

Nonlinear chiral transport phenomenaJiunn-Wei Chen,^{1,2} Takeaki Ishii,³ Shi Pu,⁴ and Naoki Yamamoto³¹*Department of Physics, Center for Theoretical Sciences, and Leung Center for Cosmology and Particle Astrophysics, National Taiwan University, Taipei 10617, Taiwan*²*Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*³*Department of Physics, Keio University, Yokohama 223-8522, Japan*⁴*Institute for Theoretical Physics, Goethe University, Max-von-Laue-Str. 1, 60438 Frankfurt am Main, Germany*

(Received 1 April 2016; published 20 June 2016)

We study the nonlinear responses of relativistic chiral matter to the external fields such as the electric field \mathbf{E} , gradients of temperature and chemical potential, ∇T and $\nabla\mu$. Using the kinetic theory with Berry curvature corrections under the relaxation time approximation, we compute the transport coefficients of possible new electric currents that are forbidden in usual chirally symmetric matter but are allowed in chirally asymmetric matter by parity. In particular, we find a new type of electric current proportional to $\nabla\mu \times \mathbf{E}$ due to the interplay between the effects of the Berry curvature and collisions. We also derive an analog of the ‘‘Wiedemann-Franz’’ law specific for anomalous nonlinear transport in relativistic chiral matter.

DOI: [10.1103/PhysRevD.93.125023](https://doi.org/10.1103/PhysRevD.93.125023)**I. INTRODUCTION**

Transport phenomena are abundant in our everyday life and are important in wide areas of physics from condensed matter physics and nuclear physics to astrophysics. A familiar example is Ohm’s law, $\mathbf{j}_e = \sigma\mathbf{E}$, where the electric current flows in the direction of the external electric field \mathbf{E} . In addition to such first-order transport, various kinds of second-order transport phenomena have already been revealed in the 19th century. The well-known examples are the Hall effect and Nernst effect, $\mathbf{j}_e = \sigma_{EB}\mathbf{E} \times \mathbf{B}$ and $\mathbf{j}_e = \sigma_{TB}(-\nabla T) \times \mathbf{B}$, respectively, where \mathbf{B} is the magnetic field and T is the temperature. One can question whether other second-order transport phenomena are possible. For example, one might imagine the electric current of the form $\mathbf{j}_e = \sigma_{E\mu}\nabla\mu \times \mathbf{E}$ with μ the chemical potential. However, such a current is not consistent with parity and is usually forbidden in a system that respects parity.

In this paper, we argue that such exotic transport phenomena become possible in relativistic matter with chirality imbalance (which we shall simply refer to as chiral matter below) where parity is explicitly violated. Examples of chiral matter are the electroweak plasma in the early Universe [1], quark-gluon plasmas created in heavy ion collisions [2,3], electromagnetic plasmas in neutron stars [4,5], neutrino matter in supernovae [6], and a new type of materials called the Weyl semimetals [7–9]. In this paper, we explicitly compute the transport coefficients of these nonlinear anomalous transports in chiral matter at low temperature, based on the kinetic theory with Berry curvature corrections [10–15] under the relaxation time approximation. For the computations of nonlinear anomalous transport coefficients using holography, see Ref. [16].

We show that the nonlinear anomalous transport above arises from the interplay between the Berry curvature and collisions. We also derive a universal relation independent of the relaxation time, which is similar to the Wiedemann-Franz law in usual metals but is specific for nonlinear anomalous transport in chiral matter. Our main results are summarized in Eqs. (18) and (24).

In this paper, we set $c = k_B = 1$ unless stated otherwise but keep e and \hbar explicitly.

II. CLASSIFICATION OF CURRENTS FROM SYMMETRIES

We first classify the possible electric currents in the presence of various external fields to the second order in derivatives. Although we limit ourselves to the electric currents in this paper, the same classification is also applicable to heat currents. The external fields we consider here are the electric field \mathbf{E} , magnetic field \mathbf{B} , gradients of temperature and chemical potential ∇T and $\nabla\mu$, and their possible combinations. (In this paper, we do not consider external fields involving the fluid velocity \mathbf{v} , such as the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$.) We also assume external fields are time independent.¹ We will see that the \mathcal{P} (parity), \mathcal{C} (charge conjugation), and \mathcal{T} (time reversal) symmetries put stringent constraints on the possible transport phenomena.

Let us first consider the electric currents in usual parity-invariant systems. To the second-order derivatives, the

¹For the time-dependent second-order anomalous transport proportional to $\dot{\mathbf{B}}$, see Ref. [17].

general expression that is consistent with \mathcal{CPT} symmetries reads $\mathbf{j}_+ = \mathbf{j}_+^{(1)} + \mathbf{j}_+^{(2)}$, where

$$\mathbf{j}_+^{(1)} = \sigma_E \mathbf{E} + \sigma_T (-\nabla T) + \sigma_\mu (-\nabla \mu), \quad (1)$$

$$\mathbf{j}_+^{(2)} = \sigma_{EB} \mathbf{E} \times \mathbf{B} + \sigma_{TB} (-\nabla T) \times \mathbf{B} + \sigma_{\mu B} (-\nabla \mu) \times \mathbf{B}. \quad (2)$$

Here and henceforth, the subscript \pm denotes the \mathcal{P} -even and \mathcal{P} -odd transport coefficients. The upper indices denote the number of derivatives.

Note that the transport coefficients in Eq. (1) are \mathcal{T} odd and must be accompanied with dissipation: under \mathcal{T} , $\mathbf{j} \rightarrow -\mathbf{j}$, $\mathbf{E} \rightarrow \mathbf{E}$, and $\mathbf{B} \rightarrow -\mathbf{B}$. On the other hand, the transport coefficients in Eq. (2) are \mathcal{T} even and do not necessarily involve dissipation (see also Ref. [18]). For the electric current, the σ_E term is called Ohm's law, the σ_T term the Seebeck effect, the σ_{EB} term the Hall effect, and the σ_{TB} term the Nernst effect.

Let us turn to a parity-violating system with nonzero chiral chemical potential, $\mu_5 = (\mu_R - \mu_L)/2$. Under \mathcal{P} , the chiral chemical potential transforms as $\mu_5 \rightarrow -\mu_5$. Then, one can write down the general expression for the parity-odd current to the second order. For simplicity, we first consider the system only with right-handed fermions at finite chemical potential μ_R (which we shall denote μ in the following). Then, the current is expressed by $\mathbf{j}_- = \mathbf{j}_-^{(1)} + \mathbf{j}_-^{(2)}$, where

$$\mathbf{j}_-^{(1)} = \sigma_B \mathbf{B}, \quad (3)$$

$$\mathbf{j}_-^{(2)} = \sigma_{ET} \mathbf{E} \times (-\nabla T) + \sigma_{E\mu} \mathbf{E} \times (-\nabla \mu) + \sigma_{T\mu} (-\nabla T) \times (-\nabla \mu), \quad (4)$$

where all the transport coefficients in Eqs. (3) and (4) are functions of μ . The σ_B term is called the chiral magnetic effect [3,19–22]. Note that the transport coefficient σ_B (which is referred to as the chiral magnetic conductivity) in Eq. (3) is \mathcal{T} even and is indeed dissipationless (see below), while those in Eq. (4) are \mathcal{T} odd and are dissipative.

It has been revealed that the coefficient σ_B is uniquely fixed from the constraint of the second law of thermodynamics and that it is related to the coefficient of the chiral anomaly [23]; its relation to the chiral anomaly also underlies that this current is dissipationless. However, it is a nontrivial question whether other parity-violating terms in Eq. (4) are fixed uniquely by the anomaly coefficients. We shall show that these new transport phenomena arise due to an interplay between the topological terms (the Berry curvature) and collisional terms.

III. KINETIC THEORY

We will be interested in the nonlinear electric currents that arise due to the explicit violation of parity symmetry in

the system, shown in Eq. (4). As there is no such a current involving the magnetic field \mathbf{B} there, it will be sufficient to consider the case only with \mathbf{E} , ∇T , and $\nabla \mu$, but without \mathbf{B} for this purpose.

A. Kinetic theory with Berry curvature

We first briefly review the kinetic theory for a single chiral fermion at $\mu \gg T$ [10–15]. (We will consider a system with both right- and left-handed fermions later.) The chiral fermions near the Fermi surface possess a Berry curvature in momentum space [10,24]. The equations of motion for chiral quasiparticles in an electric field \mathbf{E} and the Berry curvature Ω_p are [25]

$$\dot{\mathbf{x}} = \mathbf{v} + \dot{\mathbf{p}} \times \Omega_p, \quad (5)$$

$$\dot{\mathbf{p}} = e\mathbf{E}, \quad (6)$$

where $\mathbf{v} = \partial \epsilon_p / \partial \mathbf{p}$. Substituting them into the Boltzmann equation for the distribution function $n_p(\mathbf{x})$,

$$\frac{\partial n_p}{\partial t} + \dot{\mathbf{x}} \cdot \frac{\partial n_p}{\partial \mathbf{x}} + \dot{\mathbf{p}} \cdot \frac{\partial n_p}{\partial \mathbf{p}} = I_{\text{coll}}\{n_p\}, \quad (7)$$

the kinetic equation in the present case is given by [10–15]

$$\frac{\partial n_p}{\partial t} + (\mathbf{v} + e\mathbf{E} \times \Omega_p) \cdot \frac{\partial n_p}{\partial \mathbf{r}} + e\mathbf{E} \cdot \frac{\partial n_p}{\partial \mathbf{p}} = I_{\text{coll}}\{n_p\}, \quad (8)$$

where $I_{\text{coll}}\{n_p\}$ is the collision term.

B. Relaxation time approximation

For the collision term in Eq. (8), we use the relaxation time approximation,

$$I_{\text{coll}} = -\frac{\delta n_p}{\tau}, \quad (9)$$

where τ is the relaxation time (which we assume to be a constant) and $\delta n_p = n_p - n_p^{(0)}$ is the deviation from the equilibrium distribution function.

For inhomogeneous temperature T and chemical potential μ , the stationary solution of the kinetic equation can be found order by order in derivatives,

$$n_p^{(0)} = \frac{1}{e^{(\epsilon - \mu)/T} + 1}, \quad (10a)$$

$$\delta n_p^{(1)} = \tau \mathbf{v} \cdot \left(-e\mathbf{E} + \nabla \mu + \frac{\epsilon - \mu}{T} \nabla T \right) \frac{\partial n_p^{(0)}}{\partial \epsilon}, \quad (10b)$$

$$\delta n_p^{(2)} = \tau e\mathbf{E} \times \Omega_p \cdot \left(\nabla \mu + \frac{\epsilon - \mu}{T} \nabla T \right) \frac{\partial n_p^{(0)}}{\partial \epsilon}. \quad (10c)$$

Here, $n_p^{(0)}$ is the equilibrium Fermi-Dirac distribution, and the upper indices $k = 0, 1, 2$ denote the terms involving k derivatives. Precisely speaking, there are other terms not listed in $\delta n_p^{(2)}$ which are not related to the Berry curvature corrections. However, they are irrelevant for our purpose of computing the parity-violating transport coefficient to the second order, and we will ignore them below.

We pause here to remark on the relaxation time approximation in Eq. (9). Performing the momentum integral of the kinetic equation (8), one obtains the continuity equation as long as the following condition is fulfilled:

$$\int \frac{d\mathbf{p}}{(2\pi\hbar)^3} I_{\text{coll}}\{n_p\} = 0. \quad (11)$$

One can check that the collision term in Eq. (9) together with Eq. (10) indeed satisfies this condition, at least for a spherically symmetric Fermi sphere; so, the relaxation time approximation in the present case is consistent with the particle number conservation law.

IV. NONLINEAR ELECTRIC CURRENTS

A. System with a single chiral fermion

To simplify the argument, we first consider the case with a single chiral fermion. The electric current in the presence of the Berry curvature corrections is given by [10,13]

$$\mathbf{j}_e = e \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \left[\mathbf{v} n_p + (e\mathbf{E} \times \boldsymbol{\Omega}_p) n_p - \epsilon_p \boldsymbol{\Omega}_p \times \frac{\partial n_p}{\partial \mathbf{x}} \right]. \quad (12)$$

The first term in Eq. (12) is the usual convective current, the second term is the anomalous Hall current, and the third term is the magnetization current [26] that originates from the magnetic moment of chiral fermions [13,15].

The electric current to the second order in derivatives is then

$$\begin{aligned} \mathbf{j}_e^{(2)} &= e \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \left[\mathbf{v} \delta n_p^{(2)} + (e\mathbf{E} \times \boldsymbol{\Omega}_p) \delta n_p^{(1)} - \epsilon_p \boldsymbol{\Omega}_p \times \frac{\partial \delta n_p^{(1)}}{\partial \mathbf{x}} \right] \\ &\equiv \mathbf{j}_1 + \mathbf{j}_2 + \mathbf{j}_3. \end{aligned} \quad (13)$$

They can be respectively computed by substituting Eq. (10). For right-handed fermions, for example, the results read

$$\mathbf{j}_1 = -\frac{e^2 \tau}{12\pi^2 \hbar^2} \nabla \mu \times \mathbf{E}, \quad (14a)$$

$$\mathbf{j}_2 = \frac{e^2 \tau}{12\pi^2 \hbar^2} \nabla \mu \times \mathbf{E}, \quad (14b)$$

$$\mathbf{j}_3 = \frac{e^2 \tau}{12\pi^2 \hbar^2} \nabla \mu \times \mathbf{E}. \quad (14c)$$

Summing over the three contributions above, we obtain

$$\sigma_{E\mu} = \pm \frac{e^2 \tau}{12\pi^2 \hbar^2} \quad (15)$$

for right and left-handed fermions, respectively.

Note here that two contributions of the form $\nabla T \times \nabla \mu$ with the opposite signs exactly cancel out in \mathbf{j}_3 , and the electric current proportional to $\nabla T \times \nabla \mu$ is absent, at least within the present relaxation time approximation (although it is in principle allowed by the symmetry). We also find that the electric current proportional to $\nabla T \times \mathbf{E}$ disappears after the momentum integral.² At this moment, we do not have a clear understanding of the physical reason that underlies their absence.

As a comparison, let us also compute the electrical conductivity in relativistic matter. Under the relaxation time approximation, we obtain the Ohmic current from Eqs. (10) and (12) as

$$\mathbf{j}_e^{(1)} = -e^2 \tau \int \frac{d\mathbf{p}}{(2\pi\hbar)^3} \mathbf{v} (\mathbf{v} \cdot \mathbf{E}) \frac{\partial n_p^0}{\partial \epsilon} = \frac{e^2 \mu^2 \tau}{6\pi^2 \hbar^3} \mathbf{E}. \quad (16)$$

Using the number density for a single chiral fermion, $n = \mu^3 / (6\pi^2 \hbar^3)$, the electrical conductivity can be expressed as

$$\sigma_E = \frac{ne^2 \tau}{\mu}. \quad (17)$$

This takes the familiar form of the Drude-type formula if we identify the effective mass $m^* = \mu$.

By taking the ratio between Eqs. (15) and (17), we obtain the analog of the ‘‘Wiedemann-Franz law’’ specific for relativistic chiral matter,

$$\frac{\mu^2 |\sigma_{E\mu}|}{\sigma_E} = \frac{\hbar c}{2}, \quad (18)$$

where we restored c . Equation (18) shows that the ratio between the electrical conductivity σ_E and anomalous nonlinear conductivity $\sigma_{E\mu}$ (multiplied by μ^2 to match the dimension) is a universal quantity that depends only on the physical constants \hbar and c . At least within the present relaxation time approximation, this relation is independent of the microscopic details (i.e., the relaxation time τ).

It should be remarked that, in the case of usual metals, the Wiedemann-Franz-type law for the ratio between the

²Precisely speaking, the result of the momentum integral can be different in Weyl metals in condensed matter systems, where the description of chiral fermions has finite UV and IR energy cutoffs. In this case, σ_{TE} can remain nonzero at finite temperature but is suppressed exponentially as $\sigma_{TE} \propto e^{-\mu/T}$. In particular, σ_{ET} vanishes at $T = 0$.

linear and nonlinear transport coefficients of electric currents do not exist. This may be understood as follows: since all the nonlinear transport σ_{EB} , σ_{TB} , and $\sigma_{\mu B}$ in Eq. (2) are \mathcal{T} even, the transport coefficients are even functions of τ . Hence, the ratios between these nonlinear transport coefficients and the electrical conductivity $\sigma_E \propto \tau$ must depend on τ (i.e., nonuniversal). In chiral matter, on the other hand, the nonlinear transport $\sigma_{E\mu}$ is \mathcal{T} odd, and $\sigma_{E\mu}/\sigma_E$ can be independent of τ .

B. System with both right- and left-handed fermions

So far, we have considered the case with a single chiral fermion. We now consider a system with right- and left-handed fermions in the presence of finite chiral chemical potential $\mu_5 = (\mu_R - \mu_L)/2$. We denote the vector chemical potential by $\mu = (\mu_R + \mu_L)/2$. In this case, the kinetic equation (8) needs to be extended to include both right- and left-handed fermions as (see also Refs. [11,27])

$$\frac{\partial n_p^i}{\partial t} + (\mathbf{v} + e\mathbf{E} \times \boldsymbol{\Omega}_p) \cdot \frac{\partial n_p^i}{\partial \mathbf{r}} + e\mathbf{E} \cdot \frac{\partial n_p^i}{\partial \mathbf{p}} = I_{\text{coll}}^i \{n_p^i\}, \quad (19)$$

where $i = R, L$ denote the chirality of fermions. The collision term $I_{\text{coll}}^i \{n_p^i\}$ generally describes the interchiral and intrachiral scatterings. Here, we assume that the former mean free time (which we denote τ_{tr}) is much larger than the latter (which we denote τ) [11,27], and we ignore the effects of the former in the leading order in $\tau/\tau_{\text{tr}} \ll 1$. Then, the kinetic equations for right- and left-handed fermions are decoupled from each other. Analogously to the discussion above, we use the relaxation time approximation,

$$I_{\text{coll}}^i = -\frac{\delta n_p^i}{\tau}, \quad (20)$$

where the thermal relaxation time τ is assumed to be the same constant for right and left-handed fermions [11,27].³

Taking the summation and subtraction of the contributions from right- and left-handed fermions, we then obtain the nonlinear anomalous electric and axial currents,

$$\mathbf{j}_e = \mathbf{j}_e^R + \mathbf{j}_e^L = \frac{e^2 \tau}{6\pi^2 \hbar^2} \boldsymbol{\nabla} \mu_5 \times \mathbf{E}, \quad (21)$$

$$\mathbf{j}_5 = \mathbf{j}_e^R - \mathbf{j}_e^L = \frac{e^2 \tau}{6\pi^2 \hbar^2} \boldsymbol{\nabla} \mu \times \mathbf{E}, \quad (22)$$

respectively. On the other hand, one finds that the Ohmic current in this case is

³The assumption that τ is the same for right- and left-handed fermions should be approximately valid, at least in the regime $\mu_5 \ll \mu$.

$$\mathbf{j}_e = \frac{e^2 \tau (\mu^2 + \mu_5^2)}{3\pi^2 \hbar^3} \mathbf{E}. \quad (23)$$

Denoting the transport coefficients in Eqs. (21) and (23) by $\sigma_{E\mu_5}$ and σ_E , respectively, we arrive at the Wiedemann-Franz-type law in this case as

$$(\mu^2 + \mu_5^2) \frac{\sigma_{E\mu_5}}{\sigma_E} = \frac{\hbar c}{2}, \quad (24)$$

where we restored c again.

This relation may be valid in dense relativistic chiral plasmas in neutron stars [4] and supernovae [5]. In Weyl semimetals, a similar relation should hold, but c is replaced by the Fermi velocity v_F that depends on the details of the band structure.

V. CONCLUSIONS

In this paper, we explored nonlinear responses of chiral matter to external fields, based on the kinetic theory with Berry curvature corrections. The exotic transport phenomena found in this paper should be relevant to the dynamical evolution of chiral matter, such as the electro-weak plasma in the early Universe, quark-gluon plasmas created in heavy ion collisions, and supernova explosions. Our predictions may also be tested experimentally in Weyl semimetals.

In this paper, we derived the analog of the ‘‘Wiedemann-Franz’’ law for anomalous transport, as shown in Eqs. (18) and (24). To what extent this relation is universal (i.e., independent of microscopic details of systems) beyond the relaxation time approximation would be an important question to be investigated in future.

It would be interesting to study possible new nonlinear heat currents specific for chiral matter, similar to the nonlinear anomalous electric currents found in this paper. One should be able to compute such heat currents by $j_Q^i = T^{0i} - \mu j_n^i$, where j_n^i is the particle number current and $T^{\mu\nu}$ is the energy-momentum tensor including the Berry curvature corrections defined in Ref. [13]. One can also ask the possible effects of finite fermion mass (see, e.g., Ref. [28]). We defer these questions to future work.

ACKNOWLEDGMENTS

J.-W. C. would like to thank the Rudolph Peierls Centre for Theoretical Physics of the University of Oxford and Oxford Holography group for hospitality. J.-W. C. is supported in part by the Ministry of Science and Technology, Taiwan, under Grants No. 102-2112-M-002-013-MY3 and No. 105-2918-I-002 -003 and the CASTS of NTU. The author S. P. is supported by the Alexander von Humboldt Foundation, Germany. The work of N. Y. is

supported in part by JSPS KAKENHI Grant No. 26887032 and MEXT-Supported Program for the Strategic Research Foundation at Private Universities, “Topological Science” (Grant No. S1511006).

Note added.—While this work was being completed, we learned that I. Shovkovy and his collaborators also obtained the results [29] that have some overlap with our calculations.

-
- [1] M. Joyce and M. E. Shaposhnikov, *Phys. Rev. Lett.* **79**, 1193 (1997).
- [2] D. E. Kharzeev, L. D. McLerran, and H. J. Warringa, *Nucl. Phys. A* **803**, 227 (2008).
- [3] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, *Phys. Rev. D* **78**, 074033 (2008).
- [4] J. Charbonneau and A. Zhitnitsky, *J. Cosmol. Astropart. Phys.* **08** (2010) 010.
- [5] A. Ohnishi and N. Yamamoto, [arXiv:1402.4760](https://arxiv.org/abs/1402.4760).
- [6] N. Yamamoto, *Phys. Rev. D* **93**, 065017 (2016).
- [7] X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, *Phys. Rev. B* **83**, 205101 (2011).
- [8] A. A. Burkov and L. Balents, *Phys. Rev. Lett.* **107**, 127205 (2011).
- [9] G. Xu, H. Weng, Z. Wang, X. Dai, and Z. Fang, *Phys. Rev. Lett.* **107**, 186806 (2011).
- [10] D. T. Son and N. Yamamoto, *Phys. Rev. Lett.* **109**, 181602 (2012).
- [11] D. T. Son and B. Z. Spivak, *Phys. Rev. B* **88**, 104412 (2013).
- [12] M. A. Stephanov and Y. Yin, *Phys. Rev. Lett.* **109**, 162001 (2012).
- [13] D. T. Son and N. Yamamoto, *Phys. Rev. D* **87**, 085016 (2013).
- [14] J. W. Chen, S. Pu, Q. Wang, and X. N. Wang, *Phys. Rev. Lett.* **110**, 262301 (2013).
- [15] C. Manuel and J. M. Torres-Rincon, *Phys. Rev. D* **90**, 076007 (2014).
- [16] E. Megias and F. Pena-Benitez, *J. High Energy Phys.* **05** (2013) 115.
- [17] A. Jimenez-Alba and H.-U. Yee, *Phys. Rev. D* **92**, 014023 (2015).
- [18] D. E. Kharzeev and H.-U. Yee, *Phys. Rev. D* **84**, 045025 (2011).
- [19] A. Vilenkin, *Phys. Rev. D* **22**, 3080 (1980).
- [20] H. B. Nielsen and M. Ninomiya, *Phys. Lett.* **130B**, 389 (1983).
- [21] A. Y. Alekseev, V. V. Cheianov, and J. Frohlich, *Phys. Rev. Lett.* **81**, 3503 (1998).
- [22] D. T. Son and A. R. Zhitnitsky, *Phys. Rev. D* **70**, 074018 (2004).
- [23] D. T. Son and P. Surowka, *Phys. Rev. Lett.* **103**, 191601 (2009).
- [24] G. E. Volovik, *The Universe in a Helium Droplet* (Clarendon Press, Oxford, 2003).
- [25] G. Sundaram and Q. Niu, *Phys. Rev. B* **59**, 14915 (1999).
- [26] J. Y. Chen, D. T. Son, M. A. Stephanov, H.-U. Yee, and Y. Yin, *Phys. Rev. Lett.* **113**, 182302 (2014).
- [27] M. Stephanov, H.-U. Yee, and Y. Yin, *Phys. Rev. D* **91**, 125014 (2015).
- [28] J. W. Chen, J. Y. Pang, S. Pu, and Q. Wang, *Phys. Rev. D* **89**, 094003 (2014).
- [29] E. V. Gorbar, I. A. Shovkovy, S. Vilchinskii, I. Rudenok, A. Boyarsky, and O. Ruchayskiy, *Phys. Rev. D* **93**, 105028 (2016).