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An analytic approach to converting POE parameters into D-H parameters for serial-link robots

Liao Wu, Ross Crawford, and Jonathan Roberts

***Abstract*—The Denavit-Hartenberg (D-H) model and the product of exponentials (POE) model have been two popular methods for modeling the kinematics of a serial-link robot. While these two models are equivalent in essence, no study has revealed how to convert from the POE model to the D-H model. The conversion enables direct utilization of established algorithms formulated with D-H parameters or compensation of the D-H model after calibration with the POE parameters. It also provides a simpler method to determine the D-H parameters of a robot. For these reasons, this paper proposes an analytic approach to automatically convert a group of POE parameters into the associated D-H parameters. Three lemmas are proved for the derivation of the final algorithm. An implementation of the algorithm in MATLAB is provided as well.**

***Index Terms*—Denavit-Hartenberg (D-H) parameters, Product of Exponentials (POE), kinematics, serial-link robots.**

I. Introduction

KINEMATIC modeling of robots is a fundamental issue in robotics. To date, two systems of methodologies have been adopted in the robotics community. The most well known and used system is the Denavit-Hartenberg (D-H) notation, the invention of which can be dated to 1955 [1]. Generally, the D-H model uses a group of frames that are rigidly attached to the links of a robot to represent the geometric structure, and abstract the motion of the robot as the dynamic coordinate transformations of these frames. Four parameters, referred to as the D-H parameters, are used to construct the homogeneous transformations between adjacent link frames. By using these parameters, the kinematics of a robot can be uniquely, concisely and accurately described, that is, given a group of joint variables, we are able to determine the resultant transformations between the link frames, and then calculate the final position and orientation of the end-effector by concatenating these transformations. In addition, based on the D-H notation, a number of established algorithmic techniques can be employed to solve the dynamics, motion planning and control problems [2], [3].

An alternative system to model the kinematics of a robot is the use of the product of exponentials (POE) formula, which was initially proposed in 1984 [4]. Rather than treating the motion of the robot as a set of frame transformations, the POE method regards the motion as a chain of integration of joint twists over the joint variables with respect to an initial state where all the joints are in their zero positions [5]. By using this method, it is not necessary to set up link frames or cope with the associated transformations. All that is required is the relationship between the base frame and the tool frame in the initial state, as well as the joint twists evaluated with respect to the base frame. Then, the forward kinematics of the robot can be represented by the product of a cluster of exponential mappings of the joint twists and the initial twist. The advantages of the POE method include its simple and general framework of modeling a robot and its close relationship with the Lie groups theory, which facilitates the transplantation of modern theoretical developments in differential geometry to robotics (some excellent examples can be seen in [5]–[7]). Hence, there has been an emerging trend of using the POE model in robotics research recently [8], [9].

In essence, the D-H model and the POE model are equivalent in representing the kinematics of a robot. In fact, there have been studies showing how to convert the D-H model into the POE formula [5], [7]. However, to the best of the authors' knowledge, no study has revealed the reverse conversion yet. The benefits of knowing how to convert the POE model to the D-H model include: 1) since the D-H model is earlier and more widely adopted in the community, there are a lot of established algorithms formulated with the D-H parameters that can be directly utilized, while derivation of algorithms with the POE formulation may require much more extra efforts; 2) many commercial industrial robots are programmed with the D-H parameters, but the POE model is important for some procedures like calibration [8], [9]. The conversion enables the compensation of D-H parameters after the calibration with POE parameters; and 3) as the process of modeling a robot using the POE method is much easier than the D-H method, it provides a simpler way for finding the D-H parameters of a robot, that is, the robot can be first modeled with the POE formula and then converted to the D-H notation.

Therefore, the contribution of this paper lies in its first proposal of an analytic algorithm that can automatically and accurately convert a set of POE parameters to a group of D-H parameters that are more familiar to the community.

The rest of this paper is organized as follows. Section II gives an brief introduction of the D-H model and the POE model. Section III reviews the conversion from the D-H model to the POE model, and elaborates an algorithm for converting the POE model to the D-H model with three lemmas proven as the basis. Finally, the paper is concluded in Section IV.

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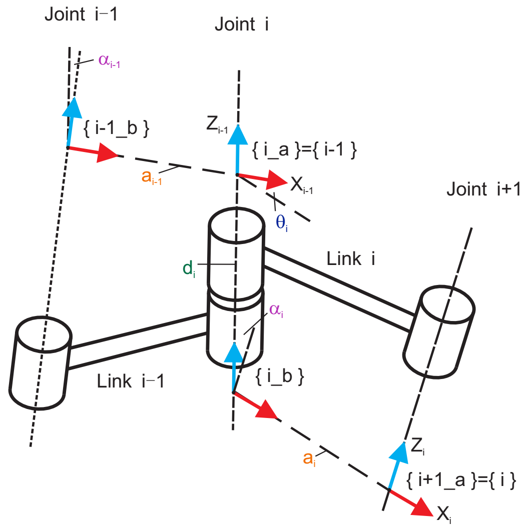
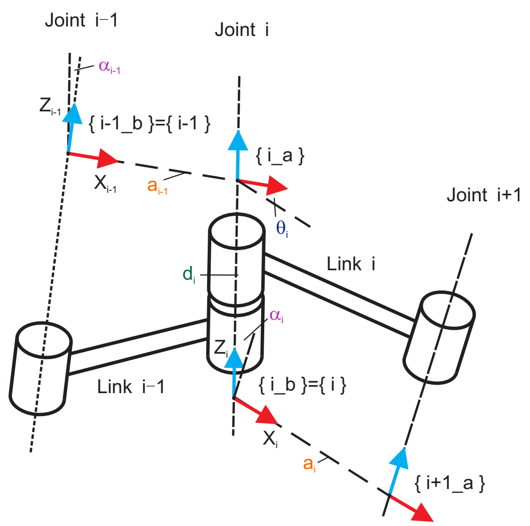


(a) Standard D-H parameters.



(b) Modified D-H parameters.

Fig. 1. The difference between standard D-H parameters and modified D-H parameters.

II. D-H Parameters and POE Parameters

A. D-H Parameters

In 1955, Denavit and Hartenberg [1] introduced a set of rules for modeling the geometric structure of a serial-link robot, which has been referred to as the D-H convention and widely adopted as a standard modeling method by the robotics community. As a fundamental issue in robotics, the D-H model is well introduced in almost every textbook of robotics nowadays [10]–[13]. Therefore, this paper will not elaborate the D-H rules in detail. Instead, the focus is on important aspects that are most relevant to the problem addressed in this paper.

Generally, the D-H convention uses two parameters, the *joint angle* θ and the *joint offset* d, to describe a joint and two other parameters, the *link twist* α and the *link length* a, to represent a link. For a revolute joint, θ is a variable and the other three parameters are constant; for a prismatic joint, d is a variable while the other three parameters are fixed. Frames are attached to the links and transformations between these frames form the forward kinematics. However, variations can be involved in the assignment of frames and the selection of parameters. Two dominant versions that are both widely adopted in the textbooks are the standard D-H model [10], [11] and the modified D-H model [12], [13], as shown in Fig. 1. Both models use the common normal between two adjacent joint axes as an abstract of the geometry of a link, and have the same definitions for the four D-H parameters. The main difference between the two models is their assignment of link frames. The standard model places the frame of link $i-1$ at joint i which connects link $i-1$ and link i, while the modified model locates the frame of link $i-1$ along joint $i-1$. A consequence of this difference is the different homogeneous transformation between two consecutive link frames. In the standard model (Fig. 1(a)), the homogeneous transformation from frame $\{i-1\}$ to frame $\{i\}$ is formed by

$$^{i-1}\boldsymbol{H}_i = R_Z(\theta_i)T_Z(d_i)R_X(\alpha_i)T_X(a_i), \tag{1}$$

where $R(\cdot)$ and $T(\cdot)$ stand for 4 by 4 rotation and translation transformations, respectively, and the subscript indicates the axis to rotate about or translate along. In contrast, in the modified model (Fig. 1(b)), the transformation from frame $\{i-1\}$ to frame $\{i\}$ is given by

$$^{i-1}\boldsymbol{H}_i = R_X(\alpha_{i-1})T_X(a_{i-1})R_Z(\theta_i)T_Z(d_i). \tag{2}$$

Therefore, the two models have different sets of four parameters to construct the transformations between adjacent link frames.

Due to the rules introduced above, both versions have restrictions on the placement of the base frame and the tool frame. However, in the POE formula as will be introduced in the next section, the base frame and the tool frame are arbitrarily located. In order to make the D-H model and the POE model equivalent for the bidirectional conversion, a uniform representation of the D-H model is proposed, in which the base frame and the tool frame are arbitrarily placed and encoded with the D-H notations, as illustrated in Fig. 2. Under this treatment, the forward kinematics can be formulated as (3) using the standard version, or as (4) using the modified version. Once a uniform representation is obtained, it is straightforward to extract the standard D-H parameters according to (3) or the modified D-H parameters according to (4).

It is worth noting that the joint variables may have zero-offsets. Taking this into account, for a revolute joint, θ should be replaced with $q+\bar{\theta}$, where q is the reading from the angle sensor and $\bar{\theta}$ is the zero-offset. Likewise, for a prismatic joint, d should be replaced with $q+\bar{d}$, where q is the reading from the displacement sensor and $\bar{d}$ is the zero-offset. If we define $Q(\cdot)$ as a function that means $R_Z(\cdot)$ for a revolute joint and $T_Z(\cdot)$ for a prismatic joint, we can rewrite (3) as

$$^{B}\boldsymbol{H}_T = {}^{B}\boldsymbol{H}_0 Q(q_1)\,{}^{0}\boldsymbol{H}_1 Q(q_2)\cdots{}^{n-2}\boldsymbol{H}_{n-1}Q(q_n)\,{}^{n-1}\boldsymbol{H}_n\,{}^{n}\boldsymbol{H}_T \tag{5}$$

based on the facts that $R_Z(q+\bar{\theta}) = Q(q)R_Z(\bar{\theta})$ and $R_Z(\theta)T_Z(q+\bar{d}) = Q(q)R_Z(\theta)T_Z(\bar{d})$. Note that the homoge-

$$\begin{aligned}{}^{B}\boldsymbol{H}_T = &\underbrace{R_Z({}^{B\_a}\theta_{B\_b})T_Z({}^{B\_a}d_{B\_b})R_X({}^{B\_b}\alpha_{1\_a})T_X({}^{B\_b}a_{1\_a})}_{{}^{B}\boldsymbol{H}_0}\underbrace{R_Z({}^{1\_a}\theta_{1\_b})T_Z({}^{1\_a}d_{1\_b})R_X({}^{1\_b}\alpha_{2\_a})T_X({}^{1\_b}a_{2\_a})}_{{}^{0}\boldsymbol{H}_1}\cdots \\ &R_X({}^{n-1\_b}\alpha_{n\_a})T_X({}^{n-1\_b}a_{n\_a})\underbrace{R_Z({}^{n\_a}\theta_{n\_b})T_Z({}^{n\_a}d_{n\_b})R_X({}^{n\_b}\alpha_{T\_a})T_X({}^{n\_b}a_{T\_a})}_{{}^{n-1}\boldsymbol{H}_n}\underbrace{R_Z({}^{T\_a}\theta_{T\_b})T_Z({}^{T\_a}d_{T\_b})}_{{}^{n}\boldsymbol{H}_T}. \end{aligned} \tag{3}$$

$$\begin{aligned}{}^{B}\boldsymbol{H}_T = &\underbrace{R_Z({}^{B\_a}\theta_{B\_b})T_Z({}^{B\_a}d_{B\_b})}_{{}^{B}\boldsymbol{H}_0}\underbrace{R_X({}^{B\_b}\alpha_{1\_a})T_X({}^{B\_b}a_{1\_a})R_Z({}^{1\_a}\theta_{1\_b})T_Z({}^{1\_a}d_{1\_b})}_{{}^{0}\boldsymbol{H}_1}R_X({}^{1\_b}\alpha_{2\_a})T_X({}^{1\_b}a_{2\_a})\cdots \\ &\underbrace{R_X({}^{n-1\_b}\alpha_{n\_a})T_X({}^{n-1\_b}a_{n\_a})R_Z({}^{n\_a}\theta_{n\_b})T_Z({}^{n\_a}d_{n\_b})}_{{}^{n-1}\boldsymbol{H}_n}\underbrace{R_X({}^{n\_b}\alpha_{T\_a})T_X({}^{n\_b}a_{T\_a})R_Z({}^{T\_a}\theta_{T\_b})T_Z({}^{T\_a}d_{T\_b})}_{{}^{n}\boldsymbol{H}_T}. \end{aligned} \tag{4}$$

neous transformations in (5) are formulated with the constant zero-offsets rather than the total joint variables as in (3).

B. POE Model

In 1984, Brockett [4] introduced an alternative approach, the POE formula, to depict the kinematics of a robot. Different from the D-H convention, the POE method only retains the base frame and the tool frame, which can be arbitrarily placed as long as they are rigidly attached to the base link and the tool link, respectively. Instead of using link frames and D-H parameters to describe the links and joints, the POE method employs twists to represent the joints, as shown in Fig. 3. When the robot is in its initial configuration (each joint is in the zero position), we record the unit 3-D vector of a revolute joint axis as $\boldsymbol{\omega}$ and the 3-D position of an arbitrary point along the joint axis as $\boldsymbol{p}$, and then calculate another 3-D vector using cross product, $\boldsymbol{v} = \boldsymbol{p} \times \boldsymbol{\omega}$ (note that all the data are measured with respect to the base frame). Then, the joint twist is a 4×4 matrix that belongs to the Lie algebra $se(3)$ and is constructed by

$$\hat{\boldsymbol{\xi}} = \begin{bmatrix} \hat{\boldsymbol{\omega}} & \boldsymbol{v} \\ \boldsymbol{0}^T & 0 \end{bmatrix} \tag{6}$$

where $\hat{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$ is the skew-symmetric matrix of $\boldsymbol{\omega}$. The joint twist can also be represented in a 6-D vector form, which is named the twist coordinates and given by $\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{\omega}^T & \boldsymbol{v}^T \end{bmatrix}^T \in \mathbb{R}^6$.

If the joint is prismatic, there is no revolute joint axis. In this case, $\boldsymbol{\omega}$ is assigned to be a zero vector and $\boldsymbol{v}$ is defined as the unit 3-D vector of the translation direction of this joint.

In addition, when the robot is in its initial configuration, the homogeneous transformation from the base frame to the tool frame, $\boldsymbol{H}_T$, is also captured and can be converted into a general twist, $\hat{\boldsymbol{\xi}}_T$ or the coordinate form, $\boldsymbol{\xi}_T = \begin{bmatrix} \boldsymbol{\omega}_T^T & \boldsymbol{v}_T^T \end{bmatrix}^T$. Note that, however, in this twist, $\boldsymbol{\omega}_T$ does not have to be a unit vector and $\boldsymbol{v}_T$ is not necessary to be a cross product of the two vectors $\boldsymbol{p}_T$ and $\boldsymbol{\omega}_T$.

The forward kinematics then can be expressed as

$${}^{B}\boldsymbol{H}_T = \exp(\hat{\boldsymbol{\xi}}_1 q_1)\exp(\hat{\boldsymbol{\xi}}_2 q_2)\cdots\exp(\hat{\boldsymbol{\xi}}_n q_n)\exp(\hat{\boldsymbol{\xi}}_T) \tag{7}$$

where $\exp(\cdot)$ stands for the exponential mapping from a twist to a homogeneous transformation, $\hat{\boldsymbol{\xi}}_i$ $(i = 1, 2, ..., n)$ stands for the joint twist, and $\hat{\boldsymbol{\xi}}_T$ stands for the initial twist of the tool frame. The variable q_i $(i = 1, 2, ..., n)$ is the joint variable; for a revolute joint, q_i is the rotation angle, and for a prismatic joint, q_i is the translation distance.

III. Conversion between D-H parameters and POE parameters

A. Conversion from D-H parameters to POE parameters

The procedure to convert from a set of given D-H parameters to the corresponding POE parameters have been described in [5], [7]. This section provides a brief review of this conversion.

It is easy to verify that $R_Z(q)$ can be written as an exponential $\exp(\hat{\boldsymbol{\xi}}' q)$ where the coordinate of $\hat{\boldsymbol{\xi}}'$ is

$$\boldsymbol{\xi}' = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]^T. \tag{8}$$

Similarly, $T_Z(q)$ can be written as $\exp(\hat{\boldsymbol{\xi}}' q)$ where the coordinate of $\hat{\boldsymbol{\xi}}'$ is

$$\boldsymbol{\xi}' = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1]^T. \tag{9}$$

Therefore, we can uniformly rewrite $Q(q)$ as $\exp(\hat{\boldsymbol{\xi}}' q)$, and change (5) into

$$\begin{aligned} {}^{B}\boldsymbol{H}_T \quad = \quad & {}^{B}\boldsymbol{H}_0 \exp(\hat{\boldsymbol{\xi}}'_1 q_1){}^{0}\boldsymbol{H}_1 \exp(\hat{\boldsymbol{\xi}}'_2 q_2)\cdots \\ & {}^{n-2}\boldsymbol{H}_{n-1}\exp(\hat{\boldsymbol{\xi}}'_n q_n){}^{n-1}\boldsymbol{H}_n{}^{n}\boldsymbol{H}_T. \end{aligned} \tag{10}$$

Then, to complete the conversion, this property can be used where $\boldsymbol{M}\exp(\hat{\boldsymbol{\xi}})\boldsymbol{M}^{-1} = \exp(\boldsymbol{M}\hat{\boldsymbol{\xi}}\boldsymbol{M}^{-1})$ for any nonsingular square matrix $\boldsymbol{M}$ and any twist $\hat{\boldsymbol{\xi}}$. Since homogeneous transformations are nonsingular square matrices, we have

$$\begin{aligned} {}^{B}\boldsymbol{H}_T &= \exp({}^{B}\boldsymbol{H}_0\hat{\boldsymbol{\xi}}'_1{}^{B}\boldsymbol{H}_0^{-1}q_1){}^{B}\boldsymbol{H}_0{}^{0}\boldsymbol{H}_1\exp(\hat{\boldsymbol{\xi}}'_2 q_2)\cdots \\ &\quad {}^{n-2}\boldsymbol{H}_{n-1}\exp(\hat{\boldsymbol{\xi}}'_n q_n){}^{n-1}\boldsymbol{H}_n{}^{n}\boldsymbol{H}_T \\ &= \exp({}^{B}\boldsymbol{H}_0\hat{\boldsymbol{\xi}}'_1{}^{B}\boldsymbol{H}_0^{-1}q_1)\exp({}^{B}\boldsymbol{H}_0{}^{0}\boldsymbol{H}_1\hat{\boldsymbol{\xi}}'_2({}^{B}\boldsymbol{H}_0{}^{0}\boldsymbol{H}_1)^{-1}q_2) \\ &\quad {}^{B}\boldsymbol{H}_0{}^{0}\boldsymbol{H}_1\cdots{}^{n-2}\boldsymbol{H}_{n-1}\exp(\hat{\boldsymbol{\xi}}'_n q_n){}^{n-1}\boldsymbol{H}_n{}^{n}\boldsymbol{H}_T \\ &\quad \vdots \\ &= \exp(\hat{\boldsymbol{\xi}}_1 q_1)\exp(\hat{\boldsymbol{\xi}}_2 q_2)\cdots\exp(\hat{\boldsymbol{\xi}}_n q_n)\exp(\hat{\boldsymbol{\xi}}_T) \end{aligned} \tag{11}$$

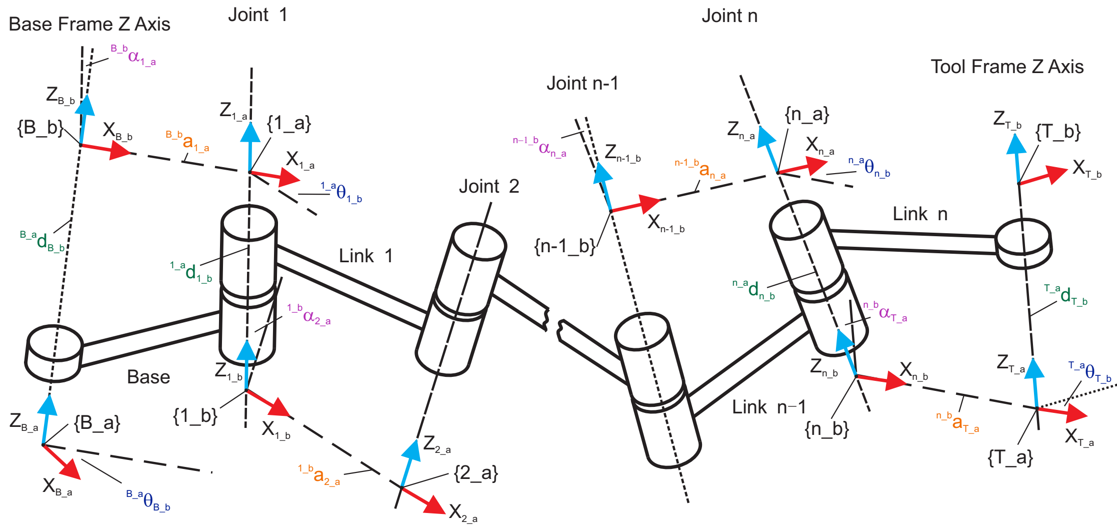


Fig. 2. D-H parameters including arbitrarily located base frame and tool frame.

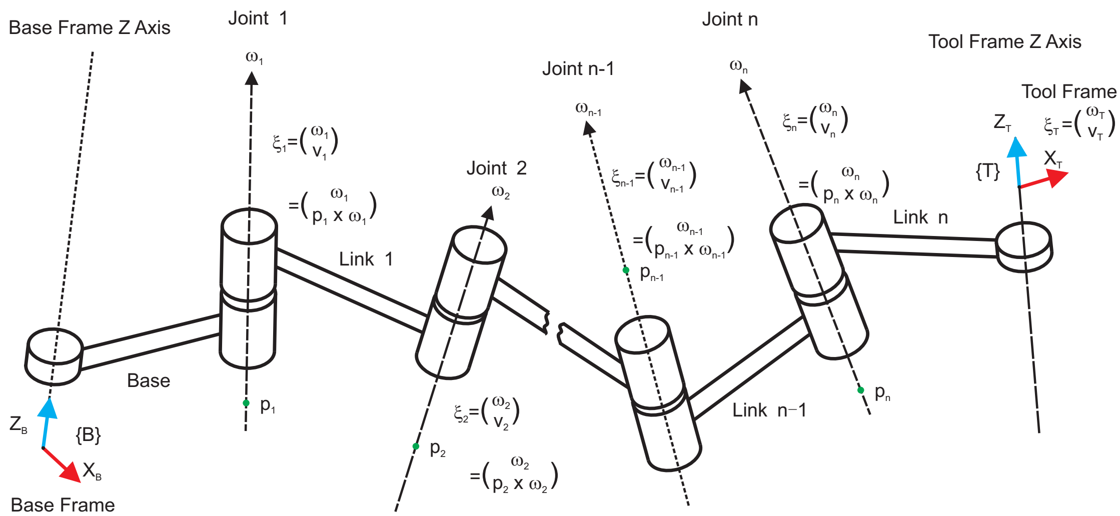


Fig. 3. Modeling of a serial-link robot using the POE method.

where, for $i=1,2,\ldots,n$,

$$\hat{\boldsymbol{\xi}}_i = ({}^{B}\boldsymbol{H}_0{}^{0}\boldsymbol{H}_1\cdots{}^{i-2}\boldsymbol{H}_{i-1})\hat{\boldsymbol{\xi}}_i'({}^{B}\boldsymbol{H}_0{}^{0}\boldsymbol{H}_1\cdots{}^{i-2}\boldsymbol{H}_{i-1})^{-1} \quad (12)$$

and

$$\exp(\hat{\boldsymbol{\xi}}_T) = {}^{B}\boldsymbol{H}_0{}^{0}\boldsymbol{H}_1\cdots{}^{n-2}\boldsymbol{H}_{n-1}{}^{n-1}\boldsymbol{H}_n{}^{n}\boldsymbol{H}_T. \quad (13)$$

Hence, the D-H model is converted into the POE formula.

B. Conversion from POE Parameters to D-H Parameters

Conversion from the POE parameters to the D-H parameters is more complicated. Before deriving the algorithm, the following lemmas should be proved first.

Lemma 1. *For a revolute joint, the exponential* $\exp(\hat{\boldsymbol{\xi}}q)$ *can be converted into the form of* $\boldsymbol{H}R_Z(q)\boldsymbol{H}^{-1}$*, where* $\boldsymbol{H} = R_Z(\theta)T_Z(d)R_X(\alpha)T_X(a)$.

Proof. As previously introduced, $R_Z(q)$ can be written as $\exp(\hat{\boldsymbol{\xi}}'q)$ where $\boldsymbol{\xi}'$ is as in (8). Hence, we have

$$\boldsymbol{H}R_Z(q)\boldsymbol{H}^{-1} = \boldsymbol{H}\exp(\hat{\boldsymbol{\xi}}'q)\boldsymbol{H}^{-1} = \exp(\boldsymbol{H}\hat{\boldsymbol{\xi}}'\boldsymbol{H}^{-1}q). \quad (14)$$

It is then clear that proving $\exp(\hat{\boldsymbol{\xi}}q) = \boldsymbol{H}R_Z(q)\boldsymbol{H}^{-1}$ is equivalent to proving $\hat{\boldsymbol{\xi}} = \boldsymbol{H}\hat{\boldsymbol{\xi}}'\boldsymbol{H}^{-1}$, or the twist coordinates form,

$$\boldsymbol{\xi} = Ad(\boldsymbol{H})\boldsymbol{\xi}' \quad (15)$$

where $Ad(\cdot)$ is the adjoint transformation and has the form

$$Ad(\boldsymbol{H}) = Ad(\begin{bmatrix}\boldsymbol{R} & \boldsymbol{t}\\ \boldsymbol{0}^T & 1\end{bmatrix}) = \begin{bmatrix}\boldsymbol{R} & \boldsymbol{0}\\ \hat{\boldsymbol{t}}\boldsymbol{R} & \boldsymbol{R}\end{bmatrix} \quad (16)$$

in which $\boldsymbol{R}$ and $\boldsymbol{t}$ are the rotation matrix and translation vector in $\boldsymbol{H}$, respectively.

Let us assume $\boldsymbol{H}$ can be represented by $R_Z(\theta)T_Z(d)R_X(\alpha)T_X(a)$. Expanding $\boldsymbol{H}$ and substituting (8) and (16) into (15), we obtain

$$\boldsymbol{\omega} := \begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix} = \begin{bmatrix}\sin(\theta)\sin(\alpha)\\ -\cos(\theta)\sin(\alpha)\\ \cos(\alpha)\end{bmatrix} \tag{17}$$

$$\boldsymbol{v} := \begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}\omega_3\sin(\theta) & -\omega_2\\ -\omega_3\cos(\theta) & \omega_1\\ \omega_2\cos(\theta)-\omega_1\sin(\theta) & 0\end{bmatrix}\begin{bmatrix}a\\ d\end{bmatrix} \tag{18}$$

where $\boldsymbol{\omega}$ and $\boldsymbol{v}$ are the components of $\boldsymbol{\xi}$.

Therefore, all that is left is to find solutions to (17) and (18). For this purpose, three possible cases of ω_3 are examined:

1) $\underline{\omega_3 = 1}$ This means $\omega_1 = \omega_2 = v_3 = 0$ since $\boldsymbol{\omega}$ and $\boldsymbol{v}$ conform to the two constraints

$$||\boldsymbol{\omega}|| = 1 \tag{19}$$

$$\boldsymbol{\omega}^T\boldsymbol{v} = 0. \tag{20}$$

It can be verified that ($\alpha = 0$, $a = \sqrt{v_1^2+v_2^2}$, $\theta = \operatorname{atan2}(v_1, -v_2)$, $d = 0$) is one of the solutions to (17) and (18);

2) $\underline{\omega_3 = -1}$ Similarly to case 1, it can be verified that ($\alpha = \pi$, $a = \sqrt{v_1^2+v_2^2}$, $\theta = \operatorname{atan2}(-v_1, v_2)$, $d = 0$) is one of the solutions to (17) and (18);

3) <u>OTHERWISE</u> According to (17), we have

$$\alpha = \pm\arccos(\omega_3) \tag{21}$$

$$\theta = \operatorname{atan2}(\omega_1/\sin(\alpha), -\omega_2/\sin(\alpha)). \tag{22}$$

Substituting (22) into the last row of (18), we have

$$a\sin(\alpha) = -v_3. \tag{23}$$

As the link length a is usually assumed to be nonnegative, the polarity of the right side of (23) can be used to disambiguate the sign of α in (21).

Then, a and d can be solved from (18) by using any technique of solving a system of equations.

Alternatively, a can be obtained from (23) as

$$a = -\frac{v_3}{\sin(\alpha)}. \tag{24}$$

Substituting (19), (20), (21), (22), and (24) into (18), we can verify that

$$d = \frac{\omega_1 v_2 - \omega_2 v_1}{\omega_1^2+\omega_2^2} \tag{25}$$

is a solution to (18).

In summary, solutions to (17) and (18) can always be found and thus the lemma is proved. □

Lemma 2. *For a prismatic joint, the exponential* $\exp(\hat{\boldsymbol{\xi}}q)$ *can be converted into the form of* $\boldsymbol{H}T_Z(q)\boldsymbol{H}^{-1}$*, where* $\boldsymbol{H} = R_Z(\theta)T_Z(d)R_X(\alpha)T_X(a)$.

Proof. Similarly to the proof of Lemma 1, all that is required is to prove that $\boldsymbol{\xi} = Ad(\boldsymbol{H})\boldsymbol{\xi}'$ where $\boldsymbol{\xi}'$ is as in (9).

By expanding $\boldsymbol{H}$, the problem is equivalent to proving that

$$\boldsymbol{v} := \begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}\sin(\theta)\sin(\alpha)\\ -\cos(\theta)\sin(\alpha)\\ \cos(\alpha)\end{bmatrix} \tag{26}$$

always has a solution. Note that $\boldsymbol{v}$ conforms to the constraint $||\boldsymbol{v}|| = 1$. Again, three possible cases of v_3 can be examined:

1) $\underline{v_3 = 1}$ This means $v_1 = v_2 = 0$. It can be verified that ($\alpha = 0, \theta = 0$) is a solution to (26);

2) $\underline{v_3 = -1}$ Similarly to case 1, ($\alpha = \pi, \theta = 0$) can be verified to be a solution to (26);

3) <u>OTHERWISE</u> The equation will have a solution as ($\alpha = \arccos(v_3)$, $\theta = \operatorname{atan2}(v_1/\sin(\alpha), -v_2/\sin(\alpha))$.

In all the three cases of v_3, the values of a and d do not affect the result. Thus, they can be set to zeros for simplicity.

Hence, since (26) always has a solution, the lemma is proved. □

Lemma 3. *For an arbitrary twist* $\hat{\boldsymbol{\xi}}$*, the exponential* $\exp(\hat{\boldsymbol{\xi}})$ *can be decomposed into a product* $\exp(\hat{\boldsymbol{\xi}}) = \boldsymbol{H}_1\boldsymbol{H}_2$*, where* $\boldsymbol{H}_1 = R_Z(\theta_1)T_Z(d_1)R_X(\alpha_1)T_X(a_1)$ *and* $\boldsymbol{H}_2 = R_Z(\theta_2)T_Z(d_2)$.

Proof. The exponential mapping is written in the matrix form,

$$\exp(\hat{\boldsymbol{\xi}}) = \begin{bmatrix}\boldsymbol{R} & \boldsymbol{t}\\ \boldsymbol{0}^T & 1\end{bmatrix}. \tag{27}$$

Thus, the problem is equivalent to finding a group of (θ_1, d_1, α_1, a_1, θ_2, d_2) that satisfies

$$\begin{bmatrix}\boldsymbol{R} & \boldsymbol{t}\\ \boldsymbol{0}^T & 1\end{bmatrix} = R_Z(\theta_1)T_Z(d_1)R_X(\alpha_1)T_X(a_1)R_Z(\theta_2)T_Z(d_2). \tag{28}$$

Expanding (28), we have

$$\boldsymbol{R} = R_Z(\theta_1)R_X(\alpha_1)R_Z(\theta_2) \tag{29}$$

$$\boldsymbol{t} = \begin{bmatrix}0 & \cos(\theta_1) & \sin(\theta_1)\sin(\alpha_1)\\ 0 & \sin(\theta_1) & -\cos(\theta_1)\sin(\alpha_1)\\ 1 & 0 & \cos(\alpha_1)\end{bmatrix}\begin{bmatrix}d_1\\ a_1\\ d_2\end{bmatrix} := \boldsymbol{A}\boldsymbol{x}. \tag{30}$$

It can be seen that (29) is the Euler angles decomposition of $\boldsymbol{R}$ in the ZXZ order. It is known that this decomposition always has at least one solution. According to the value of α_1, three cases are discussed as follows:

1) $\underline{\alpha_1 = 0}$ This causes the Euler angles decomposition to be singular as infinite solutions to (29) exist. At the same time, (30) becomes

$$\boldsymbol{t} = \begin{bmatrix}a_1\cos(\theta_1)\\ a_1\sin(\theta_1)\\ d_1+d_2\end{bmatrix}. \tag{31}$$

Therefore, we have ($a_1 = \sqrt{t_1^2+t_2^2}$, $\theta_1 = \operatorname{atan2}(t_2, t_1)$). By substituting θ_1 and α_1 into (29), θ_2 can be solved. Since d_1 and d_2 are redundant, ($d_1 = t_3$, $d_2 = 0$) can be adopted for simplicity.

2) $\underline{\alpha_1 = \pi}$ Similarly to case 1, the Euler angles decomposition has infinite solutions. In addition, (30) becomes

$$\boldsymbol{t} = \begin{bmatrix}a_1\cos(\theta_1)\\ a_1\sin(\theta_1)\\ d_1-d_2\end{bmatrix}. \tag{32}$$

Again, we have ($a_1 = \sqrt{t_1^2+t_2^2}$, $\theta_1 = \operatorname{atan2}(t_2, t_1)$) and can substitute θ_1 and α_1 into (29) to solve θ_2. Moreover, ($d_1 = t_3$, $d_2 = 0$) can be taken as the solution.

3) <u>OTHERWISE</u> The Euler angles decomposition has exactly one solution in this case. For (30), it can be verified that

$$
\begin{aligned}
{}^{B}\boldsymbol{H}_T &= \exp(\hat{\boldsymbol{\xi}}_1 q_1)\exp(\hat{\boldsymbol{\xi}}_2 q_2)\cdots\exp(\hat{\boldsymbol{\xi}}_n q_n)\exp(\hat{\boldsymbol{\xi}}_T) \\
&\overset{\text{Lemma 1 \& 2}}{=} {}^{B}\boldsymbol{H}_0 Q(q_1)\,{}^{B}\boldsymbol{H}_0^{-1}\exp(\hat{\boldsymbol{\xi}}_2 q_2)\cdots\exp(\hat{\boldsymbol{\xi}}_n q_n)\exp(\hat{\boldsymbol{\xi}}_T) \\
&= {}^{B}\boldsymbol{H}_0 Q(q_1)\exp({}^{B}\boldsymbol{H}_0^{-1}\hat{\boldsymbol{\xi}}_2\,{}^{B}\boldsymbol{H}_0 q_2)\,{}^{B}\boldsymbol{H}_0^{-1}\cdots\exp(\hat{\boldsymbol{\xi}}_n q_n)\exp(\hat{\boldsymbol{\xi}}_T) \\
&\overset{\boldsymbol{\xi}_2' := Ad({}^{B}\boldsymbol{H}_0^{-1})\boldsymbol{\xi}_2}{=} {}^{B}\boldsymbol{H}_0 Q(q_1)\exp(\hat{\boldsymbol{\xi}}_2' q_2)\,{}^{B}\boldsymbol{H}_0^{-1}\cdots\exp(\hat{\boldsymbol{\xi}}_n q_n)\exp(\hat{\boldsymbol{\xi}}_T) \\
&\overset{\text{Lemma 1 \& 2}}{=} {}^{B}\boldsymbol{H}_0 Q(q_1)\,{}^{0}\boldsymbol{H}_1 Q(q_2)\,{}^{0}\boldsymbol{H}_1^{-1}\,{}^{B}\boldsymbol{H}_0^{-1}\cdots\exp(\hat{\boldsymbol{\xi}}_n q_n)\exp(\hat{\boldsymbol{\xi}}_T) \\
&\;\;\vdots \\
&= {}^{B}\boldsymbol{H}_0 Q(q_1)\,{}^{0}\boldsymbol{H}_1 Q(q_2)\cdots{}^{n-2}\boldsymbol{H}_{n-1} Q(q_n)\,{}^{n-2}\boldsymbol{H}_{n-1}^{-1}\cdots{}^{0}\boldsymbol{H}_1^{-1}\,{}^{B}\boldsymbol{H}_0^{-1}\exp(\hat{\boldsymbol{\xi}}_T) \\
&\overset{\exp(\hat{\boldsymbol{\xi}}_T') := {}^{n-2}\boldsymbol{H}_{n-1}^{-1}\cdots{}^{0}\boldsymbol{H}_1^{-1}\,{}^{B}\boldsymbol{H}_0^{-1}\exp(\hat{\boldsymbol{\xi}}_T)}{=} {}^{B}\boldsymbol{H}_0 Q(q_1)\,{}^{0}\boldsymbol{H}_1 Q(q_2)\cdots{}^{n-2}\boldsymbol{H}_{n-1} Q(q_n)\exp(\hat{\boldsymbol{\xi}}_T') \\
&\overset{\text{Lemma 3}}{=} {}^{B}\boldsymbol{H}_0 Q(q_1)\,{}^{0}\boldsymbol{H}_1 Q(q_2)\cdots{}^{n-2}\boldsymbol{H}_{n-1} Q(q_n)\,{}^{n-1}\boldsymbol{H}_n\,{}^{n}\boldsymbol{H}_T
\end{aligned}
\tag{33}
$$

the determinant $|\boldsymbol{A}| = -\sin(\alpha_1) \neq 0$. Thus, $\boldsymbol{A}$ is invertible and (30) has exactly one solution $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{t}$.

Hence, a group of (θ_1, d_1, α_1, a_1, θ_2, d_2) can always be found to satisfy (28), and thus the lemma is proved. □

Now we look back into the original problem of converting the POE parameters to the D-H parameters. Since Lemma 1 and Lemma 2 have the same form regardless of the type of the joint, a uniform function, $Q(q)$, is used to represent the rotation or translation. When the joint is revolute, $Q(q) = R_Z(q)$, and when the joint is prismatic, $Q(q) = T_Z(q)$. Given the forward kinematics of the robot in the POE formula (7), we have (33) where ${}^{B}\boldsymbol{H}_0$, ${}^{0}\boldsymbol{H}_1$, ..., ${}^{n}\boldsymbol{H}_T$ are in the same form as in (3).

Comparing (33) to (5), we can see the POE formula is successfully converted into the standard D-H model. If the modified D-H version is to be used, we just need to regroup the parameters according to (4).

It is worth noting that we have encoded the base frame and the tool frame with D-H notations in the uniform D-H representation to allow the two frames to be arbitrarily placed. If there are restrictions on the desired location of the base frame or the tool frame after the conversion, the frame transformation for the base frame or the tool frame can be performed first in the POE setting, and then converted into the D-H model.

An implementation of the proposed algorithm in MATLAB is provided to facilitate the practical use. In the codes, an example of using the developed algorithm can be found.

IV. Conclusion

This paper has presented an analytic approach to converting the POE parameters of a robot to the D-H parameters. By formulating the base transformation and the tool transformation with the D-H notation, the proposed method applies to the POE model with an arbitrarily assigned base frame as well as an arbitrarily located tool frame. In addition, the converted D-H parameters are consistent with both the standard notation and the modified notation.

With this conversion, it is now possible to directly leverage established algorithms formulated with D-H parameters or compensate the D-H model after calibration with the POE parameters. It also provides a simpler method of determining the kinematic parameters of a robot.

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