DOI: 10.1515/auom-2016-0043
An. Şt. Univ. Ovidius Constanţa
VERSITA
Vol. 24(2),2016, 283–299

Extremal problem on $(2n, 2m-1)$-system points on the rays.

Targonskii A.

Abstract

In this work derivation of accurate estimate the production inner radius non-overlapping domains and open set. The problems arise such type in the first time in work [1]. It is late result this work generalize and strengthen in works [2 – 13]. In works [7, 8, 10] introduce the general systems points, the name n-radial systems points. In this work a success the draw generalize some results the work [7].

Let $\mathbb{N}$, $\mathbb{R}$ – the sets natural and real numbers conformity, $\mathbb{C}$ – the plain complex numbers, $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ – the Riemannian sphere, $\mathbb{R}_+ = (0, \infty)$.

For fix number $n \in \mathbb{N}$ system points

$$A_{2n,2m-1} = \{a_{k,p} \in \mathbb{C} : k = \overline{1, 2n},\ p = \overline{1, 2m-1}\},$$

we will called on the $(2n, 2m-1)$-system points on the rays, if at all $k = \overline{1, 2n}$, $p = \overline{1, 2m-1}$ the relations are executed:

$$\begin{array}{l} 0 < |a_{k,1}| < \ldots < |a_{k,2m-1}| < \infty; \\ \arg a_{k,1} = \arg a_{k,2} = \ldots = \arg a_{k,2m-1} =: \theta_k; \\ 0 = \theta_1 < \theta_2 < \ldots < \theta_n < \theta_{n+1} := 2\pi. \end{array} \tag{1}$$

For such systems of points we will consider the following sizes:

$$\alpha_k = \frac{1}{\pi}\left[\theta_{k+1} - \theta_k\right],\ k = \overline{1, 2n},\ \alpha_{n+1} := \alpha_1,\ \alpha_0 := \alpha_n, \sum_{k=1}^{2n} \alpha_k = 2.$$

Key Words: inner radius of domain, quadratic differential, piecewise-separating transformation, the Green function, radial systems of points, logarithmic capacity, variational formula.
2010 Mathematics Subject Classification: Primary 30C70, 30C75.
Received: 25.01.2014
Accepted: 20.02.2015

Let's consider system of angular domains:

$$P_k = \{w \in \mathbb{C} : \theta_k < \arg w < \theta_{k+1}\}, \quad k = \overline{1, 2n}.$$

Let's consider the following "operating" functionalities for arbitrary $(2n, 2m-1)$-system points on the rays

$$M\left(A^{(1)}_{2n,2m-1}\right) = \prod_{k=1}^{n}\prod_{p=1}^{m}\left[\chi\left(\left|a_{2k-1,2p-1}\right|^{\frac{1}{\alpha_{2k-1}}}\right)\cdot\chi\left(\left|a_{2k-1,2p-1}\right|^{\frac{1}{\alpha_{2k-2}}}\right)\right]^{\frac{1}{2}}\cdot|a_{2k-1,2p-1}|,$$

$$M\left(A^{(2)}_{2n,2m-1}\right) = \prod_{k=1}^{n}\prod_{p=1}^{m-1}\left[\chi\left(\left|a_{2k-1,2p}\right|^{\frac{1}{\alpha_{2k-1}}}\right)\cdot\chi\left(\left|a_{2k-1,2p}\right|^{\frac{1}{\alpha_{2k-2}}}\right)\right]^{\frac{1}{2}}\cdot|a_{2k-1,2p}|,$$

$$M\left(A^{(3)}_{2n,2m-1}\right) = \prod_{k=1}^{n}\prod_{p=1}^{m-1}\left[\chi\left(\left|a_{2k,2p}\right|^{\frac{1}{\alpha_{2k}}}\right)\cdot\chi\left(\left|a_{2k,2p}\right|^{\frac{1}{\alpha_{2k-1}}}\right)\right]^{\frac{1}{2}}\cdot|a_{2k,2p}|,$$

$$M\left(A^{(4)}_{2n,2m-1}\right) = \prod_{k=1}^{n}\prod_{p=1}^{m}\left[\chi\left(\left|a_{2k,2p-1}\right|^{\frac{1}{\alpha_{2k}}}\right)\cdot\chi\left(\left|a_{2k,2p-1}\right|^{\frac{1}{\alpha_{2k-1}}}\right)\right]^{\frac{1}{2}}\cdot|a_{2k,2p-1}|,$$

where $\chi(t) = \frac{1}{2}(t + \frac{1}{t})$, $t \in \mathbb{R}_+$.

Let $\{B_0, B_{k,p}\}$, $\{B_{k,p}, B_\infty\}$ — arbitrary non-overlapping domains such that

$$a_{k,p} \in B_{k,p}, \qquad B_{k,p} \subset \overline{\mathbb{C}}, \quad k = \overline{1, 2n},\ p = \overline{1, 2m-1}.$$

Let D, $D \subset \overline{\mathbb{C}}$ – arbitrary open set and $w = a \in D$, then $D(a)$ the define connected component D, the contain point a. For arbitrary $(2n, 2m-1)$-system points on the rays $A_{2n,2m-1} = \{a_{k,p} \in \mathbb{C} : k = \overline{1, 2n},\ p = \overline{1, 2m-1}\}$ and open set D, $A_{2n,2m-1} \subset D$ the define $D_k(a_{s,p})$ connected component set $D(a_{s,p}) \bigcap \overline{P_k}$, the contain point $a_{s,p}$, $k = \overline{1, 2n}$, $s = k, k+1$, $p = \overline{1, 2m-1}$, $a_{n+1,p} := a_{1,p}$.

The open set D, $A_{2n,2m-1} \subset D$ satisfied condition meets the condition of unapplied in relation to the system of points $(2n, 2m-1)$-system points on the rays $A_{2n,2m-1}$ if a condition is executed

$$D_k(a_{k,s}) \bigcap D_k(a_{k+1,p}) = \varnothing, \tag{2}$$

$k = \overline{1, 2n}$, $p, s = \overline{1, 2m-1}$ on all corners $\overline{P_k}$.

The define $r(B; a)$ inner radius domain $B \subset \overline{\mathbb{C}}$ with respect to a point $a \in B$ (look [4 – 7, 14]).

Subject of studying of our work are the following problems.

Problem 1. Let $n, m \in \mathbb{N}$, $n \geq 2$, $m \geq 2$, $\alpha \in \mathbb{R}_+$. Maximum functional be found

$$J = \prod_{k=1}^{n}\prod_{p=1}^{m} r^{\alpha}\left(B_{2k-1,2p-1}, a_{2k-1,2p-1}\right)\cdot\prod_{k=1}^{n}\prod_{p=1}^{m-1} r\left(B_{2k-1,2p}, a_{2k-1,2p}\right)\times$$

$$\times \prod_{k=1}^{n}\prod_{p=1}^{m-1} r^{\alpha}\left(B_{2k,2p}, a_{2k,2p}\right) \cdot \prod_{k=1}^{n}\prod_{p=1}^{m} r\left(B_{2k,2p-1}, a_{2k,2p-1}\right),$$

where $A_{2n,2m-1}$ – arbitrary $(2n, 2m-1)$-system points on the rays, satisfied condition (1), $\{B_{k,p}\}$ – arbitrary set non-overlapping domains, $a_{k,p} \in B_{k,p} \subset \overline{\mathbb{C}}$, and all extremal the describe ($k = \overline{1, 2n}$, $p = \overline{1, 2m-1}$).

Problem 2. Let $n, m \in \mathbb{N}$, $n \geq 2$, $m \geq 2$, $\alpha \in \mathbb{R}_{+}$. Maximum functional be found

$$I = \prod_{k=1}^{n}\prod_{p=1}^{m} r^{\alpha}\left(D, a_{2k-1,2p-1}\right) \cdot \prod_{k=1}^{n}\prod_{p=1}^{m-1} r\left(D, a_{2k-1,2p}\right) \times$$

$$\times \prod_{k=1}^{n}\prod_{p=1}^{m-1} r^{\alpha}\left(D, a_{2k,2p}\right) \cdot \prod_{k=1}^{n}\prod_{p=1}^{m} r\left(D, a_{2k,2p-1}\right),$$

where $A_{2n,2m-1}$ – arbitrary $(2n, 2m-1)$-system points on the rays, satisfied condition (1), D – arbitrary open set, the satisfied condition (2), $a_{k,p} \in D \subset \overline{\mathbb{C}}$, and all extremal the describe ($k = \overline{1, 2n}$, $p = \overline{1, 2m-1}$).

Theorem 1. *Let $n, m \in \mathbb{N}$, $n \geq 2$, $\alpha \in \mathbb{R}_{+}$. Then for all $(2n, 2m-1)$-system points on the rays $A_{2n,2m-1}$, and arbitrary set non-overlapping domains $\{B_{k,p}\}$, $a_{k,p} \in B_{k,p} \subset \overline{\mathbb{C}}$, $k = \overline{1, 2n}$, $p = \overline{1, 2m-1}$ be satisfied inequality*

$$\prod_{k=1}^{n}\prod_{p=1}^{m} r^{\alpha}\left(B_{2k-1,2p-1}, a_{2k-1,2p-1}\right) \cdot \prod_{k=1}^{n}\prod_{p=1}^{m} r\left(B_{2k-1,2p}, a_{2k-1,2p}\right) \times$$

$$\times \prod_{k=1}^{n}\prod_{p=1}^{m-1} r^{\alpha}\left(B_{2k,2p}, a_{2k,2p}\right) \cdot \prod_{k=1}^{n}\prod_{p=1}^{m} r\left(B_{2k,2p-1}, a_{2k,2p-1}\right) \leq$$

$$\leq \left(\frac{4}{(2m-1)n}\right)^{n(2m-1)(\alpha+1)} \cdot \left(M\left(A^{(1)}_{2n,2m-1}\right) \cdot M\left(A^{(3)}_{2n,2m-1}\right)\right)^{\alpha} \times$$

$$\times M\left(A^{(2)}_{2n,2m-1}\right) \cdot M\left(A^{(4)}_{2n,2m-1}\right) \cdot \left(\frac{\alpha^{\alpha}}{|\sqrt{\alpha}-1|^{|\sqrt{\alpha}-1|^2}|\sqrt{\alpha}+1|^{|\sqrt{\alpha}+1|^2}}\right)^{n\frac{2m-1}{2}}.$$

The equality obtain in this inequality, when points $a_{k,p}$ and domains $B_{k,p}$ are, conformity, the poles and the circular domains of the quadratic differential

$$Q(w)dw^2 = w^{2n-2}\left(1+w^{2n}\right)^{2m-3} \times$$

$$\times \frac{i(\alpha-1)\left((w^n+i)^{4m-2} - (w^n-i)^{4m-2}\right) - 2(1+\alpha)\left(w^{2n}+1\right)^{2m-1}}{\left((w^n+i)^{4m-2} + (w^n-i)^{4m-2}\right)^2} dw^2. \tag{3}$$

.

Theorem 2. *Let $n, m \in \mathbb{N}$, $n \geq 2$, $\alpha \in \mathbb{R}_+$. Then for all $(2n, 2m-1)$-system points on the rays, arbitrary open set D, the satisfied condition (2), $a_{k,p} \in D \subset \overline{\mathbb{C}}$, $k = \overline{1, 2n}$, $p = \overline{1, 2m-1}$ be satisfied inequality*

$$\prod_{k=1}^{n}\prod_{p=1}^{m} r^{\alpha}\left(D, a_{2k-1,2p-1}\right) \cdot \prod_{k=1}^{n}\prod_{p=1}^{m-1} r\left(D, a_{2k-1,2p}\right) \cdot \prod_{k=1}^{n}\prod_{p=1}^{m-1} r^{\alpha}\left(D, a_{2k,2p}\right) \times$$

$$\times \prod_{k=1}^{n}\prod_{p=1}^{m} r\left(D, a_{2k,2p-1}\right) \leq \left(\frac{4}{(2m-1)n}\right)^{n(2m-1)(\alpha+1)} \times$$

$$\times \left(M\left(A^{(1)}_{2n,2m-1}\right) \cdot M\left(A^{(3)}_{2n,2m-1}\right)\right)^{\alpha} \cdot M\left(A^{(2)}_{2n,2m-1}\right) \cdot M\left(A^{(4)}_{2n,2m-1}\right) \times$$

$$\times \left(\frac{\alpha^{\alpha}}{|\sqrt{\alpha}-1|^{|\sqrt{\alpha}-1|^2}|\sqrt{\alpha}+1|^{|\sqrt{\alpha}+1|^2}}\right)^{n\frac{2m-1}{2}}.$$

The equality obtain in this inequality, when

$$D = \bigcup_{k,p} B_{k,p},$$

and points $a_{k,p}$ and domains $B_{k,p}$ are, conformity, the poles and the circular domains of the quadratic differential (3).

Proof theorem 1. The theorem of the proof leans on a method of the piece-dividing transformation developed by Dubinin (look [4 – 7]).

Function

$$z_k(w) = -i\left(e^{-i\theta_k} w\right)^{\frac{1}{\alpha_k}} \tag{4}$$

realizes univalent and conformal transformations of domain P_k to the right half-plane $\mathrm{Re}z > 0$, for all $k = \overline{1, 2n}$.

Then function

$$\zeta_k(w) := \frac{1 - z_k(w)}{1 + z_k(w)} \tag{5}$$

maps univalent and conformal domain P_k on the unit circle $U = \{z : |z| \leq 1\}$, $k = \overline{1, 2n}$.

The define $\omega^{(1)}_{k,p} := \zeta_k\left(a_{k,p}\right)$, $\omega^{(2)}_{k-1,p} := \zeta_{k-1}\left(a_{k,p}\right)$, $a_{n+1,p} := a_{1,p}$, $\omega^{(2)}_{0,p} := \omega^{(2)}_{n,p}$, $\zeta_0 := \zeta_n$ $(k = \overline{1, 2n}$, $p = \overline{1, 2m-1})$.

Family of functions $\{\zeta_k(w)\}_{k=1}^{2n}$, set by equality (5), it is possible for by piece-dividing transformation (look [4 – 7]) domains $\left\{B_{k,p} : k = \overline{1, 2n},\right.$ $\left.p = \overline{1, 2m-1}\right\}$ in relation to the system of corners $\{P_k\}_{k=1}^{2n}$. For any domain $\Delta \in \mathbb{C}$ the define $(\Delta)^* := \left\{w \in \overline{\mathbb{C}} : \frac{1}{w} \in \Delta\right\}$. Let $\Omega^{(1)}_{k,p}$ the define connected component $\zeta_k\left(B_{k,p}\bigcap\overline{P}_k\right)\bigcup\left(\zeta_k\left(B_{k,p}\bigcap\overline{P}_k\right)\right)^*$, containing a point $\omega_{k,p}^{(1)}$, $\Omega_{k-1,p}^{(2)}$ – the define connected component $\zeta_{k-1}\left(B_{k,p}\bigcap\overline{P}_{k-1}\right)\bigcup\left(\zeta_{k-1}\left(B_{k,p}\bigcap\overline{P}_{k-1}\right)\right)^*$, containing a point $\omega_{k-1,p}^{(2)}$, $k=\overline{1,2n}$, $p=\overline{1,2m-1}$, $\overline{P}_0:=\overline{P}_n$, $\Omega_{0,p}^{(2)}:=\Omega_{n,p}^{(2)}$. It is clear, that, $\Omega_{k,p}^{(s)}$ generally speaking, domains are multiconnected domains, $k=\overline{1,2n}$, $p=\overline{1,2m-1}$, $s=1,2$. Pair of domains $\Omega_{k-1,p}^{(2)}$ and $\Omega_{k,p}^{(1)}$ grows out of piece-dividing transformation domains $B_{k,p}$ concerning families $\{P_{k-1},P_k\}$, $\{\zeta_{k-1},\zeta_k\}$ in point $a_{k,p}$, $k=\overline{1,2n}$, $p=\overline{1,2m-1}$.

From a formula (5) we receive the following asymptotic expressions

$$\left|\zeta_k(w)-\zeta_k(a_{k,p})\right|\sim\left[\alpha_k\cdot\chi\left(\left|a_{k,p}\right|^{\frac{1}{\alpha_k}}\right)|a_{k,p}|\right]^{-1}\cdot|w-a_{k,p}|,\quad w\to a_{k,p},\quad w\in\overline{P}_k.$$

$$\left|\zeta_{k-1}(w)-\zeta_{k-1}(a_{k,p})\right|\sim\left[\alpha_{k-1}\cdot\chi\left(\left|a_{k,p}\right|^{\frac{1}{\alpha_{k-1}}}\right)|a_{k,p}|\right]^{-1}\cdot|w-a_{k,p}|,$$

$$w\to a_{k,p},\quad w\in\overline{P}_{k-1},\quad k=\overline{1,2n},p=\overline{1,2m-1}. \tag{6}$$

From the theorem 1.9 [6] (look also [4, 5]) and formulas (6) we receive inequalities

$$r\left(B_{k,p},a_{k,p}\right)\leqslant\left\{r\left(\Omega_{k,p}^{(1)},\omega_{k,p}^{(1)}\right)\cdot r\left(\Omega_{k-1,p}^{(2)},\omega_{k-1,p}^{(2)}\right)\cdot\left[\alpha_k\cdot\chi\left(\left|a_{k,p}\right|^{\frac{1}{\alpha_k}}\right)|a_{k,p}|\right]\times\right.$$
$$\left.\times\left[\alpha_{k-1}\cdot\chi\left(\left|a_{k,p}\right|^{\frac{1}{\alpha_{k-1}}}\right)|a_{k,p}|\right]\right\}^{\frac{1}{2}},\quad k=\overline{1,2n},p=\overline{1,2m-1}. \tag{7}$$

Using formulas (7), it is received the following ratio:

$$\prod_{k=1}^{n}\prod_{p=1}^{m}r^{\alpha}\left(B_{2k-1,2p-1},a_{2k-1,2p-1}\right)\cdot\prod_{k=1}^{n}\prod_{p=1}^{m-1}r\left(B_{2k-1,2p},a_{2k-1,2p}\right)\times$$
$$\times\prod_{k=1}^{n}\prod_{p=1}^{m-1}r^{\alpha}\left(B_{2k,2p},a_{2k,2p}\right)\cdot\prod_{k=1}^{n}\prod_{p=1}^{m}r\left(B_{2k,2p-1},a_{2k,2p-1}\right)\leq$$
$$\leq\prod_{k=1}^{n}\prod_{p=1}^{m}\left[\alpha_{2k-1}\cdot\alpha_{2k-2}\cdot\left(\chi\left(\left|a_{2k-1,2p-1}\right|^{\frac{1}{\alpha_{2k-1}}}\right)\right)\times\right.$$
$$\left.\times\left(\chi\left(\left|a_{2k-1,2p-1}\right|^{\frac{1}{\alpha_{2k-2}}}\right)\right)\cdot\left|a_{2k-1,2p-1}\right|^2\right]^{\frac{\alpha}{2}}\times$$
$$\times\prod_{k=1}^{n}\prod_{p=1}^{m-1}\left[\alpha_{2k-1}\cdot\alpha_{2k-2}\cdot\left(\chi\left(\left|a_{2k-1,2p}\right|^{\frac{1}{\alpha_{2k-1}}}\right)\right)\times\right.$$

$$\times\left(\chi\left(\left|a_{2k-1,2p}\right|^{\frac{1}{\alpha_{2k-2}}}\right)\right)\cdot\left|a_{2k-1,2p}\right|^{2}\Bigg]^{\frac{1}{2}}\times$$

$$\times\prod_{k=1}^{n}\prod_{p=1}^{m-1}\left[\alpha_{2k}\cdot\alpha_{2k-1}\cdot\chi\left(\left|a_{2k,2p}\right|^{\frac{1}{\alpha_{2k}}}\right)\cdot\chi\left(\left|a_{2k,2p}\right|^{\frac{1}{\alpha_{2k-1}}}\right)\cdot\left|a_{2k,2p}\right|^{2}\right]^{\frac{\alpha}{2}}\times$$

$$\times\prod_{k=1}^{n}\prod_{p=1}^{m}\left[\alpha_{2k}\cdot\alpha_{2k-1}\cdot\chi\left(\left|a_{2k,2p-1}\right|^{\frac{1}{\alpha_{2k}}}\right)\cdot\chi\left(\left|a_{2k,2p-1}\right|^{\frac{1}{\alpha_{2k-1}}}\right)\cdot\left|a_{2k,2p-1}\right|^{2}\right]^{\frac{1}{2}}\times$$

$$\times\prod_{k=1}^{n}\prod_{p=1}^{m}\left[r\left(\Omega_{2k-1,2p-1}^{(1)},\omega_{2k-1,2p-1}^{(1)}\right)\cdot r\left(\Omega_{2k-2,2p-1}^{(2)},\omega_{2k-2,2p-1}^{(2)}\right)\right]^{\frac{\alpha}{2}}\times$$

$$\times\prod_{k=1}^{n}\prod_{p=1}^{m-1}\left[r\left(\Omega_{2k-1,2p}^{(1)},\omega_{2k-1,2p}^{(1)}\right)\cdot r\left(\Omega_{2k-2,2p}^{(2)},\omega_{2k-2,2p}^{(2)}\right)\right]^{\frac{1}{2}}\times$$

$$\times\prod_{k=1}^{n}\prod_{p=1}^{m-1}\left[r\left(\Omega_{2k,2p}^{(1)},\omega_{2k,2p}^{(1)}\right)\cdot r\left(\Omega_{2k-1,2p}^{(2)},\omega_{2k-1,2p}^{(2)}\right)\right]^{\frac{\alpha}{2}}\times$$

$$\times\prod_{k=1}^{n}\prod_{p=1}^{m}\left[r\left(\Omega_{2k,2p-1}^{(1)},\omega_{2k,2p-1}^{(1)}\right)\cdot r\left(\Omega_{2k-1,2p-1}^{(2)},\omega_{2k-1,2p-1}^{(2)}\right)\right]^{\frac{1}{2}}. \qquad (8)$$

Let's note that

$$\prod_{k=1}^{n}\prod_{p=1}^{m}\left[r\left(\Omega_{2k-1,2p-1}^{(1)},\omega_{2k-1,2p-1}^{(1)}\right)\cdot r\left(\Omega_{2k-2,2p-1}^{(2)},\omega_{2k-2,2p-1}^{(2)}\right)\right]^{\frac{\alpha}{2}}\times$$

$$\times\prod_{k=1}^{n}\prod_{p=1}^{m-1}\left[r\left(\Omega_{2k-1,2p}^{(1)},\omega_{2k-1,2p}^{(1)}\right)\cdot r\left(\Omega_{2k-2,2p}^{(2)},\omega_{2k-2,2p}^{(2)}\right)\right]^{\frac{1}{2}}\times$$

$$\times\prod_{k=1}^{n}\prod_{p=1}^{m-1}\left[r\left(\Omega_{2k,2p}^{(1)},\omega_{2k,2p}^{(1)}\right)\cdot r\left(\Omega_{2k-1,2p}^{(2)},\omega_{2k-1,2p}^{(2)}\right)\right]^{\frac{\alpha}{2}}\times$$

$$\times\prod_{k=1}^{n}\prod_{p=1}^{m}\left[r\left(\Omega_{2k,2p-1}^{(1)},\omega_{2k,2p-1}^{(1)}\right)\cdot r\left(\Omega_{2k-1,2p-1}^{(2)},\omega_{2k-1,2p-1}^{(2)}\right)\right]^{\frac{1}{2}}=$$

$$=\prod_{k=1}^{n}\left\{\prod_{p=1}^{m}r^{\alpha}\left(\Omega_{2k-1,2p-1}^{(1)},\omega_{2k-1,2p-1}^{(1)}\right)\cdot\prod_{p=1}^{m-1}r\left(\Omega_{2k-1,2p}^{(1)},\omega_{2k-1,2p}^{(1)}\right)\times\right.$$

$$\times\prod_{p=1}^{m-1}r^{\alpha}\left(\Omega_{2k-1,2p}^{(2)},\omega_{2k-1,2p}^{(2)}\right)\cdot\prod_{p=1}^{m}r\left(\Omega_{2k-1,2p-1}^{(2)},\omega_{2k-1,2p-1}^{(2)}\right)\times$$

$$\times \prod_{p=1}^{m} r^{\alpha}\left(\Omega_{2k,2p-1}^{(2)}, \omega_{2k,2p-1}^{(2)}\right) \cdot \prod_{p=1}^{m-1} r\left(\Omega_{2k,2p}^{(2)}, \omega_{2k,2p}^{(2)}\right) \times$$

$$\left. \times \prod_{p=1}^{m-1} r^{\alpha}\left(\Omega_{2k,2p}^{(1)}, \omega_{2k,2p}^{(1)}\right) \cdot \prod_{p=1}^{m} r\left(\Omega_{2k,2p-1}^{(1)}, \omega_{2k,2p-1}^{(1)}\right) \right\}^{\frac{1}{2}}, \qquad (9)$$

$$\prod_{k=1}^{n}\prod_{p=1}^{m}\left[\alpha_{2k-1}\cdot\alpha_{2k-2}\right]^{\frac{\alpha}{2}} \cdot \prod_{k=1}^{n}\prod_{p=1}^{m-1}\left[\alpha_{2k-1}\cdot\alpha_{2k-2}\right]^{\frac{1}{2}} \cdot \prod_{k=1}^{n}\prod_{p=1}^{m-1}\left[\alpha_{2k}\cdot\alpha_{2k-1}\right]^{\frac{\alpha}{2}} \times$$

$$\times \prod_{k=1}^{n}\prod_{p=1}^{m}\left[\alpha_{2k}\cdot\alpha_{2k-1}\right]^{\frac{1}{2}} = \prod_{k=1}^{2n} \alpha_k^{\frac{2m-1}{2}(\alpha+1)}, \qquad (10)$$

$$\prod_{k=1}^{n}\prod_{p=1}^{m}\left[\left(\chi\left(\left|a_{2k-1,2p-1}\right|^{\frac{1}{\alpha_{2k-1}}}\right)\right) \cdot \left(\chi\left(\left|a_{2k-1,2p-1}\right|^{\frac{1}{\alpha_{2k-2}}}\right)\right) \cdot \left|a_{2k-1,2p-1}\right|^{2}\right]^{\frac{\alpha}{2}} \times$$

$$\times \prod_{k=1}^{n}\prod_{p=1}^{m-1}\left[\left(\chi\left(\left|a_{2k-1,2p}\right|^{\frac{1}{\alpha_{2k-1}}}\right)\right) \cdot \left(\chi\left(\left|a_{2k-1,2p}\right|^{\frac{1}{\alpha_{2k-2}}}\right)\right) \cdot \left|a_{2k-1,2p}\right|^{2}\right]^{\frac{1}{2}} \times$$

$$\times \prod_{k=1}^{n}\prod_{p=1}^{m-1}\left[\chi\left(\left|a_{2k,2p}\right|^{\frac{1}{\alpha_{2k}}}\right) \cdot \chi\left(\left|a_{2k,2p}\right|^{\frac{1}{\alpha_{2k-1}}}\right) \cdot \left|a_{2k,2p}\right|^{2}\right]^{\frac{\alpha}{2}} \times$$

$$\times \prod_{k=1}^{n}\prod_{p=1}^{m}\left[\chi\left(\left|a_{2k,2p-1}\right|^{\frac{1}{\alpha_{2k}}}\right) \cdot \chi\left(\left|a_{2k,2p-1}\right|^{\frac{1}{\alpha_{2k-1}}}\right) \cdot \left|a_{2k,2p-1}\right|^{2}\right]^{\frac{1}{2}} =$$

$$= \left(M\left(A_{2n,2m-1}^{(1)}\right) \cdot M\left(A_{2n,2m-1}^{(3)}\right)\right)^{\alpha} \cdot M\left(A_{2n,2m-1}^{(2)}\right) \cdot M\left(A_{2n,2m-1}^{(4)}\right). \quad (11)$$

Then from (8) using formulas (9), (10), (11), it is received the following ratio

$$\prod_{k=1}^{n}\prod_{p=1}^{m} r^{\alpha}\left(B_{2k-1,2p-1}, a_{2k-1,2p-1}\right) \cdot \prod_{k=1}^{n}\prod_{p=1}^{m-1} r\left(B_{2k-1,2p}, a_{2k-1,2p}\right) \times$$

$$\times \prod_{k=1}^{n}\prod_{p=1}^{m-1} r^{\alpha}\left(B_{2k,2p}, a_{2k,2p}\right) \cdot \prod_{k=1}^{n}\prod_{p=1}^{m} r\left(B_{2k,2p-1}, a_{2k,2p-1}\right) \leq$$

$$\leq \prod_{k=1}^{2n} \alpha_k^{\frac{2m-1}{2}(\alpha+1)} \cdot \left(M\left(A_{2n,2m-1}^{(1)}\right) \cdot M\left(A_{2n,2m-1}^{(3)}\right)\right)^{\alpha} \cdot M\left(A_{2n,2m-1}^{(2)}\right) \cdot M\left(A_{2n,2m-1}^{(4)}\right) \times$$

$$\times \prod_{k=1}^{n} \left\{ \prod_{p=1}^{m} r^{\alpha} \left(\Omega_{2k-1,2p-1}^{(1)}, \omega_{2k-1,2p-1}^{(1)} \right) \cdot \prod_{p=1}^{m-1} r \left(\Omega_{2k-1,2p}^{(1)}, \omega_{2k-1,2p}^{(1)} \right) \times \right.$$

$$\times \prod_{p=1}^{m-1} r^{\alpha} \left(\Omega_{2k-1,2p}^{(2)}, \omega_{2k-1,2p}^{(2)} \right) \cdot \prod_{p=1}^{m} r \left(\Omega_{2k-1,2p-1}^{(2)}, \omega_{2k-1,2p-1}^{(2)} \right) \times$$

$$\times \prod_{p=1}^{m} r^{\alpha} \left(\Omega_{2k,2p-1}^{(2)}, \omega_{2k,2p-1}^{(2)} \right) \cdot \prod_{p=1}^{m-1} r \left(\Omega_{2k,2p}^{(2)}, \omega_{2k,2p}^{(2)} \right) \times$$

$$\left. \times \prod_{p=1}^{m-1} r^{\alpha} \left(\Omega_{2k,2p}^{(1)}, \omega_{2k,2p}^{(1)} \right) \cdot \prod_{p=1}^{m} r \left(\Omega_{2k,2p-1}^{(1)}, \omega_{2k,2p-1}^{(1)} \right) \right\}^{\frac{1}{2}}. \qquad (12)$$

Considering that

$$\prod_{k=1}^{2n} \alpha_k \leq \left(\frac{1}{n} \right)^{2n},$$

from the previous ratio we receive

$$\prod_{k=1}^{n} \prod_{p=1}^{m} r^{\alpha} \left(B_{2k-1,2p-1}, a_{2k-1,2p-1} \right) \cdot \prod_{k=1}^{n} \prod_{p=1}^{m-1} r \left(B_{2k-1,2p}, a_{2k-1,2p} \right) \times$$

$$\times \prod_{k=1}^{n} \prod_{p=1}^{m-1} r^{\alpha} \left(B_{2k,2p}, a_{2k,2p} \right) \cdot \prod_{k=1}^{n} \prod_{p=1}^{m} r \left(B_{2k,2p-1}, a_{2k,2p-1} \right) \leq$$

$$\leq \left(\frac{1}{n} \right)^{n(2m-1)(\alpha+1)} \cdot \left(M\Big(A_{2n,2m-1}^{(1)} \Big) \cdot M\Big(A_{2n,2m-1}^{(3)} \Big) \right)^{\alpha} \cdot M\Big(A_{2n,2m-1}^{(2)} \Big) \cdot M\Big(A_{2n,2m-1}^{(4)} \Big) \times$$

$$\times \prod_{k=1}^{n} \left\{ \prod_{p=1}^{m} r^{\alpha} \left(\Omega_{2k-1,2p-1}^{(1)}, \omega_{2k-1,2p-1}^{(1)} \right) \cdot \prod_{p=1}^{m-1} r \left(\Omega_{2k-1,2p}^{(1)}, \omega_{2k-1,2p}^{(1)} \right) \times \right.$$

$$\times \prod_{p=1}^{m-1} r^{\alpha} \left(\Omega_{2k-1,2p}^{(2)}, \omega_{2k-1,2p}^{(2)} \right) \cdot \prod_{p=1}^{m} r \left(\Omega_{2k-1,2p-1}^{(2)}, \omega_{2k-1,2p-1}^{(2)} \right) \times$$

$$\times \prod_{p=1}^{m} r^{\alpha} \left(\Omega_{2k,2p-1}^{(2)}, \omega_{2k,2p-1}^{(2)} \right) \cdot \prod_{p=1}^{m-1} r \left(\Omega_{2k,2p}^{(2)}, \omega_{2k,2p}^{(2)} \right) \times$$

$$\left. \times \prod_{p=1}^{m-1} r^{\alpha} \left(\Omega_{2k,2p}^{(1)}, \omega_{2k,2p}^{(1)} \right) \cdot \prod_{p=1}^{m} r \left(\Omega_{2k,2p-1}^{(1)}, \omega_{2k,2p-1}^{(1)} \right) \right\}^{\frac{1}{2}}. \qquad (13)$$

from the theorem 4.2.2 [7] inequalities follow

$$\prod_{p=1}^{m} r^{\alpha}\left(\Omega^{(1)}_{2k-1,2p-1}, \omega^{(1)}_{2k-1,2p-1}\right) \cdot \prod_{p=1}^{m-1} r\left(\Omega^{(1)}_{2k-1,2p}, \omega^{(1)}_{2k-1,2p}\right) \cdot \prod_{p=1}^{m-1} r^{\alpha}\left(\Omega^{(2)}_{2k-1,2p}, \omega^{(2)}_{2k-1,2p}\right) \times$$

$$\times \cdot \prod_{p=1}^{m} r\left(\Omega^{(2)}_{2k-1,2p-1}, \omega^{(2)}_{2k-1,2p-1}\right) \leq \prod_{p=1}^{2m-1} r^{\alpha}\left(G^{(1)}_{2p-1}, g^{(1)}_{2p-1}\right) \cdot r\left(G^{(1)}_{2p}, g^{(1)}_{2p}\right),$$

$$\prod_{p=1}^{m} r^{\alpha}\left(\Omega^{(2)}_{2k,2p-1}, \omega^{(2)}_{2k,2p-1}\right) \cdot \prod_{p=1}^{m-1} r\left(\Omega^{(2)}_{2k,2p}, \omega^{(2)}_{2k,2p}\right) \cdot \prod_{p=1}^{m-1} r^{\alpha}\left(\Omega^{(1)}_{2k,2p}, \omega^{(1)}_{2k,2p}\right) \times$$

$$\times \cdot \prod_{p=1}^{m} r\left(\Omega^{(1)}_{2k,2p-1}, \omega^{(1)}_{2k,2p-1}\right) \leq \prod_{p=1}^{2m-1} r\left(G^{(2)}_{2p-1}, g^{(2)}_{2p-1}\right) \cdot r^{\alpha}\left(G^{(2)}_{2p}, g^{(2)}_{2p}\right), \qquad (14)$$

where $G^{(1)}_{2p-1}$, $G^{(1)}_{2p}$, $G^{(2)}_{2p-1}$, $G^{(2)}_{2p}$ – system circular domains, $g^{(1)}_{2p-1}$, $g^{(1)}_{2p}$, $g^{(2)}_{2p-1}$, $g^{(2)}_{2p}$ – the poles of the quadratic differential

$$Q(\zeta_k)\, d\zeta_k^2 = \zeta_k^{2m-3} \cdot \frac{i(1-\alpha)\zeta_k^{4m-2} + 2(1+\alpha)\zeta_k^{2m-1} + i(\alpha-1)}{\left(\zeta_k^{4m-2}+1\right)^2} \cdot d\zeta_k^2, \quad k = \overline{1, 2n} \qquad (15)$$

From (14) we receive, using inequalities (13)

$$\prod_{k=1}^{n}\prod_{p=1}^{m} r^{\alpha}\left(B_{2k-1,2p-1}, a_{2k-1,2p-1}\right) \cdot \prod_{k=1}^{n}\prod_{p=1}^{m-1} r\left(B_{2k-1,2p}, a_{2k-1,2p}\right) \times$$

$$\times \prod_{k=1}^{n}\prod_{p=1}^{m-1} r^{\alpha}\left(B_{2k,2p}, a_{2k,2p}\right) \cdot \prod_{k=1}^{n}\prod_{p=1}^{m} r\left(B_{2k,2p-1}, a_{2k,2p-1}\right) \leq$$

$$\leq \left(\frac{1}{n}\right)^{n(2m-1)(\alpha+1)} \cdot \left(M\left(A^{(1)}_{2n,2m-1}\right) \cdot M\left(A^{(3)}_{2n,2m-1}\right)\right)^{\alpha} \cdot M\left(A^{(2)}_{2n,2m-1}\right) \cdot M\left(A^{(4)}_{2n,2m-1}\right) \times$$

$$\times \left(\prod_{p=1}^{2m-1} r^{\alpha}\left(G_{2p-1}, g_{2p-1}\right) \cdot r\left(G_{2p}, g_{2p}\right)\right)^{n}, \qquad (16)$$

where G_{2p-1}, G_{2p} – system circular domains, g_{2p-1}, g_{2p} – the poles of the quadratic differential (15).

From the last ratio, the approval of the theorem follows, using the theorem 4.1.2 [7]. **The theorem is proved.**

Proof theorem 2. At once we will note that from the condition of unapplying follows that $\operatorname{cap} \overline{\mathbb{C}} \backslash D > 0$ and set D possesses Green's generalized function $g_D(z, a)$, where $g_D(z,a) = \begin{cases} g_{D(a)}(z,a), \ z \in D(a), \\ 0, \ z \in \overline{\mathbb{C}} \backslash \overline{D(a)}, \\ \lim\limits_{\zeta \to z} g_{D(a)}(\zeta, a), \ \zeta \in D(a), z \in \partial D(a) \end{cases}$ – Green's generalized function open set D concerning a point $a \in D$, and $g_{D(a)}(z,a)$ – Green's function domain $D(a)$ concerning a point $a \in D(a)$.

Further, we will use methods of works [6, 7]. Sets we will consider $E_0 = \overline{\mathbb{C}} \backslash D$; $E(a_{k,p}, t) = \{w \in \mathbb{C} : |w - a_{k,p}| \leqslant t\}$, $k = \overline{1, 2n}$, $p = \overline{1, 2m-1}$, $n \geqslant 3$, $n, m \in \mathbb{N}$, $t \in \mathbb{R}_+$. The condenser we will enter into consideration for rather small $t > 0$

$$C\left(t,\, D,\, A_{2n,2m-1}\right) = \left\{E_0,\, E_1,\, E_2\right\},$$

where

$$E_1 = \bigcup_{k=1}^{n} \bigcup_{p=1}^{m} E(a_{2k-1,2p-1}, t) \bigcup_{k=1}^{n} \bigcup_{p=1}^{m-1} E(a_{2k,2p}, t),$$

$$E_2 = \bigcup_{k=1}^{n} \bigcup_{p=1}^{m} E(a_{2k,2p-1}, t) \bigcup_{k=1}^{n} \bigcup_{p=1}^{m-1} E(a_{2k-1,2p}, t)$$

. Capacity of the condenser $C\left(t,\, D,\, A_{2n,2m-1}\right)$ is called as (look [5])

$$\mathrm{cap}C\left(t,\, D,\, A_{2n,2m-1}\right) = \inf \int\int \left[(G'_x)^2 + (G'_y)^2\right]\, dxdy,$$

where an infimum undertakes on all continuous and to the lipschicevym in $\overline{\mathbb{C}}$ functions $G = G(z)$, such that $G\Big|_{E_0} = 0$, $G\Big|_{E_1} = \sqrt{\alpha}$, $G\Big|_{E_2} = 1$

Let is named the module of condenser C, reverse the capacity of condenser

$$|C| = \left[\mathrm{cap}C\right]^{-1}$$

From a theorem 1 [6] get

$$\left|C\left(t, D, A_{2n,2m-1}\right)\right| = \frac{1}{2\pi} \cdot \frac{1}{n(2m-1)\left(\alpha+1\right)} \cdot \log\frac{1}{t} + M(D, A_{2n,2m-1}) + o(1),\ t \to 0, \tag{17}$$

where

$$M(D, A_{2n,2m-1}) = \frac{1}{2\pi} \cdot \frac{1}{n^2(2m-1)^2 \cdot \left(\alpha+1\right)^2} \cdot \Big[\alpha \sum_{k=1}^{n} \sum_{p=1}^{m-1} \log r(D, a_{2k,2p}) +$$

$$+ \sum_{k=1}^{n} \sum_{p=1}^{m} \log r(D, a_{2k,2p-1}) + \sum_{k=1}^{n} \sum_{p=1}^{m-1} \log r(D, a_{2k-1,2p}) +$$

$$+ \alpha \sum_{k=1}^{n} \sum_{p=1}^{m} \log r(D, a_{2k-1,2p-1}) +$$

$$+\alpha \sum_{(k,p)\neq(q,s)} (g_D(a_{2k,2p}, a_{2q,2s}) + g_D(a_{2k-1,2p-1}, a_{2q-1,2s-1})) +$$

$$+2\alpha \sum_{(k,p)\neq(q,s)} g_D(a_{2k-1,2p-1}, a_{2q,2s}) + 2\sqrt{\alpha} \sum_{(k,p)\neq(q,s)} (g_D(a_{2k,2p}, a_{2q-1,2s}) +$$

$$+g_D(a_{2k,2p}, a_{2q,2s-1}) + g_D(a_{2k-1,2p-1}, a_{2q-1,2s}) + g_D(a_{2k-1,2p-1}, a_{2q,2s-1})) +$$

$$+ \sum_{(k,p)\neq(q,s)} (g_D(a_{2k-1,2p}, a_{2q-1,2s}) + g_D(a_{2k,2p-1}, a_{2q,2s-1})) +$$

$$+2 \sum_{(k,p)\neq(q,s)} g_D(a_{2k-1,2p}, a_{2q,2s-1})\Big]. \qquad (18)$$

Function (5) and definition $\omega^{(1)}_{k,p}$, $\omega^{(2)}_{k-1,p}$, $a_{n+1,p}$, $\omega^{(2)}_{0,p}$, ζ_0, Δ, $(\Delta)^*$, using, by us the theorems entered at proof 1. Let, too, $\mathbf{\Omega}^{(1)}_{k,p}$ define connected component $\zeta_k\left(D\bigcap \overline{P}_k\right)\bigcup\left(\zeta_k\left(D\bigcap \overline{P}_k\right)\right)^*$, containing a point $\omega^{(1)}_{k,p}$, $\mathbf{\Omega}^{(2)}_{k-1,p}$ – connected component $\zeta_{k-1}\left(D\bigcap \overline{P}_{k-1}\right)\bigcup\left(\zeta_{k-1}\left(D\bigcap \overline{P}_{k-1}\right)\right)^*$, containing a point $\omega^{(2)}_{k-1,p}$, $k=\overline{1,2n}$, $p=\overline{1,2m-1}$, $\overline{P}_0 := \overline{P}_{2n}$, $\mathbf{\Omega}^{(2)}_{0,p} := \mathbf{\Omega}^{(2)}_{n,p}$. It is clear, that $\mathbf{\Omega}^{(s)}_{k,p}$ generally speaking, domains are multiconnected domains, $k=\overline{1,2n}$, $p=\overline{1,2m-1}$, $s=1,2$. Pair of domains $\mathbf{\Omega}^{(2)}_{k-1,p}$ $\mathbf{\Omega}^{(1)}_{k,p}$ grows out of piece-dividing transformation open set D concerning families $\{P_{k-1}, P_k\}$, $\{\zeta_{k-1}, \zeta_k\}$ in point $a_{k,p}$, $k=\overline{1,2n}$, $p=\overline{1,2m-1}$.

Let's consider condensers

$$C_k\left(t,\, D,\, A_{2n,2m-1}\right) = \left(E_0^{(k)},\, E_1^{(k)},\, E_2^{(k)}\right),$$

where

$$E_s^{(k)} = \zeta_k\left(E_s \bigcap \overline{P}_k\right) \bigcup \left[\zeta_k\left(E_s \bigcap \overline{P}_k\right)\right]^*,$$

$k=\overline{1,2n}$, $s=0,1,2$, $\{P_k\}_{k=1}^{2n}$ – the system of corners corresponding to system of points $A_{2n,2m-1}$; operation $[A]^*$ compares to any the set $A\subset\overline{\mathbb{C}}$ a set, symmetric a set A is relative unit circle $|w|=1$. From this it follows that to the condenser $C\left(t,\, D,\, A_{2n,2m-1}\right)$, at dividing transformation is relative $\{P_k\}_{k=1}^{2n}$ and $\{\zeta_k\}_{k=1}^{2n}$, there corresponds a set of condensers the system of corners corresponding to system of points $A_{2n,2m-1}$; operation $[A]^*$ compares to any the set $A\subset\overline{\mathbb{C}}$ a set, symmetric a set A is relative unit circle $|w|=1$. From this it follows that to the condenser $C\left(t,\, D,\, A_{2n,2m-1}\right)$, at dividing transformation is relative $\{P_k\}_{k=1}^{2n}$ and $\{\zeta_k\}_{k=1}^{2n}$, there corresponds a set of condensers

$\{C_k\,(t,\,D,\,A_{2n,2m-1})\}_{k=1}^{2n}$, symmetric relatively $\{z\,:\,|z|\,=\,1\}$. According to works [6, 7], we will receive

$$\mathrm{cap}C\,(t,D,A_{2n,2m-1}) \geqslant \frac{1}{2}\sum_{k=1}^{2n}\mathrm{cap}C_k\,(t,D,A_{2n,2m-1})\,. \tag{19}$$

From here follows

$$|C\,(t,D,A_{2n,2m-1})\,| \leqslant 2\left(\sum_{k=1}^{2n}|C_k\,(t,D,A_{2n,2m-1})\,|^{-1}\right)^{-1}. \tag{20}$$

The formula (17) gives a module asymptotics $C\,(t,\,D,\,A_{2n,2m-1})$ at $t\to 0$, and $M\,(D,A_{2n,2m-1})$ is the given module of a set D relatively $A_{2n,2m-1}$. Using formulas (6) and that fact that a set D meets the condition of unapplied in relation to the system of points $A_{2n,2m-1}$, for condensers we will receive similar asymptotic representations $C_k\,(t,D,A_{2n,2m-1})$, $k=\overline{1,2n}$

$$|C_k\,(t,D,A_{2n,2m-1})\,| = \frac{1}{2\pi(2m-1)\,(\alpha+1)}\log\frac{1}{t}+M_k\,(D,A_{2n,2m-1})+o(1),\ t\to 0, \tag{21}$$

where

$$M_{2k-1}\,(D,A_{2n,2m-1}) = \frac{1}{2\pi(2m-1)^2\,(\alpha+1)^2}\times$$

$$\times\left[\alpha\sum_{p=1}^{m}\log\frac{r\left(\mathbf{\Omega}^{(1)}_{2k-1,2p-1},\omega^{(1)}_{2k-1,2p-1}\right)}{\left[\alpha_{2k-1}\cdot\chi\left(|a_{2k-1,2p-1}|^{\alpha_{2k-1}}\right)|a_{2k-1,2p-1}|\right]^{-1}}+\right.$$

$$+\alpha\sum_{p=1}^{m-1}\log\frac{r\left(\mathbf{\Omega}^{(2)}_{2k-1,2p},\omega^{(2)}_{2k-1,2p}\right)}{\left[\alpha_{2k-1}\cdot\chi\left(|a_{2k,2p}|^{\alpha_{2k-1}}\right)|a_{2k,2p}|\right]^{-1}}+$$

$$+\sum_{t=1}^{m}\log\frac{r\left(\mathbf{\Omega}^{(2)}_{2k-1,2t-1},\omega^{(2)}_{2k-1,2t-1}\right)}{\left[\alpha_{2k-1}\cdot\chi\left(|a_{2k,2t-1}|^{\alpha_{2k-1}}\right)|a_{2k,2t-1}|\right]^{-1}}+$$

$$\left.+\sum_{t=1}^{m-1}\log\frac{r\left(\mathbf{\Omega}^{(1)}_{2k-1,2t},\omega^{(1)}_{2k-1,2t}\right)}{\left[\alpha_{2k-1}\cdot\chi\left(|a_{2k-1,2t}|^{\alpha_{2k-1}}\right)|a_{2k-1,2t}|\right]^{-1}}\right],$$

$$M_{2k}\,(D,A_{2n,2m-1}) =$$

$$=\frac{1}{2\pi(2m-1)^2\,(\alpha+1)^2}\cdot\left[\alpha\sum_{p=1}^{m-1}\log\frac{r\left(\mathbf{\Omega}^{(1)}_{2k,2p},\omega^{(1)}_{2k,2p}\right)}{\left[\alpha_{2k}\cdot\chi\left(|a_{2k,2p}|^{\alpha_{2k}}\right)|a_{2k,2p}|\right]^{-1}}+\right.$$

$$+\alpha\sum_{p=1}^{m}\log\frac{r\left(\mathbf{\Omega}^{(2)}_{2k,2p-1},\omega^{(2)}_{2k,2p-1}\right)}{\left[\alpha_{2k}\cdot\chi\left(|a_{2k+1,2p-1}|^{\alpha_{2k}}\right)|a_{2k+1,2p-1}|\right]^{-1}}+$$

$$+\sum_{t=1}^{m-1}\log\frac{r\left(\mathbf{\Omega}^{(2)}_{2k,2t},\omega^{(2)}_{2k,2t}\right)}{\left[\alpha_{2k}\cdot\chi\left(|a_{2k+1,2t}|^{\alpha_{2k}}\right)|a_{2k+1,2t}|\right]^{-1}}+$$

$$+\sum_{t=1}^{m}\log\frac{r\left(\mathbf{\Omega}^{(1)}_{2k,2t-1},\omega^{(1)}_{2k,2t-1}\right)}{\left[\alpha_{2k}\cdot\chi\left(|a_{2k,2t-1}|^{\alpha_{2k}}\right)|a_{2k,2t-1}|\right]^{-1}}\Bigg],\qquad k=\overline{1,n}.$$

By means of (21), we receive

$$|C_k\left(t,D,A_{2n,2m-1}\right)|^{-1}=\frac{2\pi(2m-1)\left(\alpha+1\right)}{\log\frac{1}{t}}\times$$

$$\times\left(1+\frac{2\pi(2m-1)\left(\alpha+1\right)}{\log\frac{1}{t}}M_k\left(D,A_{2n,2m}\right)+o\left(\frac{1}{\log\frac{1}{t}}\right)\right)^{-1}=$$

$$=\frac{2\pi(2m-1)\left(\alpha+1\right)}{\log\frac{1}{t}}-\left(\frac{2\pi(2m-1)\left(\alpha+1\right)}{\log\frac{1}{t}}\right)^2M_k\left(D,A_{2n,2m-1}\right)\qquad(22)$$

$$+o\left(\left(\frac{1}{\log\frac{1}{t}}\right)^2\right),\ t\to 0.\qquad(23)$$

Further, from (22), follows that

$$\sum_{k=1}^{2n}|C_k\left(t,D,A_{2n,2m-1}\right)|^{-1}=\frac{4\pi n(2m-1)\left(\alpha+1\right)}{\log\frac{1}{t}}-$$

$$-\left(\frac{2\pi(2m-1)\left(\alpha+1\right)}{\log\frac{1}{t}}\right)^2\cdot\sum_{k=1}^{2n}M_k\left(D,A_{2n,2m-1}\right)+o\left(\left(\frac{1}{\log\frac{1}{t}}\right)^2\right),\qquad t\to 0.\qquad(24)$$

In turn, allows (24) to receive the following asymptotic representation

$$\left(\sum_{k=1}^{2n}|C_k\left(t,D,A_{2n,2m-1}\right)|^{-1}\right)^{-1}=\frac{\log\frac{1}{t}}{4\pi n(2m-1)\left(\alpha+1\right)}\times$$

$$\times\left(1-\frac{\pi(2m-1)\left(\alpha+1\right)}{n\log\frac{1}{t}}\cdot\sum_{k=1}^{2n}M_k\left(D,A_{2n,2m-1}\right)+o\left(\frac{1}{\log\frac{1}{t}}\right)\right)^{-1}=$$

$$= \frac{\log \frac{1}{t}}{4\pi n(2m-1)\,(\alpha+1)} + \frac{1}{4n^2} \cdot \sum_{k=1}^{2n} M_k\left(D, A_{2n,2m-1}\right) + o(1), \qquad t \to 0. \quad (25)$$

Inequalities, (19) and (20) taking into (17) and (25) allow to notice that

$$\frac{1}{2\pi} \cdot \frac{1}{n(2m-1)\,(\alpha+1)} \cdot \log \frac{1}{t} + M(D, A_{2n,2m-1}) + o(1) \leqslant$$

$$\leqslant \frac{\log \frac{1}{t}}{2\pi n(2m-1)\,(\alpha+1)} + \frac{1}{2n^2} \cdot \sum_{k=1}^{2n} M_k\left(D, A_{2n,2m-1}\right) + o(1). \qquad (26)$$

From (26) at $t \to 0$ we receive that

$$M(D, A_{2n,2m-1}) \leqslant \frac{1}{2n^2} \cdot \sum_{k=1}^{2n} M_k\left(D, A_{2n,2m-1}\right). \qquad (27)$$

Formulas (18), (21) and (27) lead to the following expression

$$\frac{1}{2\pi} \cdot \frac{1}{n^2(2m-1)^2 \cdot (\alpha+1)^2} \cdot \Big[\alpha \sum_{k=1}^{n} \sum_{p=1}^{m-1} \log r(D, a_{2k,2p}) +$$

$$+ \sum_{k=1}^{n} \sum_{p=1}^{m} \log r(D, a_{2k,2p-1}) + \sum_{k=1}^{n} \sum_{p=1}^{m-1} \log r(D, a_{2k-1,2p}) +$$

$$+ \alpha \sum_{k=1}^{n} \sum_{p=1}^{m} \log r(D, a_{2k-1,2p-1}) +$$

$$+ \alpha \sum_{(k,p)\neq(q,s)} \left(g_D(a_{2k,2p}, a_{2q,2s}) + g_D(a_{2k-1,2p-1}, a_{2q-1,2s-1})\right) +$$

$$+ 2\alpha \sum_{(k,p)\neq(q,s)} g_D(a_{2k-1,2p-1}, a_{2q,2s}) +$$

$$+ 2\sqrt{\alpha} \sum_{(k,p)\neq(q,s)} \left(g_D(a_{2k,2p}, a_{2q-1,2s}) + g_D(a_{2k,2p}, a_{2q,2s-1}) +\right.$$

$$+ g_D(a_{2k-1,2p-1}, a_{2q-1,2s}) + g_D(a_{2k-1,2p-1}, a_{2q,2s-1}) +$$

$$+ \sum_{(k,p)\neq(q,s)} \left(g_D(a_{2k-1,2p}, a_{2q-1,2s}) + g_D(a_{2k,2p-1}, a_{2q,2s-1})\right) +$$

$$+ 2 \sum_{(k,p)\neq(q,s)} g_D(a_{2k-1,2p}, a_{2q,2s-1})\Big] \leq \frac{1}{4\pi n^2 (2m-1)^2\,(\alpha+1)^2} \times$$

$$\times \left[\alpha \sum_{k=1}^{n} \sum_{p=1}^{m} \log \frac{r\left(\mathbf{\Omega}_{2k-1,2p-1}^{(1)}, \omega_{2k-1,2p-1}^{(1)}\right)}{\left[\alpha_{2k-1} \cdot \chi\left(|a_{2k-1,2p-1}|^{\alpha_{2k-1}}\right) |a_{2k-1,2p-1}|\right]^{-1}} + \right.$$

$$+\alpha \sum_{k=1}^{n} \sum_{p=1}^{m-1} \log \frac{r\left(\mathbf{\Omega}_{2k-1,2p}^{(2)}, \omega_{2k-1,2p}^{(2)}\right)}{\left[\alpha_{2k-1} \cdot \chi\left(|a_{2k,2p}|^{\alpha_{2k-1}}\right) |a_{2k,2p}|\right]^{-1}} +$$

$$+ \sum_{k=1}^{n} \sum_{t=1}^{m} \log \frac{r\left(\mathbf{\Omega}_{2k-1,2t-1}^{(2)}, \omega_{2k-1,2t-1}^{(2)}\right)}{\left[\alpha_{2k-1} \cdot \chi\left(|a_{2k,2t-1}|^{\alpha_{2k-1}}\right) |a_{2k,2t-1}|\right]^{-1}} +$$

$$+ \sum_{k=1}^{n} \sum_{t=1}^{m-1} \log \frac{r\left(\mathbf{\Omega}_{2k-1,2t}^{(1)}, \omega_{2k-1,2t}^{(1)}\right)}{\left[\alpha_{2k-1} \cdot \chi\left(|a_{2k-1,2t}|^{\alpha_{2k-1}}\right) |a_{2k-1,2t}|\right]^{-1}} +$$

$$+\alpha \sum_{k=1}^{n} \sum_{p=1}^{m-1} \log \frac{r\left(\mathbf{\Omega}_{2k,2p}^{(1)}, \omega_{2k,2p}^{(1)}\right)}{\left[\alpha_{2k} \cdot \chi\left(|a_{2k,2p}|^{\alpha_{2k}}\right) |a_{2k,2p}|\right]^{-1}} +$$

$$+\alpha \sum_{k=1}^{n} \sum_{p=1}^{m} \log \frac{r\left(\mathbf{\Omega}_{2k,2p-1}^{(2)}, \omega_{2k,2p-1}^{(2)}\right)}{\left[\alpha_{2k} \cdot \chi\left(|a_{2k+1,2p-1}|^{\alpha_{2k}}\right) |a_{2k+1,2p-1}|\right]^{-1}} +$$

$$+ \sum_{k=1}^{n} \sum_{t=1}^{m-1} \log \frac{r\left(\mathbf{\Omega}_{2k,2t}^{(2)}, \omega_{2k,2t}^{(2)}\right)}{\left[\alpha_{2k} \cdot \chi\left(|a_{2k+1,2t}|^{\alpha_{2k}}\right) |a_{2k+1,2t}|\right]^{-1}} +$$

$$\left. + \sum_{k=1}^{n} \sum_{t=1}^{m} \log \frac{r\left(\mathbf{\Omega}_{2k,2t-1}^{(1)}, \omega_{2k,2t-1}^{(1)}\right)}{\left[\alpha_{2k} \cdot \chi\left(|a_{2k,2t-1}|^{\alpha_{2k}}\right) |a_{2k,2t-1}|\right]^{-1}} \right].$$

Thus, we receive (12). Further, the proof of the theorem comes to an end in the same way, as well as the proof of the theorem 1. **The theorem is proved**.

Acknowledgement. The author is grateful to Prof. Bakhtin for suggesting problems and useful discussions.

References

[1] M. A. Lavrent'ev, On the theory of conformal mappings, Tr. Fiz.-Mat. Inst. Akad. Nauk SSSR, 5, 159245 (1934).

[2] G. M. Goluzin, Geometric Theory of Functions of a Complex Variable [in Russian], Nauka, Moscow (1966).

[3] G. P. Bakhtina, Variational Methods and Quadratic Differentials in Problems for Disjoint Domains [in Russian], Author's Abstract of the Candidate-Degree Thesis (Physics and Mathematics), Kiev (1975).

[4] V. N. Dubinin, Separating transformation of domains and problems of extremal division, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Ros. Akad. Nauk, 168, 4866 (1988). – 168. – P. 48 – 66.

[5] V. N. Dubinin, Method of symmetrization in the geometric theory of functions of a complex variable, Usp. Mat. Nauk, 49, No.1 (295), 376 (1994).

[6] V. N. Dubinin, Asymptotic representation of the modulus of a degenerating condenser and some its applications, Zap. Nauchn. Sem. Peterburg. Otdel. Mat. Inst., 237, 5673 (1997).

[7] Bakhtin, A. K., Bakhtina, G. P., Zelinskii, Yu. B., "Topological-algebraic structures and geometric methods in complex analysis," Proceedings of the Institute of Mathematics of NAS of Ukraine 73 (2008), 308 pp. (Russian).

[8] O. K. Bakhtin, Inequalities for the inner radii of nonoverlapping domains and open sets, Ukr. Mat. Zh., 61, No. 5, 596610 (2009); English translation: Ukr. Math. J., 61, No. 5, 716733 (2009).

[9] Dubinin, V. N., Capacities of condensers and symmetrization in geometric function theory of complex variables, Dalnayka, Vladivostok, 2009 (Russian).

[10] A. K. Bakhtin and A. L. Targonskii, Extremal problems and quadratic differentials, Nelin. Kolyvannya, 8, No. 3, 298303 (2005); English translation: Nonlin. Oscillations, 8, No. 3, 296301 (2005).

[11] A. L. Targonskii, Extremal problems of partially nonoverlapping domains on a Riemann sphere, Dopov. Nats. Akad. Nauk Ukr., No. 9, 3136 (2008).

[12] G. V. Kuz'mina, Problem of extremal division of a Riemann sphere, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Ros. Akad. Nauk, 276, 253275 (2001).

[13] E. G. Emel'yanov, On the problem of the maximum of the product of powers of conformal radii of disjoint domains, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Ros. Akad. Nauk, 286, 103-114 (2002).

[14] W. K. Hayman, Multivalent Functions, Cambridge University, Cambridge (1958).

[15] J. A. Jenkins, Univalent Functions and Conformal Mapping, Springer, Berlin (1958).

Andrey Targonskii,
Department of Mathematics,
Zhitomir State University,
Veluka Berdichivska, 40, 10008 Zhitomir, Ukraine.
Email: targonsk@zu.edu.ua, targonsk@mail.ru