Mathematical Research in High School: The PRIMES Experience

Pavel Etingof, Slava Gerovitch, and Tanya Khovanova

Consider a finite set of lines in 3-space. A *joint* is a point where three of these lines (not lying in the same plane) intersect. If there are L lines, what is the largest possible number of joints? Well, let's try our luck and randomly choose k planes. Any pair of planes produces a line, and any triple of planes, a joint. Thus, they produce $L := k(k-1)/2$ lines and $J := k(k-1)(k-2)/6$ joints. If k is large, J is about $\frac{\sqrt{2}}{3}L^{3/2}$. For many years it was conjectured that one cannot do much better than that, in the sense that if L is large, then $J \leq CL^{3/2}$, where C is a constant (clearly, $C \geq \frac{\sqrt{2}}{3}$). This was proved by Larry Guth and Nets Katz in 2007 and was a breakthrough in incidence geometry. Guth also showed that one can take $C = 10$. Can you do better? Yes! The best known result is that any number $C > 4/3$ will do. This was proved in 2014 by Joseph Zurier, an eleventh-grader from Rhode Island [Z].

Here is another problem. Let K and L be convex bodies in space, and suppose that we can hide K behind L no matter from where we look (we are allowed to translate the bodies but may not rotate them). Is it true that the volume of K is at most the volume of L? Curiously, no! Christina Chen, a tenth-grader from Massachusetts, showed in 2011 that the volume ratio can be about 1.16, the best currently known value ([Ch]; see Figure 1). So, can it be arbitrarily large? No! Christina, Tanya Khovanova, and Dan Klain showed that the volume ratio is less than 3 in any dimension [CKK].

Figure 1. PRIMES student Christina Chen is showing a picture of a convex body which can hide behind a tetrahedron of smaller volume (PRIMES Conference, 2011).

Seriously? Is it really possible for tenth- and eleventh-graders to do original mathematical research?

Yes! Christina and Joseph, as well as over a hundred other students, have done their research at PRIMES (Program for Research In Mathematics, Engineering, and Science; `web.mit.edu/primes`), which we've been running in the MIT mathematics department since January 2011. Every year we receive numerous questions about our program from prospective students and their parents and also from academics who want to organize a similar program. Here we'd like to answer some of these questions, to share our experience, and to tell a wider mathematical community how such a seemingly impossible thing as mathematical research in high school can actually be done.

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What kind of students do you look for? My son was wondering—should he even bother to apply if he doesn't have a perfect score at the IMO and hasn't yet mastered Wiles's proof of Fermat's Last Theorem?

P.E.: If he is in love with math, yes, by all means! Some background (such as calculus) is needed, but generally he will learn along the way under his mentor's guidance. Also, many gifted high-schoolers do well at math competitions, but good researchers are not always quick problem-solvers. It takes time, effort, and perseverance to learn the background and try different approaches, many of which are doomed to fail. We look for students with a talent for mathematical research and a stamina to carry it through, for avid learners, hard workers, and imaginative explorers. And, above all, for those who are crazy about mathematics!

How do you select students? My daughter asks: to get accepted, does she need to be a machine that makes coffee into theorems?

P.E.: Mathematicians are mere humans who make coffee into theorems 10 percent of the time, and into unsuccessful attempts to prove theorems the rest of their lives—and we welcome your daughter to the club!

S.G.: We carefully consider Olympiad scores, statements of purpose, recommendations, and grades, but the pivotal part of the application is the entrance problem set. We post it in mid-September, due in two months.

T.K.: These are not the kinds of problems that one can crack quickly: at first glance, some of them may puzzle even a math professor. Students are expected to think about a problem, consult books and online sources, think again the next day, then again and again, until one day they finally get it and then write a full solution with detailed proofs. This protracted engagement with mathematical problems resembles a research process.

P.E.: In fact, this is similar to my favorite hobby, picking mushrooms. You may run around the woods for hours seeing nothing, and then all of a sudden you find a real treasure. You need patience and ability to enjoy the process and forget about everything else. There is a lot in common between a good mushroom hunter and a good mathematician.

Do PRIMES students work individually or in groups?

S.G.: Most projects are individual and involve one-on-one mentoring. Freshmen and sophomores, however, usually work in groups of two on joint projects. Group discussions make research more exciting and stimulating for younger students and give them a gentler entry into the world of mathematics. Even in individual projects the students are not alone: they collaborate with their mentors and faculty who suggested projects. They form a team in which mathematicians of different levels of experience and seniority become equal collaborators. This way PRIMES students learn the art of collaboration and teamwork.

P.E.: In short, we do have options for both introverted mathematicians (who like to look at their shoes while doing research) and extroverted ones (who prefer to look at the shoes of their collaborator). And, of course, all PRIMES students are encouraged to look at the shoes of their mentors as often as they like!

How do you select projects? Can my student be told to prove the Twin Primes Conjecture in PRIMES?

P.E.: Famous open problems don't usually make good projects, but we don't assign "toy projects" with known solutions either. Students delve into real research, with all its uncertainties, disappointments, and surprises. Finding cutting-edge projects requiring a minimal background is one of the trickiest tasks in running PRIMES. Here are some features we want to see in a PRIMES project:

1. *Accessible beginning.* Presence of simple initial steps to get started.
2. *Flexibility.* A possibility to think about several related questions, switching from one to another if stuck, and to tweak the questions if they are too hard or insufficiently interesting.
3. *Computer (experimental) component.* A possibility of computer-assisted exploration aimed at finding patterns and making conjectures. This way students, who often have strong programming skills, can contribute to the project early, when they don't yet have a working knowledge of the theoretical tools. It is also easier to learn new mathematical concepts, e.g., those from algebra and representation theory, through a hands-on experience with a computer algebra system.
4. *Adviser involvement.* Availability of a research mathematician other than the mentor (usually the professor or researcher who suggested the project) to advise the project through email and occasional meetings. Such meetings make a big difference.
5. *Big picture/motivation.* Connection, at least at the level of ideas, to a wider context and to other people's work.
6. *Learning component.* The project should encourage the student to study advanced mathematics on a regular basis.
7. *Doability.* A reasonable expectation that a good student would obtain some new results in several months to present at the annual PRIMES conference in mid-May and produce publishable results in one year.
8. *Relation to the mentor's research program or area.*

T.K.: A crucial part of research is the art of asking your own questions, not just solving other people's problems. When the students realize that it is in their power to move the project in a new direction, they get very excited and start feeling ownership of the project. The ability to trust themselves and ask their own questions is very important in their future lives, independent of their career choices. That's why we try to choose projects that develop this ability.

P.E.: Sounds easy? Well, if you have a bit of free time or have nothing better to do (e.g., during an excruciatingly boring math lecture that you can't sneak out of), just try to come up with a project satisfying most of these conditions. And when you do, please send it to us!

Is it true that in PRIMES mathematics equals elementary combinatorics? Do PRIMES students work on elaborate Olympiad problems instead of learning about algebra, topology, geometry, analysis, number theory?

P.E.: Not really. We've had many projects in these fields, especially in noncommutative algebra and representation theory. Also, PRIMES students get exposed to these areas in PRIMES reading groups.

This said, it's true that many PRIMES projects are in discrete math. This is because in this field, it's easier to find interesting projects requiring relatively little initial background. However, they are not just elaborate Olympiad problems. Many of them are designed to touch upon fundamental questions and to encourage learning about other areas with which discrete math has many deep connections. In short, we try to show our students both the breadth and the unity of mathematics.

Noncommutative algebra and representation theory in high school? Touch upon fundamental questions? No kidding? Can you give some examples?

P.E.: You want me to get technical? All right, here you go.

One group of projects concerns representations of rational Cherednik algebras. Let G be a finite group and V be its finite-dimensional representation over a field k. Then one can define the rational Cherednik algebra $H(G,V)$, which is a certain remarkable deformation of the algebra $kG \ltimes D(V)$, the semidirect product of the group algebra of G with the algebra of differential operators on V. For example, if $G = \mathbb{Z}/2\mathbb{Z}$, $V = k$, and the generator $s \in \mathbb{Z}/2\mathbb{Z}$ acts on the coordinate x on V by $s(x) = -x$, then $H(G,V)$ is generated by s, x and the *Dunkl operator* $\partial_x - \frac{k}{x}s$. Representations of $H(G,V)$ are currently a subject of active research.

In [DS] Sheela Devadas and her mentor, Steven Sam, studied lowest-weight irreducible representations of $H(G,V)$ for G being the complex reflection group $G(m,r,n)$ and $V = k^n$ (where $\mathrm{char}\, k = p$) using methods of commutative algebra. They gave conjectural character formulas for some of them and proved these formulas in a number of cases. In general, this is a difficult open problem. It is not easy even in the case $n = 2$ and $m = r$ (groups of symmetries of a regular polygon); in this case, more definitive results were obtained by PRIMES student Carl Lian [Li].

In [DT] Fengning Ding and his mentor, Sasha Tsymbaliuk, considered representations of continuous Cherednik algebras, which are generalizations of $H(G,V)$ to the case when G is a reductive algebraic group (rather than a finite group). Namely, they considered the case when $G = GL(n,\mathbb{C})$ and $V = \mathbb{C}^n$. They computed the center of $H(G,V)$, classified its finite-dimensional irreducible representations, and computed their characters.

In [KL] Shashwat Kishore and his mentor, Gus Lonergan, studied signature of the canonical Hermitian form on the space $\mathrm{Hom}(M_\lambda, M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n})$, where $\lambda, \lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and M_λ is the Verma module for the Lie algebra $\mathfrak{sl}_2$. They classified the cases when this form is definite and also applied the signature formula to solve a topological problem: give lower bounds for the number of real critical points of the Gaudin model master function

$$F(t_1,\ldots,t_m,z_1,\ldots,z_n) = \prod_{1\le i<j\le m}(t_i-t_j)^2\prod_{i=1}^{m}\prod_{k=1}^{n}(t_i-z_k)^{-\lambda_k},$$

where $m = \frac{1}{2}(\lambda_1 + \cdots + \lambda_n - \lambda)$. They also generalized their results to the case of quantum group $U_q(\mathfrak{sl}_2)$ (where $|q| = 1$).

We've also had some other algebraic projects. With Yongyi Chen, Michael Zhang, and their mentor, David Jordan, we studied trace functions on the algebra $A_P := k[x,y,z]/(P)$, where P is a generic homogeneous polynomial of degree d and k is a field of characteristic p [CEJZ]. By definition, a trace function is a linear function on A_P which vanishes on Poisson brackets

$$\{f,g\} := \frac{\partial(P,f,g)}{\partial(x,y,z)}.$$

The problem was to compute the Hilbert series of the space of trace functions, i.e., $h(z) := \sum_{n\ge 0} h_n z^n$, where h_n is the dimension of the space of trace functions of degree n. It turns out that for large enough p, the function $h(z)$ is given by the following peculiar formula:

$$h(z) = \frac{(1-z^{d-1})^3}{(1-z)^3} + z^{d-3}\left(\frac{1-z^{pd}}{(1-z^p)^3} + \frac{d(d-3)z^p}{1-z^p} - 1\right).$$

We found this formula empirically on a computer and then proved it (and generalized to the quasi-homogeneous case) using some algebraic geometry and the theory of D-modules.

Another algebraic project concerned the lower central series of an associative algebra A: $L_1 = A$, $L_2 = [A, L_1]$, $L_3 = [A, L_2]$, and so on. Feigin and Shoikhet showed in 2006 that if A is free in n generators over $\mathbb{Q}$, then $B_2 = L_2/L_3$ is the space of closed differential forms of positive even degree in n variables. With Surya Bhupatiraju, Bill Kuszmaul, Jason Li, and their mentor, David Jordan, we generalized this result to the case of integer coefficients, expressing B_2 in terms of the de Rham cohomology over the integers [BEJKL]. In another project, Isaac Xia and his mentor, Yael Fregier, studied quotients $N_i := AL_i/AL_{i+1}$ and showed that if A is a free algebra in $x_1, \ldots, x_n$ over a field of characteristic p modulo relations written in terms of $x_1^{p^{m_1}}, \ldots, x_n^{p^{m_n}}$ and if the abelianization of A is finite-dimensional, then N_i have dimensions divisible by $p^{\sum m_i}$ [FX]. The proof is based on the representation theory of algebras of differential operators with divided powers.

Tired of algebra? Here is a project in combinatorics. A linear equation is r-regular if for every r-coloring of the positive integers, there exist positive integers of the same color which satisfy the equation. In 2005 Fox and Radoicic conjectured that the equation

$$x_0 + 2x_1 + \cdots + 2^{n-1}x_{n-1} - 2^n x_n = 0,$$

for any $n \geq 1$, has a degree of regularity of n, which would verify a conjecture of Rado from 1933. While Rado's conjecture was later verified with a different family of equations, the Fox-Radoicic conjecture remained open. This conjecture (in a generalized form) was proved by Noah Golowich [Go] under the mentorship of László Lovász.

S.G.: This is beautiful math, but sounds like the "prior results" section of our grant proposal. Did you copy-paste it here? This will put the readers to sleep! Tell them what our students do in the form of an exciting game or an engaging story.

P.E.: OK, let me try my best. Every day each Martian gives each of his friends one Martian peso if he is sufficiently rich to do so. What will happen?

This process is called "the parallel chip-firing game" (see Figure 2) and is an important model of dynamics on graphs. Clearly, it is eventually periodic, but as it is nonlinear, one could a priori expect complicated behavior. Yet, Ziv Scully with his mentors, Damien Jiang and Yan Zhang, were able to completely characterize the possible periodic patterns [JSZ]. This is a truly beautiful result!

S.G.: Perhaps this is a good model for funding PRIMES? I suppose this model applies not only to Martian pesos but equally to earthly hundred-dollar bills? Then we just need to make sure that PRIMES has enough sufficiently rich friends....

P.E.: Well, there is a small catch: according to this model, PRIMES would also have to give out hundred-dollar bills. The total amount of money in the system is preserved, so the salary of the program director would unfortunately have to be zero!

But surely not all your projects are at this high level. I saw that one of them is about "dessins d'enfants." Unless I am forgetting my French, this means "child's drawings." Can this possibly involve serious mathematics?

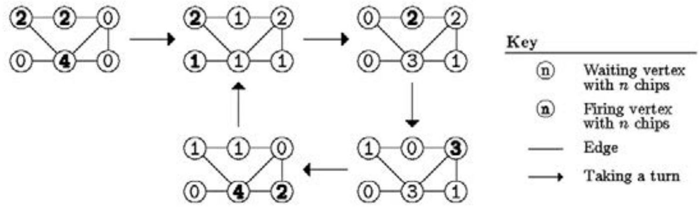

Figure 2. PRIMES project, 2011.

P.E.: In fact, this is one of our more advanced projects! The child here is Alexandre Grothendieck (1928–2014), one of the greatest mathematicians of the twentieth century. In 1984 in his famous "Esquisse d'un Programme," he proposed to study the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ through its action on the set of finite covers of the complex plane that branch at $0, 1, \infty$. He represented such covers by certain planar graphs which he called "dessins d'enfants." An important problem is to find invariants of covers (or, equivalently, Grothendieck's dessins) that allow one to show that two given covers are not equivalent under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Ravi Jagadeesan (mentored by Akhil Mathew) found a new invariant of covers which is more powerful than the previous invariants and used it to prove a new lower bound on the number of Galois orbits of a certain type [Ja]. Even though it is about "dessins d'enfants" and the author was in eleventh grade, this result is of real interest to grown-up mathematicians!

How do you match students to projects? Is the matching theory relevant here?

P.E.: Yes. We always manage to find a good matching, all thanks to the counterintuitive mathematical fact that good matchings exist and are easy to find: the Gale-Shapley Stable Marriage Theorem, which says that in the ideal world all marriages are stable. In fact, the only reason the real world is short of ideal is that people don't know enough mathematics!

T.K.: However bright, PRIMES students rarely have an idea of what a suitable project would look like. Many applicants, for example, declare on their application that they want to work on the Riemann hypothesis. Most list number theory as their top interest, which could be the result of PROMYS, Canada-USA MathCamp, Ross, and other programs teaching students advanced number theory. Yet state-of-the-art projects in number theory are usually too advanced for high school students. For this reason, it does not always make sense to follow the applicant's preferences literally.

But we try to glean from the application the true inclinations and strengths of the student and find a project that would let us play on those as much as we can.

Also, an average PRIMES student is a better programmer than an average MIT math professor. And many of our projects have a computational component. So we look at the programming background in addition to the math background when matching projects.

S.G.: Most students end up working in areas they've never heard of before, because that's where the good projects are. Also, we adjust the difficulty of each project along the way, depending on the student's abilities, preparation, and progress. This way every student discovers the joy of proving a new theorem.

How do you find mentors and match them to projects and students? Does PRIMES distract mentors from their research and ruin their careers?

S.G.: PRIMES mentors are typically math graduate students and sometimes postdocs or faculty who show desire and ability to work with high school students. We look for mentors with a knack for teaching and an inspiring personality who can be effective role models. We also try to match projects and students with the mentor's research style, whether conceptual or oriented toward problem solving.

Finally, to make sure that PRIMES does not take a toll on the mentor's career and research, we encourage mentors to suggest projects related to their own work. This not only improves the quality of mentoring but also allows mentors to combine mentoring with research, often leading to joint papers with students. Thus mentors not only receive a supplement to their stipend and acquire valuable advising experience but also add joint papers to their publication record, strengthening their position in the job market.

T.K.: Some mentors say that after teaching high school students how to do research, they finally understand it themselves!

Do you admit students from other states? Other countries? Other planets, planetary systems, galaxies?

S.G.: For the first two years, PRIMES operated as a program for local students. In 2013 we decided to do an experiment. We selected five students in a nationwide search and mentored their research projects over the Internet, using software and hardware tools for online collaboration. The experiment proved to be a success, and the following year we expanded the PRIMES-USA section to thirteen students, including two supervised by faculty from the University of Illinois at Urbana-Champaign. This year the number of out-of-state students rose to fifteen, and to meet the demand we collaborate with faculty from several universities, including CUNY and SUNY at Stony Brook.

The PRIMES-USA section not only provides research opportunities to talented math students across the nation but also serves as a laboratory for testing new methods of distance research mentoring, as well as helps spread the PRIMES approach to other universities in the United States and beyond. Arrangements to open a section of PRIMES in Europe are currently under way.

P.E.: And, yes, PRIMES-Extraterrestrial is presently under construction. Originally our down-to-earth program director was reluctant, but I've convinced him to go ahead. However, alien high school students may well turn out to know more math than MIT professors. So I envision working on proving the Riemann hypothesis under the guidance of a high school mentor from the far end of the Milky Way.…

Is it good for a math student to start research so early? Isn't it better to spend time reading and learning new mathematics?

P.E.: In many cases, it is better, yes. Reading mathematical literature and learning are vital parts of the professional life of every mathematician. They are of key importance at all ages. In fact, one of the greatest mathematicians of the twentieth century, I. M. Gelfand, said at his ninetieth birthday celebration: "I am a student of mathematics."

It is a tautology that learning is especially vital for students. For many students, guided reading is more intellectually stimulating and beneficial than an immediate plunge into research. We carefully evaluate PRIMES applicants and recommend research only for those who are ready. For most younger students we set up reading groups of 2–3 students who study an advanced mathematical book with a mentor. Devoting the first year of PRIMES to guided reading helps students build a foundation for attacking research problems in the following years.

PRIMES research projects are also designed to require learning new mathematics. Before the project starts, students devote one month to background reading, and they continue reading along with research. A research problem provides excellent motivation and environment for learning a new area of math.

S.G.: We also recommend reading groups to seniors, who spend only half a year in PRIMES before going to college. After doing a research project in junior year, PRIMES students often stay in reading groups to expand their mathematical knowledge. We encourage them to explore areas beyond the topic of their research project. This year, 40 percent of local math students at PRIMES are in reading groups, while 60 percent work on research projects.

What's the timeline of your program? Isn't a whole year too long? Do PRIMES students ever get a life?

P.E.: A year is too short! We all know it takes months and sometimes years to prove a good theorem and to write a good paper. And you get a life by getting to do math all this time! One famous mathematician said that to come up with a theorem that's any good, you have to become a sleepwalker for at least several weeks. That's what I've been trying to explain to my wife, admittedly without much success.…

S.G.: The PRIMES cycle runs through a calendar year.

January is the **reading period**: mentors give their students background reading and exercises via email or Skype. In early February we invite all local students and their parents to campus for an orientation meeting, where students meet their mentors and mingle with other students. At this meeting, Pavel shares his tips on doing mathematical research, also available at the PRIMES website.

P.E. This is a really exciting speech, especially for continuing students (given that the tips don't change from year to year, as they have eternal value).

S.G.: The spring semester is the **active research period**. At weekly meetings students and mentors discuss progress and set goals for next week. Meetings nominally last one and a half hours, but often run longer, as students and mentors get excited about new ideas. Students are encouraged to get in touch with their mentors over email midweek or any time they have a question or get stuck.

T.K.: During this period, I periodically check on every project, suggesting adjustments if necessary.

S.G.: The active research period culminates with a presentation at an annual PRIMES conference, held at MIT in May on the last weekend before Memorial Day (Figure 1, Figure 3). Prior to the conference, students prepare a research report, which includes preliminaries, previous results, statement of the problem, and new results. Since the conference comes in the middle of the annual cycle, the students present work in progress. They incorporate feedback received at the conference into their future work. The conference lasts for two days, with talks on mathematics, computer science, and computational biology before a lively audience that includes grad students, postdocs, and faculty. PRIMES students' parents, many of whom are academics or industry researchers, are also invited; they often ask interesting questions and invariably end up thoroughly impressed.

The summer break is the **independent study period**. The student and the mentor coordinate their schedules, meeting when in town, communicating by email when away, or taking a beach break when the weather is good. We also encourage PRIMES students to take advantage of other opportunities, such as attending summer math camps, which allows them to expand their scope and take a break from their project, only to return to it with renewed vigor in the fall.

Figure 3. Pavel Etingof and Tanya Khovanova with PRIMES Conference 2013 participants.

Fall is the **write-up period**. Students meet with their mentors as needed, finalize their project, and write a final paper summarizing their results. This is the time when we can teach our students to write mathematics, which is one of the important goals of PRIMES. Many PRIMES papers are submitted to national science competitions and the MAA-AMS undergraduate student poster session at the Joint Meetings in January.

T.K.: Sometimes by the end of the project the student and the mentor see a big, beautiful conjecture that generalizes their results. This conjecture is like a star shining ahead of them. When the PRIMES year is over, they can't stop, and continue working until they prove their conjecture.

Can students stay for a second year?

S.G.: Yes, every year a number of students stay for another year. This allows younger students to mature as researchers.

An example: Bill Kuszmaul was in PRIMES for four years. Having entered PRIMES in ninth grade, he did two joint projects in years 1 and 2, an individual project in year 3, and a reading group in year 4. Bill authored four papers posted on arXiv.org (two of them published in the *Journal of Algebra* and the *Electronic Journal of Combinatorics*), was a Siemens regional finalist in 2011 and 2012, a 2013 Davidson Fellow, and won Third Prize in the 2014 Intel STS. He is now a sophomore at Stanford.

P.E.: In fact, besides proving many cool theorems, Bill introduced a new English word. In his testimonial, he wrote: "It gave me an incredible feeling to have the paper come together in the final days of it being written, and I came to cherish the feeling of just putting everything in life aside and "primesing" for the rest of a day."

Is high school math research possible outside of PRIMES?

P.E.: Sure. One option is for students to work by themselves, supervised by mentors active in research. Also, there are summer programs offering such opportunities: RSI (for individual projects), PROMYS, Canada-USA MathCamp, and others (for group projects).

Sounds like you have competitors. Is your goal to put them out of business?

P.E.: In fact, our goal is to put as many of them as possible **into** business, and that's exactly why we are answering these questions here. Each year we have to turn down a growing number of strong applicants, which is a pity. We hope that soon there will be more opportunities like PRIMES. These students ought to have a chance to achieve their dream!

How is your program different from RSI, PROMYS, Canada-USA MathCamp, Ross, and other summer programs? Which one should my child choose?

S.G.: The main difference is that summer programs are compressed into a few weeks, while PRIMES operates for an entire year. This allows research at a natural pace, with sufficient time for trial and error, gaining additional background, and writing a detailed text according to professional standards.

T.K.: A few weeks are not enough. My best ideas come to me in the shower. I wouldn't be able to finish research in a summer program, as there aren't enough showers!

P.E.: Exactly. And there isn't enough hot water in the boiler. My wife complains that I leave none for anyone else, and this is expensive and environmentally unhealthy. Perhaps we should figure out a way to balance family and ecological needs with the need to do good mathematics!

S.G.: Well, Archimedes's example clearly shows that a bath, while much less wasteful, can be equally stimulating for a mathematician.

But bathroom issues aside, summer programs give students an excellent experience. They take a variety of short courses and are exposed to a wide range of mathematical topics, useful for further research. And your child doesn't really have to choose! PRIMES has a flexible schedule in the summer, which allows our students to attend summer programs, and we strongly encourage them to do so, as the two experiences reinforce each other. For instance, PRIMES students attending RSI often work there on projects related to their PRIMES projects, which magnifies the effects of both programs and often results in much stronger final papers.

How do you measure success?

S.G.: Every year PRIMES students win many prizes at national science competitions, including the very top ones. For example, in four years PRIMES has claimed twenty-four Siemens and fifteen Intel STS finalist awards. The first and second prizes at Siemens 2014, as well as the first, second, and third prizes for basic research at Intel STS 2015, went to PRIMES. Yet this is not our main criterion of success. A more important one is publications: our students have completed seventy-one papers, posted forty of them on `arXiv.org`, and published fifteen in high-level academic journals. Another criterion is matriculation record: virtually all our graduates go to top universities, where they are among the best students. Finally, the number of applications: in PRIMES-USA, it has tripled in the last two years. But above all, we feel that our mission is accomplished when our students get a taste of genuine mathematical research and fall in love with mathematics.

T.K.: The ultimate measure of success will come in a few years when these kids grow up. They are just amazing! I feel honored to work with the best mathematicians of the future generation.

Is your goal to win the largest possible number of prizes?

P.E.: Not really. We tell our students not to hyperfocus on winning science competitions and explain that mathematical research is about collaboration rather than competition. Yet, competitions are useful as an organizing and motivating factor. They need to write a paper which will be read by judges by a certain deadline, and this makes a difference. Also the Siemens and Intel STS competitions do a great job organizing activities for finalists. They meet and discuss their work with very competent judges, some of them top-level professional researchers in the field of their project. They also learn a lot from each other. And, last but not least, they have a lot of fun!

Do you expect all PRIMES math students to become research mathematicians? If they don't, do you view this as failure?

T.K.: Not necessarily. Some of them may want to do computer science, law, business, medicine, and so on. They come to us because they want to challenge their minds and try to see what math research is like, and this experience is valuable to them whatever career they choose. We had cases when students enjoyed math research so much that they changed their life plans and decided to become mathematicians. And we had other students who realized that they do not want to be mathematicians. They have a gift for mathematics, but their hearts are not there. And it is very useful to discover this before college.

So being sure that one wants to become a mathematician is not a requirement for our program. Intellectual curiosity and willingness to explore are way more important.

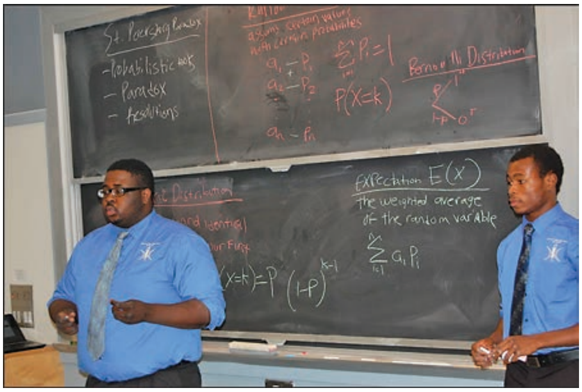

Figure 4. Primes Circle Conference 2013: PRIMES Circle students Omotoyosi Oyedeji and Tyreik Silva are giving a talk about probability theory.

Figure 5. MathROOTS students: (l to r) Josue Sican, Ben Bennington-Brandon, Trajan Hammonds, Adedoyin Olateru-Olagbegi.

What do you do to help diversify the mathematical community?

S.G.: In 2013 we set up PRIMES Circle, a math enrichment program for talented sophomores and juniors from local urban public high schools. Working in small groups under the guidance of MIT undergraduate students, PRIMES Circle participants discover the beauty of the mathematical way of thinking and the thrill of solving a challenging problem. Circle students study advanced topics in geometry, probability, combinatorics, knot theory, and so on; prepare expository papers; and make presentations at a miniconference at MIT. PRIMES Circle has expanded from eight students in 2013 to fifteen in 2015. Of current Circle students 60 percent are female, 27 percent are Hispanic, and 13 percent are African-American (see Figure 4).

In 2015 we organized a new section, MathROOTS, a twelve-day summer camp hosted by MIT for nationally selected promising high school students from underrepresented backgrounds interested in creative mathematical experiences. At MathROOTS students discover new mathematical ideas and learn problem-solving skills through a series of classes, group activities, and invited lectures led by a team of instructors with diverse experiences doing and teaching both research and competition math. (See Figure 5).

The mission of PRIMES Circle and MathROOTS is to increase diversity in the mathematical community by helping strong students from underrepresented backgrounds develop their interest in mathematics and to set them on a path toward pursuing a math-based major in college.

Do students enjoy your program?

S.G.: Every year we collect student impressions of the program and post them on the PRIMES website on the "Testimonials" page. Here are a couple of excerpts:

> "At the beginning of the PRIMES program in January last year, I was mildly nervous that I would not be able to discover anything new. However, such fears were certainly unmerited. During the first few meetings, my mentor provided my partner and me with background readings to become familiar with the common techniques. Within two months, we were formulating some of our own conjectures based on computer simulations, and before long, we were even able to find proofs of some of these conjectures."

> "I loved the feeling of being able to sit and think about problems without having anything else in my mind. It was a stress-free environment, and I thrived here. PRIMES is an excellent program—it's a remarkable way to start research at a young age with the help of incredible professionals and mentors who love the math and science that you do and will help you learn more and more. I'm very glad I chose to come to PRIMES, and it has truly changed my life as a student and a mathematician."

T.K.: Many high schools are worried about failing students and do not worry about bright students being bored. In our program no one is bored.

S.G.: Not even the program director! PRIMES has grown almost four times since its creation and currently has well over a hundred affiliates. Its administration and accounting have become as challenging as a PRIMES project!

Who pays for PRIMES?

S.G.: PRIMES is free for students, which is why it is not at all free for MIT. But it is paid for by generous people with big hearts. The biggest hearts belong to the NSF Department of Mathematical Sciences and the MIT math department (personally, its former Head Mike Sipser, currently MIT's Dean of Science, and its current Head Tom Mrowka), who have provided crucial support since the inception of PRIMES. NIH, Clay Mathematics Institute, Simons Foundation, Rosenbaum Foundation, some companies and private donors have also made major contributions. Notably, George Lusztig, MIT mathematics professor and the recipient of the 2014 Shaw Prize in mathematics, used part of his prize to make a very significant gift to PRIMES as the first contribution to its endowment. This made it possible to establish George Lusztig PRIMES mentorships. Several such mentorships are awarded each year to continuing mathematics mentors for exceptional mentor service in past years.

P.E.: In fact, while we all think hard about math, our program director has to think hard how to find more people with big hearts. And he will definitely appreciate your help!

One of the PRIMES research papers is called "Cookie Monster Plays Games." Is this a serious mathematical paper or is it really about cookies? Do you supply cookies for your research?

Figure 6. PRIMES student Leigh Marie Braswell with the Cookie Monster (PRIMES Conference, 2013).

T.K.: It is entirely serious! The Cookie Monster (Figure 6) is just a fun way to represent a certain class of combinatorial problems. Namely, there are several jars with cookies. In one move the Cookie Monster is allowed to choose some of the jars and take the same number of cookies from all of them. The question "Given the number of cookies in jars, what is the smallest number of moves needed to empty all the jars?" was studied by Leigh Marie Braswell, a PRIMES 2013 student ([BK1]; [BK2]: see Figure 6). With another student, Joshua Xiong (PRIMES 2014), we converted the Cookie Monster problem into a game and made some interesting discoveries [KX].

P.E.: After attending our conference, MIT mathematics professor Richard Stanley observed that a certain breed of Cookie Monster (the one that eats cookies only from consecutive jars) corresponds to the combinatorics of the root system A_{n-1} attached to the simple Lie algebra $\mathfrak{sl}(n)$. In fact, there are rumors that this breed was genetically engineered at my request at the MIT biology department to encourage students to learn about Lie algebras!

S.G.: Even though the Cookie Monster research does not really require cookies, we do supply them during the annual PRIMES conference in May, which you are welcome to attend. In fact, every year, the night before the conference Pavel drives to Costco and fills up his van with cookies for participants. This allows all of us to enjoy many kinds of delicious cookies. They serve as a catalyst for making coffee into theorems!

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