CONCORDANCE HOMOMORPHISMS FROM KNOT FLOER HOMOLOGY

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Abstract. We modify the construction of knot Floer homology to produce a one-parameter family of homologies tHFK for knots in S^3. These invariants can be used to give homomorphisms from the smooth concordance group $\mathcal{C}$ to $\mathbb{Z}$, giving bounds on the four-ball genus and the concordance genus of knots. We give some applications of these homomorphisms.

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1. Introduction

The signature of the symmetrized Seifert matrix gives a knot invariant $\sigma(K)$ satisfying a number of basic properties [23]: it is additive under connected sums, it changes in a controlled manner under crossing changes, and it gives a lower bound on the genus of a slice surface. Levine and Tristram [40] extend this invariant to a one-parameter family of knot invariants σ_ω indexed by points ω on the unit circle. More recently, knot invariants whose properties are similar to those of σ have been constructed using techniques such as knot Floer homology, resulting in the invariant $\tau(K)$ [28, 37]; and Khovanov homology, resulting in Rasmussen's s invariant [39]. While σ and σ_ω bound the topological slice genus, the newer invariants often give better bounds for the smooth slice genus.

The goal of the present paper is to use methods of knot Floer homology to construct a one-parameter family of knot invariants $\{\Upsilon_K(t)\}_{t\in[0,2]}$, *upsilon of K at t*, which fit together to give a real-valued function $\Upsilon_K\colon [0,2]\to\mathbb{R}$. These invariants are additive under connected sums, they behave in a controlled manner under crossing changes, and they give lower bounds on the smooth slice genus. This invariant is extracted from the filtered knot Floer complex, and it is similar to, and indeed inspired by, the work of Jen Hom [11]. (For a comparison of Υ to [11], see Section 9.)

The invariants $\{\Upsilon_K(t)\}_{t\in[0,2]}$ are extracted from a suitably modified variant of knot Floer homology [30, 37]. Recall that knot Floer homology is defined as the homology of a bigraded chain complex over the base ring $\mathbb{F}[U]$, the ring of polynomials over the field $\mathbb{F}$ of two elements. (In the following we will use coefficients in $\mathbb{F}[U]$, although, with the appropriate use of signs, the constructions and results admit extensions to give invariants over $\mathbb{Z}[U]$.) This chain complex is associated to a doubly pointed Heegaard diagram representing the knot K (equipped with some orientation). Denote the two basepoints by w and z. The generators of the knot Floer complex over $\mathbb{F}[U]$ are given combinatorially from the Heegaard diagram, the differential of the complex counts pseudo-holomorphic disks that do not cross z, while the exponent of U records the multiplicity with which the pseudo-holomorphic disk crosses w. The complex is also equipped with a pair of gradings, the *Maslov grading M* and the *Alexander grading A*, which descends to homology, endowing $\mathrm{HFK}^-(K)$ with the structure of a bigraded $\mathbb{F}[U]$-module.

The above construction of $\mathrm{HFK}^-(K)$ admits the following variation. Fix a rational number $t\in[0,2]\cap\mathbb{Q}$, and let $t=\frac{m}{n}$ where m and n are relatively prime integers. The modified complex is defined over the polynomial algebra in $v^{1/n}$. The generators of the modified theory are the same as those in the traditional knot Floer complex; and there is a single grading, now by a rational number, given by $M-t\cdot A$. The exponent of v records $(2-t)$ times the multiplicity with which the disk crosses w and t times the multiplicity with which it crosses z. Multiplication by v drops grading

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by 1, as does the differential. The modified theory provides a family $\mathrm{tHFK}(K)$ ($t \in [0,2] \cap \mathbb{Q}$) of *$t$-modified knot Floer homologies*, which is a $\mathbb{Q}$-graded module over the polynomial algebra in $v^{1/n}$.

Theorem 1.1. *For all rational $t = \frac{m}{n} \in [0,2]$ the t-modified knot Floer homology $\mathrm{tHFK}(K)$, thought of as a graded $\mathbb{F}[v^{1/n}]$-module, is an invariant of the knot K.*

A homology class ξ is said to be *homogeneous* if it is represented by a cycle in a fixed grading. It is called *non-torsion* if $v^d\xi \neq 0$ for all $d \in \frac{1}{n}\mathbb{Z}$. We define the invariant $\Upsilon_K(t)$ to be the maximal degree of any homogeneous, non-torsion homology class in $\mathrm{tHFK}(K)$. It follows immediately from Theorem 1.1 that $\Upsilon_K(t)$ is also a knot invariant (see Corollary 3.7 below).

1.1. **The behaviour of $\Upsilon_K(t)$ as a function of t.** The function Υ_K satisfies the following symmetry:

Proposition 1.2. *For any knot K, $\Upsilon_K(t) = \Upsilon_K(2-t)$.*

Υ also satisfies the following integrality properties (compare also Proposition 1.7 below).

Proposition 1.3. *The quantity $\Upsilon_K(\frac{m}{n})$ lies in $\frac{1}{n}\mathbb{Z}$.*

Indeed, the above definition of tHFK and $\Upsilon_K(t)$ for rational t can be extended to any real $t \in [0,2]$, giving a knot invariant $\Upsilon_K \colon [0,2] \to \mathbb{R}$, with the following properties:

Proposition 1.4. *For any knot K, the function Υ_K (defined on $[0,2] \cap \mathbb{Q}$) has a continuous extension to a real-valued function on $[0,2]$, which is a piecewise linear function of t, and whose derivative has finitely many discontinuities. Each slope is equal to some Alexander grading s for which $\widehat{\mathrm{HFK}}_*(K,s) \neq 0$; hence, in particular, each slope is an integer.*

The following two propositions determine the behaviour of Υ_K near 0 (and so near 2, in view of Proposition 1.2).

Proposition 1.5. $\Upsilon_K(0) = 0$.

Proposition 1.6. *The slope of $\Upsilon_K(t)$ at $t = 0$ is given by $-\tau(K)$, where $\tau(K)$ denotes the concordance invariant of the knot K defined from the knot Floer homology module $\mathrm{HFK}^-(K)$.*

Sometimes it is convenient to consider discontinuities of the derivative of $\Upsilon_K(t)$. To this end, let

$$\Delta\Upsilon'_K(t_0) = \lim_{t \searrow t_0} \Upsilon'_K(t) - \lim_{t \nearrow t_0} \Upsilon'_K(t).$$

Note that by Proposition 1.6 the quantity $\Delta\Upsilon'_K$ and τ together determine Υ_K:

$$\Upsilon_K(t) = -\tau(K)\cdot t + \sum_{0<s<t} \Delta\Upsilon'_K(s)\cdot(t-s).$$

For the function $\Upsilon_K \colon [0,2] \to \mathbb{R}$ we have the following extension of Proposition 1.3:

Proposition 1.7. *For any $t \in [0,2]$, $t \cdot \Delta\Upsilon'_K(t)$ is an even integer.*

Propositions 1.2 and 1.3 are proved in Section 4; Propositions 1.4, 1.5, and 1.6 are proved in Section 5.

1.2. **Topological properties of $\Upsilon_K(t)$.** Topological properties of $\Upsilon_K(t)$ follow from corresponding properties of knot Floer homology:

Proposition 1.8. *Υ_K is additive under connected sum of knots; i.e.*

$$\Upsilon_{K_1\#K_2}(t) = \Upsilon_{K_1}(t) + \Upsilon_{K_2}(t).$$

Proposition 1.9. *Let $m(K)$ denote the mirror of the knot K. Then*

$$\Upsilon_{m(K)}(t) = -\Upsilon_K(t).$$

The invariant Υ_K changes in a controlled manner under crossing changes:

Proposition 1.10. *Let K_+ and K_- be two knots which differ in a crossing change. Then, for $0 \leq t \leq 1$ we have that*

$$\Upsilon_{K_+}(t) \leq \Upsilon_{K_-}(t) \leq \Upsilon_{K_+}(t) + t.$$

For $1 \leq t \leq 2$, symmetry and the above inequality implies that

$$\Upsilon_{K_+}(t) \leq \Upsilon_{K_-}(t) \leq \Upsilon_{K_+}(t) + (2-t).$$

The invariant $\Upsilon_K(t)$ also provides a lower bound for the (smooth) *slice genus* $g_s(K)$ of the knot K as follows:

Theorem 1.11. *The invariants $\Upsilon_K(t)$ bound the slice genus of K; more precisely, for $0 \leq t \leq 1$,*

$$|\Upsilon_K(t)| \leq t \cdot g_s(K).$$

The bounds on the slice genus are no stronger than the bounds coming from Rasmussen's "local h invariants" [37], see also [17, 38]; in fact, the slice bounds are proven by bounding Υ_K in terms of h invariants (see Proposition 4.7 below). The bounds based on $\Upsilon_K(t)$ are convenient, though, as they come from homomorphisms:

Corollary 1.12. *For each fixed t, the map $K \mapsto \Upsilon_K(t)$ gives a homomorphism from the (smooth) knot concordance group $\mathcal{C}$ to $\mathbb{R}$; indeed, $\Upsilon_K(t)$ induces a homomorphism $\Upsilon\colon \mathcal{C} \to Cont([0,2])$ from the concordance group $\mathcal{C}$ to the vector space of continuous functions on $[0,2]$.* $\square$

Proof. It follows from Theorem 1.11 and Proposition 1.8 that if K_1 and K_2 are concordant, then $\Upsilon_{K_1} = \Upsilon_{K_2}$; i.e. Υ is a well-defined function on the concordance group. Proposition 1.8 now implies that it is a homomorphism. $\square$

In a different direction, recall that the *concordance genus* of K, written $g_c(K)$, is the minimal Seifert genus of any knot K' which is concordant to K. The invariant Υ_K can be used to bound this quantity, according to the following:

Theorem 1.13. *Let s denote the maximum of the finitely many slopes appearing in the graph of $\Upsilon_K(t)$ (c.f. Proposition 1.4). Then,*

$$s \leq g_c(K).$$

It is interesting to compare this result to [12].

Propositions 1.8, 1.9, 1.10 and Theorem 1.11 are all proved in Section 4.

1.3. Calculations.

The invariant Υ_K can be explicitly computed for some classes of knots. For alternating knots we have

Theorem 1.14. *Let K be an alternating knot (or, more generally, a quasi-alternating one) with signature σ. Then,*

$$\Upsilon_K(t) = (1 - |t-1|)\frac{\sigma}{2}.$$

In particular, the derivative of $\Upsilon_K(t)$ has at most one discontinuity, which can occur at $t = 1$.

The knot Floer homology of torus knots was determined in [33]. These computations lead to the following computation of their Υ_K invariant, which can be phrased purely in terms of their Alexander polynomial. If $K = T_{p,q}$ is the (p,q) torus knot (where p and q are positive, relatively prime integers), then the nonzero coefficients in the Alexander polynomial $\Delta_K(t)$ are all ± 1, and they alternate in sign. Write the Alexander polynomial of K as

$$\begin{aligned}\Delta_K(t) &= t^{-\frac{pq-p-q+1}{2}}\frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)} = \\ &= \sum_{k=0}^{n}(-1)^k t^{\alpha_k},\end{aligned}$$

where $\{\alpha_k\}_{k=0}^n$ is a decreasing sequence of integers. Consider a corresponding sequence $\{m_k\}_{k=0}^n$ of integers, defined inductively by the formulae

$$\begin{aligned} m_0 &= 0 \\ m_{2k} &= m_{2k-1} - 1 \\ m_{2k+1} &= m_{2k} - 2(\alpha_{2k} - \alpha_{2k+1}) + 1. \end{aligned}$$

From these integers the invariant $\Upsilon_K(t)$ is computed by the following formula:

Theorem 1.15. *Let K be a positive torus knot, and let $\{m_k, \alpha_k\}_{k=0}^n$ be the above sequences extracted from its Alexander polynomial. Then,*

$$\Upsilon_K(t) = \max_{\{i \mid 0\leq 2i\leq n\}} \{m_{2i} - t\alpha_{2i}\}.$$

In fact, we will prove a more general analogue of the above theorem (Theorem 6.2), which holds for any knot on which some positive surgery gives an L-space, in the sense of [33].

Example 1.16. Let $K = T_{3,4}$ be the $(3,4)$ torus knot. Since $\Delta_K(t) = t^3 - t^2 + 1 - t^{-2} + t^{-3}$, the function $\Upsilon_K(t)$ is given by

$$\Upsilon_K(t) = \begin{cases} -3t & t\in[0,\frac{2}{3}] \\ -2 & t\in[\frac{2}{3},\frac{4}{3}] \\ -6+3t & t\in[\frac{4}{3},2]. \end{cases}$$

Theorems 1.14 and 1.15 are proved in Section 6. For an inductive formula computing $\Upsilon_{T_{p,q}}$ in terms of the functions $\Upsilon_{T_{n,n+1}}$ see [5].

1.4. **Applications of Υ to the concordance group.** Partially computing Υ_K for an infinite family of torus knots, we get

Theorem 1.17. *The function*

$$K \mapsto \left(\frac{1}{n}\Delta\Upsilon'_K(\frac{2}{n})\right)_{n=2}^{\infty}$$

from the concordance group $\mathcal{C}$ to $\mathbb{Z}^\infty = \bigoplus_{n=2}^\infty \mathbb{Z}$ is surjective.

Remark 1.18. Implicit in the above theorem is the statement that (a) for any knot K, the invariant $\Delta\Upsilon'_K(\frac{2}{n})$ is divisible by n (as a consequence of Proposition 1.7), and that (b) for a knot K the value $\Delta\Upsilon'_K(\frac{2}{n})$ is non-zero for only finitely many n (which follows from Proposition 1.4).

Theorem 1.17 then easily implies the (well-known) existence of a direct summand of $\mathcal{C}$ isomorphic to $\mathbb{Z}^\infty$ [20].

By examining discontinuities of Υ'_K, Theorem 1.14 and Example 1.16 have the following immediate corollary (which indeed can be seen by other means, as well):

Corollary 1.19. *In the smooth concordance group $\mathcal{C}$ the torus knot $T_{3,4}$ is linearly independent from all alternating knots.* $\square$

The Levine-Tristram signature function is a powerful tool for studying the concordance group; see for instance [20, 40]. However, $\Upsilon_K(t)$ can also be used to study knots for which such topological methods yield no information: using Υ we can prove results for the subgroup $\mathcal{C}_{TS} \subset \mathcal{C}$ given by *topologically slice* knots, while the Levine-Tristram signature function vanishes on this subgroup. We illustrate this phenomenon by reproving a recent result of J. Hom [11], which states that $\mathcal{C}_{TS}$ admits a direct summand isomorphic to $\mathbb{Z}^\infty$.

For a given knot K, let $W_0^+(K)$ denote its untwisted positive Whitehead double; and let $C_{p,q}(K)$ denote its (p,q) cable (for p and q relatively prime). Consider the family of knots

$$K_n = C_{n,2n-1}(W_0^+(K))\#(-T_{n,2n-1}). \tag{1}$$

Observe that K_n are topologically slice: by a theorem of Freedman [6] the knot $W_0^+(K)$ is topologically slice, hence the cable $C_{n,2n-1}(W_0^+(K))$ is topologically concordant to the same cable of the unknot, consequently K_n is topologically slice. The partial computation of Υ_{K_n}, and the same map as used in Theorem 1.17, now yields the following:

Theorem 1.20. *Consider the topologically slice knots $\{K_n\}_{n=2}^\infty$ given in* (1)*. These form a basis for a free direct summand of the subgroup $\mathcal{C}_{TS}$ of the concordance group given by topologically slice knots. In fact, the map $\mathcal{C} \to \bigoplus_{n=2}^\infty \mathbb{Z}$ defined by*

$$K \mapsto \left(\frac{1}{2n-1}\Delta\Upsilon'_K(\frac{2}{2n-1})\right)_{n=2}^\infty$$

maps the span of $\{K_n\}_{n=2}^\infty$ isomorphically onto $\mathbb{Z}^\infty = \bigoplus_{n=2}^\infty \mathbb{Z}$.

Remark 1.21. The fact that the group $\mathcal{C}_{TS}$ of topologically slice knots contains a $\mathbb{Z}^\infty$ direct summand was first proved by Jen Hom in [11]. Her examples are very similar to the ones we have given here: only the cabling parameters are different. (We chose our parameters out of convenience for our computations.) Her homomorphisms also use the knot Floer complex, but they appear to use it differently from ours; see especially Proposition 9.4 below.

Remark 1.22. Corollary 8.15 provides a refinement of Theorem 1.20, giving lower bounds on the concordance genera of linear combinations of the K_n. Compare also [3] for a generalization of the above result.

1.5. **Outline of the paper.** In Section 2, we review some notation from knot Floer homology, as well as some of its key results. In Section 3, we spell out the definition of $\Upsilon_K(t)$ in more detail, extracted from t-modified knot Floer homology. Invariance of the t-modified theory is seen as a special case of a formal construction described in Section 4. The behaviour of $\Upsilon_K(t)$ as a function of t is studied in Section 5, where we also verify Proposition 1.4. In Section 6, we give some computations, verifying the computations for alternating and torus knots. In Section 7, we recall the essentials of bordered Floer homology, which will be used in the computations from Section 8, where we prove Theorem 1.20. In Section 9, we compare the homomorphism $\Upsilon_K(t)$ with those arising from the work of Hom [11]. Finally, in Section 10, we give a generalization to the case of links.

1.6. **Further remarks and questions.** Note that t-modified knot Floer homology has a special behaviour when we specialize to $t = 1$. In that case, one can associate moves to *unoriented* saddles. This will be further pursued in [24].

The results from Section 1.1 can be thought of as giving linear relations between the values of Υ_K at various values of t: $\Upsilon_K(t) = \Upsilon_K(2-t)$ and $\Upsilon_K(0) = 0$. It is natural to wonder if there are any further linear relations between the various values of $\Upsilon_K(t)$ for $t \in [0,1]$. We conjecture that there are none.

More explicitly, for each rational number t, consider the homomorphism

$$\phi_t \colon \mathcal{C} \to \mathbb{Z}$$

defined in terms of the expression $t = \frac{m}{n}$ where m and n are relatively prime integers by

$$\phi_{\frac{m}{n}}(K) = \begin{cases} \frac{1}{2n}\Delta\Upsilon'_K(\frac{m}{n}) & \text{if } m \text{ is odd} \\ \\ \frac{1}{n}\Delta\Upsilon'_K(\frac{m}{n}) & \text{if } m \text{ is even.} \end{cases}$$

Conjecture 1.23. *The map $K \mapsto (\phi_t(K))_{\{t\in\mathbb{Q}\big|0<t<1\}}$, where $t = \frac{m}{n}$ with (m,n) relatively prime integers, induces a surjection onto $\bigoplus_{\{t\in\mathbb{Q}\big|0<t<1\}}\mathbb{Z}$.*

A more challenging variant of the above conjecture is the following:

Conjecture 1.24. *The map $K \mapsto (\phi_t(K))_{\{t\in\mathbb{Q}\big|0<t<1\}}$, where $t = \frac{m}{n}$ with (m,n) relatively prime integers, induces a surjection from the subgroup $\mathcal{C}_{TS}$ of topologically slice knots onto $\bigoplus_{\{t\in\mathbb{Q}\big|0<t<1\}}\mathbb{Z}$.*

It is natural to wonder what the image of the above map is, when further restricted to knots with $\epsilon = 0$, in the sense of [11]; in particular, for those knots which are in the kernel of Hom's homomorphisms [11]. (For a brief discussion about ϵ, see Section 9.)

The limitations of $\Upsilon_K(t)$ become apparent when we consider alternating knots: Theorem 1.14 can be interpreted as saying that the span of all alternating knots has a one-dimensional image under Υ_K. By contrast, alternating torus knots $T_{2,2n+1}$ are linearly independent in $\mathcal{C}$; more generally, a theorem of Litherland [20] states that all torus knots are linearly indepedent in the concordance group. These limitations notwithstanding, it seems likely that one can get more information by pushing the present techniques further. For instance, in the spirit of [2], one can consider branched covers to construct further invariants. The simplest of these branched covers is the double branched cover $\Sigma(K)$ of a knot $K \subset S^3$, which is a rational homology sphere. The branch locus forms a null-homologous knot in $\Sigma(K)$. It would be natural to consider an analogue of Υ in that double branched cover to try to get further concordance information.

According to a recent result of Jen Hom [16], there are knots for which $\Upsilon \equiv 0$, but her invariant ϵ is non-zero.

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2. Notions from knot Floer homology

The knot Floer complex from [30] (which will be briefly recalled below) fits into the following formal framework:

Definition 2.1. *An **Alexander filtered, Maslov graded chain complex** C is a chain complex over $\mathbb{F}[U]$ with the following additional structure:*

- *The chain complex is a finitely generated free module over $\mathbb{F}[U]$.*
- *The complex is generated over $\mathbb{F}$ by a generating set equipped with two integer-valued functions, called the **Maslov** and the **Alexander functions**.*
- *Multiplication by U drops the Maslov function by two.*
- *Multiplication by U drops the Alexander function by one.*

The Maslov and Alexander functions induce a grading and a filtration, the ***Maslov grading*** *and the* ***Alexander filtration*** *respectively. We require the following further properties:*

- *The differential drops Maslov grading by one.*
- *The Alexander function induces a filtration and the differential respects this Alexander filtration (i.e. elements with Alexander filtration $\leq t$ are mapped to elements with Alexander filtration $\leq t$ for all t).*

A chain complex with the above properties can be associated to a doubly pointed Heegaard diagram representing a knot $K \subset S^3$ [30, 37]. Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, w, z)$ be such a Heegaard diagram for K. The chain complex $\mathcal{CFK}^-(\mathcal{H})$ is generated over $\mathbb{F}[U]$ by the same generating set $\mathfrak{S} = \mathbb{T}_\alpha \cap \mathbb{T}_\beta \subset \mathrm{Sym}^g(\Sigma)$ as the Heegaard Floer chain complex CF^- of the ambient 3-manifold S^3. The generators of the complex come equipped with two integer-valued functions, the Maslov function and the Alexander function. Up to an additive constant, these functions are characterized as follows (the additive indeterminacy will be removed later). If $\mathbf{x}$ and $\mathbf{y}$ are two generators, there is a space of homotopy classes of maps which connect them, written $\pi_2(\mathbf{x}, \mathbf{y})$. These homotopy classes are equipped with two additive functions (i.e. additive under juxtaposition): the *Maslov index*, written $\mu(\phi)$, and, for a point $p \in \Sigma - \boldsymbol{\alpha} - \boldsymbol{\beta}$ in the Heegaard surface, the *multiplicity at* p, written $n_p(\phi)$, which measures the algebraic intersection number of ϕ with the divisor $\{p\} \times \mathrm{Sym}^{g-1}(\Sigma)$.

The Maslov and Alexander functions are characterized up to overall additive shifts by the equations:

$$M(\mathbf{x}) - M(\mathbf{y}) = \mu(\phi) - 2n_w(\phi), \tag{2}$$

$$A(\mathbf{x}) - A(\mathbf{y}) = n_z(\phi) - n_w(\phi), \tag{3}$$

for any $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$.

With the generating set and grading defined as above, the differential ∂^- counts pseudo-holomorphic representatives of some $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$ with Maslov index one, and it records the multiplicity at w in the exponent of a formal variable U. Explicitly, the differential on $\mathcal{CFK}^-(\mathcal{H})$ is defined by

$$\partial^- \mathbf{x} = \sum_{\{\mathbf{y} \in \mathfrak{S}\}} \sum_{\{\phi \in \pi_2(\mathbf{x},\mathbf{y}) \big| \mu(\phi)=1\}} \#\Big(\frac{\mathcal{M}(\phi)}{\mathbb{R}}\Big) U^{n_w(\phi)} \mathbf{y}. \tag{4}$$

Here $\mathcal{M}(\phi)$ is the space of holomorphic representatives of $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, and once its dimension (which is equal to $\mu(\phi)$) is one, the symbol $\#\big(\frac{\mathcal{M}(\phi)}{\mathbb{R}}\big)$ denotes the $\mod 2$ count of the elements in the factor space $\frac{\mathcal{M}(\phi)}{\mathbb{R}}$. The Alexander function gives a filtration, ultimately resulting in the Alexander filtered, Maslov graded chain complex $\mathcal{CFK}^-(\mathcal{H})$. (Note that $\mathcal{CFK}^-(\mathcal{H})$ is a filtered complex. It is the object which was denoted $CFK^{-,*}(S^3, K)$ in [30]. The homology of the associated graded object of $\mathcal{CFK}^-(\mathcal{H})$ gives the knot Floer homology $\mathrm{HFK}^-(K)$.)

The total homology of $\mathcal{CFK}^-(\mathcal{H})$ can be shown to be isomorphic to $\mathbb{F}[U]$, cf. [32]; see Proposition 2.4 below. Equation (2) determines M only up to an overall additive constant. That indeterminacy is removed by requiring that the the generator of $\mathcal{CFK}^-(\mathcal{H})$ has Maslov grading equal to 0. (Note that this convention differs from the grading convention from [32], where this generator had grading -2; hopefully, no confusion will arise.)

Likewise, the Alexander function is specified by Equation (3) up to an overall additive constant. We remove that indeterminacy as follows. First specialize the chain complex $\mathcal{CFK}^-(\mathcal{H})$ to $U = 0$, and take the homology of the associated (Alexander) graded object, to get the knot Floer homology group $\widehat{\mathrm{HFK}}(K)$. The Maslov and Alexander gradings now descend to a bigrading on $\widehat{\mathrm{HFK}}(K) = \bigoplus_{d,s \in \mathbb{Z}} \widehat{\mathrm{HFK}}_d(K, s)$, which is a finite dimensional vector space over $\mathbb{F}$. (Here, d is induced from the Maslov grading and s from the Alexander grading.) Equivalently, we consider the $\mathbb{F}$-vector space $\widehat{\mathrm{CFK}}(\mathcal{H})$ generated over $\mathbb{F}$ by $\mathfrak{S} = \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, endowed with the differential $\widehat{\partial}$ counting only those holomorphic disks which satisfy that $n_z = n_w = 0$ (dropping the formal variable U from the formula of Equation (4)). The normalization of A is chosen so that the graded Euler characteristic $\chi = \sum_{d,s}(-1)^d \dim_{\mathbb{F}} \widehat{\mathrm{HFK}}_d(K,s)\cdot t^s$ is a symmetric polynomial in t; in fact, it is the symmetrized Alexander polynomial of K.

We can tensor a Maslov graded, Alexander filtered chain complex C (as in Definition 2.1) with $\mathbb{F}[U,U^{-1}]$ to obtain a complex C^∞ with a second $\mathbb{Z}$-filtration, given by the powers of U. More precisely, we say that for a generator $\mathbf{x}$ of C over $\mathbb{F}[U]$ an element $U^i\cdot\mathbf{x}$ has *algebraic filtration level* $-i$. There is no loss of information in doing this: C can be recovered from C^∞ by taking the $\mathbb{F}[U]$-subcomplex of C^∞ consisting of elements of algebraic filtration level ≤ 0.

In the case of knot Floer homology, by applying the above procedure to $\mathcal{CFK}^-(\mathcal{H})$ we get the chain complex $\mathcal{CFK}^\infty(\mathcal{H})$.

Theorem 2.2. *([30]) The $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain homotopy type of the resulting Alexander and algebraically filtered, Maslov graded complex $\mathcal{CFK}^\infty(\mathcal{H})$ is a knot invariant. (In particular, the $\mathbb{Z}\oplus\mathbb{Z}$ filtered chain homotopy type is independent of the orientation.)* □

Using the Alexander filtration on $\mathcal{CFK}^-(\mathcal{H})$ we can consider the homology of the associated graded object. Explicitly, we endow the same $\mathbb{F}[U]$-module freely generated by the set $\mathfrak{S}$ equipped with a differential ∂_K^- that counts only those holomorphic disks which satisfy $n_z=0$. The homology of this associated graded object is called the *knot Floer homology group* $\mathrm{HFK}^-(K)$, which is a bigraded $\mathbb{F}[U]$-module.

2.1. **Results from knot Floer homology.** We will be using several theorems about knot Floer homology in the present paper. The invariance of knot Floer homology in this setting means that for two Heegaard diagrams $\mathcal{H}_1$ and $\mathcal{H}_2$ representing the same knot K the corresponding Maslov graded, Alexander filtered chain complexes $\mathcal{CFK}^-(\mathcal{H}_1)$ and $\mathcal{CFK}^-(\mathcal{H}_2)$ are Maslov graded, Alexander filtered chain homotopy equivalent. From now on, this filtered chain homotopy type will be denoted by $\mathcal{CFK}^-(K)$; similarly, we write $\mathcal{CFK}^\infty(K)$ for the filtered chain homotopy type of the corresponding complex over $\mathbb{F}[U,U^{-1}]$. We have the following Künneth principle for connected sums:

Theorem 2.3. *([30, Theorem 7.1]) Suppose that K_1 and K_2 are two knots in S^3 and $K_1\#K_2$ is their connected sum. Then, there is a Maslov graded, Alexander filtered chain homotopy equivalence*

$$\mathcal{CFK}^-(K_1\#K_2)\sim\mathcal{CFK}^-(K_1)\otimes_{\mathbb{F}[U]}\mathcal{CFK}^-(K_2).$$

□

Another key result is the following:

Proposition 2.4. *The total homology of $\mathcal{CFK}^-(K)$ (i.e. taking the homology after forgetting the filtration) is isomorphic to $\mathbb{F}[U]$, and the Maslov grading of the generator is 0.*

Proof. This complex computes the Heegaard Floer homology of S^3, cf. [32]. The Maslov grading has been normalized so that the homology has its generator in Maslov grading 0. □

The above statement has the following restatement in terms of the structure of $\mathrm{HFK}^-(K)$ (thought of as a module over $\mathbb{F}[U]$). Consider the submodule of torsion elements

$$\mathrm{Tors}=\{x\in\mathrm{HFK}^-(K)\mid p\cdot x=0 \text{ for some polynomial } p\in\mathbb{F}[U]-\{0\}\};\tag{5}$$

and consider the quotient $\mathrm{HFK}^-(K)/\mathrm{Tors}$, which is a free $\mathbb{F}[U]$-module. Then, the free quotient is isomorphic to $\mathbb{F}[U]$ or, more succinctly, the module $\mathrm{HFK}^-(K)$ has rank one over $\mathbb{F}[U]$.

It can be shown that any non-torsion element in the $\mathbb{F}[U]$-module $\mathrm{HFK}^-(K)$ has even Maslov grading.

The symmetry in Υ_K (given in Proposition 1.2) is a consequence of the following symmetry in knot Floer homology:

Proposition 2.5. *(*[30, Proposition 3.9]*) There is a symmetry of $\mathcal{CFK}^\infty(K)$ which switches the role of the algebraic and Alexander filtrations; i.e. if $\mathcal{CFK}^\infty(K)'$ denotes the $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain complex obtained from $\mathcal{CFK}^\infty$ by exchanging the two factors in the filtration, then there if a $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain homotopy equivalence betwee $\mathcal{CFK}^\infty(K)'$ and $\mathcal{CFK}^\infty(K)$.*

For a Maslov graded, Alexander filtered chain complex (C,∂) over $\mathbb{F}[U]$ we define the *dual complex* (C^*,d) as follows. As a module, $C^* = \mathrm{Mor}_{\mathbb{F}[U]}(C,\mathbb{F}[U])$; that is, C^* is the set of $\mathbb{F}[U]$-module homomorphisms from C to the ground ring $\mathbb{F}[U]$. It is naturally an $\mathbb{F}[U]$-module, by considering the action of $p\in\mathbb{F}[U]$ on $\phi\in C^*$, defined by $(p\cdot\phi)(x)=\phi(p\cdot x)$, for all $x\in C$. The differential d is uniquely characterized by the property that $(d\phi)(x)=\phi(\partial x)$ for all $x\in C$.

The grading on C^* is defined as follows. First, equip $\mathbb{F}[U]$ with the Maslov grading and Alexander filtration so that $M(U^d)=-2d$ and $A(U^d)=-d$. Now a morphism $\phi\in C^*$ is said to be *homogeneous* if there are integers m and a so that ϕ takes any element of C with degree n and filtration level b to an element of $\mathbb{F}[U]$ with grading $m+n$ and filtration level $a+b$. This induces a grading and a filtration on C^*, where the homogeneous element $\phi\in C^*$ has grading m and filtration level a.

The following is essentially a restatement of [30, Proposition 3.7]:

Proposition 2.6. *Let K be a knot, and $m(K)$ denote its mirror. Then, there is an identification $\mathcal{CFK}^-(m(K))\cong(\mathcal{CFK}^-(K))^*$ of Alexander filtered, Maslov graded chain complexes over $\mathbb{F}[U]$.*

Proof. If $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},w,z)$ represents K, then $(-\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},w,z)$ represents $m(K)$. Reflection identifies moduli spaces of pseudo-holomorphic disks from $\mathbf{x}$ to $\mathbf{y}$ in Σ with corresponding moduli spaces of pseudo-holomorphic disks from $\mathbf{y}$ to $\mathbf{x}$ in $-\Sigma$. □

Recall that the τ-invariant $\tau(K)$ of a knot K is defined as

$$\tau(K)=-\max\{A(x)\mid x\in \mathrm{HFK}^-(K)\quad\text{is homogeneous and non-torsion}\}.$$

(Note that this is not the definition of τ from [28]; but the equivalence of the two definitions was established in [35, Lemma A.2].) This invariant provides a non-trivial lower bound for the slice genus of K:

$$|\tau(K)|\leq g_s(K).$$

The identification of $\mathcal{CFK}^-(m(K))$ as the dual of $\mathcal{CFK}^-(K)$ (together with the grading conventions applied) then easily implies that

$$\tau(m(K))=-\tau(K).$$

By considering a Heegaard diagram for a knot K adapted to a Seifert surface, a strong relation between the Seifert genus and the knot Floer homology of a knot K can be proved:

Proposition 2.7. *Let K be a knot with Seifert genus $g(K)$. Then,*

$$\max\{s\,\big|\,\widehat{\mathrm{HFK}}_*(K,s)\neq 0\}\leq g(K).$$

□

In fact, the above inequality is sharp [29], but that is not of importance to the present applications.

2.2. **Computations.** Knot Floer homology groups can be easily computed for certain special classes of knots. We will use the following computation of knot Floer homology for alternating knots [27]:

Theorem 2.8. *([27]) Let K be an alternating knot. Then, $\widehat{\mathrm{HFK}}_d(K,s)\neq 0$ only if $d-s=\frac{\sigma(K)}{2}$, where $\sigma(K)$ denotes the knot signature. In particular, $\tau(K)=-\frac{\sigma(K)}{2}$.* □

Remark 2.9. The normalization of the signature in the above theorem is such that σ of the right-handed trefoil knot is -2. Since the graded Euler characteristic of knot Floer homology is the Alexander polynomial, the above theorem determines $\widehat{\mathrm{HFK}}(K)$ for an alternating knot K in terms of its signature and Alexander polynomial. As explained in [25, Corollary 10.3.2], $\mathrm{HFK}^-(K)$ of an alternating knot K is also determined by its signature and Alexander polynomial.

Finally, we will use the computation of knot Floer homology for torus knots. We state a slightly more general version, as follows. An *L-space* is a three-manifold Y that is a rational homology sphere (i.e. $b_1(Y)=0$), with the additional property that the total rank of its Floer homology $\widehat{\mathrm{HF}}(Y)$ coincides with the number of elements in $H_1(Y;\mathbb{Z})$. All lens spaces are L-spaces.

The knot K is called an *L-space knot* if some positive surgery on K gives a 3-manifold that is an L-space. (Since $(pq-1)$-surgery on the torus knot $T_{p,q}$ is the lens space $L(pq-1,p^2)$, any positive torus knot is an L-space knot.) Let K be an L-space knot. The invariants of K are heavily constrained [33]. Specifically, the non-zero coefficients of the Alexander polynomial are all ±1, and they alternate in sign, hence there is a decreasing sequence of integers $\{\alpha_k\}_{k=0}^n$ with the property that the symmetrized Alexander polynomial of K can be written

$$\Delta_K(t)=\sum_{k=0}^{n}(-1)^k t^{\alpha_k}. \tag{6}$$

Define another sequence of integers $\{m_k\}_{k=0}^n$ by

$$\begin{aligned} m_0&=0\\ m_{2k}&=m_{2k-1}-1\\ m_{2k+1}&=m_{2k}-2(\alpha_{2k}-\alpha_{2k+1})+1. \end{aligned} \tag{7}$$

The polynomial $\Delta_K(t)$ determines a Maslov graded, Alexander filtered chain complex $C^\infty(\Delta_K)$ as follows. The complex is generated over $\mathbb{F}[U,U^{-1}]$ by generators $\{x_k\}_{k=0}^n$ with grading and filtration level specified by

$$M(x_k)=m_k \qquad \text{and} \qquad A(x_k)=\alpha_k,$$

and differential (for all $0\leq 2k-1\leq n$):

$$\partial x_{2k-1}=U^{\alpha_{2k-2}-\alpha_{2k-1}}x_{2k-2}+x_{2k}, \qquad \partial x_{2k}=0. \tag{8}$$

Theorem 2.10. *Suppose that K is an L-space knot. Then, the chain complex $C^\infty(\Delta_K(t))$ represents the $\mathbb{Z}\oplus\mathbb{Z}$-filtered chain homotopy type $\mathcal{CFK}^\infty(K)$.*

Proof. This is equivalent to the main result from [33]. □

2.3. **Slice genus bounds.** The slice genus bounds for $\Upsilon_K(t)$ will come from certain slice genus bounds from knot Floer homology. First, we make a formal definition:

Definition 2.11. *For a chain complex C over $\mathbb{F}[U]$ equipped with a (Maslov) grading, let $\delta(C)$ denote the maximal grading of any homogenous non-torsion class in the homology $H_*(C)$ of C.*

Starting from the knot Floer complex $\mathcal{CFK}^-(K)$, we can consider a new subcomplex $\mathcal{A}(K,s)$, generated by all elements $c \in \mathcal{CFK}^-(K)$ with $A(c) \leq s$.

It is perhaps easiest to think of $\mathcal{A}(K,s)$ as generated over $\mathbb{F}$ by elements $U^i\mathbf{x}$, where $\mathbf{x}$ is a preferred generator of $\mathcal{CFK}^-(K)$ over $\mathbb{F}[U]$ with $i \geq \max(A(\mathbf{x}) - s, 0)$. The complexes $\mathcal{A}(K,s)$ govern the behaviour of the Heegaard Floer homologies $\mathrm{HF}^-(S^3_n(K))$ of the 3-manifolds $S^3_n(K)$ obtained by sufficiently large surgeries on K. Functorial properties of the cobordism map then allow one to extract slice genus bounds from these subcomplexes; see especially [37, Corollary 7.4]. Here we use a formulation akin to that of Hom and Wu [17].

Definition 2.12. *Let $\nu^-(K)$ be the minimal s so that $\delta(\mathcal{A}(K,s)) = 0$.*

Strictly speaking, Hom and Wu formulate their invariant $\nu^+(K)$ in terms of HF^+, rather than HF^-. The two definitions give the same result:

Proposition 2.13. *The invariant $\nu^-(K)$ agrees with the invariant $\nu^+(K)$ defined by Hom and Wu in* [17].

Proof. For a chain complex C over $\mathbb{F}[U]$, let C^+ denote the cokernel of the localization map $C \to C \otimes_{\mathbb{F}} \mathbb{F}[U, U^{-1}]$. Write $\mathrm{CF}^-(S^3)$, $\mathrm{CF}^\infty(S^3)$ and $\mathrm{CF}^+(S^3)$ for $\mathcal{CFK}^-(K)$, $\mathcal{CFK}^-(K) \otimes \mathbb{F}[U, U^{-1}]$, and $(\mathcal{CFK}^-(K))^+$ respectively. Let $\mathcal{A}^+(K,s)$ denote the cokernel of the natural inclusion $\mathcal{A}(K,s)$ to $\mathrm{CF}^\infty(S^3)$. The definition also induces a map $v_s^+ : \mathcal{A}^+(K,s) \to \mathrm{CF}^+(S^3)$. In fact, there is a map of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{A}(K,s) \arrow[r] \arrow[d, "v_s^-"'] & \mathrm{CF}^\infty(S^3) \arrow[r] \arrow[d] & \mathcal{A}^+(K,s) \arrow[r] \arrow[d, "v_s^+"] & 0 \\
0 \arrow[r] & \mathrm{CF}^-(S^3) \arrow[r] & \mathrm{CF}^\infty(S^3) \arrow[r] & \mathrm{CF}^+(S^3) \arrow[r] & 0
\end{tikzcd}$$

In [17], the invariant $\nu^+(K)$ is defined to be the minimal s for which v_s^+ takes the image of $\mathrm{CF}^\infty(S^3)$ in $H(\mathcal{A}^+(K,s))$ isomorphically onto $H(\mathrm{CF}^+(S^3))$. Now this condition on s is equivalent to the condition that v_s^- is surjective, which in turn is equivalent to the condition that v_s^- contains the generator $1 \in \mathbb{F}[U] \cong H(\mathrm{CF}^-(S^3))$. But v_s^- is a Maslov graded map; so this latter condition in turn is equivalent to the condition that $\delta(\mathcal{A}(K,s)) = 0$. This establishes the desired equality. □

Theorem 2.14. *Let $K \subset S^3$ be a knot. Then, $\nu^-(K) \leq g_s(K)$.*

Proof. This is [17, Proposition 2.4]; see also [37, Corollary 7.4]. □

3. Definitions of t-modified knot Floer homology and $\Upsilon_K(t)$

The aim of this section is to describe the definition of t-modified knot Floer homology of a knot $K \subset S^3$ and its corresponding numerical invariant $\Upsilon_K(t)$. (See [21] for an alternative description of these constructions.)

We describe the definition first for rational t, and then extend the definition for the general case. For rational t the base ring can be chosen to be a polynomial ring, while for the general case we need to work with a slightly larger ring $\mathcal{R}$, which will be described below.

Fix a rational number $0 \leq t = \frac{m}{n} \leq 2$, and consider the chain complex over $\mathbb{F}[v^{1/n}]$, generated by the same generators that were used to generate CFK^- over $\mathbb{F}[U]$. Equip this module with the grading

$$\mathrm{gr}_t(\mathbf{x}) = M(\mathbf{x}) - tA(\mathbf{x})$$

on the generators and take $\mathrm{gr}_t(v^\alpha \mathbf{x}) = \mathrm{gr}_t(\mathbf{x}) - \alpha$ for $\alpha \in \frac{1}{n}\mathbb{Z}$, that is, multiplication by v drops the grading by one. Define the differential

$$\partial_t \mathbf{x} = \sum_{\mathbf{y}\in\mathfrak{S}} \sum_{\{\phi\in\pi_2(\mathbf{x},\mathbf{y})\big|\mu(\phi)=1\}} \#\left(\frac{\mathcal{M}(\phi)}{\mathbb{R}}\right) v^{tn_z(\phi)+(2-t)n_w(\phi)}\mathbf{y}. \tag{9}$$

This construction makes sense even when $t \in [0,2]$ is real, once we choose a little more complicated base ring. The ring described below was chosen so that the definition of ∂_t makes sense, while the ring retains a convenient property of $\mathbb{F}[U]$: finitely generated modules decompose as direct sums of cyclic modules.

Definition 3.1. *Let $\mathbb{R}_{\geq 0}$ denote the set of nonnegative real numbers. The* ring of long power series *$\mathcal{R}$ defined as follows. As an abelian group, $\mathcal{R}$ is the group of formal sums*

$$\{\sum_{\alpha\in A} v^\alpha \mid A \subset \mathbb{R}_{\geq 0},\ A \text{ well-ordered}\},$$

where we use the order on A induced from $\mathbb{R}$. Note that if A and B are well ordered subsets of $\mathbb{R}$, then so is their sum

$$A + B = \{\gamma \in \mathbb{R}\big|\gamma = \alpha+\beta \text{ for some } \alpha\in A \text{ and } \beta \in B\}.$$

The product in $\mathcal{R}$ is given by the formula

$$(\sum_{\alpha\in A} v^\alpha)\cdot(\sum_{\beta\in B} v^\beta) = \sum_{\gamma\in A+B} \#\{(\alpha,\beta)\in A\times B\big|\alpha+\beta=\gamma\}\cdot v^\gamma,$$

where the count appearing as the coefficient of v^γ is of course to be interpreted as a number modulo 2.

It is straightforward to verify that the above defined product is well-defined. The field of fractions $\mathcal{R}^*$ of the ring $\mathcal{R}$ above can be identified with

$$\{\sum_{\alpha\in A} v^\alpha \mid A \subset \mathbb{R}, A \text{ well-ordered}\}.$$

Define the *rank* of a module M over $\mathcal{R}$ as the dimension of the $\mathcal{R}^*$-vector space $M\otimes_{\mathcal{R}}\mathcal{R}^*$.

In the interest of uniformity, we will henceforth always consider the t-modified knot complex over $\mathcal{R}$, bearing in mind that $\mathbb{F}[v^{1/n}]$ (used in the definition for rational t) is a subring of $\mathcal{R}$, so we can naturally extend the base ring in the rational case. This does not affect what we mean by Υ; see Proposition 4.9.

Remark 3.2. The ring $\mathcal{R}$ is the unique valuation ring with valuation group $\mathbb{R}$ and quotient field $\mathbb{Z}/2\mathbb{Z}$. For more information on this ring, see [1, Section 11] and [7].

Lemma 3.3. *The endomorphism defined in Equation* (9) *is a differential. The differential drops the grading* gr_t *by one.*

Proof. The fact that the endomorphism is a differential can be seen directly from the fact that n_z and n_w are additive under juxtaposition of flows and that ∂ in CFK^- is a differential.

The grading properties follow quickly from Equations (2) and (3). Specifically, if $v^\alpha\mathbf{y}$ appears with non-zero multiplicity in $\partial_t\mathbf{x}$, then there is a homotopy class $\phi\in\pi_2(\mathbf{x},\mathbf{y})$ with

$$\mu(\phi) = 1,$$
$$tn_z(\phi) + (2-t)n_w(\phi) = \alpha.$$

In this case,

$$\begin{aligned}\mathrm{gr}_t(\mathbf{x}) - \mathrm{gr}_t(v^{tn_z(\phi)+(2-t)n_w(\phi)}\mathbf{y}) &= M(\mathbf{x}) - M(\mathbf{y}) - tA(\mathbf{x}) + tA(\mathbf{y}) + t(n_z(\phi) - n_w(\phi)) + 2n_w(\phi) \\ &= \mu(\phi) = 1.\end{aligned}$$

□

Definition 3.4. *We call the resulting* gr_t*-graded chain complex the* t***-modified knot Floer complex****, and denote it by* $\mathrm{tCFK}(K)$*. Its homology, denoted by* $\mathrm{tHFK}(K)$*, is called the* t***-modified knot Floer homology****; it is a finitely generated* gr_t*-graded module over* $\mathcal{R}$*.*

The construction of $\mathrm{tCFK}(K)$ can be thought of as coming from a formal construction associated to Alexander filtered, Maslov graded complexes, as it will be explained in Section 4.

Theorem 3.5. $\mathrm{tHFK}(K)$*, thought of as an isomorphism class of* gr_t*-graded module over* $\mathcal{R}$*, is a knot invariant.*

One could repeat the invariance proof for knot Floer homology (relying on handle slide and stabilization invariances) to prove Theorem 3.5. We prefer instead to appeal directly to the invariance of $\mathrm{CFK}^\infty(K)$, combined with functoriality properties of the formal construction. This proof will be given in Section 4.1.

Next we give the definition of $\Upsilon_K(t)$:

Definition 3.6. $\Upsilon_K(t) \in \mathbb{R}$ *is the maximal* gr_t*-grading of any homogeneous non-torsion element of* $\mathrm{tHFK}(K)$*.*

Theorem 3.5 has the following immediate consequence:

Corollary 3.7. $\Upsilon_K(t)$ *is a knot invariant.*

4. t-modified knot Floer homology as a formal construction

In this section we describe a way to associate new chain complexes to a given Maslov graded, Alexander filtered chain complex C over $\mathbb{F}[U]$, in the sense of Definition 2.1. The t-modified knot complexes can be thought of as associated to $\mathrm{CFK}^-(K)$ in this manner. Since the association is functorial under filtered chain homotopy equivalences (of C), the invariance of the t-modified homology groups are quickly seen to follow from the invariance of $\mathrm{CFK}^-(K)$.

Suppose that C is a finitely generated, Maslov graded, Alexander filtered chain complex over $\mathbb{F}[U]$. Let $\mathbf{x}$ be a generator of C over $\mathbb{F}[U]$, with Maslov grading $M(\mathbf{x})$. Since multiplication by U decreases the Maslov grading by 2, elements of Maslov grading $M(\mathbf{x}) - 1$ are linear combinations of elements of the form $U^{\frac{M(\mathbf{y})-M(\mathbf{x})+1}{2}}\mathbf{y}$, where $\mathbf{y}$ is a generator. In particular, $M(U^{\frac{M(\mathbf{y})-M(\mathbf{x})+1}{2}}\mathbf{y}) = M(\mathbf{x})-1$ implies that $M(\mathbf{y}) \geq M(\mathbf{x})-1$, and $M(\mathbf{x})$ and $M(\mathbf{y})$ have opposite parity. The differential on C can be written as

$$\partial \mathbf{x} = \sum_{\mathbf{y}} c_{\mathbf{x},\mathbf{y}} \cdot U^{\frac{M(\mathbf{y})-M(\mathbf{x})+1}{2}}\mathbf{y}, \tag{10}$$

where $c_{\mathbf{x},\mathbf{y}} \in \mathbb{F}$.

Definition 4.1. *Suppose that* C *is a finitely generated, Maslov graded, Alexander filtered chain complex over* $\mathbb{F}[U]$*, and let* $\mathcal{R}$ *be the ring of Definition 3.1 (containing* $\mathbb{F}[U]$ *by* $U = v^2$*). For* $t \in [0,2]$ *the* t*-modified complex* C^t *of* C *is defined as follows:*

- *As an* $\mathcal{R}$*-module,* C^t *is equal to* $C_{\mathcal{R}} = C \otimes_{\mathbb{F}[U]} \mathcal{R}$*.*
- *For each generator* $\mathbf{x}$ *of* C *over* $\mathbb{F}[U]$*, define* $\mathrm{gr}_t(\mathbf{x}) = M(\mathbf{x}) - tA(\mathbf{x})$*, and extend this to* C^t *by the convention that* $\mathrm{gr}_t(v^\alpha \mathbf{x}) = \mathrm{gr}_t(\mathbf{x}) - \alpha$*. Thus,* gr_t *induces a real-valued grading on* C^t *with the property that multiplication by* v *drops grading by 1.*

- *Endow the graded module C^t with a differential*
$$\partial_t \mathbf{x} = \sum_{\mathbf{y}} c_{\mathbf{x},\mathbf{y}} \cdot v^{\mathrm{gr}_t(\mathbf{y}) - \mathrm{gr}_t(\mathbf{x}) + 1} \mathbf{y},$$
where the coefficients $c_{\mathbf{x},\mathbf{y}} \in \mathbb{F}$ are taken from the differential of C through Equation (10)*.*

The exponent of v is chosen so that the differential drops gr_t by exactly one. The relevance of the construction is the following:

Proposition 4.2. *Starting from the Maslov graded, Alexander filtered chain complex* $(\mathrm{CFK}^-(K), \partial^-)$ *of a knot K over $\mathbb{F}[U]$, the associated t-modified complex* $(\mathrm{CFK}^-(K))^t$ *(following Definition 4.1) agrees with the t-modified knot Floer complex* $\mathrm{tCFK}(K)$ *(in the sense of Definition 3.4).*

Proof. After identifying the generators and their gradings, we only need to check that if $c_{\mathbf{x},\mathbf{y}} \neq 0$, then
$$tn_z(\phi) + (2-t)n_w(\phi) = \mathrm{gr}_t(\mathbf{y}) - \mathrm{gr}_t(\mathbf{x}) + 1;$$
but this was verified in the proof of Lemma 3.3. □

We give the t-modified complex C^t the following second, more transparently functorial description. As before, let C be a finitely generated, Maslov graded, Alexander filtered chain complex over $\mathbb{F}[U]$, and think of $\mathbb{F}[U]$ as a subring of $\mathcal{R}$ (with variable v) where $U = v^2$. Consider the tensor product of C now with the field $\mathcal{R}^*$ of fractions:
$$C_{\mathcal{R}^*} = C \otimes_{\mathbb{F}[U]} \mathcal{R}^*.$$
The Maslov grading and Alexander filtration on C induce real-valued Maslov gradings and Alexander filtrations on $C_{\mathcal{R}^*}$ by the convention that
$$A(v^\alpha \mathbf{x}) = A(\mathbf{x}) - \frac{\alpha}{2} \text{ and } M(v^\alpha \mathbf{x}) = M(\mathbf{x}) - \alpha,$$
where $\mathbf{x}$ is a homogeneous generator of C as a $\mathbb{F}[U]$-module Just as in the discussion preceding Subsection 2.1, $C_{\mathcal{R}^*}$ admits and algebraic filtration (given by $-\frac{\alpha}{2}$ for $v^\alpha \cdot \mathbf{x}$), and $C_{\mathcal{R}} = C \otimes_{\mathbb{F}[U]} \mathcal{R}$ can be recovered from $C_{\mathcal{R}^*}$ by taking the elements with algebraic filtration level ≤ 0.

Rewrite the boundary operator from Equation (10) as
$$\partial \mathbf{x} = \sum_{\mathbf{y}} c_{\mathbf{x},\mathbf{y}} \cdot v^{M(\mathbf{y}) - M(\mathbf{x}) + 1} \mathbf{y}.$$

For each $t \in [0, 2]$, there is a new filtration F^t on $C_{\mathcal{R}^*}$ defined by $\frac{t}{2}$ times the Alexander filtration plus $(1 - \frac{t}{2})$ times the algebraic filtration. Clearly, this filtration depends on t. Observe that multiplication by v drops the algebraic filtration by $\frac{1}{2}$ and the Alexander filtration by $\frac{1}{2}$, and hence it drops the F^t filtration level by $\frac{1}{2}$. Consider the subcomplex E^t of $C_{\mathcal{R}^*}$ (as an $\mathcal{R}$-module) with filtration level $F^t \leq 0$. This subcomplex retains a Maslov grading (and multiplication by v drops the Maslov grading by one).

Lemma 4.3. *The chain complex E^t with its induced Maslov grading is isomorphic to the chain complex C^t from Definition 4.1.*

Proof. Consider the $\mathcal{R}$-module map $\phi\colon C^t \to C_{\mathcal{R}^*}$ defined by $\mathbf{x} \mapsto v^{tA(\mathbf{x})}\mathbf{x}$. It is straightforward to check the image of ϕ lies in E^t, and indeed $\phi\colon C^t \to E^t$ induces an $\mathcal{R}$-module isomorphism. This isomorphism respects grading, since
$$M(\phi(\mathbf{x})) = M(v^{tA(\mathbf{x})}\mathbf{x}) = M(\mathbf{x}) - tA(\mathbf{x}) = \mathrm{gr}_t(\mathbf{x}).$$

Write ∂' for the differential on E^t and ∂ for the differential on C^t. We verify that ϕ is a chain map:

$$\begin{aligned}\partial'\phi(\mathbf{x}) &= \partial'(v^{tA(\mathbf{x})}\mathbf{x}) = v^{tA(\mathbf{x})}\partial'\mathbf{x} = v^{tA(\mathbf{x})}\sum_{\mathbf{y}} c_{\mathbf{x},\mathbf{y}} v^{M(\mathbf{y})-M(\mathbf{x})+1}\cdot \mathbf{y} \\ &= \sum_{\mathbf{y}} c_{\mathbf{x},\mathbf{y}} v^{M(\mathbf{y})-M(\mathbf{x})+1+tA(\mathbf{x})-tA(\mathbf{y})}\cdot v^{tA(\mathbf{y})}\mathbf{y} = \sum_{\mathbf{y}} c_{\mathbf{x},\mathbf{y}} v^{\mathrm{gr}_t(\mathbf{y})-\mathrm{gr}_t(\mathbf{x})+1}\phi(\mathbf{y}) = \phi(\partial\mathbf{x}),\end{aligned}$$

since $tA(\mathbf{x}) - tA(\mathbf{y}) + M(\mathbf{y}) - M(\mathbf{x}) + 1 = \mathrm{gr}_t(\mathbf{y}) - \mathrm{gr}_t(\mathbf{x}) + 1$. □

We state functoriality in terms of maps between Alexander filtered, Maslov graded chain complexes. A morphism $\phi\colon C \to C'$ of degree $m \in \mathbb{Z}$ between two Alexander filtered, Maslov graded chain complexes (in the sense of Definition 2.1) is an $\mathbb{F}[U]$-module map from C to C' that respects filtration levels (i.e. if $\xi \in C$ has filtration level $\leq t$, then $\phi(\xi) \in C'$ has filtration level $\leq t$, as well) and that sends elements in C_d to elements in C'_{d+m}. A *homomorphism* $f\colon C \to C'$ between two Alexander filtered, Maslov graded chain complexes is a morphism of degree 0 that also satisfies $\partial' \circ f + f \circ \partial = 0$. For instance, the identity map from C to itself is a homomorphism. Two homomorphisms $f, g\colon C \to C'$ are said to be *homotopic* if there is a morphism $h\colon C \to C'$ of degree 1 with $f + g = \partial' \circ h + h \circ \partial$. As usual, C and C' are called *filtered chain homotopy equivalent* if there are homomorphisms $f\colon C \to C'$ and $g\colon C' \to C$ so that $f \circ g$ and $g \circ f$ are homotopic to the respective identity maps.

With the above definitions in place, functoriality follows immediately from the second version of the t-modified construction (given in Lemma 4.3):

Proposition 4.4. *Let $f\colon C \to C'$ be a Maslov graded, Alexander filtered chain map between chain complexes over $\mathbb{F}[U]$. There is a corresponding graded chain map $f^t\colon C^t \to (C')^t$, with the following properties:*

- *If $f\colon C \to C'$ and $g\colon C' \to C''$ are two Maslov graded, Alexander filtered chain maps, then*
$$(g \circ f)^t = g^t \circ f^t.$$
- *If $f, g\colon C \to C'$ Maslov graded, Alexander filtered chain maps are chain homotopic to each other, then f^t and g^t are chain homotopic to one another. In particular, filtered chain homotopy equivalent complexes are transformed by the construction $C \mapsto C^t$ into homotopy equivalent complexes.*
- *For C and C' Maslov graded, Alexander filtered chain complexes over $\mathbb{F}[U]$ we have*
$$(C \otimes_{\mathbb{F}[U]} C')^t \cong (C^t) \otimes_{\mathcal{R}} (C')^t.$$

□

The dual complex of a chain complex C over $\mathcal{R}$ can be defined by a simple adaptation of the definitions we had earlier for chain complexes over $\mathbb{F}[U]$. In particular, if C is a finitely generated chain complex over $\mathcal{R}$, we can consider its dual complex $C^* = \mathrm{Mor}_{\mathcal{R}}(C, \mathcal{R})$, as the module of maps

$$\phi\colon C \to \mathcal{R}$$

which commute with the $\mathcal{R}$-action, i.e. for $x \in C$ and $r \in \mathcal{R}$ we have

$$\phi(r \cdot x) = r \cdot \phi(x).$$

There is a natural Kronecker pairing

$$C \otimes_{\mathcal{R}} \mathrm{Mor}_{\mathcal{R}}(C, \mathcal{R}) \to \mathcal{R},$$

denoted $\langle \cdot, \cdot \rangle$ and defined as $\langle x, \phi \rangle = \phi(x)$. The dual complex $\mathrm{Mor}_{\mathcal{R}}(C, \mathcal{R})$ is equipped with the differential $d\colon \mathrm{Mor}_{\mathcal{R}}(C, \mathcal{R}) \to \mathrm{Mor}_{\mathcal{R}}(C, \mathcal{R})$ determined by

$$\langle x, d\phi \rangle = \langle \partial x, \phi \rangle.$$

Equipping the ring $\mathcal{R}$ with the grading $\mathrm{gr}(v^\alpha) = -\alpha$, we define the degree of a morphism in C^* to be m if it takes elements in C of degree n to algebra elements of degree $m+n$.

As the results of the above construction, for a graded chain complex C over $\mathcal{R}$ we get the dual chain complex C^*, which is also graded. (Note that in this way we get the usual cochain complex, only equipped with (-1)-times its usual grading.) With this notion in place, we get

Proposition 4.5. *For a Maslov graded, Alexander filtered chain complex C over $\mathbb{F}[U]$ and for its dual complex $C^* = \mathrm{Hom}(C, \mathbb{F}[U])$ we have that*

$$(C^*)^t \cong (C^t)^*.$$

Proof. The proof follows quickly from the definitions. □

4.1. **Consequences for $\Upsilon_K(t)$.** Some basic properties of $\Upsilon_K(t)$ enumerated in Section 1 are consequences of corresponding properties of knot Floer homology, and the formal properties of t-modification. Before turning to the proofs, however, we complete the discussion of Section 3 by verifying invariance of t-modified knot Floer homology.

Proof of Theorem 3.5. As shown in [30], the Maslov graded, Alexander filtered chain complexes over $\mathbb{F}[U]$ associated to two Heegaard diagrams representing the same knot K are filtered homotopy equivalent. (Independence of the Heegaard diagram and the knot orientation are verified in [30, Theorem 3.1 and Proposition 3.9] respectively.) According to Proposition 4.4, the filtered homotopy equivalence induces homotopy equivalence of t-modified complexes, concluding the proof. □

Notice that this result then proves Theorem 1.1.

Proof of Corollary 3.7. According to Theorem 3.5, the graded $\mathcal{R}$-module $\mathrm{tHFK}(K)$ is a knot invariant. Since $\Upsilon_K(t)$ is extracted from the graded $\mathcal{R}$-module structure of $\mathrm{tHFK}(K)$, the claim of the corollary follows. □

Having established the invariance of $\mathrm{tHFK}(K)$ and Υ_K, we turn to the basic properties of Υ_K stated in Section 1.

Proof of Proposition 1.2. Suppose that C is a chain complex for knot Floer homology derived from a Heegaard diagram representing the knot K with two basepoints w and z. Let C' be the chain complex with the roles of the two basepoints switched. As stated in Proposition 2.5, there is a filtered quasi-isomorphism between C and C'. The image of a generator $\mathbf{x}$ of C is mapped to a generator $\mathbf{x}'$ of C' with

$$\begin{aligned} M'(\mathbf{x}') &= M(\mathbf{x}) - 2A(\mathbf{x}) \\ A'(\mathbf{x}') &= -A(\mathbf{x}). \end{aligned}$$

Thus, $\mathrm{gr}_t(\mathbf{x}) = \mathrm{gr}'_{2-t}(\mathbf{x}')$, and since the algebraic structure of C and C' is the same, the equality

$$\Upsilon_K(t) = \Upsilon_K(2-t) \tag{11}$$

follows. □

Proof of Proposition 1.5. Observe that when $t=0$, then $\mathrm{tCFK}(K)$ agrees with the usual differential on $\mathrm{CF}^-(S^3)$ (the Heegaard Floer chain complex of S^3), with its standard Maslov grading. In turn, $\mathrm{CF}^-(S^3)$ is graded so that its generator has grading 0, so $\Upsilon_K(0) = 0$. □

Proof of Proposition 1.8. This follows from the Künneth formula for connected sums (c.f. [30, Theorem 7.1], restated here as Theorem 2.3), together with Propositions 4.2 and 4.4. □

Proof of Proposition 1.9. Combining Propositions 4.2, 4.5, and 2.6, we have

$$\begin{aligned}\mathrm{tCFK}(m(K)) &= (\mathrm{CFK}^-(m(K)))^t\\ &= (\mathrm{CFK}^-(K)^*)t\\ &= (\mathrm{CFK}^-(K)^t)^*\\ &= (\mathrm{tCFK}(K))^*.\end{aligned}$$

It follows from the universal coefficient theorem, together with our grading conventions on dual complexes, that

$$\Upsilon_{m(K)}(t) = -\Upsilon_K(t),$$

concluding the proof. □

4.2. **Slice genus bounds.** The slice genus bound of Theorem 1.11 (and of Proposition 1.10) will be seen as consequences of the slice genus bounds coming from Theorem 2.14, and a simple algebraic principle.

Recall that if C is a finitely generated, graded chain complex over $\mathbb{F}[U]$, then $\delta(C)$ is by definition the maximal grading of any non-torsion element in the homology $H_*(C)$.

Lemma 4.6. *Let $C \to C'$ be a grading-preserving map of finitely generated, graded chain complexes over $\mathbb{F}[U]$ with the property that the induced map $H(C)\otimes_{\mathbb{F}[U]}\mathbb{F}[U,U^{-1}] \to H(C')\otimes_{\mathbb{F}[U]}\mathbb{F}[U,U^{-1}]$ is an isomorphism. Then, $\delta(C)\leq\delta(C')$.*

Proof. If $c\in C$ represents a non-torsion homology class, then so does its image in $H(C')$ (by the hypothesis). Thus $\delta(C)$, which coincides with the grading of some $c\in C$, is less than or equal to $\delta(C')$, as needed. □

Obviously, a similar inequality holds for complexes over the ring $\mathcal{R}$ (after we replace $\mathbb{F}[U,U^{-1}]$ with $\mathcal{R}^*$ in the hypothesis).

Proposition 4.7. *For $0\leq t\leq 1$, there is an inequality*

$$-t\nu^-(K)\leq \Upsilon_K(t).$$

Proof. This claim follows from the second construction of the t-modified complex, from Lemma 4.3. Adapting the corresponding notion for $\mathbb{F}[U]$-modules, let $\mathcal{A}_{\mathcal{R}}(K,s)$ denote the subcomplex of $C_{\mathcal{R}}$ generated by the elements of $C_{\mathcal{R}}$ satisfying $A\leq s$. There is an inclusion (of subcomplexes over $\mathcal{R}$)

$$v^{ts}\cdot\mathcal{A}_{\mathcal{R}}(K,s)\subset \mathrm{tCFK}(K)$$

which induces isomorphisms after we tensor with $\mathcal{R}^*$. Then

$$\delta(\mathcal{A}_{\mathcal{R}}(K,s)) - ts \leq \delta(\mathrm{tCFK}(K));$$

so if $\delta(\mathcal{A}_{\mathcal{R}}(K,s))=0$, then

$$-ts\leq\delta(\mathrm{tCFK}(K)).$$

Minimizing over all s with $\delta(\mathcal{A}(K,s))=0$, we obtain the claimed inequality. □

Proof of Theorem 1.11. By taking also the mirror $m(K)$ of K, and using the fact that $\Upsilon_{m(K)}(t) = -\Upsilon_K(t)$ from Proposition 1.9, we conclude that

$$|\Upsilon_K(t)|\leq t\max(\nu^-(K),\nu^-(m(K)).$$

The theorem now follows from Theorem 2.14. □

Proof of Proposition 1.10. Since $K_-\#(m(K_+))$ has slice genus less than or equal to one, Theorem 1.11 gives $\Upsilon_{K_-\#(m(K_+))}(t)\leq t$, so

$$\Upsilon_{K_-}(t)\leq \Upsilon_{K_+}(t)+t$$

follows from Proposition 1.8.

To see that $\Upsilon_{K_+}(t)\leq \Upsilon_{K_-}(t)$, we proceed as follows. The triangle counting map used in the proof of the skein exact sequence [30, Theorem 10.2] induces a filtered map $\mathrm{CFK}^\infty(K_+)\to \mathrm{CFK}^\infty(K_-)$. This is a sum over Spin^c structures (on the cobordism W) of maps; but restricting to either Spin^c structure with minimal $|c_1(\mathfrak{s})|$ (evaluated on the generator of $H_2(W)$), we get an isomorphism on HF^-. Apply t-modification to this map, as in Proposition 4.4, and notice that Lemma 4.6 applies. $\square$

Remark 4.8. The above proposition could alternatively be seen as a consequence of the skein inequality for $\nu^-(K)$. This follows quickly from the behaviour of the maps associated to negative definite cobordisms, see [26].

4.3. **Special behaviour for $t\in[0,2]\cap\mathbb{Q}$.** The following proposition indicates that we obtain the same Υ-invariant, regardless of the base ring used in the definition. Indeed, for rational $t=\frac{m}{n}$, the complex $\mathrm{tCFK}(K)$ can be defined over the subring $\mathbb{F}[v^{1/n}]$ of $\mathcal{R}$ (this is how we defined Υ in the introduction).

Proposition 4.9. *Let C be a finitely generated, free, graded chain complex over $\mathbb{F}[v^{1/n}]$, and consider the induced chain complex $C\otimes_{\mathbb{F}[v^{1/n}]}\mathcal{R}$. Then, the maximal grading of any homogeneous non-torsion element of $H(C)$ agrees with the maximal grading of any homogeneous non-torsion element of $H(C\otimes_{\mathbb{F}[v^{1/n}]}\mathcal{R})$. In particular, for rational t, the invariant Υ_K, defined using* $\mathrm{tCFK}(K)$ *with coefficients in $\mathbb{F}[v^{1/n}]$, coincides with Υ_K, defined using* $\mathrm{tCFK}(K)$ *with coefficients in $\mathcal{R}$.*

Proof. For a graded module M over $\mathbb{F}[v^{1/n}]$, let $\mathrm{Tors}(M)$ denote its torsion submodule. We argue first that

$$(M\otimes\mathcal{R})/\mathrm{Tors}(M\otimes\mathcal{R})=(M/\mathrm{Tors}(M))\otimes\mathcal{R}. \tag{12}$$

To see this, consider the short exact sequnece

$$0\to\mathrm{Tors}(M)\to M\to M/\mathrm{Tors}(M)\to 0,$$

where $M/\mathrm{Tors}(M)$ is a free module. Since $\mathcal{R}$ is torsion-free as a module over $\mathbb{F}[v^{1/n}]$, we can tensor the above short exact sequence with $\mathcal{R}$ to get

$$0\to\mathrm{Tors}(M)\otimes\mathcal{R}\to M\otimes\mathcal{R}\to (M/\mathrm{Tors}(M))\otimes\mathcal{R}\to 0.$$

Since the image of $\mathrm{Tors}(M)\otimes\mathcal{R}$ is contained in $\mathrm{Tors}(M\otimes\mathcal{R})$, and $(M/\mathrm{Tors}(M))\otimes\mathcal{R}$ is torsion-free, we conclude that $\mathrm{Tors}(M)\otimes\mathcal{R}=\mathrm{Tors}(M\otimes\mathcal{R})$ and hence Equation (12) holds.

By the universal coefficients theorem (with coefficient ring $\mathbb{F}[v^{1/n}]$), there is an isomorphism

$$H(C\otimes_{\mathbb{F}[v^{1/n}]}\mathcal{R})\cong H(C)\otimes_{\mathbb{F}[v^{1/n}]}\mathcal{R},$$

since $\mathcal{R}$ is a torsion-free module over $\mathbb{F}[v^{1/n}]$. Applying Equation (12), we conclude that

$$H(C\otimes_{\mathbb{F}[v^{1/n}]}\mathcal{R})/\mathrm{Tors}(H(C\otimes_{\mathbb{F}[v^{1/n}]}\mathcal{R}))\cong (H(C)/\mathrm{Tors}(H(C)))\otimes\mathcal{R}.$$

The maximal grading of any non-torsion element in $H(C)$ is, in fact, the maximal grading of a generator of the free module $H(C)/\mathrm{Tors}(H(C))$, which of course coincides with the maximal grading of $H(C\otimes\mathcal{R})/\mathrm{Tors}(H(C\otimes\mathcal{R}))$. $\square$

Proof of Proposition 1.3. The grading on $\mathrm{tCFK}(K)$, when considered over $\mathbb{F}[v^{1/n}]$, lies in $\frac{1}{n}\mathbb{Z}$. Therefore $\Upsilon_K(\frac{m}{n}) \in \frac{1}{n}\mathbb{Z}$ follows from Proposition 4.9. $\square$

5. $\Upsilon_K(t)$ AS A FUNCTION OF t

5.1. **Continuously varying homologies.** Proposition 1.4 will be seen as the special case of a general construction. As before, let $\mathcal{R}$ denote the ring of long power series, defined in Definition 3.1. We grade this ring (by real numbers) so that v has grading -1; i.e. v^α has grading $-\alpha$.

Let C be a finitely generated complex over $\mathcal{R}$. Define $\Upsilon(C)$ to be the maximal grading of any non-torsion element in $H_*(C)$.

Note that $\mathcal{R}$ has a unique maximal ideal denoted $v^{>0}\mathcal{R}$, which is the union $\cup_{\alpha>0} v^\alpha \cdot \mathcal{R}$. If C is a finitely generated complex over $\mathcal{R}$, let $C/v^{>0}C$ denote the induced complex

$$C \otimes_{\mathcal{R}} (\mathcal{R}/v^{>0}\mathcal{R}) = C/\bigcup_{\alpha>0} v^\alpha \cdot C.$$

Definition 5.1. *A continuously varying family of finitely generated chain complexeses $\{C^t\}$ over $\mathcal{R}$, indexed by $t \in [0,2]$, is the following data:*

- *Generators $\{x_i\}_{i=1}^n$ (which generate each C^t as $\mathcal{R}$-modules), so that the grading $\mathrm{gr}_t(x_i) \in \mathbb{R}$ is a continuous function of t.*
- *Differentials $D^t \colon C^t \to C^t$ which drop the grading gr_t by one, and which vary continuously in t; i.e.*
$$D^t x_i = \sum_j a_{i,j}(t) x_j,$$
where $a_{i,j}(t)$ is either zero for all t, or it is of the form $a_{i,j}(t) = v^{g_{i,j}(t)}$ for some continuous function $g_{i,j}$ of t. In fact, grading considerations ensure $g_{i,j}(t) = \mathrm{gr}_t(x_j) - \mathrm{gr}_t(x_i) + 1$.

Proposition 5.2. *Let $\{C^t\}_{t\in[0,2]}$ be a continuously varying family of finitely generated chain complexes over $\mathcal{R}$. Suppose moreover that the rank of $H_*(C^t)$ is one. Then, $\Upsilon(C^t)$ is a continuous function of t. Moreover, for each t, there is a corresponding generator $x(t)$ in the finite generating set with the property that*

$$\Upsilon(C^t) = \mathrm{gr}_t(x(t)).$$

In fact, there is some non-zero homology class in $H_*(C^t/v^{>0}C^t)$ whose grading agrees with $\Upsilon(C^t)$.

Before proving the statement, recall the following:

Lemma 5.3. *Let C be a graded, finitely generated module over $\mathcal{R}$. Then, the homology of C splits as a direct sum of graded cyclic modules; i.e. modules of the form $\mathcal{R}$ or $\mathcal{R}/v^\alpha\mathcal{R}$ for some $\alpha \in \mathbb{R}_{\geq 0}$ (with a possible shift in degree).*

Proof. Although the ring $\mathcal{R}$ is not a principal ideal domain, it is a valuation ring, so every finitely generated ideal in $\mathcal{R}$ is principal, see [1, Section 11]. (In fact, the proof of this fact for $\mathcal{R}$ is so simple that we sketch it here. Suppose that $f_1, \dots, f_n$ generate the ideal I, and write $f_i = v^{\alpha_i} q_i$ where $q_i \in \mathcal{R}$ is a unit. Choosing $\alpha = \min\{\alpha_1, \dots, \alpha_n\}$, it is easy to see that the element v^α generates the ideal I.) Adapting the proof of the usual classification of modules over a principal ideal domain, it follows immediately that any finitely generated module is a sum of cyclic modules. $\square$

Recall the definition of the dual complex $C^* = \mathrm{Mor}_{\mathcal{R}}(C, \mathcal{R})$ for a complex C over $\mathcal{R}$, with the Kronecker pairing

$$\langle \cdot, \cdot \rangle \colon C \otimes_{\mathcal{R}} \mathrm{Mor}_{\mathcal{R}}(C, \mathcal{R}) \to \mathcal{R},$$

defined by the formula $\langle c, \phi \rangle = \phi(c)$. The module C^* is equipped with a differential d that is dual to the differential on C.

Lemma 5.4. *Let C be a finitely generated chain complex over $\mathcal{R}$, generated by the elements $\{x_1,\dots,x_k\}$, and suppose that the rank of $H_*(C)$ is one. Then there is a morphism $\phi\colon C\to\mathcal{R}$ with $d\phi=0$ and an element $x\in C$ with $\partial x=0$, so that $\langle x,\phi\rangle=1$. In fact, for any such pair (x,ϕ), the degree of x is $\Upsilon(C)$ and the degree of ϕ is $-\Upsilon(C)$, and there is some generator x_i of C with the property that*

$$\operatorname{gr} x_i=\operatorname{gr} x.$$

Proof. By the universal coefficients theorem, $H_*(C)$ contains a direct summand which is isomorphic to $\mathcal{R}$. The grading of the generator x of this $\mathcal{R}$-summand is $\Upsilon(C)$. Consider the splitting of $H_*(C)$ as the sum of cyclic modules, and take the map to $\mathcal{R}$ which takes x to 1. By the universal coefficients theorem in cohomology, there is a cohomology class $[\phi]$ with the property that Kronecker pairing with ϕ realizes this map; i.e. there is a cocycle so that

$$\langle x,\phi\rangle=1.$$

It follows that ϕ cannot be realized as v times any other cocycle, therefore $\operatorname{gr}(\phi)=\Upsilon(\operatorname{Mor}_{\mathcal{R}}(C,\mathcal{R}))$. Since the grading of 1 is zero, it also follows that $\operatorname{gr}(\phi)+\operatorname{gr}(x)=0$, implying the statement.

To show that $\operatorname{gr}(x)=\operatorname{gr}(x_i)$ for some $i\in\{1,\dots,k\}$, we express x in terms of the basis for C:

$$x=\sum_{i\in I}v^{\alpha_i}\cdot x_i,$$

where $I\subset\{1,\dots k\}$, and the α_i are real numbers with $\alpha_i\geq 0$, and $\operatorname{gr}(x)=\operatorname{gr}(x_i)-\alpha_i$. Let $\alpha=\min_{i\in I}\alpha_i$. Clearly, $x=t^{\alpha}\cdot x'$, where x' is a cycle representing a non-torsion homology class, with $\operatorname{gr}_t(x)=\operatorname{gr}(x')-\alpha$. It follows that $\alpha=0$, as desired. ☐

Proof of Proposition 5.2. Fix some $s\in[0,2]$, and let t_i be any sequence with $\lim_{i\to\infty}t_i=s$. Lemma 5.4 gives sequences of cycles x^{t_i} and ϕ^{t_i} with

$$\langle x^{t_i},\phi^{t_i}\rangle=1$$

and $\operatorname{gr}_{t_i}(x^{t_i})\in\{\operatorname{gr}_{t_i}(x_1),\dots,\operatorname{gr}_{t_i}(x_k)\}$. It follows that there is a uniform bound on $\operatorname{gr}_{t_i}(x^{t_i})$ or, equivalently, on the exponents of v in the expression of x^{t_i} in terms of the basis. Thus, we can find a subsequence $\{n_i\}_{i=1}^{\infty}\subset\mathbb{N}$ so that the $x^{t_{n_i}}$ converge to $x^s\in C^s$. Passing to a further subequence if needed, we can assume that the $\phi^{t_{n_i}}$ converge to some $\phi^s\in\operatorname{Mor}_{\mathcal{R}}(C^s,\mathcal{R})$. Since $\partial_t(x^{t_{n_i}})=0$, by continuity we conclude that $\partial_s(x^s)=0$. Similarly, $d_{t_{n_i}}\phi^{t_{n_i}}=0$ imply $d_s\phi^s=0$. Now, by continuity, $\lim_{i\to\infty}\operatorname{gr}_{t_{n_i}}(x^{t_{n_i}})=\operatorname{gr}_s(x^s)$, and we conclude from Lemma 5.4 that $\Upsilon(C^{t_{n_i}})\to\Upsilon(C^s)$. Since this holds for any sequence of $\{t_i\}$ which converges to s, we conclude that the function $\Upsilon(C^t)$ is continuous at s. Since s is arbitrary, we conclude that $\Upsilon(C^t)$ is a continuous function.

Now, there are n continuous functions $\operatorname{gr}(x_i^t)$, and for any t, the value $\Upsilon(C^t)$ agrees with at least one of them (again, according to Lemma 5.4).

Finally, observe that if x^{t_i} represents a boundary in $C^{t_{n_i}}/v^{>0}C^{t_{n_i}}$, then in fact x^{t_i} would be homologous (in $C^{t_{n_i}}$) to v^{α} times a different cycle in $C^{t_{n_i}}$. But this would contradict the statement that x^{t_i} is a maximal grading, non-torsion homogeneous element. ☐

5.2. **Applications to $\Upsilon_K(t)$.** Proposition 1.4 is an immediate consequence of Proposition 5.2:

Proof of Proposition 1.4. Consider the complexes $\operatorname{tCFK}(K)$ over $\mathcal{R}$. These have a fixed generating set, and the differential is specified by

$$\partial_t\mathbf{x}=\sum_{\mathbf{y}\in\mathfrak{S}}\ \sum_{\{\phi\in\pi_2(\mathbf{x},\mathbf{y})\big|\mu(\phi)=1\}}\#\left(\frac{\mathcal{M}(\phi)}{\mathbb{R}}\right)v^{tn_z(\phi)+(2-t)n_w(\phi)}\mathbf{y}.$$

We clearly obtain a continously varying family of chain complexes C^t (over $\mathcal{R}$) in the sense of Definition 5.1.

We can now apply Proposition 5.2 (whose hypotheses are satisfied, thanks to Proposition 2.4) to conclude that $\Upsilon_K(t)$ is a continuous function of t, which agrees, at any t, with one of the finitely many linear functions $\{\mathrm{gr}_t(\mathbf{x})\}_{\mathbf{x}\in\mathfrak{S}}$ (recall that $\mathfrak{S}=\mathbb{T}_\alpha\cap\mathbb{T}_\beta$ is the set of generators and $\mathrm{gr}_t(\mathbf{x})=M(\mathbf{x})-tA(\mathbf{x})$), as stated in Proposition 5.2. Note that since the various slopes of the functions $\{\mathrm{gr}_t(\mathbf{x})\}_{\mathbf{x}\in\mathfrak{S}}$ are (-1)-times the Alexander gradings of those elements, it follows at once that the finitely many slopes of $\Upsilon_K(t)$ are all integers. □

Proof of Theorem 1.13. This follows from Proposition 1.4, together with Proposition 2.7 and the concordance invariance of Υ. □

Proof of Proposition 1.6. Consider a sequence of cycles x^t indexed by $t>0$ satisfying the following properties:

(1) *(homogeneity)* x^t is homogeneous with grading gr_t;
(2) *(non-torsion)* x^t is non-torsion;
(3) *(maximality)* x^t maximizes gr_t among all gr_t-homogeneous, non-torsion elements.

Write x^t in terms of a basis of generators

$$x^t=\sum a_i(t)v^{\epsilon_i(t)}\mathbf{x}_i,$$

(with $a_i(t)\in\mathbb{F}$ and $\epsilon_i(t)\in\mathbb{R}$). By passing to a subsequence in t, we can make the following further assumptions:

(4) The $a_i(t)=a_i$ are constant; i.e. the $\mathbf{x}^t$ converge as $t\to\infty$. Equivalently, there is some fixed set I with the property that
$$\mathbf{x}^t=\sum_{i\in I}v^{\epsilon_i(t)}\mathbf{x}_i.$$
(5) There is some $i_0\in I$ with the property that $\epsilon_{i_0}(t)\equiv 0$. (This final property follows from maximality.)

Maximality further ensures that

$$\Upsilon_K(t)=\mathrm{gr}_t(\mathbf{x}^t)=M(\mathbf{x}_i)-tA(\mathbf{x}_i)-\epsilon_i(t)$$

for $t>0$. Since $\Upsilon_K(0)=0$ (Proposition 1.5), continuity of Υ (Proposition 5.2) ensures that for those j with $\epsilon_j(0)=0$ (and those exist by Property (5)), we have

$$\mathrm{gr}_t(\mathbf{x}^t)=-tA(\mathbf{x}_j).$$

This ensures the limiting cycle $\mathbf{x}^0=\lim_{t\to 0}\mathbf{x}^t$ is a sum of chains with fixed Alexander grading, which in fact is $A(\mathbf{x}_j)$ (for any j with $\epsilon_j(0)=0$). Observe that $\mathrm{tCFK}(K)/v^{>0}\cdot\mathrm{tCFK}(K)=\widehat{\mathrm{CF}}(S^3)$, a chain complex whose homology is $\widehat{\mathrm{HF}}(S^3)\cong\mathbb{F}$. The image of $\mathbf{x}^0$ in the homology of this quotient complex generates $\widehat{\mathrm{HF}}(S^3)$. We conclude at once that $A(\mathbf{x}_j)=A(\mathbf{x}^0)\geq\tau(K)$. Thus,

$$\Upsilon_K(t)\leq -t\cdot\tau(K) \tag{13}$$

for all sufficiently small t.

For the converse, we find it convenient to work in the model for C^t considered in Lemma 4.3. Take a chain $y_0\in\widehat{\mathrm{CF}}(S^3)$ with the following properties:

- The chain y_0 is a cycle, which represents the non-trivial homology class in $\widehat{\mathrm{HF}}(S^3)$.
- The chain y_0 is homogeneous in Maslov and Alexander gradings; in particular, $M(y_0)=0$.
- The Alexander grading of y_0 is $\tau(K)$.

We can extend y_0 to a Maslov-homogeneous cycle y representing the generator $\mathrm{HF}^-(S^3)$ by adding only terms with non-zero U powers in them.

Write $y = y_0 + U \cdot y_1$. Next, consider y as a cycle in C^t. Since $Uy_1 = v^2 y_1$ has algebraic filtration less than 2, and there is a uniform upper bound on the Alexander gradings of any element, we conclude that for all $0 \leq t$ sufficiently small, $F^t(y) \leq 0$. Indeed, by making the upper bound smaller if needed, for all $0 \leq t$ sufficiently small, $F^t(v^{t\cdot\tau(K)} \cdot y) \leq 0$, so we can view $v^{t\cdot\tau} \cdot y$ as an element of E^t. Since y represents a non-zero class in $\mathrm{HF}^-(S^3)$, the class $v^{t\cdot\tau(K)} \cdot y$ represents a non-torsion homology class in E^t.

According to Lemma 4.3, in the model for E^t the Maslov grading of $v^{t\cdot\tau(K)} \cdot y$, which is $-t\cdot\tau(K)$, corresponds to the grading in C^t.

We conclude that, for all sufficiently small $t \geq 0$,

$$\Upsilon_K(t) \geq -t \cdot \tau(K).$$

Combining this with Equation (13), we conclude that for all sufficiently small ≥ 0,

$$\Upsilon_K(t) = -t \cdot \tau(K),$$

from which Proposition 1.6 follows immediately. □

Proof of Propositions 1.7. Suppose that t is a point where $\Delta\Upsilon'_K(t) \neq 0$. By Proposition 5.2, there are two different generators x and y with $\mathrm{gr}_t(x) = \mathrm{gr}_t(y)$, but $A(x) \neq A(y)$ and

$$\Delta\Upsilon'_K(t) = A(x) - A(y).$$

The condition $\mathrm{gr}_t(x) = \mathrm{gr}_t(y)$ (that is, $M(x) - tA(x) = M(y) - tA(y)$) ensures that

$$t\Delta\Upsilon'_K(t) = t(A(x) - A(y)) = M(x) - M(y)$$

is an even integer. (Recall that a non-torison element has even Maslov grading; cf. Proposition 2.4.) □

6. Computations

Theorems 1.14 and 1.15 are quick consequences of the corresponding knot Floer homology computations.

Proof of Theorem 1.14. Apply Theorem 2.8. In view of Lemma 1.9, we can assume without loss of generality that $\tau(K) = -\sigma(K)/2$ is non-negative.

In this case, there is a sequence of elements $x_0, \dots, x_n$, and $y_0, \dots, y_{n-1}$ in $\mathrm{CFK}^\infty(K)$, with $\partial y_i = Ux_i + x_{i+1}$, and

$$\begin{aligned} A(x_i) &= \tau(K) - 2i \\ M(x_i) &= -2i. \end{aligned}$$

Clearly, x_0 represents a non-torsion generator. In fact, any non-torsion class must contain at least one of the x_i. Moreover, for $0 \leq t \leq 1$, $\mathrm{gr}_t(x_0)$ is maximal among $\mathrm{gr}_t(x_i)$. Thus,

$$\Upsilon_K(t) = \mathrm{gr}_t(x_0) = -A(x_0) \cdot t = \frac{\sigma(K)}{2} \cdot t.$$

The values for $1 \leq t \leq 2$ now follow from Proposition 1.2. □

Remark 6.1. Theorem 2.8 can be generalized immediately to *quasi-alternating knots* in the sense of [33], using the appropriate generalization of Theorem 1.14 from [22].

Theorem 6.2. *Let K be an L-space knot, and let $\{\alpha_i\}_{i=0}^n$ and $\{m_i\}_{i=0}^n$ be the associated sequence of integers, as defined in Equation* (6) *and* (7) *respectively. Then,*

$$\Upsilon_K(t) = \max_{\{i \,|\, 0\leq 2i\leq n\}} \{m_{2i} - t\alpha_{2i}\}.$$

Proof. According to Theorem 2.10, we can consider the model complex specified by Equation (8) in place of the knot Floer complex. Glancing at the differential, it is clear that the non-torsion part is generated (over $\mathcal{R}$) by one of the even generators x_{2k} with $0 \leq 2k \leq n$ (and k is an integer). It follows that

$$\Upsilon_K(t) = \max_{0\leq 2k\leq n}\{\mathrm{gr}_t(x_{2k})\} = \max_{0\leq 2k\leq n}\{m_{2k} - t\alpha_{2k}\}.$$

□

The above result immediately implies:

Proof of Theorem 1.15. The statement follows immediately from Theorem 6.2, since lens spaces are L-spaces [31, Proposition 3.1], and $pq \pm 1$ surgery on the torus knot $T_{p,q}$ is a lens space. □

Example 1.16 from the introduction follows immediately. We generalize it to the family $T_{n,n+1}$, as follows:

Proposition 6.3. *Consider the torus knot $T_{n,n+1}$. Then, $\Upsilon_{T_{n,n+1}}(t)$ is the piecewise linear function whose values for $t \in [\frac{2i}{n}, \frac{2i+2}{n}]$ (for $i = 0, \dots n-1$) are given by*

$$\Upsilon_{T_{n,n+1}}(t) = -i(i+1) - \frac{1}{2}n(n-2i-1)t.$$

In particular,

$$\frac{1}{2}t \cdot \Delta\Upsilon'_{T_{n,n+1}}(t) = \begin{cases} 1 & \text{for } t = \frac{2i}{n} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The Alexander polynomial $\Delta_{n,n+1}(t)$ of $T_{n,n+1}$ is $t^{-\frac{1}{2}n(n-1)}\frac{(t^{n(n+1)}-1)(t-1)}{(t^n-1)(t^{n+1}-1)}$. Let

$$p(t) = \sum_{i=0}^{n-1} t^{ni} - t\sum_{i=0}^{n-2} t^{(n+1)i}.$$

Since

$$(t^n-1)(t^{n+1}-1)p(t) = (t-1)(t^{n(n+1)}-1),$$

we conclude that

$$p(t) = t^{\frac{1}{2}n(n-1)} \cdot \Delta_{T_{n,n+1}}(t).$$

Thus,

$$\alpha_{2i} = n(n-i) - \frac{1}{2}n(n-1) = \frac{1}{2}n(n-2i-1),$$

and the formula for $\Upsilon_{T_{n,n+1}}(t)$ now follows from Theorem 1.15. □

The above examples show that for each rational number t, the homomorphism

$$\frac{1}{2}t \cdot \Delta\Upsilon'_K(t)\colon \mathcal{C} \to \mathbb{Z}$$

is surjective. As the next lemma shows, the existence of this map implies the existence of the stated direct summand in the concordance group.

Lemma 6.4. *Let G be an Abelian group, and $H \subset G$ be a subgroup generated by the elements $(h_i)_{i=1}^{\infty}$. Suppose that $(\lambda_n \colon G \to \mathbb{Z})_{n=1}^{\infty}$ is a collection of homomorphisms with the property that $\lambda_n(h_n) = 1$ and $\lambda_m(h_n) = 0$ for $m > n$. Then, H is a $\mathbb{Z}^{\infty}$ direct summand of G.*

Proof. Consider the map $\Lambda \colon G \to \mathbb{Z}^{\infty}$ given by

$$g \mapsto (\lambda_n(g))_{n=1}^{\infty}.$$

Consider the linear transformation $\mathbb{Z}^{\infty} \to \mathbb{Z}^{\infty}$ given by

$$(a_n)_{n=1}^{\infty} \mapsto \big(\lambda_n(\sum_{i=1}^{\infty} a_i h_i)\big)_{n=1}^{\infty}.$$

The hypothesis ensures that this is the identity map plus a nilponent transformation. Such a map is necessarily invertible, concluding the proof. $\square$

Proof of Theorem 1.17. Consider the homomorphisms $K \mapsto (\lambda_n(K) = \frac{1}{n}\Delta\Upsilon'_K(\frac{2}{n}))_{n=1}^{\infty}$ and the elements $\{[T_{n,n+1}]\}_{n=1}^{\infty} \subset \mathcal{C}$. According to Proposition 6.3, $\lambda_n(T_{n,n+1}) = 1$ and $\lambda_m(T_{n,n+1}) = 0$ for $m > n$. Thus, Lemma 6.4 applies and concludes the proof. $\square$

7. Generalities on bordered Floer homology (with torus boundary)

The proof of Theorem 1.20 involves computations of knot invariants for satellite knots. This problem is well suited to bordered Floer homology [18].

Bordered Floer homology is an invariant for three-manifolds with parameterized (bordered) boundary. To a parameterized surface, this invariant associates a differential graded algebra, to a three-manifold with boundary it associates two kinds of modules over this algebra, called the *type* D and *type* A modules. Let Y be a connected, closed, oriented three-manifold equipped with a parameterized separating surface F, expressing $Y = Y_1 \cup_F Y_2$. A pairing theorem expresses the Heegaard Floer homology $\widehat{\mathrm{HF}}(Y)$ as an algebraic pairing (the "box tensor product") of the type D structure of Y_1 and of the type A structure of Y_2.

For the reader's convenience we collect here some useful facts about bordered Floer homology (in the case of three-manifolds with torus boundary). This material can be found in [18, Chapter 11]; see also [19] for a general overview of the theory.

7.1. The torus algebra.

In this section we follow [18, Section 11.1]. The algebra $\mathbb{A}(\mathbb{T})$ associated to a torus has two minimal idempotents ι_0 and ι_1, and six other basic generators:

$$\rho_1 \qquad \rho_2 \qquad \rho_3 \qquad \rho_{12} \qquad \rho_{23} \qquad \rho_{123}.$$

The differential is zero, and the non-zero products are

$$\rho_1\rho_2 = \rho_{12} \qquad \rho_2\rho_3 = \rho_{23} \qquad \rho_1\rho_{23} = \rho_{123} \qquad \rho_{12}\rho_3 = \rho_{123}.$$

(All other products of two non-idempotent basic generators vanish identically.) There are also compatibility conditions with the idempotents:

$$\begin{aligned} \rho_1 &= \iota_0\rho_1\iota_1 & \rho_2 &= \iota_1\rho_2\iota_0 & \rho_3 &= \iota_0\rho_3\iota_1 \\ \rho_{12} &= \iota_0\rho_{12}\iota_0 & \rho_{23} &= \iota_1\rho_{23}\iota_1 & \rho_{123} &= \iota_0\rho_{123}\iota_1. \end{aligned}$$

This algebra is graded by a non-commutative group G. One model for G is a group generated by triples $(j; p, q)$ where $j, p, q \in \frac{1}{2}\mathbb{Z}$ and $p + q \in \mathbb{Z}$. The group law is

$$(j_1; p_1, q_1) \cdot (j_2; p_2, q_2) = \left(j_1 + j_2 + \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}; p_1 + p_2, q_1 + q_2\right).$$

This group has a distinguished central element $\lambda = (1;0,0)$. (The group $G(\{\mathbb{T}\})$, introduced in [18, Section 11.1] naturally grades the torus algebra; $G(\{\mathbb{T}\})$ can be defined as a certain subgroup of G we discussed above.)

The gradings of the algebra elements are specified by the following formulae:

$$\mathrm{gr}(\rho_1) = \big(-\frac{1}{2};\frac{1}{2},-\frac{1}{2}\big) \qquad \mathrm{gr}(\rho_2) = \big(-\frac{1}{2};\frac{1}{2},\frac{1}{2}\big) \qquad \mathrm{gr}(\rho_3) = \big(-\frac{1}{2};-\frac{1}{2},\frac{1}{2}\big). \tag{14}$$

This is extended to all other group elements by the rule $\mathrm{gr}(ab) = \mathrm{gr}(a)\mathrm{gr}(b)$.

7.2. **Gradings on modules.** Let Y_R be a torus-bordered three-manifold, and assume for simplicity that Y_R is a homology knot complement; that is, $H_1(Y_R;\mathbb{Z}) \cong \mathbb{Z}$.

According to [18, Chapter 6], this bordered manifold has an associated *type D structure* in the sense of [18, Section 2.3], denoted $\widehat{CFD}(Y_R)$. As a vector space, this is graded by a homogeneous G-space. In fact, there is an element p which is homogeneous with grading $\langle p\rangle$ and with the property that $\widehat{CFD}(Y_R)$ is graded by the space of left cosets $G/\langle p\rangle$. The type D structure is equipped with a structure map

$$\delta^1\colon \widehat{CFD}(Y_R) \to \mathbb{A}\otimes\widehat{CFD}(Y_R),$$

(where $\mathbb{A}$ is the torus algebra $\mathbb{A}(\mathbb{T})$ recalled above) which respects the grading, in the sense that if $\mathbf{x}$ is some generator and $a\otimes\mathbf{y}$ appears with non-zero multiplicity in $\delta^1(\mathbf{x})$, then

$$\lambda^{-1}\mathrm{gr}(\mathbf{x}) = \mathrm{gr}(a)\cdot\mathrm{gr}(\mathbf{y}),$$

where $\mathrm{gr}(a)$ is the grading in the algebra and $\mathrm{gr}(\mathbf{x})$ and $\mathrm{gr}(\mathbf{y})$ denote gradings in the module.

Let Y_L be a torus-bordered three-manifold, and assume again the Y_L is a homology knot complement. According to [18, Chapter 7], there is a right $\mathcal{A}_\infty$ module associated to Y_L, the *type A invariant of* Y_L, denoted $\widehat{CFD}(Y_L)$. Moreover, there is an element q with the property that the type A invariant $\widehat{CFA}(Y_L)$ is graded by the right coset space $\langle q\rangle\backslash G$.

The $\mathcal{A}_\infty$ operations respect these gradings, in the sense that if $\mathbf{x}$ is some generator, and $\mathbf{y}$ appears with non-zero multiplicity in $m_n(\mathbf{x},a_1,\dots,a_{n-1})$, then

$$\lambda^{n-2}\mathrm{gr}(\mathbf{x})\mathrm{gr}(a_1)\cdots\mathrm{gr}(a_{n-1}) = \mathrm{gr}(\mathbf{y}). \tag{15}$$

The pairing theorem [18, Theorem 1.3] identifies the quasi-isomorphism type of $\widehat{\mathrm{CF}}(Y_L\cup Y_R)$ with that of the tensor product $\widehat{CFA}(Y_L)\boxtimes\widehat{CFD}(Y_R)$, as defined in [18, Section 2.4]. The grading set of $\widehat{\mathrm{CF}}(Y_L\cup Y_R)$ is some cyclic group (given by the Maslov grading). The pairing theorem also identifies this grading set with a subset of the double coset space $\langle q\rangle\backslash G/\langle p\rangle$. The latter space has an action by $\mathbb{Z}$, induced from translation by the central element λ.

7.3. **The pairing theorem and knots.** Suppose now that Y_L contains a knot, in addition to a bordered boundary. In this case, the diagram for Y_L contains yet another basepoint (w), and $\widehat{CFA}(Y_L)$ can correspondingly be thought of as a type A structure (for example) over the torus algebra, where the base ring is $\mathbb{F}[U]$. The grading group can be correspondingly enriched to $G\times\mathbb{Z}$, where the additional $\mathbb{Z}$-factor is called the *Alexander factor*.

According to [18, Theorem 11.19] the pairing $\widehat{CFA}(Y_L,z,w)\boxtimes\widehat{CFD}(Y_R)$ now represents the knot Floer homology HFK^- of $Y_L\cup Y_R$, equipped with the knot (supported in Y_L). The tensor product is graded by a double coset space $\langle p\rangle\backslash G\times\mathbb{Z}/\langle q\rangle$. Translation on the Alexander factor now corresponds to changing the Alexander grading for the induced knot in $Y_L\cup Y_R$. We will use this (as in [18, Chapter 11]) to study satellite operations (where Y_L is a solid torus).

8. Linear independence

Using bordered Floer homology computations, in this section we will determine parts of the knot Floer chain complex of the knots of Equation (1) from the introduction. These computations will enable us to give a proof of Theorem 1.20. In this proof we need to consider cables of the Whitehead double $W_0^+(T_{2,3})$ of the trefoil knot $T_{2,3}$. We start with a simpler computation of considering some cables of the trefoil, and then turn to cables of the Whitehead double.

See also [3, 14, 36] for similar computations.

8.1. **A warm-up: cables of the trefoil knot.** Given a knot K and relatively prime integers (p,q), let $C_{p,q}(K)$ denote the (p,q) cable of K. Let $T_{p,q}$ be the (p,q) torus knot ($C_{p,q}$ of the unknot). For integers $n \geq 2$ consider the family of knots $C_{n,2n-1}(T_{2,3})$. As a warm-up to our future calculations, we prove the following:

Lemma 8.1. *The values of $\Upsilon_{C_{n,2n-1}(T_{2,3})}$ on the interval $[0, \frac{1}{n-1}]$ are determined by*

$$\Upsilon_{C_{n,2n-1}(T_{2,3})}(t) = \begin{cases} -(n^2-n+1)\cdot t & t \in [0, \frac{2}{2n-1}] \\ 2-(n^2-3n+2)\cdot t & t \in [\frac{2}{2n-1}, \frac{1}{n-1}]. \end{cases}$$

In fact, it is not difficult to describe $\Upsilon_{C_{n,2n-1}(T_{2,3})}(t)$ completely; but the above partial computation will be sufficient for our immediate needs.

We prove Lemma 8.1 after a little preparation. The proof relies on a computation of knot Floer homology, which can be done by a number of different techniques; see for example [8]. In fact, according to [10, Theorem 1.10], $C_{n,2n-1}(T_{2,3})$ has an L-space surgery, so one could apply Theorem 6.2; see also [13]. We prefer instead to proceed using bordered Floer homology (see [18, Chapter 11] for $n=2$ and [36] for general n; see also [14]), as the computation will serve as a warm-up to a later computation given in Lemma 8.8, where Theorem 6.2 does not apply.

Lemma 8.2. *The type D module of the $+2$-framed right-handed trefoil knot complement has grading set given by $G/\lambda\mathrm{gr}(\rho_{12})\mathrm{gr}(\rho_{23})^2$. It has five generators, I, J, K, P, and Q, with gradings specified by:*

$$\begin{aligned} &\mathrm{gr}(I) = \lambda^{-2}\mathrm{gr}(\rho_{23})^{-1} \qquad \mathrm{gr}(J) = \lambda^{-1} \qquad \mathrm{gr}(K) = \mathrm{gr}(\rho_{23}) \\ &\mathrm{gr}(P) = \lambda^{-2}\mathrm{gr}(\rho_3)^{-1} \qquad \mathrm{gr}(Q) = \lambda^{-2}\mathrm{gr}(\rho_1)^{-1} \end{aligned} \tag{16}$$

The differential is specified by:

$$\begin{tikzcd} I \arrow[ddrr, "\rho_{12}"'] & P \arrow[l, "\rho_2"'] & J \arrow[l, "\rho_3"'] \arrow[d, "\rho_1"] \\ & & Q \\ & & K \arrow[u, "\rho_{123}"'] \end{tikzcd} \tag{17}$$

where the arrows connecting generators represent terms in δ^1, and the labels specify algebra elements; e.g.

$$\delta^1 J = \rho_3 \otimes P + \rho_1 \otimes Q.$$

Proof. Recall that the knot Floer homology group $\widehat{\mathrm{HFK}}(T_{2,3})$ of the right-handed trefoil has three generators, which we label $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$; with gradings $A(\mathbf{i}) = 1$, $A(\mathbf{j}) = 0$, $A(\mathbf{k}) = -1$, $M(\mathbf{i}) = 0$, $M(\mathbf{j}) = -1$, $M(\mathbf{k}) = -2$; and a differential (in CFK^∞) with $\partial\mathbf{j} = U\cdot\mathbf{i} + \mathbf{k}$. The type D module of the lemma follows from the HFK-to-type D algorithm, given in [18, Theorem 11.27]. $\square$

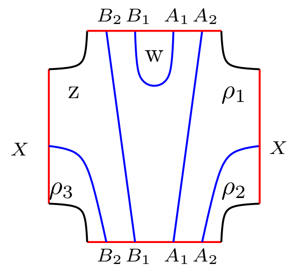

FIGURE 1. **Heegaard diagram for the $n = 2$ cabling piece.** This is taking place on the punctured torus, with the usual opposite sides identifications. The Heegaard diagram represents a bordered diagram for an $(n, -1)$ cabling piece.

To compute the cable, we tensor with the type A module for the $(n, -1)$ cabling module; see [14, 36]. This graded module can be described as follows:

Lemma 8.3. *The $(n, -1)$ cabling module has grading set $\lambda\mathrm{gr}(U^n)\mathrm{gr}(\rho_3)\mathrm{gr}(\rho_2)\backslash G$. Its generators are X and $\{A_i, B_i\}_{i=1}^n$, with gradings specified by*

$$\mathrm{gr}(X) = e \tag{18}$$

$$\mathrm{gr}(A_i) = \lambda^{i-n}\mathrm{gr}(\rho_2)^{-1}(\mathrm{gr}(\rho_2)\mathrm{gr}(\rho_1))^{i-n} \tag{19}$$

$$\mathrm{gr}(B_i) = = \lambda^{i-n}\mathrm{gr}(U)^{n-i}\mathrm{gr}(\rho_3)(\mathrm{gr}(\rho_2)\mathrm{gr}(\rho_1))^{i-n}. \tag{20}$$

The operations are specified by the following graph:

$$\begin{tikzcd}
 & A_n \arrow[dl, "\rho_2"'] \arrow[dd, dashed, "U^n"] & A_{n-1} \arrow[l, "\rho_2\otimes\rho_1"'] \arrow[dd, dashed, "U^{n-1}"] & \cdots \arrow[l, "\rho_2\otimes\rho_1"'] & A_i \arrow[l, "\rho_2\otimes\rho_1"'] \arrow[dd, dashed, "U^i"] & \cdots \arrow[l, "\rho_2\otimes\rho_1"'] & A_1 \arrow[l, "\rho_2\otimes\rho_1"'] \arrow[dd, dashed, "U"] \\
X \arrow[dr, "\rho_3"'] & & & & & & \\
 & B_n & B_{n-1} \arrow[l, "U\rho_2\otimes\rho_1"] & \cdots \arrow[l, "U\rho_2\otimes\rho_1"] & B_i \arrow[l, "U\rho_2\otimes\rho_1"] & \cdots \arrow[l, "\rho_2\otimes\rho_1"'] & B_1 \arrow[l, "U\rho_2\otimes\rho_1"]
\end{tikzcd} \tag{21}$$

Proof. The cabling module is the type A module associated to a doubly-pointed Heegaard diagram. Since that diagram has genus one, the holomorphic curve counting can be done combinatorially; see Figure 1 for a picture with $n = 2$. The computation was done in [36] (see also [14]); we recall here highlights for the reader's convenience.

The diagram appearing in the statement of the lemma is a shorthand: dashed arrows represent m_1 actions (labelled by their outputs in $\mathbb{F}[U]$), and all other operations are obtained by concatenating undashed paths (labelled by elements in $\mathbb{A}$ or $\mathbb{A}\otimes\mathbb{A}$). If there is a sequence of (undashed) arrows connecting some generator P to some generator Q, there is a corresponding algebra operation from P to Q whose sequence of input algebra elements is obtained from the sequences appearing on the edges by multiplying the last algebra element on some arrow with the first algebra element on the next arrow. For instance, concatenating the path from A_{n-1} to A_n (which is labelled $\rho_2\otimes\rho_1$) with the path from A_n to X (which is labelled by ρ_2) we obtain an operation

$$m_3(A_{n-1}, \rho_2\otimes\rho_1\cdot\rho_2) = X.$$

After verifying Equation (21), Equations (18), (19), and (20) follow from Equation (15).

The verification of the grading set follows immediately from Equations (19) and (20) for $i = 1$, and the fact that $\lambda^{-1}\mathrm{gr}(A_1) = \mathrm{gr}(U)\mathrm{gr}B_1$. $\square$

For simplicity, we have also reproduced the above answer in the special case where $n = 2$, see Equation (22). The corresponding Heegaard diagram is pictured in Figure 1. (Note that our numbering is slightly different from the one from [36].)

$$
\begin{tikzcd}
 & A_2 \arrow[dl, "\rho_2"'] \arrow[dd, dashed, "U^2"] & A_1 \arrow[l, "\rho_2 \otimes \rho_1"'] \arrow[dd, dashed, "U^1"] \\
X \arrow[dr, "\rho_3"'] & & \\
 & B_2 & B_1 \arrow[l, "U\rho_2 \otimes \rho_1"]
\end{tikzcd}
\tag{22}
$$

The next lemma describes the chain homotopy type of the bigraded chain complex $\mathrm{CFK}^-(C_{n,2n-1}(T_{2,3}))$ over $\mathbb{F}[U]$, whose homology is the knot Floer homology $\mathrm{HFK}^-(C_{n,2n-1}(T_{2,3}))$. This is the chain complex obtained by taking the associated graded object for $\mathcal{CFK}^-(C_{n,2n-1}(T_{2,3}))$.

Lemma 8.4. *The chain homotopy type of the complex* $\mathrm{CFK}^-(C_{n,2n-1}(T_{2,3}))$ *has a representative with generators* $\{A_i \otimes P\}_{i=1}^n$, $\{B_i \otimes P\}_{i=1}^n$ $\{A_i \otimes Q\}_{i=1}^n$, $\{B_i \otimes Q\}_{i=1}^n$ *and three more generators* $\{X \otimes I, X \otimes J, X \otimes K\}$. *The differential is specified by*

$$
\begin{aligned}
\partial(A_i \otimes Q) &= U^i B_i \otimes Q \\
\partial(A_i \otimes P) &= \begin{cases} U^i B_i \otimes P & \text{if } i < n-2 \\ U^{n-2} B_{n-2} \otimes P + B_n \otimes Q & \text{if } i = n-2 \\ U^{n-1} B_{n-1} \otimes P + X \otimes K & \text{if } i = n-1 \\ U^n B_n \otimes P + X \otimes I & \text{if } i = n \end{cases} \\
\partial(X \otimes J) &= B_n \otimes P \\
\partial(X \otimes I) &= 0 \\
\partial(X \otimes K) &= 0 \\
\partial(B_i \otimes Q) &= 0 \\
\partial(B_i \otimes P) &= 0.
\end{aligned}
$$

Relative bigradings are specified by

$$
\begin{aligned}
\mathbf{M}(B_1 \otimes Q) - \mathbf{M}(A_1 \otimes Q) &= 1 & \mathbf{A}(B_1 \otimes Q) - \mathbf{A}(A_1 \otimes Q) &= 1 \\
\mathbf{M}(A_1 \otimes Q) - \mathbf{M}(B_2 \otimes Q) &= 1 & \mathbf{A}(A_1 \otimes Q) - \mathbf{A}(B_2 \otimes Q) &= 2n-2.
\end{aligned}
\tag{23}
$$

For $i = 2, \dots, n-1$

$$
\begin{aligned}
\mathbf{M}(B_i \otimes Q) - \mathbf{M}(A_i \otimes Q) &= 2i-1 & \mathbf{A}(B_i \otimes Q) - \mathbf{A}(A_i \otimes Q) &= i \\
\mathbf{M}(A_i \otimes Q) - \mathbf{M}(B_{i-1} \otimes P) &= 1 & \mathbf{A}(A_i \otimes Q) - \mathbf{A}(B_{i-1} \otimes P) &= n-i+1 \\
\mathbf{M}(B_{i-1} \otimes P) - \mathbf{M}(A_{i-1} \otimes P) &= 2i-3 & \mathbf{A}(B_{i-1} \otimes P) - \mathbf{A}(A_{i-1} \otimes P) &= i-1 \\
\mathbf{M}(A_{i-1} \otimes P) - \mathbf{M}(B_{i+1} \otimes Q) &= 1 & \mathbf{A}(A_{i-1} \otimes P) - \mathbf{A}(B_{i+1} \otimes Q) &= n-i-1
\end{aligned}
\tag{24}
$$

and

$$
\begin{aligned}
\mathbf{M}(B_n \otimes Q) - \mathbf{M}(A_n \otimes Q) &= 2n-1 & \mathbf{A}(B_n \otimes Q) - \mathbf{A}(A_n \otimes Q) &= n \\
\mathbf{M}(A_n \otimes Q) - \mathbf{M}(B_{n-1} \otimes P) &= 1 & \mathbf{A}(A_n \otimes Q) - \mathbf{A}(B_{n-1} \otimes P) &= 1 \\
\mathbf{M}(B_{n-1} \otimes P) - \mathbf{M}(A_{n-1} \otimes P) &= 2n-3 & \mathbf{A}(B_{n-1} \otimes P) - \mathbf{A}(A_{n-1} \otimes P) &= n-1 \\
\mathbf{M}(A_{n-1} \otimes P) - \mathbf{M}(X \otimes K) &= 1 & \mathbf{A}(A_{n-1} \otimes P) - \mathbf{A}(X \otimes K) &= 0 \\
\mathbf{M}(X \otimes J) - \mathbf{M}(B_n \otimes P) &= 1 & \mathbf{A}(B_n \otimes P) - \mathbf{A}(X \otimes J) &= 0 \\
\mathbf{M}(A_n \otimes P) - \mathbf{M}(X \otimes I) &= 1 & \mathbf{A}(A_n \otimes P) - \mathbf{A}(X \otimes I) &= 0
\end{aligned}
\tag{25}
$$

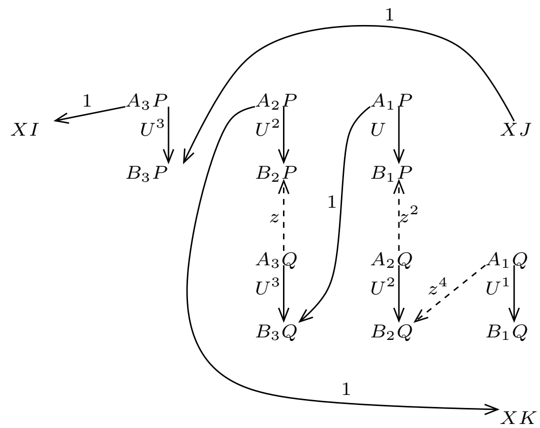

FIGURE 2. **Knot Floer complex** $\mathrm{CFK}^-(C_{3,5}(T_{2,3}))$ **of** $C_{3,5}(T_{2,3})$**.** Solid arrows (which are all labelled with U-powers) indicate differentials (and the labels indicate the coefficients); dashed arrows are not differentials, but they connect pairs of generators of Maslov grading difference 1 and Alexander grading difference recorded in the z exponent of the labels. The complex for $C_{n,2n-1}(T_{2,3})$ with $n > 3$ has very similar structure; the case $n = 2$ is slightly degenerate; see Figure 3.

and finally

$$\begin{aligned} \mathbf{M}(X \otimes J) - \mathbf{M}(X \otimes K) &= 2n-1 \qquad \mathbf{A}(X \otimes J) - \mathbf{A}(X \otimes K) = n \\ \mathbf{M}(B_n \otimes P) - \mathbf{M}(A_n \otimes P) &= 2n-1 \qquad \mathbf{A}(B_n \otimes P) - \mathbf{A}(A_n \otimes P) = n. \end{aligned} \tag{26}$$

These are calibrated by

$$\mathbf{M}(B_1 \otimes Q) = 0 \qquad \mathbf{A}(B_1 \otimes Q) = n^2 - n + 1. \tag{27}$$

Remark 8.5. The equations are stated in the above order in order to draw attention to the ordering of the generators by Alexander grading; e.g. the following sequence of generators have decreasing Alexander grading:

$$B_1 \otimes Q, \qquad A_1 \otimes Q, \qquad B_2 \otimes Q,$$

then (for $i = 2, \dots, n-1$)

$$B_i \otimes Q, \qquad A_i \otimes Q, \qquad B_{i-1} \otimes P, \qquad A_{i-1} \otimes P, \qquad B_{i+1} \otimes Q,$$

and finally

$$A_n \otimes Q, \qquad B_{n-1} \otimes P, \qquad A_{n-1} \otimes P.$$

(The gradings of other generators will be irrelevant, as they do not represent homology classes in $\widehat{\mathrm{HFK}}$.)

For future reference, notice that all other generators of $\mathrm{CFK}^-(C_{n,2n-1}(W_0^+(T_{2,3})))$ have Alexander grading $< g - 2n + 1$ and Maslov grading < -2. See Figure 4 for an illustration of the portion of $\mathcal{CFK}^-$ (with Alexander grading $\geq \mathbf{A}(B_2 \otimes Q^r)$).

Remark 8.6. See Figure 2 for an illustration of the chain complex with $n = 3$ (the general $n > 2$ case looks similar); see Figure 3 for the degenerate case where $n = 2$.

Proof of Lemma 8.4. The lemma is a straightforward pairing of the module from Lemma 8.3 with the one from Lemma 8.2 (in view of the pairing theorem, [18, Theorem 11.19]).

To illustrate this, we verify that

$$\partial(A_{n-2} \otimes P) = U^{n-2} B_{n-2} \otimes P + B_n \otimes Q.$$

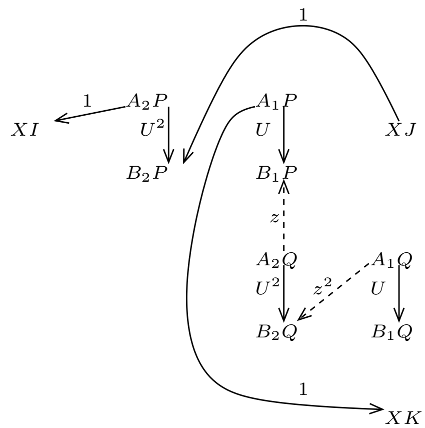

FIGURE 3. **Knot Floer complex of $C_{2,3}(T_{2,3})$.** This is the complex for $\mathrm{CFK}^-(C_{2,3}(T_{2,3}))$, with the notational conventions from Figure 2.

Lemma 8.3 states that $m_1(A_{n-2}) = U^{n-2}B_{n-2}$. This gives rise to the first term in the above boundary map. For the second term, we pair the sequence

$$\delta^1 P = \rho_2 \otimes I, \quad \delta^1 I = \rho_{12} \otimes K, \quad \delta^1 K = \rho_{123} \otimes Q.$$

with the action

$$m_4(A_{n-2}, \rho_2 \otimes \rho_{12} \otimes \rho_{123}) = B_n$$

coming from the concatenation of four arrows in Equation (21). It is easy to see that there are no other terms in the differential. The other differentials are verified similarly.

The pairing theorem can also be used to compute the stated bigradings. We illustrate this by computing the grading of $A_i \otimes P$, as follows:

$$\begin{aligned}
\mathrm{gr}(A_i \otimes P) &= \mathrm{gr}(A_i) \cdot \mathrm{gr}(P) \\
&= \lambda^{i-n}\mathrm{gr}(\rho_2)^{-1}(\mathrm{gr}(\rho_2)\mathrm{gr}(\rho_1))^{i-n} \cdot \lambda^{-2}\mathrm{gr}(\rho_3)^{-1} \\
&\sim (\lambda \mathrm{gr}(U^n)\mathrm{gr}(\rho_3)\mathrm{gr}(\rho_2))^{2i+1-2n} \\
&\qquad \left(\lambda^{i-n}\mathrm{gr}(\rho_2)^{-1}(\mathrm{gr}(\rho_2)\mathrm{gr}(\rho_1))^{i-n} \cdot \lambda^{-2}\mathrm{gr}(\rho_3)^{-1}\right) \\
&\qquad \left(\lambda \mathrm{gr}(\rho_{12})\mathrm{gr}(\rho_{23})^2\right)^{n-i} \\
&= \lambda^{-1-4i-2i^2+4n+4in-2n^2} u^{n(-1-2i+2n)}.
\end{aligned}$$

In the above, $\sim$ denotes the equivalence relation of double cosets; the exponent $n-i$ of $\lambda\mathrm{gr}(\rho_{12})\mathrm{gr}(\rho_{23})$ is chosen to cancel all factors of $\mathrm{gr}(\rho_{12})$ (up to overall factors of λ); and the exponent $(2i+1-2n)$ of $\lambda\mathrm{gr}(U^n)\mathrm{gr}(\rho_3)\mathrm{gr}(\rho_2)$ is chosen to cancel the factors of $\mathrm{gr}(\rho_{23})$ (up to factors of u and λ). The final step is a straightforward computation in the grading group, using the formulas recalled in Section 7.1. As in [18, Section 11.9], the pairing theorem interprets this double coset element as computing the Maslov/Alexander bigrading of generators. Specifically, the exponent of λ computes $\mathbf{M} - 2\mathbf{A}$, while the exponent of u computes $-\mathbf{A}$. Thus the above computation shows that, the Maslov/Alexander bigrading of $A_i \otimes P$ (up to overall shifts) is

$$\mathbf{M}(A_i \otimes P) = -1 - 4i - 2i^2 + 2n + 2n^2 \qquad \mathbf{A}(A_i \otimes P) = -(n(-1-2i+2n)).$$

Proceeding in a similar manner, we find:

$$\begin{aligned}
\mathbf{M}(B_i\otimes P) &= -2(1+i+i^2-n-n^2) & \mathbf{A}(B_i\otimes P) &= (i-n)(-1+2n)\\
\mathbf{M}(A_i\otimes Q) &= -2i^2+2i+2n^2+2n-1 & \mathbf{A}(A_i\otimes Q) &= -2n(-i+n+1)\\
\mathbf{M}(B_i\otimes Q) &= -2\left(i^2+i-n^2-n+1\right) & \mathbf{A}(B_i\otimes Q) &= (2n-1)(i-n)\\
\mathbf{M}(X\otimes I) &= -2n-2 & \mathbf{A}(X\otimes I) &= n\\
\mathbf{M}(X\otimes J) &= -1 & \mathbf{A}(X\otimes J) &= 0\\
\mathbf{M}(X\otimes K) &= 2n & \mathbf{A}(X\otimes K) &= -n.
\end{aligned}$$

The relative bigrading statements in the statement of the lemma are a direct consequence of these computations.

The non-trivial homology class in $\widehat{\mathrm{HFK}}$ with minimal Alexander grading is represented by $B_{n-1}\otimes P$; in fact, that class descends to a non-torsion class in HFK^-. By symmetry, it follows that the cycle $B_1\otimes Q$ with maximal Alexander grading represents the τ-invariant of $C_{n,2n-1}(T_{2,3})$ (in the sense that it descends to a generator for $\widehat{\mathrm{CF}}(S^3)$); in particular $\mathbf{M}(B_1\otimes Q)=0$. Its Alexander grading can be read off from the Alexander polynomial. □

By Lemma 8.4, the tensor product of these two modules has generating set

$$\{A_i\otimes P, B_i\otimes P, A_i\otimes Q, B_i\otimes Q, X\otimes I, X\otimes J, X\otimes K\}_{i=1}^n.$$

When $n>2$, there are four differentials (not decorated by U): from $X\otimes J$ to $B_n\otimes P$; from $A_n\otimes P$ to $X\otimes I$; and from $A_{n-2}\otimes P$ to $X\otimes K$; and from $A_{n-1}\otimes P$ to $B_n\otimes Q$.

Lemma 8.7. *Let K be a knot so that $\widehat{\mathrm{CFK}}(K)$ has three generators a, b, and c with the property that there are integers $1\leq k$ and $0\leq \ell$ with*

$$\begin{aligned}
\mathbf{M}(a)-\mathbf{M}(b) &= 2k-1 & \mathbf{A}(a)-\mathbf{A}(b) &= k\\
\mathbf{M}(b)-\mathbf{M}(c) &= 1 & \mathbf{A}(b)-\mathbf{A}(c) &= \ell.
\end{aligned}$$

Then, for any integer $n>\max(2,\ell)$, if $t<\frac{1}{n-1}$, then

$$gr_t(a)>\mathrm{gr}_t(b)>\mathrm{gr}_t(c).$$

Proof. This is straightforward arithmetic. □

Proof of Lemma 8.1. Let $L_n=C_{n,2n-1}(T_{2,3})$.

By the computation of the differentials in Lemma 8.4, it follows that the set

$$\{A_i\otimes P\}_{i=1}^{n-3},\{B_i\otimes P\}_{i=1}^{n-1},\{A_i\otimes Q\}_{i=1}^{n},\{B_i\otimes Q\}_{i=1}^{n-1},\{A_n\otimes Q+U^nA_{n-1}\otimes P\}$$

of cycles in $\mathrm{CFK}^-(L_n)$ generate the homology $\mathrm{HFK}^-(L_n)$. Note the ranges of the indices: the computation of the differential allows us to remove the chain complex generators $X\otimes I$, $X\otimes J$, $X\otimes K$, $A_{n-1}\otimes P$, $B_{n-1}\otimes P$, and $A_{n-2}\otimes P$. The final generator takes into account the differential which eliminates $B_n\otimes Q$. Similarly, the corresponding subset of generators in $\mathrm{tCFK}(L_n)$ (where the last generator is replaced by $v^{c-\mathrm{gr}_t(A_n\otimes Q)}A_n\otimes Q+v^{c-\mathrm{gr}_t(A_{n-1}\otimes P)}A_{n-1}\otimes P$ with $c=\max(\mathrm{gr}_t(A_{n-1}\otimes Q),\mathrm{gr}_t(A_n\otimes P)))$ span a quasi-isomorphic subcomplex of $\mathrm{tCFK}(L_n)$.

In view of Lemma 8.4, we can apply Lemma 8.7 repeatedly to conclude that for $t\leq\frac{1}{n-1}$, we have that for all $i=2\dots n-1$,

$$\mathrm{gr}_t(B_i\otimes Q)>\mathrm{gr}_t(A_i\otimes Q)>\mathrm{gr}(B_{i-1}\otimes P)>\mathrm{gr}_t(A_{i-1}\otimes P)>\mathrm{gr}_t(B_{i+1}\otimes Q);$$

and also

$$\mathrm{gr}_t(B_n\otimes Q)>\mathrm{gr}_t(A_n\otimes Q)>gr_t(B_{n-1}\otimes P)>\mathrm{gr}_t(A_{n-1}\otimes P).$$

Using the grading computations from Lemma 8.4, we also see that for $t \leq \frac{1}{n-1}$,

$$\mathrm{gr}_t(B_1 \otimes Q) > \mathrm{gr}_t(A_2 \otimes Q) \qquad \text{and} \qquad \mathrm{gr}_t(A_1 \otimes Q) > \mathrm{gr}_t(A_2 \otimes Q).$$

Thus, the generators $B_1 \otimes Q$, $A_1 \otimes Q$, and $B_2 \otimes Q$ are the three homology generators with maximal gr_t.

We verify next that $B_1 \otimes Q$ is a cycle, representing a non-torsion homology class in $\mathrm{tHFK}(L_n)$. From the gradings computed in Lemma 8.4, it follows that the Maslov grading of $B_1 \otimes Q$ is greater by at least 2 than the Maslov gradings of all other elements, except for $A_1 \otimes Q$. But $\partial(B_1 \otimes Q)$ cannot equal $A_1 \otimes Q$, because that would violate $\partial^2 = 0$. It follows that $B_1 \otimes Q$ represents a cycle in $\mathcal{CFK}^-(L_n)$. Moreover, since its Alexander grading is greater than the Alexander grading of all other generators, it follows that $B_1 \otimes Q$ represents a non-trivial homology class in $H(\mathcal{CFK}^-(L_n)) \cong \mathbb{F}[U]$. We conclude that $B_1 \otimes Q$, now thought of as an element of $\mathrm{tHFK}(L_n)$, has non-trivial image in $H(\mathrm{tCFK}(L_n \otimes \mathcal{R}^*)) \cong \mathcal{R}^*$; i.e. it is a non-torsion homology class.

Next, we claim that

$$\partial(A_1 \otimes Q) = U \cdot B_1 \otimes Q + B_2 \otimes Q \tag{28}$$

in $\mathcal{CFK}^-(C_{n,2n-1}(K))$. Observe first that the Maslov gradings of $U \cdot B_1 \otimes Q$ and $B_2 \otimes Q$ are strictly greater than the Maslov gradings of all generators of $\mathcal{CFK}^-$, other than $A_1 \otimes Q$. It follows that $\partial(A_1 \otimes Q)$ can contain no other terms. By the computation of $\mathrm{CFK}^-(C_{n,2n-1}(K))$, it follows that $U \cdot B_1 \otimes Q$ appears with non-zero coefficient in $\partial(A_1 \otimes Q)$. We wish to verify that $\partial(A_1 \otimes Q)$ also contains $B_2 \otimes Q$, an element whose Alexander filtration level is $2n-2$ less than that of $A_1 \otimes Q$. Since the Alexander filtration levels are different, the existence of this term in the differential is not visible directly from the differential in the associated graded graded object $\mathrm{CFK}^-(L_n)$; rather, its existence is verified by the following indirect argument.

When $n > 2$, we argue as follows. By Lemma 8.4, the three homology classes in $\widehat{\mathrm{HFK}}(L_n)$ with minimal Alexander grading are represented by $B_{n-1} \otimes P$, $A_n \otimes Q + U^n A_{n-2} \otimes P$, and $B_{n-2} \otimes P$ (noting that $B_n \otimes Q$ is homologous to $U^{n-2} B_{n-2} \otimes P$); and that lemma gives a differential in $\mathrm{CFK}^-(L_n)$ from $A_n \otimes Q + U^n A_{n-2} \otimes P$ to $U^{2n-2} \cdot B_{n-2} \otimes P$.

Consider now the complex $\mathcal{C}' = \mathcal{CFK}^-(L_n)'$ appearing in Proposition 2.5, which is obtained from $\mathcal{CFK}^-(L_n)$ by reversing the roles of the algebraic and Alexander filtrations. Generators for $\mathcal{C}'$ are of the form $\mathbf{x}' = U^{A(\mathbf{x})} \cdot \mathbf{x}$, where $\mathbf{x}$ is a generator for $\mathcal{CFK}^-(L_n)$. The fact that $U^{2n-2} \cdot B_{n-2} \otimes P$ appears in $\partial(A_n \otimes Q + U^n \cdot A_{n-2} \otimes P)$ ensures that the element $(B_{n-2} \otimes P)'$, whose Alexander grading is $2n-2$ smaller than that of $(A_n \otimes Q + U^n \cdot A_{n-2} \otimes P)'$, appears in $\partial(A_n \otimes Q + U^n \cdot A_{n-2} \otimes P)'$. The filtered chain homotopy equivalence from $\mathcal{C}'$ to $\mathcal{CFK}^-(L_n)$ sends $(B_{n-1} \otimes P)'$, $(A_n \otimes Q + U^n \cdot A_{n-2} \otimes P)'$, and $(B_{n-2} \otimes P)'$ to $B_1 \otimes Q$, $A_1 \otimes Q$, and $B_2 \otimes Q$ respectively, as those are the only generators in the corresponding bigradings. It follows that there is a non-zero term in the differential from $A_1 \otimes Q$ to $B_2 \otimes Q$, which drops Alexander filtration level by $2n-2$, i.e. establishing Equation (28) when $n > 2$. We say that this differential from $A_1 \otimes Q$ to $B_2 \otimes Q$ is symmetric to the differential from $A_n \otimes Q + U^n A_{n-2} \otimes P$ to $U^{2n-2} \cdot B_{n-2} \otimes P$.

When $n = 2$, there is no generator $A_{n-2} \otimes P$. Instead, the three generators in $\widehat{\mathrm{HFK}}(L_n)$ with minimal Alexander grading are $B_1 \otimes P$, $A_2 \otimes Q$, and $B_2 \otimes Q$. In this case, the differential from $A_2 \otimes Q$ to $U^2 \cdot B_2 \otimes Q$ is symmetric to the differential from $A_1 \otimes Q$ to $B_2 \otimes Q$ which drops Alexander grading by $2n - 2 = 2$, completing the verification of Equation (28).

Since $B_1 \otimes Q$ represents a non-torsion class in $\mathrm{tHFK}(L_n)$ and Equation (28) holds, we conclude that $B_2 \otimes Q$ also represents a non-torsion class in $\mathrm{tHFK}(L_n)$. Since

$$\begin{aligned}
\mathrm{gr}_t(B_1 \otimes Q) &= -(n^2 - n + 1)t, \\
\mathrm{gr}_t(B_2 \otimes Q) &= -2 - (n^2 - 3n + 2)t,
\end{aligned}$$

the computation of $\Upsilon_{L_n}(t)$ for $t \in [0, \frac{1}{n-1}]$ now follows. $\square$

8.2. **Cables of the Whitehead double.** Now we turn to the (partial) computation of the knot Floer complex of the knot $C_{n,2n-1}(W_0^+(T_{2,3}))$. Our goal is to determine Υ of this knot on the interval $[0, \frac{1}{n-1}]$.

Lemma 8.8. *The values of* $\Upsilon_{C_{n,2n-1}(W_0^+(T_{2,3}))}(t)$ *on the interval* $[0, \frac{1}{n-1}]$ *are determined by*

$$\Upsilon_{C_{n,2n-1}(W_0^+(T_{2,3}))}(t) = \begin{cases} -(n^2 - n + 1) \cdot t & t \in [0, \frac{2}{2n-1}] \\ 2 - (n^2 - 3n + 2) \cdot t & t \in [\frac{2}{2n-1}, \frac{1}{n-1}]. \end{cases}$$

Recall that for a knot K, its 0-twisted Whitehead double (with a positive clasp) is denoted by $W_0^+(K)$. The knot Floer homology for this knot was computed (in terms of the knot Floer complex for K) in [9]; see also [4]. In the special case where K is the right-handed trefoil knot $T_{2,3}$, his result specializes to the following:

Theorem 8.9. *(Hedden, [9]) For the 0-twisted Whitehead double of the right-handed trefoil (with its positive clasp), the knot Floer homology has 15 generators, which we denote* $\mathbf{i}^r$, $\mathbf{j}^r$, $\mathbf{k}^r$ *for* $r = 0, 1, 2, 3$ *and* $\mathbf{l}^s$ *for* $s = 1, 2, 3$. *The Alexander gradings of these elements are given (for* $r = 0, 1, 2, 3$ *and* $s = 1, 2, 3$*) by*

$$A(\mathbf{i}^r) = A(\mathbf{l}^s) = 0 \qquad A(\mathbf{j}^r) = 1 \qquad A(\mathbf{k}^r) = -1.$$

The Maslov gradings are given by

$$\begin{aligned} M(\mathbf{i}^0) &= -1, & M(\mathbf{j}^0) &= 0, & M(\mathbf{k}^0) &= -2, \\ M(\mathbf{i}^1) = -1 &= M(\mathbf{l}^1), & M(\mathbf{j}^1) &= 0, & M(\mathbf{k}^1) &= -2, \\ M(\mathbf{i}^r) = -2 &= M(\mathbf{l}^s), & M(\mathbf{j}^s) &= -1, & M(\mathbf{k}^s) &= -2, \end{aligned}$$

for $s = 2, 3$. *Moreover, for* $r = 0, 1, 2, 3$ *and* $s = 1, 2, 3$

$$\partial \mathbf{i}^r = U\mathbf{j}^r, \qquad \partial \mathbf{k}^s = \mathbf{l}^s;$$

similarly, if we let ∂_z^1 *denote the component of the differential which crosses the* z *basepoint exactly once, but not the* w *basepoint, then*

$$\partial_z^1 \mathbf{i}^r = \mathbf{k}^r, \qquad \partial_z^1 \mathbf{j}^s = \mathbf{l}^s.$$

□

Informally, Theorem 8.9 says that the knot Floer complex splits as a sum of a component which looks like the knot Floer complex for the right-handed trefoil, and three further "boxes": four generators connected with four arrows, two vertical and two horizontal. This direct sum description is a little misleading: there might in principle be further horizontal arrows which cross both w and z basepoints. However, these are not relevant in the algorithm for reconstructing the corresponding type D structure.

Proposition 8.10. *The type D structure of the complement of the 0-framed positive Whitehead double of the right-handed trefoil knot, with framing +2, splits as a direct sum of four summands; one of these is the type D structure of the right-handed trefoil, spelled out in Lemma 8.2 (though we will now keep the superscript* 0 *in the notation for the five generators,* I^0, P^0, J^0, Q^0, *and* K^0*).*

There are three further summands, with eight generators apiece $\{I^t, J^t, K^t, P^t, Q^t, R^t, S^t\}$ *with* $t = 1, 2, 3$ *and differential*

(29)
$$\begin{tikzcd}
I^t & P^t \arrow[l, "\rho_2"'] & J^t \arrow[l, "\rho_3"'] \arrow[d, "\rho_1"] \\
R^t \arrow[from=1-1, "\rho_1"'] & & Q^t \\
L^t \arrow[u, "\rho_{123}"] & S^t \arrow[l, "\rho_2"'] & K^t \arrow[l, "\rho_3"'] \arrow[u, "\rho_{123}"']
\end{tikzcd}$$

Gradings for these generators, thought of as elements of $G/\lambda \mathrm{gr}(\rho_{12})\mathrm{gr}(\rho_{23})^2$*, are given by:*
(30)
$$\begin{array}{llll}
\mathrm{gr}(I^1) = \lambda^{-2}\mathrm{gr}(\rho_{23})^{-1} & \mathrm{gr}(J^1) = \lambda^{-1} & \mathrm{gr}(K) = \mathrm{gr}(\rho_{23}) & \mathrm{gr}(L^1) = \lambda^{-1} \\
\mathrm{gr}(P^1) = \lambda^{-2}\mathrm{gr}(\rho_3)^{-1} & \mathrm{gr}(Q^1) = \lambda^{-2}\mathrm{gr}(\rho_1)^{-1} & \mathrm{gr}(S^1) = \lambda^{-1}\mathrm{gr}(\rho_3)\mathrm{gr}(\rho_{23}) & \mathrm{gr}(R^1) = \lambda^{-2}\mathrm{gr}(\rho_1)^{-1}.
\end{array}$$

For $s = 2, 3$*, corresponding eight generators have grading* λ^{-1} *times their* $s = 1$ *counterparts; e.g.* $\mathrm{gr}(I^s) = \lambda^{-3}\mathrm{gr}(\rho_{23})^{-1}$*. (For* $s = 0$*, the gradings are as specified in Lemma 8.2; note that for those five generators, the gradings are the same as the gradings of the corresponding* $s = 1$ *generators.)*

Proof. This is a straightforward combination of Theorem 8.9 with the HFK-to-type D module result [18, Theorem 11.27]. $\square$

Thus, to compute the knot Floer homology of $C_{n,2n-1}(W_0^+(T_{2,3}))$, it remains to compute the pairing of the cabling type A module with a "square" (on the eight generators I^t, J^t, K^t, L^t, P^t, Q^t, S^t, R^t). This computation was done by Petkova [36]. Those results can be summarized as follows:

Lemma 8.11. *(See* [36]*) Consider the square type* D *module with eight generators and differentials according to the following diagram:*

(31)
$$\begin{tikzcd}
I & P \arrow[l, "\rho_2"'] & J \arrow[l, "\rho_3"'] \arrow[d, "\rho_1"] \\
R \arrow[from=1-1, "\rho_1"'] & & Q \\
L \arrow[u, "\rho_{123}"] & S \arrow[l, "\rho_2"'] & K \arrow[l, "\rho_3"'] \arrow[u, "\rho_{123}"']
\end{tikzcd}$$

Gradings for these generators, thought of as elements of $G/\lambda \mathrm{gr}(\rho_{12})\mathrm{gr}(\rho_{23})^2$*, are given by:*
(32)
$$\begin{array}{llll}
\mathrm{gr}(I) = \lambda^{-2}\mathrm{gr}(\rho_{23})^{-1} & \mathrm{gr}(J) = \lambda^{-1} & \mathrm{gr}(K) = \mathrm{gr}(\rho_{23}) & \mathrm{gr}(L) = \lambda^{-1} \\
\mathrm{gr}(P) = \lambda^{-2}\mathrm{gr}(\rho_3)^{-1} & \mathrm{gr}(Q) = \lambda^{-2}\mathrm{gr}(\rho_1)^{-1} & \mathrm{gr}(S) = \lambda^{-1}\mathrm{gr}(\rho_3)^{-1}\mathrm{gr}(\rho_{23}) & \mathrm{gr}(R) = \lambda^{-2}\mathrm{gr}(\rho_1)^{-1}.
\end{array}$$

The pairing of this type D *module with the cabling type* A *module from Lemma 8.3 gives a chain complex with generators*

$$\{A_i \otimes P, A_i \otimes Q, A_i \otimes R, A_i \otimes S, B_i \otimes P, B_i \otimes Q, B_i \otimes R, B_i \otimes S, X \otimes I, X \otimes J, X \otimes K, X \otimes L\}_{i=1}^n.$$

Let

- *i denote any integer between $1,\dots,n$,*
- *j denote any integer between $1,\dots,n-1$,*
- *k any integer between $1,\dots,n-2$;*

then the differential is specified by:

$$\begin{aligned}
\partial(A_j\otimes P) &= A_{j+1}\otimes R + U^j\cdot B_j\otimes P\\
\partial(A_n\otimes P) &= U^n\cdot B_n\otimes P + J\otimes X\\
\partial(A_i\otimes Q) &= U^i\cdot B_i\otimes Q\\
\partial(A_i\otimes R) &= U^i\cdot B_i\otimes R\\
\partial(A_k\otimes S) &= U^k\cdot B_k\otimes S\\
\partial(A_{n-1}\otimes S) &= B_n\otimes R + U^{n-1}\cdot B_{n-1}\otimes S\\
\partial(A_n\otimes S) &= U^nB_n\cdot\otimes S + L\otimes X\\
\partial(B_j\otimes P) &= U\cdot B_{j+1}\otimes R\\
\partial(X\otimes J) &= B_n\otimes P\\
\partial(X\otimes I) &= 0\\
\partial(X\otimes K) &= B_n\otimes S.
\end{aligned}$$

Relative gradings are specified as follows. The relative bigradings of the generators $A_i\otimes P$, $B_i\otimes P$, $A_i\otimes Q$, and $B_i\otimes Q$, $X\otimes I$, $X\otimes J$, and $X\otimes K$ are as in Lemma 8.4. For $j=1,\dots,n-1$

$$\begin{aligned}
\mathbf{M}(B_{j+1}\otimes Q)-\mathbf{M}(A_j\otimes Q) &= 1 \qquad & \mathbf{A}(B_{j+1}\otimes Q)-\mathbf{M}(A_j\otimes Q) &= 1\\
\mathbf{M}(B_{j+1}\otimes P)-\mathbf{M}(B_j\otimes R) &= 1 \qquad & \mathbf{A}(B_{j+1}\otimes P)-\mathbf{A}(B_j\otimes R) &= 1\\
\mathbf{M}(B_i\otimes S)-\mathbf{M}(A_i\otimes S) &= 2i-1 \qquad & \mathbf{A}(B_i\otimes S)-\mathbf{A}(A_i\otimes S) &= i.
\end{aligned}\tag{33}$$

Remark 8.12. The results of the above lemma can be paraphrased as follows. If a and b are two generators with $\mathbf{M}(a)-\mathbf{M}(b)=1$ and $\mathbf{A}(a)-\mathbf{A}(b)=i$, we write a dashed line from a to b labelled by z^i. (Note that this does not indicate a differential, hence the dashing on the line.) With this notation, the tensor product of the box with the cabling bimodule produces a cyclic summand and further n distinct summands; one of these has the form

$$\begin{tikzcd}
B_1\otimes Q \arrow[d, dashed, "z^n"] & A_1\otimes Q \arrow[l, "U"'] \arrow[d, dashed, "z^n"] \\
B_1\otimes R & A_1\otimes R \arrow[l, "U"']
\end{tikzcd}\tag{34}$$

For j between $1, \dots, n-1$, the additional summands are

$$
\begin{tikzcd}
B_{j+1} \otimes Q \arrow[d, dashed, "z"'] & & A_{j+1} \otimes Q \arrow[ll, "U^{j+1}"'] \arrow[dd, dashed, "z^{n-j}"] & \\
B_j \otimes S & A_j \otimes S \arrow[l, "U^j"'] \arrow[d, dashed, "z^{n-j-1}"'] & & \\
& B_{j+1} \otimes R & B_j \otimes P \arrow[l, "U"'] & A_j \otimes P \arrow[l, "U^j"'] \arrow[d] \\
& & & A_{j+1} \otimes R \arrow[ull, "U^{j+1}"]
\end{tikzcd}
\tag{35}
$$

Proof of Lemma 8.11. Again, this is a straightforward computation in the spirit of Lemma 8.4. □

Now we are in the position to give a partial computation of the Υ-invariant of $C_{n,2n-1}(W_0^+(T_{2,3}))$.

Proof of Lemma 8.8. Let $K'_n = C_{n,2n-1}(W_0^+(T_{2,3}))$. Consider the eighteen generators of $\mathrm{CFK}^-(K'_n)$ indexed by $r = 0,1,2,3,\ s = 1,2,3$:

$$B_1 \otimes Q^r, \quad A_1 \otimes Q^r, \quad B_2 \otimes Q^r, \quad B_1 \otimes R^s, \quad A_1 \otimes R^s, \tag{36}$$

obtained by tensoring the generators A_1 and B_2 of the cabling piece with type D generators Q^r and R^s of the type D structure of the complement of $W_0^+(T_{2,3})$ from Proposition 8.10 (and the superscripts are as in the statement of that proposition).

It will be useful to identify the bigradings of these generators. To this end, let

$$g = n^2 - n + 1, \qquad \text{and} \qquad \epsilon_i = \begin{cases} 0 & i = 0,1 \\ -1 & i = 2,3. \end{cases}$$

For $r = 0,1$ and $s = 1,2,3$, we have that

$$
\begin{aligned}
\mathbf{M}(B_1 \otimes Q^r) &= 0 + \epsilon_r & \mathbf{A}(B_1 \otimes Q^r) &= g \\
\mathbf{M}(A_1 \otimes Q^r) &= -1 + \epsilon_r & \mathbf{A}(A_1 \otimes Q^r) &= g - 1 \\
\mathbf{M}(B_1 \otimes R^s) &= -1 + \epsilon_s & \mathbf{A}(B_1 \otimes R^s) &= g - n \\
\mathbf{M}(A_1 \otimes R^s) &= -2 + \epsilon_s & \mathbf{A}(A_1 \otimes R^s) &= g - n - 1 \\
\mathbf{M}(B_2 \otimes Q^r) &= -2 + \epsilon_r & \mathbf{A}(B_2 \otimes Q^r) &= g - 2n + 1.
\end{aligned}
$$

The computation of Υ will follow from partial information about $\mathcal{CFK}^-$ that can be extracted from the above computations. This partial information is broken into a sequence of successively verified claims.

Claim 1. For $t < \frac{1}{n-1}$, the generators of K'_n enumerated in Equation (36) have gr_t strictly greater than all of its other generators. This follows from the argument of Lemma 8.1 and Equation (33).

Claim 2. There are non-zero elements x_1, y, and x_2 in $\mathcal{CFK}^-$, which are in the span (over $\mathbb{F}$) of the eighteen generators from Equation (36), satisfying the following further properties:

(X-1) x_1 is in the same bigrading as $B_1 \otimes Q^0$
(X-2) y is in the same bigrading is $A_1 \otimes Q^0$
(X-3) x_2 is in the same bigrading as $B_2 \otimes Q^2$

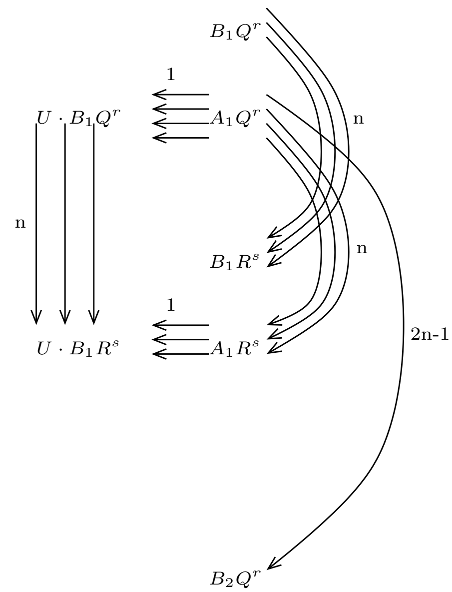

FIGURE 4. **Portion of the knot Floer complex of** $C_{n,2n-1}(W_0^+(T_{2,3}))$**.** We have illustrated generators in Alexander gradings $\geq \mathbf{A}(B_2 \otimes Q^r)$, and appearing with U multiplies with exponent ≤ 1. The convention here is that $r = 0,1,2,3$ and $s = 1,2,3$. The horizontal coordinate represents the number of U powers, and the vertical coordinate indicates the Alexander grading. We have also illustrated all vertical and horizontal differentials connecting these elements; more precisely, a collection of parallel arrows indicates a linear map connecting spans of generators, and the number of parallel arrows indicates the dimension of its image. The integers indicate the lengths of these arrows.

(X-4) $\partial_v x_1 = 0$
(X-5) $\partial_h y = x_1$
(X-6) $\partial_v y = x_2$.

We find these elements as follows. First we find some element y in the same bigrading as $A_1 \otimes Q$ with the property that $\partial_v y = x_2$, where x_2 is a non-zero element in the same bigrading as $B_2 \otimes Q$. The element y corresponds, under the conjugation symmetry of knot Floer homology, to the element of $\mathcal{CFK}^-$ represented by

$$\xi = A_n \otimes Q^0 + U^n A_{n-2} \otimes P^0$$

(in case $n > 2$; when $n = 2$, take $\xi = A_2$).

Since $\partial \xi = U^{2n-2} B_{n-2} \otimes P^0$, there is a symmetric differential $\partial_h y = U x_1$, for some non-zero element x_1 in the same bigrading as $B_1 \otimes Q^0$. In fact, the bordered computation shows that all non-zero elements in the bigrading of y have non-zero ∂_h in the bigrading of $U x_1$. Thus, the fact that $\partial_v x_1 = 0$ is a consequence of $\partial^2 = 0$. This completes the construction of x_1, x_2, and y satisfying Properties (X-1)-(X-6).

Claim 3. The element x_1 constructed above is a cycle in $\mathcal{CFK}^-(K_n')$, which represents a non-trivial homology class. The fact that x_1 is a cycle follows from the fact that $\partial_v(x_1) = 0$, so ∂x_1 contains terms with non-zero U power, and, according to the above grading computations, all such elements have Maslov grading < -2; but $\mathbf{M}(x_1) = 0$. Moreover, since x_1 has maximal Alexander grading among all generators, it follows that x_1 represents a homologically non-trivial class in $\widehat{\mathrm{CF}}(S^3) = \mathcal{CFK}^-/(U = 0)$ and hence also in $\mathcal{CFK}^-$, whose homology is $\mathbb{F}[U]$. It follows that x_1 represents a non-torsion homology class in $\mathrm{tHFK}(K_n')$.

Claim 4. The following equation holds in $\mathcal{CFK}^-(K'_n)$,

$$\partial y = Ux_1 + x_2. \tag{37}$$

To see this, observe that the definitions of ∂_v and ∂_h ensure that

$$\partial y = Ux_1 + x_2 + Uz,$$

where z is some element in Alexander filtration level is less than or equal to that of y. But the above grading computations show that such an element z has Maslov grading < 0, and so $\mathbf{M}(Uz) < -2$. Since $\mathbf{M}(y) = -1$, we conclude that $z = 0$, verifying Equation (37).

Claim 5. The elements x_1 and x_2 represent non-torsion homology classes in $\mathrm{tHFK}(K'_n)$ The statement for x_1 follows immediately from Claim 3, and the statement for x_2 follows from that, together with Claim 4.

Claim 6. For $t < \frac{1}{n-1}$, the elements x_1 and x_2 are the two non-torsion elements of $\mathrm{tHFK}(K'_n)$ with maximal gr_t. Elements with the same bigrading as $A_1 \otimes Q^r$ have $\mathrm{gr}_t(A_1 \otimes Q^r) > \mathrm{gr}_t(B_1 \otimes Q^r)$ for all t. However, the differential ∂_h is injective on the span of $A_1 \otimes Q^r$, which implies also that the cycles in $\mathrm{tCFK}(K'_n)$ cannot contain components among the $A_1 \otimes Q^r$. Similarly, the differential ∂_v is injective on the span of $B_1 \otimes Q^r$ so cycles in $\mathrm{tCFK}(K'_n)$ cannot contain components among the $B_1 \otimes Q^r$. Finally, if a cycle in $\mathrm{tCFK}(K'_n)$ contains a component among $B_1 \otimes R^s$, then that cycle is homologous to another one, obtained by adding $\partial(B_1 \otimes Q^s)$. It follows from Claim 1 now that x_1 and x_2 are two non-torsion elements with maximal gr_t for $t \in [0, \frac{1}{n-1}]$.

In view of Claim 6, the result follows from the fact that

$$\mathrm{gr}_t(x_1) = -(n^2 - n + 1)t \qquad \text{and} \qquad \mathrm{gr}_t(x_2) = -2 - (n^2 - 3n + 2).$$

□

In the proof of Theorem 1.20 we need to compare the above result with $\Upsilon_{T_{n,2n-1}}$.

Lemma 8.13.

$$\Upsilon_{T_{n,2n-1}}(t) = -(n-1)^2 \cdot t$$

for $t \leq \frac{2}{n}$.

That latter function can be computed explicitly from the Alexander polynomials, as in Theorem 1.15. Thus, we could obtain the theorem by playing around with coefficients of the Alexander polynomial; we prefer instead to obtain these bounds via bordered Floer homology, in the spirit of the previous computations.

Proof of Lemma 8.13. Recall [18, Theorem A.11] that the 2-framed unknot complement has type D module with three generators which we write as P, Q, and I, and coefficient maps

$$\begin{tikzcd} & P \arrow[dd, "\rho_{23}"] \\ I \arrow[ur, "\rho_{123}"] & \\ & Q \arrow[ul, "\rho_2"] \end{tikzcd} \tag{38}$$

By the pairing theorem [18, Theorem 11.19], the tensor product of this with the cabling type A module computes $\mathrm{CFK}^-(T_{n,2n-1})$. In the tensor product, we obtain a sequence starting with

$$B_i \otimes P \stackrel{U^i}{\leftarrow} A_i \otimes P \stackrel{z^{n-i}}{\dashrightarrow} B_i \otimes Q \stackrel{U^i}{\leftarrow} A_i \otimes Q \stackrel{z^{n-i-1}}{\dashrightarrow} B_{i+1} \otimes P,$$

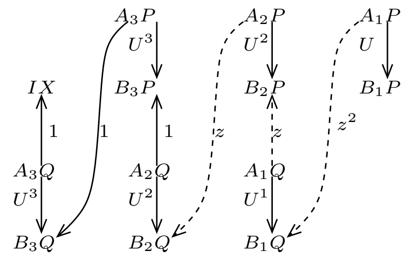

FIGURE 5. **Knot Floer complex of $T_{3,5}$.** The complex for $T_{3,5}$, as computed by the pairing theorem.

for $i = 1, \dots, n-2$, terminating at

$$B_{n-1} \otimes P \overset{U^{n-1}}{\leftarrow} A_{n-1} \otimes P \overset{z}{\dashrightarrow} B_{n-1} \otimes P.$$

Note that the remaining generators cancel in homology. See Figure 5 for an example.

The generators $B_i \otimes P$ and $B_i \otimes Q$ are the ones that can represent torsion homology classes, since they are the ones with even Maslov gradings. Moreover, it follows from the above computations that for all $i = 1, \dots, 2n-2$, if $t < \frac{2}{n}$

$$\mathrm{gr}_t(B_i \otimes P) - \mathrm{gr}_t(B_i \otimes Q) = 2i - nt > 0$$
$$\mathrm{gr}_t(B_i \otimes Q) - \mathrm{gr}_t(B_{i+1} \otimes Q) = 2i - (n-1)t > 0,$$

so $B_1 \otimes Q$ is a non-torsion class with maximal gr_t for $t < \frac{2}{n}$; and $M(B_1 \otimes Q) = 0$ and $A(B_1 \otimes Q) = (n-1)^2$. It follows that

$$\Upsilon_{T_{n,2n-1}}(t) = -(n-1)^2 \cdot t$$

for $t \leq \frac{2}{n}$. ☐

Putting the (partial) computations of $\Upsilon_{C_{n,2n-1}(W_0^+(T_{2,3}))}$ and of $\Upsilon_{T_{n,2n-1}}$ together, we get

Proof of Theorem 1.20. Observe that $K \mapsto \Delta\Upsilon'_K(t)$ is a concordance homomorphism. For $t \leq \frac{2}{2n-1}$, this homomorphism vanishes for $T_{n,2n-1}$ (by Lemma 8.13); thus,

$$\Delta\Upsilon'_{K_n}(t) = \begin{cases} 0 & \text{for } t < \frac{2}{2n-1} \\ 2n-1 & \text{for } t = \frac{2}{2n-1}. \end{cases}$$

We can now apply Lemma 6.4 to the homomorphisms $\{\frac{1}{2n-1}\Delta\Upsilon'_K(\frac{2}{2n-1})\}_{n=2}^{\infty}$ and the knots K_n. ☐

Remark 8.14. When $n = 1$, the knot K_n is simply the Whitehead double of the trefoil. Using Theorem 8.9 directly, we can see that the family of knots K_n for all $n \geq 1$ is linearly independent. But for this linear independence result, we use the homomorphism $\frac{1}{2}\Delta\Upsilon'_K(1)$, as well as the $\frac{1}{2n-1}\Delta\Upsilon'_K(\frac{2}{2n-1})$ for $n \geq 2$.

The above linear independence result can be stated in terms of the concordance genus.

Corollary 8.15. *Let $\{a_n\}_{n=2}^{\infty}$ be a sequence of integers with finitely many non-zero terms. Consider the knot $K = \#_{n=2}^{\infty} a_n K_n$. Let*

$$c = \sum a_n \tau(K_n) = \sum a_n \left(\frac{n^2 - 3n + 2}{2} \right);$$

and let $m = \max\{n \mid a_n \neq 0\}$. Then, the concordance genus of K is bounded below by

$$\max\{|c|, |c + a_m \cdot 2m + 1|\}.$$

Proof. This is a direct application of Theorem 1.13, combined with computations in the proof of Theorem 1.20. ☐

9. Comparison with Hom's homomorphisms

It is interesting to compare the concordance homomorphisms constructed here with those defined by Hom [11]. By a recent result of Hom [16] there are knots for which our invariant $\Upsilon_K(t) \equiv 0$, but for which her invariant ϵ (which she uses to construct concordance homomorphisms) is non-zero. We expect conversely that there are also knots K with $\Upsilon_K(t) \not\equiv 0$ but $\epsilon = 0$. In this section, we give a formal construction which shows that there is no algebraic obstruction to the existence of such knots.

Just like Υ_K, Hom's homomorphisms are constructed from invariants of (suitable) Maslov graded, Alexander filtered chain complexes over $\mathbb{F}[U]$. By construction, her homomorphisms vanish on a particular subset of such complexes. Let us recall this set.

Definition 9.1. *Let C be a Maslov graded, Alexander filtered chain complex which is free over $\mathbb{F}[U]$. Suppose moreover that $H_*(C) \cong \mathbb{F}[U]$, with generator in Maslov grading 0. Let $\mathcal{A}(C)$ be the subcomplex of C generated by all elements with Alexander filtration $A \leq 0$. Let $\mathcal{A}'(C)$ be subcomplex of $C \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]$ which is generated by $C \subset C \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]$ and all elements of $C \otimes_{\mathbb{F}[U]} \mathbb{F}[U, U^{-1}]$ with $A \leq 0$.*

*We say that C is **strongly trivial** if the map $\mathcal{A}(C) \to \mathcal{A}'(C)$ induces an isomorphism*

$$H_*(\mathcal{A}(C))/\mathrm{Tors} \to H_*(\mathcal{A}'(C))/\mathrm{Tors}.$$

(Here Tors *denotes the torsion part of an $\mathbb{F}[U]$-module.) In the case where the rank of $H_*(C)$ is equal to one, this is equivalent to the condition that $\delta(\mathcal{A}(C)) = \delta(C) = \delta(\mathcal{A}'(C))$, in the notation of Definition 2.11.*

*We say that C is **ϵ-trivial** if the map*

$$\mathcal{A}(C)/U \to \mathcal{A}'(C)/U$$

on $\mathcal{A}(C)/U = \mathcal{A}(C)/(U \cdot \mathcal{A}(C))$ given by the embedding induces a non-zero map in homology.

Note that the map $\mathcal{A}(C) \to \mathcal{A}'(C)$ naturally factors through C itself; similarly $\mathcal{A}(C)/U \to \mathcal{A}'(C)/U$ factors through C/U. In [15, Definition 3.1] Hom defines a complex C to have $\epsilon(C) = 0$ if both maps

$$H_*(\mathcal{A}(C)/U) \to H_*(C/U) \text{ and } H_*(C/U) \to H_*(\mathcal{A}'(C)/U)$$

are non-trivial. Since, $H_*(C/U) = \mathbb{F}$, this condition is equivalent to the condition that C is ϵ-trivial, in the above sense. The relevance of strong triviality is the following:

Proposition 9.2. *If C is strongly trivial, then $\Upsilon_C(t) \equiv 0$.*

Proof. There are inclusions $\mathcal{A}(C_{\mathcal{R}}) \subset E^t \subset \mathcal{A}'(C_{\mathcal{R}})$ for all t, hence by Lemma 4.6 we get that $\delta(\mathcal{A}(C)) \leq \delta(E^t) \leq \delta(\mathcal{A}'(C))$. (Note that $\delta(\mathcal{A}(C_{\mathcal{R}})) = \delta(\mathcal{A}(C))$ by Proposition 4.9.) Since C is strongly trivial, it follows that $\delta(\mathcal{A}(C)) = \delta(\mathcal{A}'(C))$, implying that $\delta(E^t)$ is constant. Since for $t = 0$ we have that $\delta(E^t) = 0$, the claim of the lemma follows. ☐

Proposition 9.3. *If C is strongly trivial, then it is also ϵ-trivial.*

Proof. Consider a generator of $H_*(\mathcal{A}(C))/\mathrm{Tors}$. This can be lifted to an element ξ of $H_*(\mathcal{A}(C))$ which is in the cokernel of U, i.e. which injects into $H_*(\mathcal{A}(C)/U)$. Call the image $\widehat{\xi}$. Moreover,

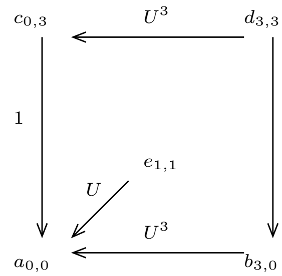

FIGURE 6. **An ϵ-trivial chain complex with non-trivial Υ.** Arrows represent terms in the differentials, and the labels represent the algebra element appearing in the corresponding term.

since its image in $H_*(\mathcal{A}'(C))$ induces a generator of $H_*(\mathcal{A}'(C))/\mathrm{Tors}$, it follows that its image also injects in $H_*(\mathcal{A}'(C)/U)$. By commutativity of the diagram

$$\begin{tikzcd}
\mathcal{A}(C) \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & \mathcal{A}'(C) \arrow[d] \\
\mathcal{A}(C)/U \arrow[r] & C/U \arrow[r] & \mathcal{A}'(C)/U,
\end{tikzcd}$$

we conclude that $\widehat{\xi}$ is mapped non-trivially into $H_*(\mathcal{A}'(C)/U)$, as desired. $\square$

The converse of the above proposition is not true. An ϵ-trivial complex with Υ_C not identically zero (hence C not strongly trivial) can be given as follows.

Consider the $\mathbb{Z}\oplus\mathbb{Z}$-filtered complex C^∞ over $\mathbb{F}[U,U^{-1}]$ with five generators $a_{0,0}$, $b_{3,0}$, $c_{0,3}$, $d_{3,3}$, and $e_{1,1}$, satisfying the grading conditions

$$\begin{aligned}
M(a_{0,0}) &= 1\\
M(b_{3,0}) &= -4\\
M(c_{0,3}) &= 2\\
M(d_{3,3}) &= -3\\
M(e_{1,1}) &= 0
\end{aligned}$$

and

$$\begin{aligned}
A(a_{0,0}) = A(d_{3,3}) = A(e_{1,1}) &= 0\\
A(c_{0,3}) = -A(b_{3,0}) &= 3;
\end{aligned}$$

and equipped with the differential

$$\begin{aligned}
\partial a_{0,0} &= 0\\
\partial b_{3,0} &= U^3\cdot a_{0,0}\\
\partial c_{0,3} &= a_{0,0}\\
\partial d_{3,3} &= b_{3,0} + U^3\cdot c_{0,3}\\
\partial e_{1,1} &= U\cdot a_{0,0},
\end{aligned}$$

pictured in Figure 6.

Proposition 9.4. *The above complex is ϵ-trivial, but*

$$\Upsilon_C(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq \frac{2}{3} \\ 2-3t & \text{for } \frac{2}{3} \leq t \leq 1 \\ -4+3t & \text{for } 1 \leq t \leq \frac{4}{3} \\ 0 & \text{for } \frac{4}{3} \leq t \leq 2. \end{cases}$$

Proof. It is easy to see that $H_*(\mathcal{A}(C)/U) \cong \mathbb{F}^3$, generated by the elements $a_{0,0}, e_{1,1}$ and $b_{3,0}$. A similar computation shows that $H_*(\mathcal{A}'(C)/U) \cong \mathbb{F}^3$, generated by $d_{3,3}$, $e_{1,1}$ and $c_{0,3} + U^{-3}b_{3,0}$. The map on homology induced by the map $\mathcal{A}(C)/U \to \mathcal{A}'(C)/U$ maps $e_{1,1}$ to $e_{1,1}$, hence it is non-zero, showing that C is ϵ-trivial.

Since

$$\begin{aligned} \partial_t a_{0,0} &= 0 \\ \partial_t b_{3,0} &= v^{3(2-t)} \cdot a_{0,0} \\ \partial c_{0,3} &= v^{3t} \cdot a_{0,0} \\ \partial d_{3,3} &= v^{3(2-t)} \cdot b_{3,0} + v^{3t} \cdot c_{0,3} \\ \partial e_{1,1} &= v^2 \cdot a_{0,0}, \end{aligned}$$

we can easily see that for $t \leq \frac{2}{3}$ the elements

$$z_1 = b_{3,0} + v^{4-3t} \cdot e_{1,1} \quad \text{and} \quad z_2 = v^{2-3t} \cdot c_{0,3} + e_{1,1}$$

generate the torsion-free quotient, while for $t \in [\frac{2}{3}, 1]$ the elements

$$z_1 = b_{3,0} + v^{4-3t} \cdot e_{1,1} \quad \text{and} \quad z_2' = c_{0,3} + v^{3t-2} \cdot e_{1,1}$$

play the same role. Since $\mathrm{gr}_t(z_1) = -4 + 3t$, $\mathrm{gr}_t(z_2) = 0$, and $\mathrm{gr}_t(z_2') = 2 - 3t$, we get the desired shape of $\Upsilon_C(t)$ on $[0, 1]$. A similar computation (or the symmetry $\Upsilon_C(t) = \Upsilon_C(2-t)$) then computes Υ_C on $[0, 2]$, concluding the proof of the proposition. $\square$

10. The case of links

Knot Floer homology can be generalized to links in several ways; see for instance [34]. There are analogous generalizations of the t-modified theory to links. We describe here one such generalization, which will be useful in [24].

An ℓ-component oriented link $L = (L_1, \dots, L_\ell)$ can be represented by a Heegaard diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathbf{w}, \mathbf{z})$, where:

- Σ is a surface of genus g,
- $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are $g + \ell - 1$-tuples of pairwise disjoint, simple closed curves,
- and the pair $(\mathbf{w}, \mathbf{z}) = \{(w_i, z_i)\}_{i=1}^{\ell}$ is an ℓ-tuple of pairs of basepoints;

see [34, Section 3]. The diagram equips each component of L with an orientation; we assume that this orientation matches with the given orientation of L.

The generating set of the free $\mathcal{R}$-module $\mathrm{tCFL}(\mathcal{H})$ is given by the intersection points $\mathfrak{S} = \mathbb{T}_\alpha \cap \mathbb{T}_\beta \in \mathrm{Sym}^{g+\ell-1}(\Sigma)$. For $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, let

$$n_{\mathbf{w}}(\phi) = \sum_{i=1}^{\ell} n_{w_i}(\phi) \qquad \text{and} \qquad n_{\mathbf{z}}(\phi) = \sum_{i=1}^{\ell} n_{z_i}(\phi).$$

The Maslov and Alexander functions are once again characterized up to an overall additive shift by the equations

$$M(\mathbf{x}) - M(\mathbf{y}) = \mu(\phi) - 2n_{\mathbf{w}}(\phi), \tag{39}$$

$$A(\mathbf{x}) - A(\mathbf{y}) = n_{\mathbf{z}}(\phi) - n_{\mathbf{w}}(\phi), \tag{40}$$

for any $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$. Before pinning down the additive indeterminacy on these functions, we consider the differential on $\mathrm{tCFL}(\mathcal{H})$:

$$\partial_t \mathbf{x} = \sum_{\mathbf{y}\in\mathfrak{S}} \sum_{\{\phi\in\pi_2(\mathbf{x},\mathbf{y}) \big| \mu(\phi)=1\}} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}}\right) v^{tn_{\mathbf{z}}(\phi)+(2-t)n_{\mathbf{w}}(\phi)} \mathbf{y}. \tag{41}$$

Lemma 10.1. *The homology of the $t = 0$ specialization of the above complex is a free $\mathcal{R}$-module of rank $2^{\ell-1}$. In fact, up to an overall shift in gradings, there is a graded isomorphism*

$$H_*(\mathrm{tCFL}|_{t=0}(\mathcal{H})) \cong (\mathcal{R}_{-\frac{1}{2}} \oplus \mathcal{R}_{\frac{1}{2}})^{\ell-1}. \tag{42}$$

The same holds when $t = 2$:

$$H_*(\mathrm{tCFL}|_{t=2}(\mathcal{H})) \cong (\mathcal{R}_{-\frac{1}{2}} \oplus \mathcal{R}_{\frac{1}{2}})^{\ell-1}. \tag{43}$$

Proof. The $t = 0$ specialization is independent of the placement of $\mathbf{z}$. (This specialization is equipped with the Maslov grading.) Consider the chain complex over $\mathbb{F}[U_1, \dots, U_\ell]$ with the same generators as before, but with differential specified by

$$\partial \mathbf{x} = \sum_{\mathbf{y}\in\mathfrak{S}} \sum_{\{\phi\in\pi_2(\mathbf{x},\mathbf{y}) \big| \mu(\phi)=1\}} \# \left(\frac{\mathcal{M}(\phi)}{\mathbb{R}}\right) U_1^{n_{w_1}(\phi)} \cdots U_\ell^{n_{w_\ell}(\phi)} \mathbf{y}.$$

According to [34, Theorem 4.4], this chain complex computes $\mathrm{HF}^-(S^3) \cong \mathbb{F}[U]$, where all the U_i act as translations by U. Thus, if we set them equal to one another, the resulting complex is $\mathbb{F}[U] \otimes V^{\ell-1}$. The $t = 0$ complex is gotten by changing the base ring to $\mathcal{R}$ with variable v (and with the understanding of $U = v^2$), equipped with the Maslov grading.

The $t = 2$ specialization is independent of the placement of $\mathbf{w}$ (even in the defintion of $\mathrm{gr}_2 = M - 2A$). □

Definition 10.2. *Let L be an oriented link. Eliminate the additive indeterminacy in M by the requirement that Equation (42) holds without shifting the grading. Next, eliminate the additive indeterminacy in A by the requirement that Equation (43) holds without shifting the grading. Using these normalizations, we define the grading gr_t on the generator $\mathbf{x}$ of $\mathrm{tCFL}(\mathcal{H})$ by the usual formula*

$$\mathrm{gr}_t(\mathbf{x}) = M(\mathbf{x}) - tA(\mathbf{x}),$$

*and extend it to the $\mathcal{R}$-module by $\mathrm{gr}_t(v^\alpha \mathbf{x}) = \mathrm{gr}_t(\mathbf{x}) - \alpha$. The homology $\mathrm{tHFL}(\mathcal{H})$ of the resulting graded chain complex is a graded $\mathcal{R}$-moduli, called the t-**modified link homology** of L.*

We have the following analogue of Theorem 1.1:

Theorem 10.3. *The t-modified link homology $\mathrm{tHFL}(\mathcal{H})$ of the Heegaard diagram $\mathcal{H}$ is an invariant of the underlying oriented link L, and is denoted by $\mathrm{tHFL}(L)$.*

Proof. In [34], the link complex $\mathrm{CFL}^-(\mathcal{H})$ is a $\mathbb{Z}^\ell$-filtered chain complex which is also equipped with a Maslov grading. The Alexander multi-grading is specified by the vector

$$\mathbf{A} = (A_1, \dots, A_\ell),$$

and the underlying algebra is $\mathbb{F}[U_1, \dots, U_\ell]$.

We can specialize the link complex by setting $U_1 = \cdots = U_\ell$ to get a variant which is defined over $\mathbb{F}[U]$, endowed with the $\mathbb{Z}$-filtration $A = \sum_{i=1}^{\ell} A_i$ (specified again up to an overall shift). We call the resulting complex the *algebraically collapsed link complex* $\mathrm{cCFL}^-(\mathcal{H})$. It follows that $\mathrm{tCFL}(\mathcal{H})$ is simply the t-modification (in the sense of Section 4) of the algebraically collapsed link complex.

It is easy to see that $\mathbb{Z}^\ell$-filtered homotopy equivalences between $\mathrm{CFL}^-(\mathcal{H})$'s for different Heegaard diagrams representing L induce $\mathbb{Z}$-filtered homotopy equivalences of the corresponding collapsed complex. Thus, since the filtered homotopy type of $\mathrm{CFL}^-(\mathcal{H})$ is a link invariant [34, Theorem 4.7], functoriality of the t-modification (Proposition 4.4) implies the result. $\square$

The definition of the knot invariant $\Upsilon_K(t)$ extends to links as follows.

Definition 10.4. *For $t \in [0,2]$ choose a homogeneous basis $\{e_i(t)\}_{i=1}^n$ for the free $\mathcal{R}$-module $\mathrm{tHFL}(L)/\mathrm{Tors}$. The* ***$\Upsilon$-set of the oriented link*** *L* ***at*** *t is the set $\{\mathrm{gr}_t(e_i(t))\}_{i=1}^n$ (a set with possible repetitions).*

Theorem 10.5. *The Υ-set of L at any $t \in [0,2]$ is a set with $2^{\ell-1}$ elements (counted with repetitions). It is an invariant of the oriented link L.*

Proof. The proof consists of two parts. First, we must show that $\mathrm{tHFL}(L)/\mathrm{Tors}$ is a free module of rank $2^{\ell-1}$. Second, we must show that the set is a link invariant.

To see that $\mathrm{tHFL}(L)/\mathrm{Tors}$ is a free module of rank $2^{\ell-1}$, we use the fact that $\mathrm{HFK}^\infty(L) \cong \mathbb{F}[U,U^{-1}] \otimes (\mathbb{F}\oplus\mathbb{F})^{\otimes(\ell-1)}$. This follows from an application of [34, Theorem 4.7] (exactly as in the proof of Lemma 10.1).

Invariance follows immediately from Theorem 10.3. $\square$

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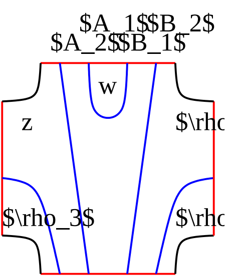

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