathematics education, is teaching how to think correctly, which can be realized by problem solving and use of exploratory teaching methods. According to earlier research [29], [17], [24], knowledge of heuristic methods plays an important role in this learning process. They also show that related skills can be developed in varying capacity. As a representative example, it is known that problem solving via change of representation, a method in which the context of the problem is significantly altered, is a very difficult skill to master. This observation led me to investigate the extent to which it is possible to teach change of representation as an effective problem solving strategy. As part of my research, I worked with a group of high school students and a group of undergraduate students, to see if these skills can be taught effectively.

1.2 The goal of my research

My **goals** were as follows:

1. overview of the theoretical background of mathematical thinking and problem solving
2. measuring the problem solving skills of students. More precisely I focused on the following:
    - the success rate of visual representation in solving problems
    - the effectiveness of the transformation principle as problem solving strategy
    - ability to connect and synthesize different parts of the curriculum
3. to introduce a curriculum that helps develop the above mentioned skills
4. to prove my **hypothesis**, formulated before my study:
    - with the appropriate curriculum one can develop the problem solving skills of the students. More precisely:
        - the ability of the students to connect and synthesize different parts of the curriculum can be increased
        - the ability of the students to use visual representation and the transformation principle successfully can be increased.
    - by explicitly teaching specific problem solving heuristics, the available problem solving skill set of students can be significantly increased.

2 The theoretical background of my research

Before carrying out my study I made an assessment of the current mathematics education literature regarding the goals of my project. After an overview of the goals of mathematics education [4], [39], [37], I examined in detail the structure of problem solving: the process of problem solving [10],[25],[36],[18], the most useful heuristic methods [26], [30], and the related metacognitive elements [19]. For sake of better understanding, I spent time with a few aspects of mathematical thinking: the inner and outer network of mathematical concepts [8], [13], the role of visual representation [9], [28], the features of critical and creative thinking [16], [23], [35], and the complex cognitive model of problem solving [33]. Lastly, I did an overview of the possibilities regarding development of problem solving skills, with focus on cognitive [9], [5], [37], metacognitive [11] and affective [32] elements.

3 The methods and content of my research

3.1 The preliminary evaluation

At the beginning of my research, the high school and undergraduate students participated in a preliminary assessment. They were given three unusual problems to solve within one and a half hours, and they were also instructed to write down their feelings and thoughts parallel to solving the problems.

3.1.1 The goal of the preliminary evaluation

My goal was to measure the following:

- what problem solving strategies are the students familiar with
- are they trying to use change of representation in their approach to solving the problems
- to what extent are the students capable to use knowledge from different parts of mathematics and to combine different ideas
- the presence of metacognitive thoughts during the problem solving process, what emotions do they feel, and how can they communicate these

As an additional goal of the assessment, I planned to compare the results of the high school and undergraduate students.

3.1.2 The problems

1. Find the real solutions of the following system: $\begin{cases} x+y+z=3 \\ x^2+y^2+z^2=3 \end{cases}$

2. Show that for all $a, b, c > 0$ we have
$$\sqrt{a^2-ab+b^2}+\sqrt{a^2-ac+c^2} \geq \sqrt{b^2+bc+c^2}.$$

3. After arriving at a deserted island, the pirates hanged the uprisers at the gallows (A), and then buried their treasure in the following manner. They measured the distance from the gallows A to the spring (F), then turned right, walked the same amount as they did from A to F, and then took up the point K_1. Similarly, they measured the distance from the gallows A to the cave (B), then they turned left, walked the same amount as they did from A to B, and took up the point K_2. At the end they buried the treasure at the midpoint of $[K_1K_2]$. Twenty years later the captain returned for the treasure, but to his surprise he could not find the gallows. Can he still find the treasure?

3.1.3 The evaluation of the solutions

My goal with the first problem was to see if the students are familiar with methods for solving systems of equations that go beyond the standard techniques of linear equations. More precisely, will they recognize the isolation of complete squares (they may have seen similar tricks in 7th or 8th grade), will they recognize the possibility of using inequalities between different means, or maybe Cauchy's inequality. Unfortunately, the squared mean is not taught any more in high-school, so most students did not think of it. Also, three dimensional analytic geometry is also not part of the high school curriculum any more, so only the undergraduate students had a chance of using this. The solutions are summarized in the following table:

	High school students	Undergraduate students
Complete solution	$3/23 \approx 13\%$	$8/24 \approx 33{,}34\%$
Correct conjecture without proof	$10/23 \approx 43{,}5\%$	$9/24 \approx 37{,}5\%$
Geometric thinking	$0/23 \approx 0\%$	$3/24 \approx 12{,}5\%$
Flawed solution	$10/23 \approx 43{,}5\%$	$7/24 \approx 29{,}16\%$

With the second problem I wanted to see if the students would try to solve an algebra problem geometrically. Will they recognize that the terms containing roots can be expressed as lengths of segments, so what they need to show is that the sum of the lengths of two segments is bigger then the length of a third one? If they get this far, will they draw a diagram and see that they need to use the triangle inequality? Their solutions are summarized in the following table:

	High school students	Undergraduate students
Complete solution	$3/23 \approx 13\%$	$4/24 \approx 16{,}67\%$
Geometric solution	$3/23 \approx 13\%$	$2/24 \approx 8{,}33\%$
Geometric thinking	$4/23 \approx 17{,}4\%$	$6/24 \approx 25\%$
Did not solve	$20/23 \approx 87\%$	$20/24 \approx 83{,}33\%$

With the third problem I wanted to see if the students are able to make a conjecture, possibly after sketching multiple different diagrams. Will they think of using geometric transformations or complex numbers to prove their conjecture?

	High school students	Undergraduate students
Complete solution	$0/23 \approx 0\%$	$0/24 \approx 0\%$
Good conjecture without proof	$7/23 \approx 30{,}43\%$	$5/24 \approx 20{,}83\%$
Usage of complex numbers	$0/23 \approx 0\%$	$0/24 \approx 0\%$
Usage of transformations	$0/23 \approx 0\%$	$0/24 \approx 0\%$
No conjecture	$16/23 \approx 69{,}57\%$	$19/24 \approx 79{,}17\%$

3.2 The goals and method of the skill development

Based on the outcome I can say the following:

- both groups were lacking skills in solving non-routine mathematical problems
- in most cases, even guessing or conjecturing the right solution was deemed too difficult
- speculation regarding change of representation appeared in only few cases
- metacognitive activity is rarely present during problem solving, and communication of such activity seems very difficult
- the results of undergraduate students were only slightly better compared to the results of the high school students

3.2.1 The goals of the skill development

Based on this assessment I determined the following main lines of skill development:

- learning the steps of problems solving consciously
- teaching the basic heuristic procedures and strategies via problem solving
- developing thinking in terms of analogies

- using experimentation to make conjectures
- teach them how to change representation, and the ability recognize hints in the wording of the problem that indicates the correct usage of this method
- to recognize what representation sheds more light on the structure of mathematical objects
- the usage of the transformation principle as basic heuristic strategy
- analysis of different visual representations for a problem
- coming up with new problems by modifying a given model
- finding different solutions for the same problem

3.2.2 The method of the skill development

In accordance with my goals, the development of skills took place over a three month period. I met with the students every week for a session of one and a half hours. Before every session, the students received a worksheet with problems up for discussion. I encouraged individual work within groups of 2-3 students. I only interfered with their work when I was asked specifically to do so. I put emphasis on trying to give multiple solutions with different methods to the same problems. When different solutions were found, we discussed the effectiveness of each method. At last, I asked the students to formulate problems that can be solved by the same method.

4 The process of the skill development and its evaluation

The thematics of each meeting revolved mostly around three main strategies:

1. Algebraic formulation - geometric solution
   solutions involving classical, analytic and trigonometric methods
2. Geometric formulation - algebraic solution
   usage of Cartesian coordinates
   usage of the Gaussian complex plane
3. Introduction of appropriate functions
   usage of properties of basic functions
   usage of function calculus

Solving the given problems required finding an appropriate context that was different from the wording of the problem. The corresponding strategy of course was different given the different nature of each problem, corresponding to the above thematics.

4.1 Geometric representation of algebraic problems

The first strategy revolved around approaching algebraic equations, inequalities, systems of equations and extremum problems using methods that were different from the usual curriculum.

My goal was to make the students use their geometric skill set to solve these problems that were worded using algebra, effectively combining different themes in the standard curriculum that are considered far apart from each other. We worked with geometric models that allowed students to find solutions via trigonometry, as well as analytic, classical and vector geometry. The students realized that using geometric models it is possible to solve a much wider class of problems than we discussed, and I encouraged them to formulate such problems on their own. In all cases we discussed the effectiveness of our strategies.

4.1.1 Classical plane geometry

When the wording of problem contains expressions similar to the length, area or volume of a geometric object, then one might become "suspicious". Of course recognizing such expressions requires adequate mathematical knowledge, in addition to being able to connect relatively distant parts of the mathematical curriculum.

Some of the more typical algebraic expressions suggesting a classical plane geometric approach are as follows:

- $\sqrt{x^2+y^2}$ - diagonal of rectangle, hypotenuse of right triangle
- $x\sqrt{2}$ - diagonal of square, hypotenuse of isosceles right triangle
- $\sqrt{x^2-y^2}$ - one of the sides of a right triangle
- $x^2+y^2 \pm xy$ - squared length of a triangle's side
- $A \pm B \cdot \cos\alpha$ or $A \pm B \cdot \sin\alpha$ - squared length of some triangle's side
- $\sqrt{x^2+y^2+z^2}$ - diagonal of rectangular parallelepiped
- $x\sqrt{3}$ - diagonal of cube
- $x \cdot y$ - area of rectangle
- $x \cdot y \cdot z$ - volume of rectangular parallelepiped

4.1.2 Coordinate geometry

Similar to the previous subsection, one needs to look for expressions resembling the length, area or volume of some geometric object in the wording of the problem.
Some of the more typical algebraic expressions suggesting a coordinate geometric approach are as follows:

- $\sqrt{x^2+y^2}$ - distance of point from the origin
- $\sqrt{(a-b)^2+(c-d)^2}$ - distance of two points
- $a \cdot x + b \cdot y = c$ - equation of a line
- $x^2+y^2=c$ - equation of circle centered at the origin
- $\sqrt{1-x^2}$ - the y component of a point on a circle circle centered at the origin of radius 1
- $x^2+y^2+z^2=c$ - equation of sphere centered at the origin
- $a \cdot x + b \cdot y + c \cdot z = d$ - equation of a plane

It may be beneficial to turn to vectors, if it is possible to construct expressions appearing in the wording of problem using operations on vectors. Commonly appearing expressions are length of vectors, dot product of vectors, or inequalities involving vectors. In our activities, we dealt with situations when the dot product of vectors needed to be expressed in two different ways, we studied triangle inequalities, and also the Cauchy-Bunyakovszkij-Schwarz and Minkowski inequalities.

4.1.3 Trigonometry

With some equations and systems of equations it may be beneficial to carry out a trigonometric substitution, and then studying the resulting equations trigonometrically. Usually, when expressions like $\sqrt{1-x^2}$ or x^2+y^2 appear, where $|x|, |y| \leq 1$, then one may try to substitute $x = \sin\alpha$, $y = \cos\alpha$. On the other hand, when dealing with $\sqrt{1+x^2}$, the substitution $x = \mathrm{tg}\alpha$ or $x = \mathrm{ctg}\alpha$ maybe the successful one.

4.1.4 Representative example

We illustrate the first strategy with the following problem. In the spirit of the activities, I will indicate questions giving away hints about the solution, and we will ultimately solve the problem following the provided hints.

Problem 4.1 *Suppose* $a, b, c, d \in \mathbb{R}$ *satisfies* $4a + 3b = 12$ *and* $3d - 4c = 12$. *Show that*

$$\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} + \sqrt{(a-c)^2 + (b-d)^2} \geq 7.68.$$

Aiding questions:

- What do the assumptions remind of? How about the expressions under the square root?
- Can you represent the assumptions using some geometric model?
- How to represent the square roots? How about their sums?
- How to represent the minimal sum?
- When do we have equality in the inequality?
- Given a fixed triangle, what is the minimal perimeter of an inscribed triangle? How to prove this?

Solution: In the assumption we have equations of two lines. Geometrically, the expression $\sqrt{a^2 + b^2}$ represents the distance of $A(a, b)$ from the origin. Similarly, $\sqrt{c^2 + d^2}$ represents the distance of $B(c, d)$ from the origin, and the third expression in the inequality we need to prove represents the length of the segment AB.

We consider a coordinate system with origin at $O(0, 0)$. By the assumptions of the problem the point $A(a, b)$ is on the line $d_1 : \ 4x + 3y = 12$, and the point $B(c, d)$ is on the line $d_2 : -4x + 3y = 12$. (see figure 1.) In this context we need to prove that

$$OA + OB + AB \geq 7.68,$$

or equivalently, we need to argue that the perimeter of the OAB triangle is greater then 7.68. Our problem is similar to that of Fagnano, in which one has to determine the inscribed triangle

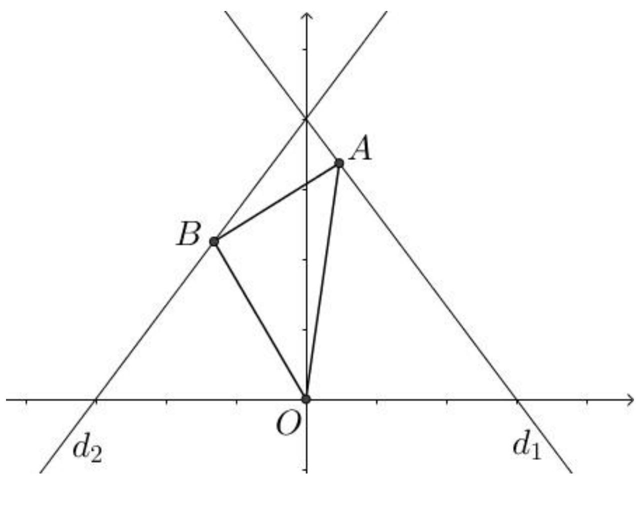

Figure 1: The triangle with minimal perimeter

in a fixed triangle with the least perimeter. For this reason, let us "unpack" the sides of the triangle OAB in such a manner that the perimeter is optimzed by the distance between two fixed points. Let O_1 be the projection of O across d_1, and O_2 be the projection of O across d_2 (see figure 2.). Because of this $OA = O_1A$ and $OB = O_2B$, hence

$$OA + OB + AB = O_1A + O_2B + AB.$$

One the other hand, $O_1A + O_2B + AB \geq O_1O_2$, hence $OA + OB + AB \geq O_1O_2$.

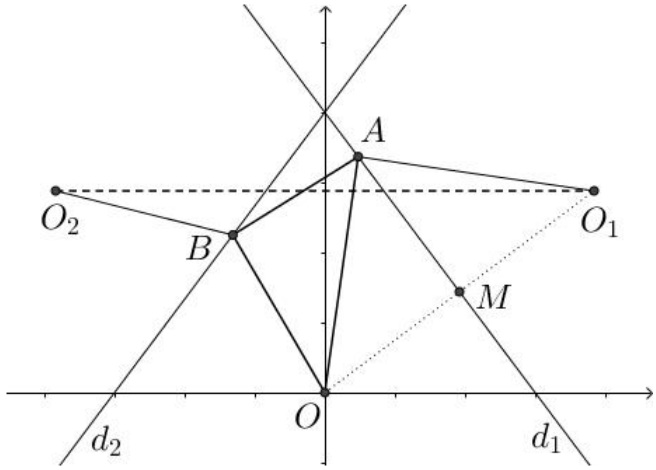

Figure 2: Fagnano's problem

If the length of $[O_1O_2]$ is 7.68, then the proof is complete. If it is bigger, then the inequality of the problem can be sharpened. If it is smaller, then the inequality cannot hold. We need to compute the length of $[O_1O_2]$.

We notice that if the coordinates of O_1 are (x_1, y_1), then the coordinates of O_2 are $(-x_1, y_1)$, because these points are symmetric with respect to the Oy axis. As a result, $|O_1O|_2 = 2x_1$. Consequently, it is enough to determine x_1.

The equation of the line OO_1 is of the form $y = mx$, hence the condition $OO_1 \perp d_1$ implies that $m = \frac{3}{4}$. if $OO_1 \cap d_1 = \{M\}$, with coordinates (x_2, y_2), then $x_1 = 2x_2$ and $y_1 = 2y_2$, because M is the midpoint of $[OO_1]$. Consequently the coordinate of M satisfies

$$\begin{cases} 3x_2 + 4y_2 = 12 \\ y_2 = \frac{3}{4}x_2 \end{cases}$$

From here $x_2 = \frac{48}{25}$, hence $O_1O_2 = 4x_2 = \frac{192}{25} = 7.68$. As a result, the inequality is true, since we have

$$\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} + \sqrt{(a - c)^2 + (b - d)^2} = OA + OB + AB \geq O_1O_2 = 7.68.$$

We have equality when A and B are at the intersection of d_1 and d_2 with O_1O_2. Then $a = 0.84$, $b = 2.88$, $c = -0.84$ and $d = 2.88$. The resulting triangle OAB is the orthic triangle of the triangle formed by the Ox axis and the lines d_1 and d_2.

Observations: 1. Associating lengths of certain segments with the square roots in the wording of the problem to was quickly recognized by the students. However the process of "unpacking" the sides of the triangle was only noticed by 1-2 students. After giving them this hint, most of them where able to carry out the related calculations one way or the other.

2. Because none of the students where familiar with Fagnano's problem, we used Fejer's method involving transformations to solve this problem.

4.2 Geometric problems - algebraic methods

The second strategy was the opposite of the previous one. Given a problem with geometric wording, try to find a solution using algebra.

My goal was to illustrate that problems that appear in a geometric context can be partially of completely solved using algebraic methods, as these allow to discuss the relation between the different objects more precisely. We highlighted two methods specifically: usage of Cartesian coordinates, or complex numbers from the Gaussian plane. For each problem we discussed the specific hints in wording that suggest one geometric approach of the other. For example, problems that involve rotation suggest a treatment using complex numbers. According to my experiences the students found it easier to use this second strategy than the first strategy discussed above.

4.2.1 The use of the Cartesian coordinate system

Coordinate geometry is perhaps the closest part of geometry to algebra. Indeed, after adequate choice of coordinate system, the relationship between geometric objects can be studied using algebraic methods. To avoid triviality, I used problems whose statement did not directly suggest use of coordinates. Because of this, one of the biggest difficulty the students experienced consisted of placing the objects in an appropriate coordinate system. After this, the algebraic description of the relationships between the different geometric objects, and subsequent proof seemed easier for the students.

4.2.2 The use of the Gaussian complex plane

Problems that can be solved using complex numbers almost always can be solved using coordinates as well. Indeed, the Gaussian complex plane, along with its complex numbers, coincides with the Cartesian plane, along with its vectors.

Despite this, especially in the case of problems involving rotations, it is better to use complex numbers, as multiplication by complex numbers corresponds to dilated rotations. This often allows for an easier proof of regularity of objects, perpendicularity or parallelism.

4.2.3 Representative example

Problem 4.2 *On one side of a* 60° *degree angle with vertex* V *we take up to the points* A *and* A_1 *so that* $|VA| = p$ *and* $|VA_1| = 2q$*. On the other side, we take up the points* B *and* B_1 *such that* $|VB| = q$ *and* $|VB_1| = 2p$*. Let* C *be the midpoint of* $[A_1B_1]$*. What type of triangle is* ABC*?*

Aiding questions:

- What is given and what does the problem want? Make a diagram!
- Could you compute the coordinates of the points using the information given?
- Could you conjecture what type of triangle ABC might be?
- How could you prove this? What methods are available? Which is more advantageous?

Solution: For sake of simplicity we take up the origin O at the vertex V, and let $[OA$ represent the Ox axis (see Figure 3.) Rotation by $+60^\circ$ degrees is represented by the unit length complex number

$$\epsilon = \frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad \epsilon^3 = -1.$$

As a result, we get that

$$a = p, \quad a_1 = 2q, \quad b = q \cdot \epsilon, \quad b_1 = 2p \cdot \epsilon.$$

As C is the midpoint of $[A_1 B_1]$, we obtain that $c = p \cdot \epsilon + q$.
1st method. The triangle ABC is regular if B is the rotation of C around A by $+60°$ degrees, i.e.,

$$b - a = (c - a) \cdot \epsilon.$$

Substituting the real and imaginary parts:

$$q \cdot \epsilon - p = (p \cdot \epsilon + q - p) \cdot \epsilon,$$

$$p(\epsilon^2 - \epsilon + 1) = 0,$$

which is true since $\epsilon^3 = -1$ implies $\epsilon^2 - \epsilon + 1 = 0$.
2nd method. We substitute the real and imaginary parts into the characterizing condition of ABC to be a regular triangle. This gives

$$p^2 + q^2 \cdot \epsilon^2 + p^2 \cdot \epsilon^2 + 2pq \cdot \epsilon + q^2 = pq \cdot \epsilon + pq \cdot \epsilon^2 + q^2 \cdot \epsilon + p^2 \cdot \epsilon + pq.$$

Reorganizng terms we get that

$$p^2(\epsilon^2 - \epsilon + 1) + q^2(\epsilon^2 - \epsilon + 1) = pq(\epsilon^2 - \epsilon + 1),$$

$$(p^2 - pq + q^2)(\epsilon^2 - \epsilon + 1) = 0,$$

which is again true since $\epsilon^3 = -1$ implies $\epsilon^2 - \epsilon + 1 = 0$.

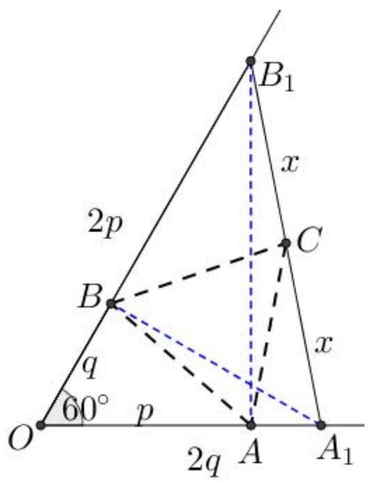

Figure 3: Triangle in an angular sector

Observations: Generally speaking, the students did well on this problem, and both of the methods appeared among the correct solutions. Those who where not comfortable with using rotations preferred the second method. Lastly, we mentioned a solution to the problem using the notion of similarity from classical plane geometry.

4.3 Finding an appropriate function

The last strategy involved solution of problems using the identification of an appropriate function, and studying its properties.

My goal with this strategy was to show the students that central element in the Romanian mathematical curriculum, the calculus of functions, is not as isolated as they might think, and it can be used as an effective strategy in solving a vast set of problems. Let the students see that combining different parts of mathematics is useful, and often indispensable in solving non-routine problems.

4.3.1 Using properties of basic functions

We mostly tried to find non-algorithmic solutions to equations, inequalities, systems of equations and geometric extremum problems. Solutions to such problems can often be guessed, but showing uniqueness often requires a more involved analysis. This is often carried out by introducing an appropriate function, studying its domain of definition and image, injectivity, surjectivity, monotonicity and convexity. In case of geometric extremum problems, guessing a basic estimate can be often difficult, however this can be made easier with the introduction of an appropriate function and proving its boundedness.

4.3.2 Using elements of mathematical analysis

In addition to the above, we studied formulas involving sums. Such sums can be often computed after differentiating or integrating an appropriate function.

4.3.3 Representative problem

Problem 4.3 *Solve the following system of equations:* $\begin{cases} x - \sqrt{y} = 1 \\ y - \sqrt{z} = 1 \\ z - \sqrt{x} = 1 \end{cases}$

Aiding questions:

- What is the structure of each equation? What do you recognize?
- Could you rewrite the system differently? Could you express one of the variables in terms of the others?
- Could you write up one equation with only one variable that is equivalent to the whole system?
- What form does this equation have? What does this imply?

Solution: Since each equation has only two variables, a sequence of substitutions allows to write the following equation for x:

$$\begin{cases} x = 1 + \sqrt{y} \\ y = 1 + \sqrt{z} \\ z = 1 + \sqrt{x} \end{cases} \implies x = 1 + \sqrt{1 + \sqrt{1 + \sqrt{x}}}.$$

Next we introduce the function

$$f : \mathbb{R}_+ \to \mathbb{R},\ f(x) = 1 + \sqrt{x}.$$

Using this function one can write the above system in the form$f(f(f(x))) = x$, i.e., we are looking for the fixed point of $f \circ f \circ f$. Since f is striclty increasing, we have that

$$f(f(f(x))) = x \quad \Longleftrightarrow \quad f(x) = x \quad \Longleftrightarrow \quad 1 + \sqrt{x} = x, \ x \geq 1.$$

From here, after rearranging terms and then squaring both sides we obtain that $x_{1,2} = \frac{3 \pm \sqrt{5}}{2}$, but only $x = \frac{3+\sqrt{5}}{2}$ solves the equation. Hence the solution is $M = \left\{ \left(\frac{3+\sqrt{5}}{2}; \frac{3+\sqrt{5}}{2}; \frac{3+\sqrt{5}}{2} \right) \right\}$.

Observations: In the solution of the problem we used a particular case of the following fact: the fixed point of a strictly increasing function f is the same as the fixed point of $f \circ f \circ \ldots \circ f$, i.e.,

$$(\underbrace{f \circ f \circ \ldots \circ f}_{n\text{times}})(x) = x \quad \Longleftrightarrow \quad f(x) = x.$$

During the proof we saw that monotonicity of f is a sufficiency condition for existence of a unique fixed point.

5 The material of the final assessment

5.1 The goals and method of the assessment

We followed up our activities with a final assessment. Similar to the preliminary assessment, students had 90 minutes at their disposal to solve three problems. Each sheet was split into two parts. On the left hand side the students had to write up their solution. On the right hand side I asked the students to document their thoughts and metacognitive activity.
The goal was to measure the following:

- the ability to formulate conjectures using experimentation, and later proving these conjectures
- the ability to use different representation in thinking, creating and analyzing visual representations
- the ability to use the transformation principle as a heuristic procedure

Besides the above, I was also curious to learn if there was any change in metacognitive activity compared to the preliminary assessment.

5.2 The problems

1. Find the minimal value of the expression $x^2 + y^2$ subject to the constraint $5x + y = 7$ with $x, y \in \mathbb{R}$.
2. Find the value of $2xy+2yz+xz$ if $x, y, z > 0$ satisfies $3x^2+4y^2+6xy = 169$, $4y^2+z^2-2yz = 25$, and $3x^2 + z^2 + 3xz = 144$.
3. Solve the equation $(2^x - 1)^2 = \log_2 (\sqrt{x} + 1)$, for $x \geq 0$.

5.3 The evaluation of the solutions

In case of the first problem I wanted to know if the students think of solving an algebraic extremum problem using geometric methods. Also, I wanted to see how many students will try both algebraic and geometric methods. The results were as follows:

	High school students	Undergraduate students
Complete solution	$10/23 \approx 43{,}48\%$	$14/24 \approx 58{,}34\%$
Geometric solution	$4/23 \approx 17{,}4\%$	$6/24 \approx 25\%$
Geometric thinking	$13/23 \approx 56{,}52\%$	$16/24 \approx 66{,}67\%$
Incorrect solution	$13/23 \approx 56{,}52\%$	$10/24 \approx 41{,}66\%$

With the second problem I wanted to measure if the students recognize the possibility of using a geometric model for an algebra problem, and if they can subsequently provide a complete solution using this method. The results were as follows:

	High school students	Undergraduate students
Complete solution	$8/23 \approx 34{,}78\%$	$10/24 \approx 41{,}67\%$
Correct geometric model	$11/23 \approx 47{,}82\%$	$13/24 \approx 54{,}16\%$
Geometric thinking	$16/23 \approx 69{,}56\%$	$18/24 \approx 75\%$
No model	$7/23 \approx 30{,}44\%$	$6/24 \approx 25\%$

With the third problem I wanted to measure the students ability to use function theoretic arguments when dealing with equations. Will they try to prove strict convexity of the expressions on both sides of the equation, with this recognizing that there could be at most two solutions, which they may be able to guess? Will the recognize that the two expressions are inverses of each other, hence the equation can be brought into a more simple form? The results were as follows:

	High school students	Undergraduate students
Complete solution	$6/23 \approx 26{,}1\%$	$10/24 \approx 41{,}67\%$
Correct conjecture	$19/23 \approx 82{,}6\%$	$18/24 \approx 75\%$
Noticing the inverses	$4/23 \approx 17{,}4\%$	$14/24 \approx 58{,}33\%$
No conjecture	$4/23 \approx 17{,}4\%$	$6/24 \approx 25\%$

5.4 Statistical comparison of the assessments

Based on the tests we can say that both groups of students increased their problem solving skill set significantly, and they had more confidence in using different, previously not seen methods in their approach.

Complete solutions	High school students	Undergraduate students
Preliminary assessment	$6/69 \approx 8{,}7\%$	$12/72 \approx 16{,}67\%$
Final assessment	$24/69 \approx 34{,}78\%$	$34/72 \approx 47{,}22\%$

In case of all three problems there were more complete solutions, moreover some students gave more than one solution to the same problem. Also, there was significant increase in the number of students that could not completely solve a problem, but were able to find an effective approach using a different representation. There was also increased metacognitive activity, and many students were able to effectively communicate their background thought process in much detail.

Different representations	High school students	Undergraduate students
Preliminary assessment	$4/69 \approx 5{,}8\%$	$9/72 \approx 12{,}5\%$
Final assessment	$33/69 \approx 47{,}83\%$	$48/72 \approx 66{,}67\%$

The effectiveness of our activities was measured by statistical comparison of the two assessments. In case of partial or complete solution, students received 1 or 2 points for each problem. If at least two solutions were given, a student received 3 points for the problem. Consequently, the total score of each student was between 0 and $3 \cdot (2+1) = 9$ points. Since sample size was small, and the F-test indicated no significant difference in the variance of scores for each of the assessments, I compared the significant difference in the average scores using a paired samples T-test. This indicated a significant increase in score for both group of students. Moreover, I also saw significant increase in number of complete solutions, and number of partial solutions were the correct visual representation was identified. Absence of complete solution in these latter cases is likely due to lack of specific content knowledge. Comparing the problem solutions of the two assessments, I noticed a clear difference in metacognitive activity as well, especially noticeable in the case of the undergraduate students.

6 Final conclusion

6.1 Proof of my hypothesis

The marked positive effect of my activities was proved by statistical methods, and the research proved my preliminary hypothesis according to which

- with the appropriate curriculum it is possible to develop the problem solving skills of students
  - it is possible to increase the skill of students to connect relatively distant parts of the curriculum
  - students can effectively learn how to use visual representation and the transformation principle
- by teaching problem solving heuristics, it is possible to significantly increase the problem solving skill set of students.

6.2 Further research

The development of problem solving strategies can only be accomplished by a long and persistent process. As a result, I think it is important to teach problem solving heuristics to students explicitly, not only during material specific workshops, but also during regular classroom hours on a daily basis. This would be one of my goals in the next step of my research. As another goal, I would like to integrate the problems of my activities into the classroom setting, by possibly extending the thematics with additional topics.

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2. *Semmi sem az, aminek látszik* - Magyar Tudomány Napja Erdélyben, Matematikadidaktika szekció, 2012. nov. 9-11., Kolozsvár

3. *Számítógépes kísérletezés, sejtés és bizonyítás a matematika tanításában* - Magyar Tudomány Napja Erdélyben, Matematikadidaktika szekció, 2011. nov. 4-6., Székelyudvarhely

4. *Kíváncsiságvezérelt matematikaoktatás* - Új utak és módok az oktatásban, BBTE Neveléstudományi konferencia, 2010. május, Kolozsvár

5. *Geometriai struktúrák számítógépes modellezése* - Minőségfejlesztés a matematikaoktatásban, BBTE tanártovábbképző, 2009, 2010, 2011. szept., Kolozsvár

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