# Chern-Simons five-form and holographic baryons

Pak Hang Chris Lau[1,2,*] and Shigeki Sugimoto[1,3,†]

[1] *Center for Gravitational Physics, Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan*

[2] *Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*

[3] *Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo, Kashiwanoha, Kashiwa 277-8583, Japan*

(Received 13 January 2017; published 16 June 2017)

In the top-down holographic model of QCD based on D4/D8-branes in type IIA string theory and some of the bottom-up models, the low energy effective theory of mesons is described by a five-dimensional Yang-Mills-Chern-Simons theory in a certain curved background with two boundaries. The five-dimensional Chern-Simons term plays a crucial role in reproducing the correct chiral anomaly in four-dimensional massless QCD. However, there are some subtle ambiguities in the definition of the Chern-Simons term for the cases with topologically nontrivial gauge bundles, which include the configurations with baryons. In particular, for the cases with three flavors, it was pointed out by Hata and Murata that the naive Chern-Simons term does not lead to an important constraint on the baryon spectrum, which is needed to pick out the correct baryon spectrum observed in nature. In this paper, we propose a formulation of a well-defined Chern-Simons term which can be used for the cases with baryons, and show that it recovers the correct baryon constraint as well as the chiral anomaly in QCD.

DOI: 10.1103/PhysRevD.95.126007

## I. INTRODUCTION

The gauge/gravity duality provides a powerful method to study strongly coupled gauge theories using theories with gravity [1–3]. One of its surprising features is that the space-time dimensions of the gravity side are higher than those of the corresponding gauge theory. For this reason this type of duality is called holographic duality. It has been applied to QCD and there have been a lot of successes in revealing the properties of QCD and physics of hadrons.[1] The holographic dual description of QCD (or QCD-like theory) is called holographic QCD. A common feature of the holographic models is that the meson effective action is given as a five-dimensional gauge theory embedded in a certain curved background.

In this paper, our main focus is on the five-dimensional Chern-Simons (CS) term[2]

$$S_{\mathrm{CS}} = C \int_{M_5} \omega_5(A), \tag{1.1}$$

where $C$ is a constant and $\omega_5(A)$ is the CS five-form that satisfies $d\omega_5(A) = \mathrm{tr}(F^3)$. The explicit form of the CS five-form is

$$\begin{aligned} \omega_5(A) &\equiv \mathrm{tr}\left(AF^2 - \frac{1}{2}A^3F + \frac{1}{10}A^5\right) \\ &= \mathrm{tr}\left(AdAdA + \frac{3}{2}A^3dA + \frac{3}{5}A^5\right). \end{aligned} \tag{1.2}$$

It appears in the meson effective action in the top-down holographic model of QCD proposed in [5][3] and some of the bottom-up models (see, e.g., [7–10]). In these models, the effective theory of mesons is described by a five-dimensional $U(N_f)$ Yang-Mills-Chern-Simons (YM-CS) action on a curved space-time $M_5$, where $N_f$ is the number of massless quarks, and the coefficient of the CS term is related to the number of colors $N_c$ by

$$C = \frac{iN_c}{24\pi^2}. \tag{1.3}$$

The normalizable modes of the five-dimensional $U(N_f)$ gauge field $A$ correspond to the degrees of freedom of a tower of vector and axial vector mesons (such as rho meson, omega meson, $a_1$ meson, etc.) as well as the massless pions.[4] It has been shown that the masses as well as coupling constants for low-lying mesons read off from the five-dimensional YM-CS theory turn out to be in

*pakhang.lau@yukawa.kyoto-u.ac.jp
†sugimoto@yukawa.kyoto-u.ac.jp

[1] See [4] for a recent review.

[2] Here, the gauge field $A$ is a one-form and its field strength $F = dA + A \wedge A$ is a two-form that take values in the anti-Hermitian matrices. We often omit the symbol "∧" for the wedge products of the differential forms.

[3] See [6] for a review.

[4] In this paper, we consider the cases with massless quarks. See [11] for the proposals to include quark masses.

 

reasonably good agreement with the experimental data and provide some predictions for the unknown parameters.

The CS term plays crucial roles in many aspects in holographic QCD. First of all, the chiral anomaly in QCD is correctly reproduced due to the CS term. In fact, the five-dimensional expression of the Wess-Zumino-Witten (WZW) term in QCD [12–14] has a direct physical interpretation in terms of the five-dimensional CS term in holographic QCD [5]. Furthermore, some of the decay modes of the omega meson ($\omega \to \pi^0\gamma$ and $\omega \to \pi^0\pi^+\pi^-$) are induced by terms generated from the CS term. Surprisingly, the structure of the interaction terms for these decay modes predicted by holographic QCD agrees with that of the Gell-Mann–Sharp–Wagner model [15], which is a phenomenological model proposed to reproduce the experimental data of the omega meson decay [16] (see also [9]). The CS term is also important in the analysis of baryons. Due to the CS term, it can be shown that the baryon number is equal to the instanton number defined on a time slice [5]. When the vector (and axial-vector) mesons are integrated out, the five-dimensional YM-CS action reduces to the action of the Skyrme model [5,16]. The Skyrme model was proposed by Skyrme to describe baryons as topological solitons called Skyrmions [17]. The pion field in the soliton has a nontrivial winding number representing an element of the homotopy group $\pi_3(U(N_f)) \simeq \mathbb{Z}$. The relation between the instanton number for the five-dimensional gauge field and the winding number carried by the pion field is precisely that proposed by Atiyah and Manton [18] in an attempt to obtain approximate Skyrmion solutions by using instanton solutions.

However, there are some subtle ambiguities in the definition of the CS term. In the explicit expression of the CS term in (1.1) with (1.2), we have implicitly assumed that the gauge field $A$ is a globally well-defined one-form on the five-dimensional space-time $M_5$. This is, however, not always possible when the gauge configuration with a given boundary condition is topologically nontrivial, including the cases with baryons. In such cases, it is necessary to cover the five-dimensional space-time $M_5$ by multiple patches on which the gauge field is well defined. One might naively think that the CS term can be defined as just a sum of the CS term defined on each patch. However, this approach does not work, because it depends on the choice of the gauge, and some additional terms are needed to make it well defined. Related to this issue, a problem was pointed out by Hata and Murata in [19]. They tried to analyze the spectrum of baryons in the case with $N_f = 3$, generalizing the analysis for $N_f = 2$ in [20], and claimed that a constraint needed to get the correct baryon spectrum [see (2.29)] cannot be obtained by using the naive CS term. They proposed a new CS term that gives the correct constraint, but it does not reproduce the chiral anomaly of QCD. Our main goal is to propose a well-defined CS term that solves all these problems.

The paper is organized as follows. We start with reviewing the problems in more detail while fixing our notation in Sec. II. Our proposal for the well-defined CS term is given in Sec. III. In Sec. IV, we revisit the analysis of the effective action for the collective coordinates of the soliton solution representing baryons and show that the correct constraint is obtained from the new CS term. Section V gives a summary and outlook.

## II. PUZZLE

### A. The model

Our starting point is the five-dimensional $U(N_f)$ YM-CS action given by

$$S_{5\,\mathrm{dim}} = S_{\mathrm{YM}} + S_{\mathrm{CS}}, \tag{2.1}$$

with $S_{\mathrm{CS}}$ as defined in (1.1) and the kinetic term for the gauge field

$$S_{\mathrm{YM}} = -\frac{\kappa}{2}\int_{M_5} \mathrm{tr}(F \wedge *F), \tag{2.2}$$

where $\kappa$ is a constant and $*$ is the Hodge star in five-dimensional space-time $M_5$. Although the details of the metric on $M_5$ are not important in our main purpose, we use the following form of the metric for explicit calculations:

$$ds^2 = 4(k(z)\tilde{k}(z)\eta_{\mu\nu}dx^\mu dx^\nu + \tilde{k}(z)^2 dz^2), \tag{2.3}$$

where $x^\mu$ ($\mu = 0, 1, 2, 3$) are the coordinates for the four-dimensional Minkowski space-time and $z$ is the coordinate for the fifth direction. Then, the Hodge dual of the field strength two-form $F$ is

$$*F = -\frac{k(z)}{3}F^\mu{}_z\epsilon_{\mu\nu\rho\sigma}dx^\nu dx^\rho dx^\sigma + \frac{\tilde{k}(z)}{2}F^{\mu\nu}\epsilon_{\mu\nu\rho\sigma}dx^\rho dx^\sigma dz, \tag{2.4}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor in four-dimensional Minkowski space with $\epsilon_{0123} = +1$, and the Lorentz indices are raised and lowered by the Minkowski metric $(\eta_{\mu\nu}) = (\eta^{\mu\nu}) = \mathrm{diag}(-1, 1, 1, 1)$. Then, the YM action (2.2) is written as

$$S_{\mathrm{YM}} = \kappa\int d^4x dz\,\mathrm{tr}\left(\frac{1}{2}\tilde{k}(z)F_{\mu\nu}F^{\mu\nu} + k(z)F_{\mu z}F^\mu{}_z\right). \tag{2.5}$$

The meson effective action in [5] is given by (2.1) with $\tilde{k}(z) = (1 + z^2)^{-1/3}$ and $k(z) = 1 + z^2$.

The boundary of $M_5$ is a disjoint union of the four-dimensional edges at $z \to +\infty$ and $z \to -\infty$[5]:

[5]Note that the asymptotic region at $|x^\mu| \to \infty$ is not regarded as the boundary. In order to avoid confusion, we compactify the $x^\mu$ directions in the following discussion.

$$\partial M_5 = M_4^{(+\infty)} \cup (-M_4^{(-\infty)}), \tag{2.6}$$

where $M_4^{(\pm\infty)} \equiv M_5|_{z\to\pm\infty}$ and the minus sign in front of $M_4^{(-\infty)}$ means the orientation is reversed. The boundary values of the gauge field pulled back on $M_4^{(\pm\infty)}$, denoted as $A|_{z\to\pm\infty} (= \lim_{z\to\pm\infty} A_\mu dx^\mu)$, are interpreted as the external gauge fields associated with the chiral symmetry $U(N_f)_L \times U(N_f)_R$ in QCD.[6] More precisely, we set $\hat{A}_\pm = A|_{z\to\pm\infty}$, where $\hat{A}_+$ and $\hat{A}_-$ are the external gauge fields associated with $U(N_f)_R$ and $U(N_f)_L$, respectively. Because the gauge field at the boundary is fixed, the gauge symmetry of the system consists of the gauge transformation that acts trivially at the boundaries. The gauge transformation at $z \to \pm\infty$ corresponds to that of the chiral symmetry. Note that the CS term (1.1) is not invariant under the gauge transformation that acts nontrivially at the boundary. In fact, the infinitesimal gauge transformation of the CS term with $\delta_\Lambda A = d\Lambda + [A, \Lambda] \equiv D_A\Lambda$ is

$$\delta_\Lambda S_{\rm CS} = C\left(\int_{M_4^{(+\infty)}} \omega_4^1(\hat{\Lambda}_+, \hat{A}_+) - \int_{M_4^{(-\infty)}} \omega_4^1(\hat{\Lambda}_-, \hat{A}_-)\right), \tag{2.7}$$

where $\hat{\Lambda}_\pm \equiv \Lambda|_{z\to\pm\infty}$ and

$$\omega_4^1(\Lambda, A) \equiv {\rm tr}\left(\Lambda d\left(AdA + \frac{1}{2}A^3\right)\right). \tag{2.8}$$

Here, we have used the formula

$$\delta_\Lambda \omega_5(A) = d\omega_4^1(\Lambda, A) + \mathcal{O}(\Lambda^2), \tag{2.9}$$

and the Stokes' theorem.[7] (2.7) precisely agrees with the chiral anomaly in QCD.[8]

### B. Problems of the CS term

In order to illustrate the problem clearly, let us compactify the time and $x^{1\sim3}$ directions, and consider the case that the topology of the space-time is equivalent to

$$M_5 \simeq S^1 \times S^3 \times \mathbb{R}, \tag{2.10}$$

where $S^1$ is the compactified time direction, $S^3$ is the compactified $x^{1\sim3}$ directions and $\mathbb{R}$ is the $z$ direction.[9] As shown in [5], the baryon number $n_B$ is given by the instanton number on a time slice (see also Sec. III D for a derivation):

$$n_B = \frac{1}{8\pi^2}\int_{S^3\times\mathbb{R}} {\rm tr}(F^2). \tag{2.11}$$

When the gauge field $A$ is a globally well-defined one-form on $M_5$, using the formula

$${\rm tr}(F^2) = d\omega_3(A), \tag{2.12}$$

with the CS three-form

$$\omega_3(A) \equiv {\rm tr}\left(AF - \frac{1}{3}A^3\right) = {\rm tr}\left(AdA + \frac{2}{3}A^3\right), \tag{2.13}$$

and the Stokes' theorem, (2.11) can be rewritten as

$$n_B = \frac{1}{8\pi^2}\left(\int_{S^3} \omega_3(A)|_{z\to+\infty} - \int_{S^3} \omega_3(A)|_{z\to-\infty}\right). \tag{2.14}$$

This expression inevitably vanishes if we impose the boundary condition $A|_{z\to\pm\infty} = 0$. Therefore, if we adopt the identification $\hat{A}_\pm = A|_{z\to\pm\infty}$ in the previous subsection, the globally well-defined gauge field $A$ can describe only the $n_B = 0$ sector of the gauge configuration, when the external gauge fields $\hat{A}_\pm$ are turned off. This is clearly restricting the gauge configurations too much. As usual in gauge theory, we should include the gauge configurations defined on topologically nontrivial gauge bundles.

In order to describe gauge configurations with nonzero baryon number, we cover the space-time manifold $M_5$ with two patches as

$$M_5 = M_5^- \cup M_5^+, \tag{2.15}$$

where $M_5^\pm$ are chosen to be $M_5^\pm \equiv \{(x^\mu, z) \in M_5 | \pm z > -\epsilon\}$ with a small positive parameter $\epsilon$. The intersection of the two patches is

$$M_5^- \cap M_5^+ \simeq M_4^{(0)} \times (-\epsilon, +\epsilon), \tag{2.16}$$

where $M_4^{(0)} \equiv \{(x^\mu, z) \in M_5 | z = 0\} \simeq S^1 \times S^3$. In the following, we understand $\epsilon$ as an infinitesimal parameter and take the limit $\epsilon \to 0$ at the end of the calculations. The picture in the $\epsilon \to 0$ limit is depicted in Fig. 1. The gauge configuration is defined by the gauge field $A_\pm$ defined on

[6] The axial $U(1)$ subgroup of $U(N_f)_L \times U(N_f)_R$ is anomalous. This anomaly can also be seen in string theory as discussed in [5], but we will not discuss it here.

[7] See the Appendix for our notations and useful formulas.

[8] See, e.g., a textbook [21] for a review of anomaly.

[9] To be more precise, we add the boundary points $\{z \to \pm\infty\}$ to $\mathbb{R}$ and treat the $z$ direction as a closed interval $I = [-\infty, +\infty]$.

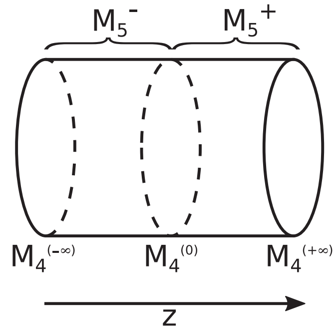


FIG. 1. The five-dimensional space-time $M_5$.

each patch $M_5^{\pm}$[10] and connected by the gluing condition on the intersection as

$$A_+ = A_-^h \equiv hA_-h^{-1} + hdh^{-1} \quad (\text{on } M_5^- \cap M_5^+), \tag{2.17}$$

where $h$ is a $U(N_f)$-valued function defined on the intersection $M_5^- \cap M_5^+$. The external gauge fields $\hat{A}_\pm$ are now related to the boundary values of the gauge fields $A_\pm$ as

$$\hat{A}_\pm \equiv A_\pm|_{z\to\pm\infty}. \tag{2.18}$$

The gauge transformation is given by

$$A_\pm \to A_\pm^{g_\pm} \equiv g_\pm A_\pm g_\pm^{-1} + g_\pm dg_\pm^{-1}, \quad h \to g_+ h g_-^{-1}, \tag{2.19}$$

where $g_\pm$ are $U(N_f)$-valued functions on $M_5^\pm$. The boundary values of the gauge functions $\hat{g}_\pm \equiv g_\pm|_{z\to\pm\infty}$ correspond to those of the (gauged) chiral symmetry as $(\hat{g}_-, \hat{g}_+) \in U(N_f)_L \times U(N_f)_R$.

In this setup, it is possible to have gauge configurations with nonzero baryon number. In fact, (2.11) gives

$$n_B = \frac{1}{24\pi^2}\int_{S^3} \mathrm{tr}((hdh^{-1})^3)|_{z=0} \tag{2.20}$$

for the case with $\hat{A}_\pm = 0$. The baryon number (2.20) is equivalent to the winding number given as an element of $\pi_3(U(N_f)) \simeq \mathbb{Z}$ represented by the $U(N_f)$-valued function $h|_{z=0}$ restricted at a time slice.

[10]For simplicity, we have assumed here that $A_\pm$ are well-defined $U(N_f)$-valued one-forms on $M_5^\pm$. This is always the case for a static gauge configuration and a small perturbation around it, because the gauge bundle over $S^3$ is trivial due to $\pi_2(U(N_f)) \simeq 0$. A counterexample is a gauge configuration with nonzero instanton number defined on $S^1 \times S^3$, which looks like a baryon configuration with the time and $z$ directions interchanged. General gauge configurations may be described by introducing more patches to have good covering of $M_5$, though we will not discuss the details here.

The question now is how to define the CS term in this setup. While the CS term is supposed to give the correct chiral anomaly, we should make sure that it is invariant (up to a $2\pi$ shift) under the gauge transformations with $\hat{g}_\pm = 1$ that act trivially at the boundary. One can immediately see that a naive expression like

$$C\left(\int_{M_5^-} \omega_5(A_-) + \int_{M_5^+} \omega_5(A_+)\right), \tag{2.21}$$

does not work. This is one of the reasons that the naive CS term has to be modified.

Another approach is to insist on a globally well-defined gauge field $A$, and modify the relation between the boundary values of the gauge field and the external gauge field associated with the chiral symmetry. This can be achieved from the above description by the gauge transformation (2.19) with $g_\pm = h_\pm$ satisfying $h_+ h h_-^{-1} = 1$ on $M_5^- \cap M_5^+$. Then, the gauge field $A$ defined as

$$A \equiv A_\pm^{h_\pm} \quad \text{on} \quad M_5^\pm \tag{2.22}$$

is a globally well-defined one-form on $M_5$, because the gluing condition (2.17) implies $A_+^{h_+} = A_-^{h_-}$ on the intersection $M_5^- \cap M_5^+$. In this case, because of the relation (2.18), the boundary values of the gauge field $A$ are not equal to the external gauge fields $\hat{A}_\pm$, but related by the gauge transformation as

$$A|_{z\to\pm\infty} = \hat{A}_\pm^{\hat{h}_\pm}, \tag{2.23}$$

where $\hat{h}_\pm \equiv h_\pm|_{z\to\pm\infty}$. It is important to note that a gauge configuration is specified by the pair $(A, \hat{h}_\pm)$. Two gauge configurations with the same gauge field $(A, \hat{h}_\pm)$ and $(A, \hat{h}'_\pm)$ can be physically inequivalent when $\hat{h}_\pm$ and $\hat{h}'_\pm$ are different.

It is easy to see that, with the identification (2.23), the expressions for the baryon number (2.14) and (2.20) are identical. When the external gauge fields are turned off, the boundary values of the gauge field are given by $A|_{z\to\pm\infty} = \hat{h}_\pm d\hat{h}_\pm^{-1}$ and the baryon number (2.14) is given by the difference of the winding number carried by $\hat{h}_+$ and $\hat{h}_-$ as

$$n_B = -\frac{1}{24\pi^2}\int_{S^3} (\mathrm{tr}((\hat{h}_+ d\hat{h}_+^{-1})^3) - \mathrm{tr}((\hat{h}_- d\hat{h}_-^{-1})^3)). \tag{2.24}$$

Therefore, for the gauge configurations with nonzero baryon number, $\hat{h}_\pm$ cannot be trivial and the gauge field $A$ does not vanish at the boundaries.

One might think that the naive CS term (1.1) can be used for this globally well-defined gauge field $A$. However, this CS term depends on the choice of the gauge, since (1.1) is not invariant under the gauge transformation that changes the boundary values. To see this, consider a gauge transformation

$$A \to A^g, \qquad \hat{h}_\pm \to (g\hat{h}_\pm)|_{z\to\pm\infty}, \tag{2.25}$$

with a $U(N_f)$-valued function $g$ on $M_5$. This gauge transformation does not act on the external gauge fields $\hat{A}_\pm$ and hence the gauge configurations $(A, \hat{h}_\pm)$ and $(A^g, g\hat{h}_\pm)$ are physically equivalent. The problem is that $\omega_5(A)$ and $\omega_5(A^g)$ are not equal [see (A16)] and it is not clear which one we should use. Moreover, the naive CS term (1.1) does not reproduce the expression (2.7) for the chiral anomaly. Because of the boundary condition (2.23), the relation between the boundary values of the gauge function $g$ in the gauge transformation $A \to A^g$ and the gauge function for the gauged chiral symmetry $\hat{g}_\pm$ is modified as

$$\hat{g}_\pm = (\hat{h}_\pm^{-1} g \hat{h}_\pm)|_{z\to\pm\infty}. \tag{2.26}$$

Then, the transformation $(A,\hat{h}) \to (A^g,\hat{h})$ induces $\hat{A}_\pm \to \hat{A}_\pm^{\hat{g}_\pm}$ as desired. For the infinitesimal gauge transformation with $g \simeq 1-\Lambda$ and $\hat{g}_\pm \simeq 1-\hat{\Lambda}_\pm$, (2.26) gives $\hat{\Lambda}_\pm = (\hat{h}_\pm^{-1}\Lambda\hat{h}_\pm)|_{z\to\pm\infty}$ and hence the infinitesimal gauge transformation of the naive CS term (1.1) is

$$\delta_\Lambda S_{\rm CS} = C\left(\int_{M_4^{(+\infty)}} \omega_4^1(\hat{h}_+\hat{\Lambda}_+\hat{h}_+^{-1}, \hat{A}_+^{\hat{h}_+}) - \int_{M_4^{(-\infty)}} \omega_4^1(\hat{h}_-\hat{\Lambda}_-\hat{h}_-^{-1}, \hat{A}_-^{\hat{h}_-})\right), \tag{2.27}$$

which does not agree with (2.7) in general.

In addition to these issues, there is a more practical problem of the CS term pointed out by Hata and Murata in [19]. They studied the spectrum of baryons in holographic QCD with $N_f = 3$. The analysis is similar to that for the three-flavor Skyrme model. In the Skyrme model, baryons are represented as topological solitons called Skyrmions in a theory of pions. There are collective coordinates corresponding to the $SU(3)$ rotation (for $N_f = 3$) of the Skyrmion solution, which are denoted by $a \in SU(3)$. (See Sec. IV C.) It has been shown that the WZW term gives

$$S_{\rm WZW} = -i\frac{N_c n_B}{\sqrt{3}} \int dt\, {\rm tr}(t_8 a^{-1}\partial_t a), \tag{2.28}$$

which leads to a constraint

$$\psi(a e^{it_8\theta}) = \psi(a) \exp\left(i\frac{N_c n_B}{2\sqrt{3}}\theta\right), \tag{2.29}$$

on the wave function $\psi(a)$ for the quantum mechanics of the collective coordinates [22–28].[11] Here,

$$t_8 \equiv \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \tag{2.30}$$

is the eighth generator of the $SU(3)$ algebra. This constraint is crucial to obtain the baryon spectrum consistent with the experiments. Since the WZW term can be derived from the CS term in holographic QCD [5], it is natural to expect that the CS term plays a similar role. However, it was claimed that the contribution from the CS term vanishes and the constraint (2.29) cannot be reproduced, by using the naive CS term (1.1) in a certain gauge. In order to get the correct constraint (2.29), they proposed to use the CS term of the form

$$S_{\rm CS}^{\rm HM} = C\int_{M_6} {\rm tr}(F^3), \tag{2.31}$$

where $M_6$ is a six-dimensional manifold with $\partial M_6 = M_5$. Although they succeeded in recovering the correct constraint by using this new CS term, it is also problematic. First, as emphasized above, $M_5$ has boundaries and the meaning of "$\partial M_6 = M_5$" is not clear, because $\partial M_5 = \emptyset$ is a necessary condition to have such $M_6$. Furthermore, this term is manifestly gauge invariant and it does not recover the chiral anomaly (2.7).

## III. PROPOSAL

In this section, we propose a new CS term that solves all the problems discussed in the previous section.

### A. Proposal for the CS term

Using the notation introduced in Sec. II B, our proposal for the CS term is given by

$$S_{\rm CS}^{\rm new} \equiv C\left(\int_{M_5^-}\omega_5(A_-) + \int_{M_5^+}\omega_5(A_+) + \frac{1}{10}\int_{N_5^{(0)}} {\rm tr}((\tilde{h}d\tilde{h}^{-1})^5) + \int_{M_4^{(0)}} \alpha_4(dh^{-1}h, A_-)\right), \tag{3.1}$$

where $N_5^{(0)}$ is a five-dimensional manifold satisfying $\partial N_5^{(0)} = M_4^{(0)}$, $\tilde{h}$ is a $U(N_f)$-valued function on $N_5^{(0)}$ satisfying $\tilde{h}|_{\partial N_5^{(0)}} = h$, and

$$\begin{aligned}\alpha_4(V,A) &\equiv \frac{1}{2}{\rm tr}\left(V(A^3 - AF - FA) + \frac{1}{2}VAVA + V^3A\right) \\ &= -\frac{1}{2}{\rm tr}\left(V(AdA + dAA + A^3) - \frac{1}{2}VAVA - V^3A\right).\end{aligned} \tag{3.2}$$

[11] See also a textbook [21] for a review.

Useful formulas for the CS five-form $\omega_5(A)$ and the four-form $\alpha_4(V,A)$ can be found in Appendix A 3. Note that the last term in (3.1) can be replaced with

$$-C\int_{M_4^{(0)}}\alpha_4(dhh^{-1},A_+), \tag{3.3}$$

using (A19). The third and fourth terms in (3.1) are added to the naive expression (2.21). The motivation for adding these terms will soon become clear.

A few comments are in order. In (3.1), we have assumed the existence of $N_5^{(0)}$ and $\tilde{h}$.[12] For the case with $M_4^{(0)}\simeq S^1\times S^3$ and $h\in SU(N_f)$, which is the case of our main interest, one can choose $N_5^{(0)}$ to be $N_5^{(0)}\simeq D\times S^3$, where $D$ is a disk satisfying $\partial D=S^1$, and then $\tilde{h}$ exists because the image of $h$, as a map from $S^1$ to $SU(N_f)$ at each point in $S^3$, is contractible in $SU(N_f)$. The choice of $N_5^{(0)}$ and $\tilde{h}$ does not matter, due to the standard argument for the WZW term [13].

This new CS term has the following desired properties:

1. It reduces to (1.1) when $h$ is topologically trivial.
2. It is invariant (up to a $2\pi\mathbb{Z}$ shift) under the gauge transformation (2.19) with $g_\pm|_{z\to\pm\infty}\to 1$.
3. It reproduces the correct chiral anomaly in QCD (2.7) with the identification $\hat{A}_\pm=A_\pm|_{z\to\pm\infty}$ and $\hat{g}_\pm=e^{-\hat{\Lambda}_\pm}=g_\pm|_{z\to\pm\infty}$.
4. It reduces to the Hata-Murata's proposal (2.31) when $M_5$ does not have boundaries [i.e. $M_4^{(\pm\infty)}=\emptyset$], and there exists a six-dimensional manifold $M_6$ such that $\partial M_6=M_5$ and $M_6=M_6^+\cup M_6^-$ with $M_6^+\cap M_6^-\simeq N_5^{(0)}\times(-\epsilon,\epsilon)$ and $\partial M_6^\pm\simeq M_5^\pm\cup(\pm N_5^{(0)})$ (see Fig. 2 for the picture in the limit $\epsilon\to 0$).

Let us show these properties one by one.

1. When $h$ is topologically trivial, i.e. $h$ can be continuously deformed to $h=1$, there exists a $U(N_f)$-valued function $\tilde{h}$ on $M_5^-$ such that $\tilde{h}=h$ on the intersection $M_5^-\cap M_5^+$ and satisfies the boundary condition $\tilde{h}|_{z\to-\infty}\to 1$. Then, we can obtain a globally well-defined one-form $A$ on $M_5$ by defining

$$A\equiv\begin{cases}A_-^{\tilde{h}} & (\text{on } M_5^-)\\ A_+ & (\text{on } M_5^+)\end{cases}. \tag{3.4}$$

We choose $N_5^{(0)}=M_5^-\cup N_5^{(-\infty)}$, where $N_5^{(-\infty)}$ is a five-dimensional manifold with $\partial N_5^{(-\infty)}=M_4^{(-\infty)}$, and extend $\tilde{h}$ to $N_5^{(0)}$ by setting $\tilde{h}|_{N_5^{(-\infty)}}=1$. Then, we obtain

$$\begin{aligned}S_{\mathrm{CS}}^{\mathrm{new}}&=C\bigg(\int_{M_5^-}\omega_5(A_-)+\int_{M_5^+}\omega_5(A_+)\\&\quad+\int_{M_5^-}\bigg[\frac{1}{10}\mathrm{tr}((\tilde{h}d\tilde{h}^{-1})^5)+d\alpha_4(d\tilde{h}^{-1}\tilde{h},A_-)\bigg]\bigg)\\&=C\bigg(\int_{M_5^-}\omega_5(A_-^{\tilde{h}})+\int_{M_5^+}\omega_5(A_+)\bigg)\\&=C\int_{M_5}\omega_5(A),\end{aligned} \tag{3.5}$$

where (A16) is used.

2. Under the gauge transformation (2.19), the CS term (3.1) is transformed as

$$\begin{aligned}S_{\mathrm{CS}}^{\mathrm{new}}\to C\bigg(&\int_{M_5^-}\omega_5(A_-^{g_-})+\int_{M_5^+}\omega_5(A_+^{g_+})\\&+\frac{1}{10}\int_{N_5^{(0)}}\mathrm{tr}((\tilde{h}'d\tilde{h}'^{-1})^5)\\&+\int_{M_4^{(0)}}\alpha_4(dh'^{-1}h',A_-^{g_-})\bigg),\end{aligned} \tag{3.6}$$

where $h'\equiv g_+hg_-^{-1}$ and $\tilde{h}'$ are $U(N_f)$-valued functions on $M_5^-\cap M_5^+$ and $N_5^{(0)}$, respectively, satisfying $\tilde{h}'|_{\partial N_5^{(0)}}=h'|_{z=0}$. Note that since $g_\pm|_{z=0}$ are topologically trivial due to the boundary conditions $g_\pm|_{z\to\pm\infty}\to 1$, there exist $U(N_f)$-valued functions

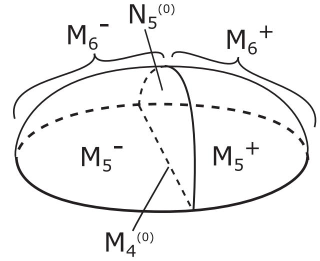


FIG. 2. The six-dimensional space-time $M_6$.

[12] For a generic choice of $M_4^{(0)}$ and $h$, the existence of $N_5^{(0)}$ and $\tilde{h}$ is not guaranteed. For example, for $M_4^{(0)}=\mathbf{CP}^2$, which is known to be a nontrivial element of the cobordism group for oriented closed four-manifolds, $N_5^{(0)}$ does not exist. On the other hand, when $M_4^{(0)}=S^1\times M_3$ with $M_3$ a closed oriented three-manifold $M_3$, there always exists a four-manifold $N_4$ satisfying $\partial N_4=M_3$ and $N_5^{(0)}$ can be either $D\times M_3$ or $S^1\times N_4$. If $h$ is topologically nontrivial on $M_3$, like the examples with $n_B\neq 0$ considered in Sec. II B, we should choose $N_5^{(0)}=D\times M_3$ so that $\tilde{h}$ defined on $N_5^{(0)}$ can be found. However, if $h$ has a nontrivial winding number as a map from $S^1$ to $U(N_f)$ at each point in $M_3$, this is not possible. For this reason, we consider the cases that $h$ does not wind around a nontrivial one-cycle in $U(N_f)$ along the $S^1$ direction. For the case of $M_4^{(0)}\simeq S^4$, we can choose $N_5^{(0)}$ to be a five-dimensional ball and then $\tilde{h}$ always exists for $N_f\geq 3$, because $\pi_4(U(N_f))$ is trivial.

$\tilde{g}_\pm$ on $N_5^{(0)}$ satisfying $\tilde{g}_\pm|_{\partial N_5^{(0)}} = g_\pm|_{z=0}$ and $\tilde{h}'$ can be constructed by $\tilde{h}' = \tilde{g}_+ \tilde{h} \tilde{g}_-^{-1}$. Then, using (A16), (A20), (A28) and (A29), one can show that (3.6) is equal to

$$
\begin{aligned}
&C\left(\int_{M_5^-}\omega_5(A_-)+\int_{M_5^+}\omega_5(A_+)+\frac{1}{10}\int_{N_5^{(0)}}\mathrm{tr}((\tilde{h}d\tilde{h}^{-1})^5)\right.\\
&\left.+\int_{M_4^{(0)}}\alpha_4(dh^{-1}h,A_-)\right),\\
&+\frac{C}{10}\left(\int_{M_5^+}\mathrm{tr}(G_+^5)+\int_{N_5^{(0)}}\mathrm{tr}(\tilde{G}_+^5)\right)\\
&+\frac{C}{10}\left(\int_{M_5^-}\mathrm{tr}(G_-^5)-\int_{N_5^{(0)}}\mathrm{tr}(\tilde{G}_-^5)\right),
\end{aligned}
\tag{3.7}
$$

where $G_\pm \equiv dg_\pm^{-1}g_\pm$ and $\tilde{G}_\pm \equiv d\tilde{g}_\pm^{-1}\tilde{g}_\pm$. The first and second lines are $S_{\mathrm{CS}}^{\mathrm{new}}$ defined in (3.1). The third and forth lines can be omitted because they take values in $2\pi\mathbb{Z}$.

3. Here, we consider the infinitesimal gauge transformation with $\hat{g}_\pm \simeq 1 - \Lambda_\pm$.[13] In this case, $g_\pm|_{z=0}$ is again topologically trivial and it suffices to show property 3 for the cases with $g_\pm = 1$ on $M_5^- \cap M_5^+$, because of the property 2 shown above. Then, since the third and fourth terms in (3.1) do not change under the gauge transformation, the proof of (2.7) is the same as that reviewed in Sec. II A.
4. Using the relations $\partial M_6^\pm = M_5^\pm \cup (\pm N_5^{(0)})$ and the Stokes' theorem, we obtain

$$
\begin{aligned}
S_{\mathrm{CS}}^{\mathrm{HM}} &= C\left(\int_{M_6^-} d\omega_5(A_-) + \int_{M_6^+} d\omega_5(A_+)\right)\\
&= C\left(\int_{M_5^-}\omega_5(A_-)+\int_{M_5^+}\omega_5(A_+)\right.\\
&\quad\left.+\int_{N_5^{(0)}}(\omega_5(A_+)-\omega_5(A_-))\right).
\end{aligned}
\tag{3.8}
$$

Now, $A_+$ and $A_-$ are related by $A_+ = A_-^{\tilde{h}}$ on $M_6^- \cap M_6^+ \simeq N_5^{(0)} \times (-\epsilon, +\epsilon)$. Then, it is easy to check, using (A16),

$$
\begin{aligned}
&\int_{N_5^{(0)}}(\omega_5(A_+)-\omega_5(A_-))\\
&\quad=\int_{N_5^{(0)}}\frac{1}{10}\mathrm{tr}((\tilde{h}d\tilde{h}^{-1})^5)+\int_{\partial N_5^{(0)}}\alpha_4(d\tilde{h}^{-1}\tilde{h},A_-),
\end{aligned}
\tag{3.9}
$$

which shows that $S_{\mathrm{CS}}^{\mathrm{HM}}$ (2.31) agrees with $S_{\mathrm{CS}}^{\mathrm{new}}$ (3.1).

[13] See Sec. III B for the finite transformation.

### B. Other useful expressions

It is often more useful to use the globally well-defined gauge field $A$ defined in (2.22) to describe the CS term. A similar analysis as in (3.6)–(3.7) shows that the new CS term (3.1) can be rewritten as

$$
\begin{aligned}
S_{\mathrm{CS}}^{\mathrm{new}} = C\Bigg(&\int_{M_5}\omega_5(A)+\int_{N_5^{(+\infty)}}\frac{1}{10}\mathrm{tr}((h_+^{-1}dh_+)^5)\\
&+\int_{M_4^{(+\infty)}}\alpha_4(d\hat{h}_+\hat{h}_+^{-1},A)-\int_{N_5^{(-\infty)}}\frac{1}{10}\mathrm{tr}((h_-^{-1}dh_-)^5)\\
&-\int_{M_4^{(-\infty)}}\alpha_4(d\hat{h}_-\hat{h}_-^{-1},A)\Bigg),
\end{aligned}
\tag{3.10}
$$

where $N_5^{(\pm\infty)}$ are five-dimensional manifolds with $\partial N_5^{(\pm\infty)} = M_4^{(\pm\infty)}$ and $h_\pm$ are the $U(N_f)$-valued function on $N_5^{(\pm\infty)}$ satisfying $h_\pm|_{\partial N_5^{(\pm\infty)}} = \hat{h}_\pm$. The relation between the boundary values of the gauge field $A$ and the external gauge fields $\hat{A}_\pm$ is given by (2.23). The boundary terms in (3.10) can also be written in terms of the external gauge fields as

$$
\begin{aligned}
S_{\mathrm{CS}}^{\mathrm{new}} = C\Bigg(&\int_{M_5}\omega_5(A)+\int_{N_5^{(+\infty)}}\frac{1}{10}\mathrm{tr}((h_+^{-1}dh_+)^5)\\
&-\int_{M_4^{(+\infty)}}\alpha_4(d\hat{h}_+^{-1}\hat{h}_+,\hat{A}_+)-\int_{N_5^{(-\infty)}}\frac{1}{10}\mathrm{tr}((h_-^{-1}dh_-)^5)\\
&+\int_{M_4^{(-\infty)}}\alpha_4(d\hat{h}_-^{-1}\hat{h}_-,\hat{A}_-)\Bigg),
\end{aligned}
\tag{3.11}
$$

where we have used (A19). This expression makes it clear that we do not have to modify the CS term for $N_f = 2$ and $\hat{A}_\pm = 0$, because the additional terms in (3.11) vanish in that case.

The expressions (3.10) and (3.11) can be written in a more compact notation as

$$
\begin{aligned}
S_{\mathrm{CS}}^{\mathrm{new}} &= C\left(\int_{M_5}\omega_5(A)+\int_{N_5}\frac{1}{10}\mathrm{tr}((h^{-1}dh)^5)\right.\\
&\quad\left.+\int_{\partial M_5}\alpha_4(dhh^{-1},A)\right)\\
&= C\left(\int_{M_5}\omega_5(A)+\int_{N_5}\frac{1}{10}\mathrm{tr}((h^{-1}dh)^5)\right.\\
&\quad\left.-\int_{\partial M_5}\alpha_4(dh^{-1}h,\hat{A})\right),
\end{aligned}
\tag{3.12}
$$

where $N_5$ is a five-dimensional manifold with two connected components $N_5 = N_5^{(+\infty)} \cup (-N_5^{(-\infty)})$ satisfying

$$
\partial N_5 = \partial M_5 = M_4^{(+\infty)} \cup (-M_4^{(-\infty)}),
\tag{3.13}
$$

and $h$ is a $U(N_f)$-valued function on $N_5$ with $\hat{h}_\pm = h|_{M_4^{(\pm\infty)}}$. The external gauge field $\hat{A}$ in (3.12) is defined on the boundary $\partial M_5$ with the identification $\hat{A}_\pm = \hat{A}|_{M_4^{(\pm\infty)}}$. The relation to the boundary value (2.23) is written as

$$A|_{\partial M_5} = \hat{A}^h. \tag{3.14}$$

It is not difficult to show, using (A16), (A21) and (A28), that this CS term is invariant (up to a $2\pi\mathbb{Z}$ shift) under the transformation (2.25), which can be written as

$$A \to A^g, \qquad h \to gh, \qquad \hat{A} \to \hat{A}, \tag{3.15}$$

assuming that $g$ can be extended to $N_5$.

The transformation corresponding to the chiral symmetry discussed around (2.26) is given by

$$A \to A^g, \qquad h \to h, \qquad \hat{A} \to \hat{A}^{\hat{g}}, \tag{3.16}$$

with

$$\hat{g} = (h^{-1}gh)|_{\partial M_5}, \tag{3.17}$$

where $\hat{g}_\pm \equiv \hat{g}|_{M_4^{(\pm\infty)}}$ corresponds to the chiral symmetry. Combining this with the inverse of (3.15), we find that the chiral transformation is also induced by

$$A \to A, \qquad h \to g^{-1}h, \qquad \hat{A} \to \hat{A}^{\hat{g}}. \tag{3.18}$$

It is also straightforward to show that the CS term (3.12) transforms under the transformation (3.16) with (3.17) as

$$S_{\mathrm{CS}}^{\mathrm{new}} \to S_{\mathrm{CS}}^{\mathrm{new}} + C\left(\int_{N_5} \frac{1}{10}\mathrm{tr}((\hat{g}d\hat{g}^{-1})^5) + \int_{\partial M_5} \alpha_4(d\hat{g}^{-1}\hat{g}, \hat{A})\right), \tag{3.19}$$

up to the $2\pi\mathbb{Z}$ shift, where we have assumed that $\hat{g}$ can be extended to $N_5$. If we consider an infinitesimal chiral transformation with $\hat{g} \simeq 1 - \hat{\Lambda}$, then (3.19) reduces to the formula for chiral anomaly (2.7).

There is another useful expression that generalizes (2.31) to the cases with a boundary. Note that $M_5 \cup (-N_5)$ is a five-dimensional manifold without boundary. Suppose there exists a six-dimensional manifold $M_6$ with $\partial M_6 = M_5 \cup (-N_5)$ and the gauge field $A$ can be extended to $M_6$. Then, we have

$$\int_{M_6} \mathrm{tr}(F^3) = \int_{M_5} \omega_5(A) - \int_{N_5} \omega_5(A). \tag{3.20}$$

Next, we extend the external gauge field $\hat{A}$ to $N_5$ by defining $\hat{A} \equiv A^{h^{-1}}$ (on $N_5$), which reduces to (3.14) at $\partial N_5 = \partial M_5$. Then, using (A16), we find

$$\int_{N_5} \omega_5(\hat{A}) = \int_{N_5} \left(\omega_5(A) + \frac{1}{10}\mathrm{tr}((h^{-1}dh)^5)\right) + \int_{\partial N_5} \alpha_4(dhh^{-1}, A). \tag{3.21}$$

Comparing (3.20) and (3.21) with (3.12), we obtain a simple formula[14]

$$S_{\mathrm{CS}}^{\mathrm{new}} = C\left(\int_{M_6} \mathrm{tr}(F^3) + \int_{N_5} \omega_5(\hat{A})\right). \tag{3.22}$$

### C. Pion field

The relation between the $U(N_f)$-valued pion field $U(x^\mu)$ in the chiral Lagrangian and the five-dimensional gauge field was proposed in [5,7,18]:

$$U(x^\mu) = \mathrm{P}\exp\left(-\int_{-\infty}^{+\infty} dz A_z(x^\mu, z)\right). \tag{3.23}$$

This formula should be modified as follows.

For the gauge field considered in Sec. III A, the correct expression is

$$U(x^\mu) = \mathrm{P}\exp\left(-\int_0^{+\infty} dz A_{+z}(x^\mu, z)\right) h(x^\mu)|_{z=0} \times \mathrm{P}\exp\left(-\int_{-\infty}^{0} dz A_{-z}(x^\mu, z)\right). \tag{3.24}$$

For the gauge field $A$ in (2.22), this is equivalent to

$$U(x^\mu) = \hat{h}_+^{-1}(x^\mu)\mathrm{P}\exp\left(-\int_{-\infty}^{+\infty} dz A_z(x^\mu, z)\right)\hat{h}_-(x^\mu). \tag{3.25}$$

This expression is invariant under the gauge transformation (2.25).

On the other hand, (3.24) transforms under the gauge transformation (2.19) as

$$U(x^\mu) \to \hat{g}_+(x^\mu)U(x^\mu)\hat{g}_-(x^\mu)^{-1}, \tag{3.26}$$

where $\hat{g}_\pm \equiv g_\pm|_{z\to\pm\infty}$, which is nothing but the chiral transformation of the pion field. In terms of (3.25), (3.26) can be easily seen by the transformation (3.16) or (3.18).

[14] A similar expression was suggested in [19] as a quick remedy to recover the chiral anomaly. Our derivation gives its precise meaning.

### D. Equations of motion and current

For later use, let us write down the equations of motion and currents with our new CS term. Since the additional terms in our new CS term do not affect these equations, the results in this subsection are not new. Nevertheless, it will be instructive to show them explicitly. The action (2.1) is replaced with

$$S_{5\,\mathrm{dim}} = S_{\mathrm{YM}} + S_{\mathrm{CS}}^{\mathrm{new}}. \tag{3.27}$$

Here, we use the expression (3.12) for the CS term $S_{\mathrm{CS}}^{\mathrm{new}}$. Using (A22) and (A26), an infinitesimal variation of the action is computed as[15]

$$\delta S = \int_{M_5} \mathrm{tr}(\delta A(-\kappa D_A * F + 3CF^2)) + \int_{\partial M_5} \mathrm{tr}\left(\delta\hat{A}\left(-\kappa\widehat{*F} + C\left(\hat{F}\hat{A} + \hat{A}\hat{F} - \frac{1}{2}\hat{A}^3\right)\right)\right), \tag{3.28}$$

where $D_A$ is the covariant derivative defined in (A2), $\hat{A}$ is the external gauge field related to the boundary value of the gauge field as (3.14), and

$$\hat{F} \equiv (h^{-1}Fh)|_{\partial M_5}, \qquad \widehat{*F} \equiv (h^{-1} * Fh)|_{\partial M_5}, \qquad \delta\hat{A} \equiv (h^{-1}\delta A h)|_{\partial M_5}. \tag{3.29}$$

Note here that $\widehat{*F}$ is different from the Hodge dual of $\hat{F}$ defined on $\partial M_5$. Its explicit form with (2.4) is

$$\widehat{*F} = \left(-\frac{k(z)}{3}(h^{-1}F^{\mu}{}_{z}h)\epsilon_{\mu\nu\rho\sigma}dx^{\nu}dx^{\rho}dx^{\sigma}\right)\Big|_{\partial M_5}. \tag{3.30}$$

The first term in (3.28) gives the equations of motion

$$-\kappa D_A * F + 3CF^2 = 0, \tag{3.31}$$

which is consistent with the boundary condition $\delta\hat{A} = 0$. The variation with respect to the external gauge field $\hat{A}$ at the boundary in (3.28) gives the current three-form:

$$J_{\pm} \equiv \pm i\left(-\kappa\widehat{*F}_{\pm} + C\left(\hat{F}_{\pm}\hat{A}_{\pm} + \hat{A}_{\pm}\hat{F}_{\pm} - \frac{1}{2}\hat{A}_{\pm}^3\right)\right), \tag{3.32}$$

where (Hodge dual of) $J_-$ and $J_+$ correspond to the currents of $U(N_f)_L$ and $U(N_f)_R$, respectively [29–31]. Then, it is straightforward to check, using the equations of motion (3.31), that it satisfies the (consistent) anomaly equation[16]:

$$D_{\hat{A}_{\pm}} J_{\pm} = \pm\frac{N_c}{24\pi^2} d\left(\hat{A}_{\pm}d_{\pm}\hat{A}_{\pm} + \frac{1}{2}\hat{A}_{\pm}^3\right). \tag{3.33}$$

The baryon number current is defined as

$$J_B = \frac{1}{N_c}(\mathrm{tr}J_+ + \mathrm{tr}J_-), \tag{3.34}$$

and the baryon number (for $\hat{A} = 0$) is

$$n_B = \int_{S^3} J_B = \frac{i}{N_c}\int_{S^3}[\mathrm{tr}(-\kappa\widehat{*F})]_{z=-\infty}^{z=+\infty} = \frac{1}{8\pi^2}\int_{S^3\times\mathbb{R}} \mathrm{tr}(F^2), \tag{3.35}$$

where we have used the equations of motion (3.31) and Stokes' theorem in the last step, reproducing the expression in (2.11).

## IV. APPLICATION TO BARYONS

In this section, we analyze the effective action for the collective coordinates of the soliton solution corresponding to baryons. We show that the term (2.28) needed to obtain the correct constraint (2.29) is reproduced by using the CS term proposed in the previous section. This statement was already shown in [19] using (2.31) for the $n_B = 1$ case. As we have seen in Sec. III B that our CS term reduces to (2.31) when $\hat{A} = 0$, we should recover their result. In our derivation, we will not use an explicit solution corresponding to a baryon so that it can be generalized to the cases with $n_B > 1$.

### A. Collective coordinates

In this subsection, we work in the $A_0 = 0$ gauge. We assume there exists a solution of the equations of motion (3.31) with nonzero baryon number $n_B$, denoted as

$$A^{\mathrm{cl}} = A_M^{\mathrm{cl}} dx^M, \tag{4.1}$$

where "cl" refers to a classical solution and $M = 1, 2, 3, z$ is the spatial index. We also assume that this gauge field is globally well defined and regular everywhere in $M_5$.

Here, we consider the cases with $\hat{A}_{\pm} = 0$. Then, for a finite energy solution, the gauge field approaches a pure gauge configuration near the boundary as

$$A^{\mathrm{cl}} \to h_{\pm}^{\mathrm{cl}} dh_{\pm}^{\mathrm{cl}-1}, \qquad (z \to \pm\infty). \tag{4.2}$$

[15] The variation with respect to $h$ can be absorbed in $\delta A$, using the transformation (3.15).

[16] See [31] for a detailed discussion on the currents and the anomaly equations in holographic QCD.

Because of the condition $A_0 = 0$, $h_\pm^{\rm cl}$ are time independent. Without loss of generality, we can assume $h_-^{\rm cl}|_{z\to-\infty} = 1$ and $h_+^{\rm cl}|_{z\to+\infty} \equiv h_0(\vec{x})$, where $h_0$ is a $U(N_f)$-valued function on the $S^3$ parametrized by $\vec{x} = (x^1, x^2, x^3)$ satisfying

$$n_B = \frac{1}{24\pi^2}\int_{S^3} \mathrm{tr}((h_0^{-1}dh_0)^3). \tag{4.3}$$

Following [20], we consider a gauge configuration

$$A_M = VA_M^{\rm cl}V^{-1} + V\partial_M V^{-1} \tag{4.4}$$

with a globally well-defined $SU(N_f)$-valued function $V$.[17] The idea is as follows. If $V$ is time independent, it can be regarded as the collective coordinates (coordinates of the instanton moduli space) corresponding to the global gauge rotation, since $A_M$ is again a classical solution with the same energy. A standard procedure of the moduli space quantization method[18] is to promote the collective coordinates to be time-dependent variables and reduce the system to a quantum mechanics of these variables. To this end, one should also make a compensating gauge transformation so that the gauge configuration satisfies the Gauss law equation, which is the equation of motion for $A_0$:

$$dt \wedge (-\kappa D_A * F + 3CF^2) = 0. \tag{4.5}$$

$V$ in (4.4) contains both the collective coordinates and the compensating gauge transformation, and it can depend on the five-dimensional space-time coordinates. We assume that the initial value of $V$ is 1 and hence its value at a fixed time is connected to $V = 1$ by a continuous deformation.

With this choice of the gauge configuration, the asymptotic value of the gauge field is

$$A_M \to Vh_\pm^{\rm cl}\partial_M(Vh_\pm^{\rm cl})^{-1}, \qquad (z \to \pm\infty). \tag{4.6}$$

The electric fields $F_{0i}$ $(i = 1, 2, 3)$ are assumed to vanish at the boundaries $z \to \pm\infty$. Then, $F_{0i}|_{z\to\pm\infty} = \partial_0 A_i|_{z\to\pm\infty} = 0$ implies that the asymptotic values of $A_i$ should be time independent, and therefore, since the initial value of $V$ is assumed to be 1, one has

$$A_i \to h_\pm^{\rm cl}\partial_i h_\pm^{\rm cl-1}, \qquad (z \to \pm\infty) \tag{4.7}$$

for all time. This implies that $V$ has the following asymptotic values,

$$V|_{z\to-\infty} = a_-(t), \quad V|_{z\to+\infty} = h_0(\vec{x})a_+(t)h_0^{-1}(\vec{x}), \tag{4.8}$$

with $a_\pm(t)$ being $SU(N_f)$-valued functions that depend only on time.

With the asymptotic expression of the gauge field in (4.7), $\hat{h}_\pm$ in (2.23) can be chosen as

$$\hat{h}_- = 1, \qquad \hat{h}_+ = h_0(\vec{x}), \tag{4.9}$$

and the CS term (3.11) is simply

$$S_{\rm CS}^{\rm new} = C\int_{M_5} \omega_5(A). \tag{4.10}$$

Therefore, the naive CS term is actually the correct one in this gauge choice.

Let us now consider the Gauss law equation (4.5). With the expression (4.4), one can easily show that $F_{MN} = VF_{MN}^{\rm cl}V^{-1}$ and

$$F_{0M} = \dot{A}_M = V(F_{0M}^{\rm cl} - D_M^{\rm cl}\Phi)V^{-1}, \tag{4.11}$$

where the dot denotes the time derivative, and we have defined $\Phi \equiv V^{-1}\dot{V}$ and $D_M^{\rm cl}\Phi \equiv \partial_M\Phi + [A_M^{\rm cl}, \Phi]$. Using these relations and the fact that $A_M^{\rm cl}$ is a classical solution, (4.5) becomes

$$dt \wedge (D_A^{\rm cl} * (D_A^{\rm cl}\Phi dt)) = 0, \tag{4.12}$$

where the covariant derivative acting on $\Phi$ is $D_A^{\rm cl}\Phi \equiv D_M^{\rm cl}\Phi dx^M$. In components, (4.12) is given by

$$D_M^{\rm cl}(\sqrt{-g}g^{MN}g^{00}D_N^{\rm cl}\Phi) = 0. \tag{4.13}$$

For the background with the metric (2.3), this is written explicitly as

$$\delta^{ij}D_i^{\rm cl}D_j^{\rm cl}\Phi + \tilde{k}(z)^{-1}D_z^{\rm cl}(k(z)D_z^{\rm cl}\Phi) = 0, \tag{4.14}$$

where $i, j = 1, 2, 3$.

With the expression (4.8), $\Phi$ has the following asymptotic values

$$\begin{aligned}\Phi|_{z\to-\infty} &= a_-(t)^{-1}\dot{a}_-(t),\\ \Phi|_{z\to+\infty} &= h_0(\vec{x})a_+(t)^{-1}\dot{a}_+(t)h_0^{-1}(\vec{x}).\end{aligned} \tag{4.15}$$

Therefore, $\Phi$ is determined as the solution of the Gauss law equation (4.12) with the boundary condition (4.15).

[17] One could consider $V$ to be a $U(N_f)$-valued function. However, we only consider the configurations of $V$ that do not wind around a nontrivial one-cycle of $U(N_f)$ along the time direction in the following (see the footnote on p. 12 for a related issue) and, at least for such configurations, it is possible to show that the diagonal $U(1)$ part of the $U(N_f)$ does not contribute to the effective action studied in Sec. IV B and we can restrict $V$ to be an $SU(N_f)$-valued function.

[18] See, e.g., [32] for a review of this method explained for the magnetic monopoles.

### B. Effective action

To obtain the effective action for $a_\pm(t)$, it turns out to be more convenient to make a gauge transformation (2.25) using $g = V^{-1}$. Then, the configuration in (4.4) is mapped to

$$A_0 = V^{-1}\dot{V} \equiv \Phi, \qquad A_M = A_M^{\rm cl}, \tag{4.16}$$

and $\hat{h}_\pm$ in (2.23) is given by

$$\hat{h}_- = a_-(t)^{-1}, \qquad \hat{h}_+ = h_0(\vec{x})a_+(t)^{-1}. \tag{4.17}$$

Then, the CS term (3.11) is

$$S_{\rm CS}^{\rm new} = C\left(\int_{M_5}\omega_5(A) + \int_{N_5^{(+\infty)}}\frac{1}{10}{\rm tr}((a_+h_0^{-1}d(h_0a_+^{-1}))^5)\right). \tag{4.18}$$

Here, $N_5^{(+\infty)}$ is assumed to be $N_5^{(+\infty)} \simeq D\times S^3$, and $h_0$ and $a_+$ are extended to be functions on it. We can choose $h_0$ and $a_+$ to be constant along the $D$ and $S^3$ directions, respectively. Using the relation (A28), one can show that (4.18) is equivalent to

$$S_{\rm CS}^{\rm new} = C\left(\int_{M_5}\omega_5(A) - \frac{1}{2}\int_{M_4^{(+\infty)}}dt\,{\rm tr}(a_+^{-1}\dot{a}_+(h_0^{-1}dh_0)^3)\right). \tag{4.19}$$

Although it is a bit more tedious, it is also possible to derive (4.19) directly from (4.10) by using (A16) with $g = V^{-1}$.[19]

The first term on the right-hand side of (4.19) can be evaluated as follows. The relation (A22) with $\delta A = \Phi dt$ implies

$$\omega_5(A) = \omega_5(A^{\rm cl}) + 3{\rm tr}(\Phi dt(F^{\rm cl})^2) + d\beta_4(\Phi dt, A^{\rm cl}), \tag{4.20}$$

where $\beta_4$ is defined in (A23). The contribution from the collective coordinates to the CS five-form is

$$\begin{aligned}\int_{M_5}&\omega_5(A) - \int_{M_5}\omega_5(A^{\rm cl})\\ &= \int_{M_5}3{\rm tr}(\Phi dt(F^{\rm cl})^2) + \int_{M_4^{(+\infty)}}\beta_4(\Phi dt, A^{\rm cl})|_{z=+\infty}\\ &\quad - \int_{M_4^{(-\infty)}}\beta_4(\Phi dt, A^{\rm cl})|_{z=-\infty}\\ &= \int_{M_5}3{\rm tr}(\Phi dt(F^{\rm cl})^2) + \frac{1}{2}\int_{M_4^{(+\infty)}}dt\,{\rm tr}(a_+^{-1}\dot{a}_+(h_0^{-1}dh_0)^3).\end{aligned} \tag{4.21}$$

[19] The integral of ${\rm tr}((V^{-1}dV)^5)$ over $M_5$ can be evaluated by using (A31).

Substituting this back to (4.19), one obtains

$$S_{\rm CS}^{\rm new} = C\int_{M_5}\omega_5(A^{\rm cl}) + 3C\int_{M_5}dt\,{\rm tr}(\Phi(F^{\rm cl})^2). \tag{4.22}$$

The field strength for the gauge field (4.16) is

$$F = F^{\rm cl} + D_A^{\rm cl}\Phi dt, \tag{4.23}$$

and the YM part is given as

$$\begin{aligned}S_{\rm YM} = S_{\rm YM}(A^{\rm cl}) &- \frac{\kappa}{2}\int_{M_5}{\rm tr}(D_A^{\rm cl}\Phi dt\wedge *(D_A^{\rm cl}\Phi dt))\\ &- \kappa\int_{M_5}dt\,{\rm tr}(\Phi D_A^{\rm cl} * F^{\rm cl}) - \kappa\int_{\partial M_5}dt\,{\rm tr}(\Phi * F^{\rm cl}).\end{aligned} \tag{4.24}$$

Using the fact that $A^{\rm cl}$ satisfies the equations of motion (3.31), the total action (3.27) becomes

$$S_{5\,\rm dim} = S_{5\,\rm dim}(A^{\rm cl}) + S_1 + S_2, \tag{4.25}$$

where $S_{5\,\rm dim}(A^{\rm cl})$ is the action evaluated with $A = A^{\rm cl}$, and $S_1$ and $S_2$ are the terms including $\Phi$:

$$S_1 = -\kappa\int_{\partial M_5}dt\,{\rm tr}(\Phi * F^{\rm cl}), \tag{4.26}$$

$$S_2 = -\frac{\kappa}{2}\int_{M_5}{\rm tr}(D_A^{\rm cl}\Phi dt\wedge *(D_A^{\rm cl}\Phi dt)). \tag{4.27}$$

Using the Gauss law equation (4.12), $S_2$ can also be written as

$$S_2 = -\frac{\kappa}{2}\int_{\partial M_5}dt\,{\rm tr}(\Phi * (D_A^{\rm cl}\Phi dt)). \tag{4.28}$$

For the background with the metric (2.3), (4.26) and (4.28) can be written as

$$S_1 = 2\kappa\int d^4x[k(z){\rm tr}(\Phi F_{0z}^{\rm cl})]_{z\to-\infty}^{z\to+\infty}, \tag{4.29}$$

$$S_2 = \kappa\int d^4x[k(z){\rm tr}(\Phi D_z^{\rm cl}\Phi)]_{z\to-\infty}^{z\to+\infty}. \tag{4.30}$$

Substituting the asymptotic expressions of $\Phi$ (4.15) into (4.26), one obtains

$$S_1 = -i\int dt\,{\rm tr}(a_+^{-1}\dot{a}_+n_+^{\rm cl} + a_-^{-1}\dot{a}_-n_-^{\rm cl}), \tag{4.31}$$

with $n_\pm^{\rm cl}$ defined by

$$n_{\pm}^{\rm cl} \equiv \int_{S^3} J_{\pm}^{\rm cl} = \mp i\kappa \int_{S^3} \widehat{*F^{\rm cl}}|_{z\to\pm\infty}, \tag{4.32}$$

where $J_{\pm}^{\rm cl}$ are the classical current three-forms given by (3.32) with $A = A^{\rm cl}$ and $\hat{A}_{\pm} = 0$. The classical quark number matrix is defined as $n_Q^{\rm cl} \equiv n_+^{\rm cl} + n_-^{\rm cl}$. Its diagonal elements are interpreted as the number of up quarks, down quarks, strange quarks, etc., carried by the classical solution and the trace is the total quark number:

$$\mathrm{tr} n_Q^{\rm cl} = N_c n_B. \tag{4.33}$$

### C. Relation to Skyrmions

The action of the Skyrme model is written in terms of the pion field $U(x^\mu)$ discussed in Sec. III C. The classical solution corresponding to the baryon carries nonzero winding number as an element of $\pi_3(U(N_f)) \simeq \mathbb{Z}$. In the standard approach for $N_f = 3$, the ansatz for the field configuration is

$$U(x^\mu) = a(t) U^{\rm cl}(\vec{x}) a(t)^{-1}, \tag{4.34}$$

where $U^{\rm cl}(\vec{x}) \in SU(3)$ is a classical solution representing a baryon and $a(t) \in SU(3)$ is the collective coordinates corresponding to the $SU(3)$ rotation. The classical solution is assumed to be of the form

$$U^{\rm cl}(\vec{x}) = \begin{pmatrix} U_0(\vec{x}) & \\ & 1 \end{pmatrix}, \tag{4.35}$$

where $U_0(\vec{x})$ is the Skyrmion solution for $N_f = 2$. The form of the solution (4.35) is natural in the sense that exciting the components of the mesons with a strange quark costs more energy than those with only up and down quarks, when we include the mass term to the Lagrangian.

The pion field (3.25) for our gauge configuration (4.16) is given by

$$U(x^\mu) = a_+(t) h_0^{-1}(\vec{x}) \mathrm{P}\exp\left(-\int_{-\infty}^{+\infty} dz A_z^{\rm cl}(x^\mu, z)\right) a_-(t)^{-1}, \tag{4.36}$$

and it corresponds to the above ansatz (4.34) with the identification $a_+(t) = a_-(t) = a(t)$ and

$$U^{\rm cl}(x^\mu) = h_0^{-1}(\vec{x}) \mathrm{P}\exp\left(-\int_{-\infty}^{+\infty} dz A_z^{\rm cl}(x^\mu, z)\right). \tag{4.37}$$

Note that, in the infinite volume limit, the pion field is supposed to approach its vacuum value at spatial infinity, i.e. $U(x^\mu)|_{|\vec{x}|\to\infty} = 1$. Since the modes with $a_+ \neq a_-$ change the vacuum configuration, they are unphysical in the infinite volume limit. For this reason, we impose $a_+ = a_-$ hereafter.

Motivated by the ansatz (4.35), we consider embedding a classical solution for $N_f = 2$ into the $U(3)$ gauge field to obtain $A^{\rm cl}$ for $N_f = 3$, as it was done in [19]. Decomposing the $U(2)$ gauge field into the $SU(2)$ part and $U(1)$ part as

$$A^{U(2)} = A^{SU(2)} + A^{U(1)}, \tag{4.38}$$

the equations of motion (3.31) for $N_f = 2$ can be written as

$$-\kappa D_A * F^{SU(2)} + 6C F^{U(1)} F^{SU(2)} = 0, \tag{4.39}$$

$$-\kappa d * F^{U(1)} + 3C((F^{U(1)})^2 + (F^{SU(2)})^2) = 0. \tag{4.40}$$

These equations can be consistently truncated by restricting $F_{0M}^{SU(2)} = 0$ and $F_{MN}^{U(1)} = 0$ for $M, N = 1, 2, 3, z$. In this case, only the $U(1)$ part of the gauge field contributes in (4.32) and the classical quark number matrix $n_Q^{\rm cl}$ for $N_f = 2$ is proportional to the unit matrix. When the solution for $N_f = 2$ is embedded into the $U(3)$ gauge field, $n_Q^{\rm cl}$ is of the form

$$n_Q^{\rm cl} = \frac{N_c n_B}{2} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \tag{4.41}$$

which means that, before quantization of the collective modes $a(t)$, the classical configuration represents a state with no strangeness and equal number of up and down quarks.

Imposing $a_+ = a_- \equiv a \in SU(3)$, (4.31) becomes

$$\begin{aligned} S_1 &= -i \int dt\, \mathrm{tr}(a^{-1} \dot{a} n_Q^{\rm cl}) \\ &= -i \frac{N_c n_B}{\sqrt{3}} \int dt\, \mathrm{tr}(t_8 a^{-1} \dot{a}), \end{aligned} \tag{4.42}$$

which precisely agrees with (2.28). Note that $\mathrm{tr}(t_8 a^{-1} \dot{a})$ does not appear in $S_2$. To see this, let us assume that $a$ is of the form $a = e^{it_8\theta(t)}$. For this, since $t_8$ commutes with $A^{\rm cl}$ and $h_0$, $\Phi = a^{-1}\dot{a} = it_8\dot{\theta}$ solves the equations (4.12) and (4.15). Then, it is clear that $S_2$ vanishes. Because $\dot{\theta}$ appears only in (4.42), the momentum conjugate to $\theta$ is

$$P_\theta = \frac{N_c n_B}{2\sqrt{3}}, \tag{4.43}$$

and hence the correct baryon constraint (2.29) is recovered.

## V. CONCLUSION AND OUTLOOK

In this paper, we reexamined a puzzle concerned with the CS term in the five-dimensional meson effective theory of holographic QCD. We proposed a modified CS term and demonstrated that the new action successfully reproduces the required baryon constraint as well as the chiral anomaly.

Although we obtained a CS term that can be used for the topologically nontrivial gauge configurations corresponding to baryons, our construction is not completely general. For example, the expression (3.12) is applicable only when $N_5$ and $h$ can be constructed and the gauge field can be treated as a globally well-defined one-form field on $M_5$. For the expression (3.22), we have to assume the existence of $M_6$ and $N_5$ as well as an extension of the gauge fields to these spaces. (See the footnote on p. 12 for further comments.) It would be interesting to investigate an expression of the CS term that works for more generic situations, as it was done in [33] for the three-dimensional CS term.

The main motivation for the present work is to solve a puzzle concerned with baryons in holographic QCD with $N_f = 3$ and make it applicable to the physics of baryons including strange quarks. In order to be more realistic, it would be important to include the mass of the strange quark. There are already some works along this direction. (See, e.g., [19,34–38].) We hope our work removes possible concerns on the validity of the formulation and provides some new insight into application of holographic QCD to hyperons.

## ACKNOWLEDGMENTS

We thank K. Hashimoto and H. Hata for discussion, and Y. Kikuchi for the collaboration in the early stage of the project. P. H. C. L. acknowledges support as an International Research Fellow of the Japan Society for the Promotion of Science (JSPS). The final stage of the research of P. H. C. L. was supported by the Croucher Fellowship. The work of S. S. was supported by JSPS KAKENHI [Grant-in-Aid for Scientific Research (C)] Grant No. JP16K05324.

## APPENDIX: NOTATIONS AND USEFUL FORMULAS

### 1. Gauge field, covariant derivative, etc.

In our convention, the gauge field $A$ and its field strength $F = dA + A^2$ are the anti-Hermitian one-form and two-form, respectively. The gauge transformation is

$$\begin{aligned} &A \to A^g \equiv gAg^{-1} + gdg^{-1} = g(A + dg^{-1}g)g^{-1}, \\ &F \to F^g \equiv gFg^{-1}. \end{aligned} \tag{A1}$$

For a general (matrix-valued) $n$-form $\alpha_n$, we define $D_A\alpha_n$ as

$$D_A\alpha_n \equiv d\alpha_n + A\alpha_n - (-1)^n \alpha_n A. \tag{A2}$$

It satisfies the Leibniz rule

$$D_A(\alpha_n\beta_m) = (D_A\alpha_n)\beta_m + (-1)^n\alpha_n D_A\beta_m. \tag{A3}$$

One can show

$$D_A F = dF + AF - FA = 0. \tag{A4}$$

Note that $d$ and $D_A$ are the same in the trace:

$$d\mathrm{tr}\alpha_n = \mathrm{tr}d\alpha_n = \mathrm{tr}(D_A\alpha_n). \tag{A5}$$

The infinitesimal variation of the field strength is

$$\delta F = d\delta A + \delta A A + A\delta A + \delta A^2 = D_A\delta A + \mathcal{O}(\delta A^2). \tag{A6}$$

The infinitesimal gauge transformation with $g = e^{-\Lambda}$ is

$$\delta_\Lambda A \equiv (A^g - A)|_{\mathcal{O}(\Lambda)} = d\Lambda + [A, \Lambda] = D_A\Lambda. \tag{A7}$$

The following trivial relations that follow from $\mathrm{tr}((\text{odd form})^{2n}) = 0$ are sometimes useful:

$$\mathrm{tr}(A^2) = \mathrm{tr}(A^4) = 0, \qquad \mathrm{tr}(AFAF) = 0. \tag{A8}$$

### 2. CS three-form

The CS three-form is defined as

$$\omega_3(A) \equiv \mathrm{tr}\left(AF - \frac{1}{3}A^3\right) = \mathrm{tr}\left(AdA + \frac{2}{3}A^3\right), \tag{A9}$$

which satisfies

$$d\omega_3(A) = \mathrm{tr}(F^2). \tag{A10}$$

The gauge transformation is

$$\omega_3(A^g) = \omega_3(A) - \frac{1}{3}\mathrm{tr}((gdg^{-1})^3) - d\mathrm{tr}(dg^{-1}gA). \tag{A11}$$

The infinitesimal gauge transformation with $g = e^{-\Lambda}$ and $\delta_\Lambda A = D_A\Lambda$ is

$$\delta_\Lambda\omega_3(A) = d\mathrm{tr}(\Lambda dA) + \mathcal{O}(\Lambda^2). \tag{A12}$$

The infinitesimal variation is

$$\delta\omega_3(A) = 2\mathrm{tr}(\delta A F) + d\mathrm{tr}(\delta A A) + \mathcal{O}(\delta A^2). \tag{A13}$$

### 3. CS five-form

The definition of the CS five-form is

$$\begin{aligned} \omega_5(A) &\equiv \mathrm{tr}\left(AF^2 - \frac{1}{2}A^3F + \frac{1}{10}A^5\right) \\ &= \mathrm{tr}\left(AdAdA + \frac{3}{2}A^3dA + \frac{3}{5}A^5\right), \end{aligned} \tag{A14}$$

which satisfies

$$d\omega_5(A) = \mathrm{tr}(F^3). \tag{A15}$$

The gauge transformation is

$$\omega_5(A^g) = \omega_5(A) + \frac{1}{10}\mathrm{tr}((gdg^{-1})^5) + d\alpha_4(dg^{-1}g, A), \tag{A16}$$

where

$$\begin{aligned}\alpha_4(V,A) &= -\frac{1}{2}\mathrm{tr}\left(V(AdA + dAA + A^3) - \frac{1}{2}VAVA - V^3A\right)\\ &= \frac{1}{2}\mathrm{tr}\left(V(A^3 - AF - FA) + \frac{1}{2}VAVA + V^3A\right).\end{aligned} \tag{A17}$$

This $\alpha_4(V,A)$ satisfies the following relations:

$$\alpha_4(V, \pm V) = 0 \tag{A18}$$

for any one-form $V$,

$$\alpha_4(dgg^{-1}, A^g) = -\alpha_4(dg^{-1}g, A), \tag{A19}$$

and

$$\begin{aligned}\alpha_4(d(gh)(gh)^{-1}, A^g) &= \alpha_4(g(H-G)g^{-1}, A^g)\\ &= \alpha_4(H,A) - \alpha_4(G,A)\\ &\quad - \frac{1}{2}\mathrm{tr}\left(G^3H + GH^3 - \frac{1}{2}GHGH\right)\\ &\quad + \frac{1}{2}d\mathrm{tr}((H-G)(AG - GA)),\end{aligned} \tag{A20}$$

where $G = dg^{-1}g$ and $H = dhh^{-1}$. Using (A19) and (A20), one can also show

$$\begin{aligned}\alpha_4(d(gh)^{-1}(gh), A) &= \alpha_4(dh^{-1}h, A) + \alpha_4(G, A^h)\\ &\quad + \frac{1}{2}\mathrm{tr}\left(G^3H + GH^3 - \frac{1}{2}GHGH\right)\\ &\quad - \frac{1}{2}d\mathrm{tr}((H-G)(A^hG - GA^h))),\end{aligned} \tag{A21}$$

where $G = dg^{-1}g$ and $H = dhh^{-1}$.

The infinitesimal variation is

$$\delta\omega_5(A) = 3\mathrm{tr}(\delta AF^2) + d\beta_4(\delta A, A) + \mathcal{O}(\delta A^2), \tag{A22}$$

where

$$\beta_4(\delta A, A) \equiv \mathrm{tr}\left(\delta A\left(FA + AF - \frac{1}{2}A^3\right)\right). \tag{A23}$$

The infinitesimal gauge transformation with $g = e^{-\Lambda}$ and $\delta_\Lambda A = D_A\Lambda$ is

$$\begin{aligned}\delta_\Lambda\omega_5(A)|_{\mathcal{O}(\Lambda)} &= d\alpha_4(d\Lambda, A)|_{\mathcal{O}(\Lambda)} = d\omega_4^1(\Lambda, A)\\ &= d(3\mathrm{tr}(\Lambda F^2) + \beta_4(D_A\Lambda, A)),\end{aligned} \tag{A24}$$

where

$$\begin{aligned}\omega_4^1(\Lambda, A) &\equiv \mathrm{tr}\left(\Lambda d\left(AdA + \frac{1}{2}A^3\right)\right)\\ &= \frac{1}{2}\mathrm{tr}(\Lambda(2F^2 - FA^2 - AFA - A^2F + A^4)).\end{aligned} \tag{A25}$$

The infinitesimal variation of $\alpha_4(A)$ is

$$\begin{aligned}\delta\alpha_4(V,A) &\equiv \alpha_4(V, A+\delta A) - \alpha_4(V,A)\\ &= \frac{1}{2}\mathrm{tr}(\delta A(2FV + 2VF - (A+V)^3 + A^3))\\ &\quad - \frac{1}{2}d\mathrm{tr}(\delta A[V,A]) + \mathcal{O}(\delta A^2).\end{aligned} \tag{A26}$$

## 4. WZW

When $U = gh$, where $g$ and $h$ are $U(N_f)$-valued functions, we have

$$\mathrm{tr}((U^{-1}dU)^3) = -\mathrm{tr}(G^3) + \mathrm{tr}(H^3) + 3d\mathrm{tr}(GH), \tag{A27}$$

and

$$\begin{aligned}\mathrm{tr}((U^{-1}dU)^5) &= -\mathrm{tr}(G^5) + \mathrm{tr}(H^5)\\ &\quad + 5d\mathrm{tr}\left(G^3H + GH^3 - \frac{1}{2}GHGH\right),\end{aligned} \tag{A28}$$

where $G = dg^{-1}g$, $H = dhh^{-1}$. This formula can also be shown from (A16) by setting $U^{-1}dU = A^{h^{-1}}$ with $A = g^{-1}dg = -G$.

When $U = gfh$, where $g$, $f$ and $h$ are $U(N_f)$-valued functions, we have

$$\begin{aligned}&\mathrm{tr}((U^{-1}dU)^5)\\ &= -\mathrm{tr}(G^5) + \mathrm{tr}(F^5) + \mathrm{tr}(H^5)\\ &\quad + 5d\mathrm{tr}(f^{-1}(G-F)^3fH + f^{-1}(G-F)fH^3\\ &\quad - \frac{1}{2}\left(f^{-1}(G-F)fH)^2 + G^3F + GF^3 - \frac{1}{2}GFGF\right)\\ &= -\mathrm{tr}(G^5) - \mathrm{tr}(\hat{F}^5) + \mathrm{tr}(H^5)\\ &\quad + 5d\mathrm{tr}(G^3f(H-\hat{F})f^{-1} + Gf(H-\hat{F})^3f^{-1}\\ &\quad - \frac{1}{2}\left(Gf(H-\hat{F})f^{-1})^2 + \hat{F}^3H + \hat{F}H^3 - \frac{1}{2}\hat{F}H\hat{F}H\right),\end{aligned} \tag{A29}$$

where $G = dg^{-1}g$, $F = dff^{-1}$, $\hat{F} = df^{-1}f$ and $H = dhh^{-1}$.

An important property is that when $M_5$ is a five-dimensional closed manifold, the integral

$$\frac{C}{10}\int_{M_5} \mathrm{tr}((U^{-1}dU)^5) \tag{A30}$$

takes values in $2\pi\mathbb{Z}$ and its contribution in the action can be dropped. When $M_5$ has a boundary, a useful trick to evaluate this integral is to find $N_5$ such that $\partial N_5 = \partial M_5$, i.e. $M_5 \cup (-N_5)$ is a closed manifold, and extend $U$ to be a $U(N_f)$-valued function on $M_5 \cup (-N_5)$. If such $N_5$ and $U$ exist, $M_5$ can be replaced with $N_5$ by using

$$\frac{C}{10}\int_{M_5} \mathrm{tr}((U^{-1}dU)^5) = \frac{C}{10}\int_{N_5} \mathrm{tr}((U^{-1}dU)^5), \quad (\mathrm{mod}\, 2\pi\mathbb{Z}). \tag{A31}$$


[1] J. M. Maldacena, The large N limit of superconformal field theories and supergravity, Int. J. Theor. Phys. **38**, 1113 (1999) [Adv. Theor. Math. Phys. **2**, 231 (1998)].
[2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Gauge theory correlators from noncritical string theory, Phys. Lett. B **428**, 105 (1998).
[3] E. Witten, Anti-de Sitter space and holography, Adv. Theor. Math. Phys. **2**, 253 (1998).
[4] A. Gúijosa, QCD, with strings attached, Int. J. Mod. Phys. E **25**, 1630006 (2016).
[5] T. Sakai and S. Sugimoto, Low energy hadron physics in holographic QCD, Prog. Theor. Phys. **113**, 843 (2005).
[6] A. Rebhan, The Witten-Sakai-Sugimoto model: A brief review and some recent results, EPJ Web Conf. **95**, 02005 (2015).
[7] D. T. Son and M. A. Stephanov, QCD and dimensional deconstruction, Phys. Rev. D **69**, 065020 (2004).
[8] S. K. Domokos and J. A. Harvey, Baryon Number-Induced Chern-Simons Couplings of Vector and Axial-Vector Mesons in Holographic QCD, Phys. Rev. Lett. **99**, 141602 (2007).
[9] A. Pomarol and A. Wulzer, Baryon physics in holographic QCD, Nucl. Phys. **B809**, 347 (2009).
[10] S. K. Domokos, H. R. Grigoryan, and J. A. Harvey, Photoproduction through Chern-Simons term induced interactions in holographic QCD, Phys. Rev. D **80**, 115018 (2009).
[11] O. Aharony and D. Kutasov, Holographic duals of long open strings, Phys. Rev. D **78**, 026005 (2008); K. Hashimoto, T. Hirayama, F. L. Lin, and H. U. Yee, Quark mass deformation of holographic massless QCD, J. High Energy Phys. 07 (2008) 089; R. McNees, R. C. Myers, and A. Sinha, On quark masses in holographic QCD, J. High Energy Phys. 11 (2008) 056; P. C. Argyres, M. Edalati, R. G. Leigh, and J. F. Vazquez-Poritz, Open Wilson lines and chiral condensates in thermal holographic QCD, Phys. Rev. D **79**, 045022 (2009); For other approaches, see R. Casero, E. Kiritsis, and A. Paredes, Chiral symmetry breaking as open string tachyon condensation, Nucl. Phys. **B787**, 98 (2007); K. Hashimoto, T. Hirayama, and A. Miwa, Holographic QCD and pion mass, J. High Energy Phys. 06 (2007) 020; N. Evans and E. Threlfall, Quark mass in the Sakai-Sugimoto model of chiral symmetry breaking, arXiv:0706.3285; O. Bergman, S. Seki, and J. Sonnenschein, Quark mass and condensate in HQCD, J. High Energy Phys. 12 (2007) 037; A. Dhar and P. Nag, Sakai-Sugimoto model, tachyon condensation and chiral symmetry breaking, J. High Energy Phys. 01 (2008) 055; A. Dhar and P. Nag, Tachyon condensation and quark mass in modified Sakai-Sugimoto model, Phys. Rev. D **78**, 066021 (2008).
[12] J. Wess and B. Zumino, Consequences of anomalous Ward identities, Phys. Lett. **37B**, 95 (1971).
[13] E. Witten, Global aspects of current algebra, Nucl. Phys. **B223**, 422 (1983).
[14] O. Kaymakcalan, S. Rajeev, and J. Schechter, Nonabelian anomaly and vector meson decays, Phys. Rev. D **30**, 594 (1984).
[15] M. Gell-Mann, D. Sharp, and W. G. Wagner, Decay Rates of Neutral Mesons, Phys. Rev. Lett. **8**, 261 (1962).
[16] T. Sakai and S. Sugimoto, More on a holographic dual of QCD, Prog. Theor. Phys. **114**, 1083 (2005).
[17] T. H. R. Skyrme, A non-linear field theory, Proc. R. Soc. A **260**, 127 (1961); Particle states of a quantized meson field, Proc. R. Soc. A **262**, 237 (1961); T. H. R. Skyrme, A unified field theory of mesons and baryons, Nucl. Phys. **31**, 556 (1962).
[18] M. F. Atiyah and N. S. Manton, Skyrmions from instantons, Phys. Lett. B **222**, 438 (1989).
[19] H. Hata and M. Murata, Baryons and the Chern-Simons term in holographic QCD with three flavors, Prog. Theor. Phys. **119**, 461 (2008).
[20] H. Hata, T. Sakai, S. Sugimoto, and S. Yamato, Baryons from instantons in holographic QCD, Prog. Theor. Phys. **117**, 1157 (2007).
[21] V. P. Nair, *Quantum Field Theory: A Modern Perspective* (Springer, New York, 2005).
[22] E. Witten, Current algebra, baryons, and quark confinement, Nucl. Phys. **B223**, 433 (1983).
[23] E. Guadagnini, Baryons as solitons and mass formulae, Nucl. Phys. **B236**, 35 (1984).
[24] P. O. Mazur, M. A. Nowak, and M. Praszalowicz, SU(3) extension of the Skyrme model, Phys. Lett. **147B**, 137 (1984).
[25] M. Chemtob, Skyrme model of baryon octet and decuplet, Nucl. Phys. **B256**, 600 (1985).

[26] S. Jain and S. R. Wadia, Large *N* baryons: Collective coordinates of the topological soliton in SU(3) chiral model, Nucl. Phys. **B258**, 713 (1985).

[27] A. V. Manohar, Equivalence of the chiral soliton and quark models in large N, Nucl. Phys. **B248**, 19 (1984).

[28] A. P. Balachandran, F. Lizzi, V. G. J. Rodgers, and A. Stern, Dibaryons as chiral solitons, Nucl. Phys. **B256**, 525 (1985).

[29] K. Hashimoto, T. Sakai, and S. Sugimoto, Holographic baryons: Static properties and form factors from gauge/string duality, Prog. Theor. Phys. **120**, 1093 (2008).

[30] A. Rebhan, A. Schmitt, and S. A. Stricker, Meson supercurrents and the Meissner effect in the Sakai-Sugimoto model, J. High Energy Phys. 05 (2009) 084.

[31] A. Rebhan, A. Schmitt, and S. A. Stricker, Anomalies and the chiral magnetic effect in the Sakai-Sugimoto model, J. High Energy Phys. 01 (**2010**) 026.

[32] J. A. Harvey, Magnetic monopoles, duality and supersymmetry, arXiv:hep-th/9603086.

[33] R. Dijkgraaf and E. Witten, Topological gauge theories and group cohomology, Commun. Math. Phys. **129**, 393 (1990).

[34] K. Hashimoto, N. Iizuka, T. Ishii, and D. Kadoh, Three-flavor quark mass dependence of baryon spectra in holographic QCD, Phys. Lett. B **691**, 65 (2010).

[35] T. Ishii, Toward bound-state approach to strangeness in holographic QCD, Phys. Lett. B **695**, 392 (2011).

[36] S. Aoki, K. Hashimoto, and N. Iizuka, Matrix theory for baryons: An overview of holographic QCD for nuclear physics, Rep. Prog. Phys. **76**, 104301 (2013).

[37] K. Matsumoto, Y. Nakagawa, and H. Suganuma, A study of the H-dibaryon in holographic QCD, J. Phys. Soc. Jpn. Conf. Proc. **13**, 020014 (2017).

[38] H. Suganuma and K. Matsumoto, Holographic QCD for H-dibaryon (uuddss), EPJ Web Conf. **137**, 13018 (2017).