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COMMON FIXED POINT THEOREMS FOR A FAMILY OF MULTIVALUED F-CONTRACTIONS WITH AN APPLICATION TO SOLVE A SYSTEM OF INTEGRAL EQUATIONS

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Abstract. Inspired by the work of Wardowski in [33] and Samet et al. in [26], in this article, we introduce some new contractive conditions for sequence of multi functions. We have constructed non-trivial examples to validate our results. We have applied our results to find a solution of a system of integral equations.

1. Introduction

The Banach contraction principle is a famous theorem in the field of fixed point theory and it is not wrong to say that it brought about a new era in metric fixed point theory. Since its inception, major and minor developments have been made regarding its generalization. In the recent past Wardowski ([33]) categorized some mappings into a new family and called it F or $\mathfrak{F}$ family. Using the mappings from $\mathfrak{F}$ family he introduced a new contraction condition namely the F-contractions, which effectively generalized the famous Banach contraction condition. Several researchers studying metric fixed point theory have comprehensively generalized the Banach contraction condition, see for example [2,30,25,18,13,29,22,24,28,20,1,26,6,21,7,19,14,3–5,15–17,27,12,31, 11,9,10,8,23,32,33]. Semat *et al.* in [26] also succeeded in generalizing Banach contraction condition by introducing α-ψ-contraction. Many authors appreciated these two conditions which can be seen in [6, 21, 7, 19, 14, 3–5, 15, 16].

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Keeping in view both of these ideas, in this paper we introduce new contraction conditions for a sequence of multifunction and prove corresponding fixed point theorem. We also give a common fixed point theorem for sequence of bounded multifunctions by using the δ-distance. To conclude our findings we establish an existence theorem for a system of integral equations.

We gather some common results, notations and definitions, which are required for this paper. Let (X, d) be a metric space. We denote the set of all nonempty subsets of X by $N(X)$, the class of all nonempty closed subsets of X by $C(X)$ and the class of all nonempty bounded subsets of X by $B(X)$. For $b \in N(X)$, $d(a, B) = \inf\{d(a, b) : b \in N(X)\}$. For $A, B \in B(X)$, $\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$. Note that δ satisfies all conditions of a metric, except $A = B \Rightarrow \delta(A, B) = 0$. For $A, B \in C(X)$, the generalized Hausdorff metric on $C(X)$ is given as,

$$H(A,B) = \begin{cases} \max\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\} & \text{if the maximum exists} \\ \infty & \text{otherwise} \end{cases}$$

Wardowski [33] introduced the following definition.

Definition 1.1. *Let $\mathfrak{F}$ be the class of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying:*

- (F_1) *F is increasing, that is, for each $a_1, a_2 \in (0, \infty)$ with $a_1 < a_2$, we have $F(a_1) < F(a_2)$.*
- (F_2) *For each sequence $\{\mathfrak{d}_n\}$ of positive real numbers we have $\lim_{n\to\infty} \mathfrak{d}_n = 0$ if and only if $\lim_{n\to\infty} F(\mathfrak{d}_n) = -\infty$.*
- (F_3) *There exists $k \in (0, 1)$ such that $\lim_{\mathfrak{d}\to 0^+} \mathfrak{d}^k F(\mathfrak{d}) = 0$.*

Following are some examples of such functions.

- (i) $F_a = \ln a$ for each $a \in (0, \infty)$.
- (ii) $F_b = b + \ln b$ for each $b \in (0, \infty)$.
- (iii) $F_c = -\frac{1}{\sqrt{c}}$ for each $c \in (0, \infty)$.

Wardowski ([33]) introduced F-contraction and proved corresponding fixed point theorem as,

Definition 1.2 ([33]). *Let (X, d) be a metric space. A mapping $T : X \to X$ is F-contraction if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that for each $x, y \in X$ with $d(Tx, Ty) > 0$, we have*

$$\tau + F(d(Tx, Ty)) \le F(d(x, y)).$$

Note that if T is F_a-contraction, then it is also Banach contraction. This it is not in the case for F_b-contraction.

Theorem 1.3 ([33]). *Let (X, d) be a complete metric space and let $T : X \to X$ be F-contraction. Then T has a unique fixed point.*

Sgroi and Vetro [29] introduced the following theorem.

THEOREM 1.4 ([29]). *Let (X, d) be a complete metric space and let $T : X \to CB(X)$. Assume that there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that*

$$2\tau + F(H(Tx, Ty)) \leq F(a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + L d(y, Tx)), \tag{1.1}$$

for each $x, y \in X$ with $Tx \neq Ty$, where $a_1, a_2, a_3, a_4, L \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$. Then T has a fixed point.

2. Main results

We begin this section by introducing the following definitions.

DEFINITION 2.1. *Let $\alpha : X \times X \to [0, \infty)$. A sequence of mappings $\{T_i : X \to N(X)\}_{i=1}^{\infty}$ is α-admissible sequence if for each $x \in X$ and $y \in T_i x$ for some $i \in \mathbb{N}$ such that $\alpha(x, y) \geq 1$, then we have $\alpha(y, z) \geq 1$ for each $z \in T_{i+1} y$. A sequence of mappings $\{T_i : X \to N(X)\}_{i=1}^{\infty}$ is α_*-admissible sequence if for each $x, y \in X$ with $\alpha(x, y) \geq 1$, we have $\alpha_*(T_i x, T_j y) \geq 1$ for each $i, j \in \mathbb{N}$, where $\alpha_*(T_i x, T_j y) = \inf\{\alpha(u, v) : u \in T_i x$ and $v \in T_j y\}$.*

The sequence of mappings is said to be strictly α-admissible and strictly α_*-admissible if we have strict inequality in the above definition.

REMARK 2.2. (*i*) Note that if a sequence of mappings $\{T_i : X \to N(X)\}_{i=1}^{\infty}$ is strictly α_*-admissible sequence, then it is strictly α-admissible sequence.
(*ii*) When $\{T_i\}_{i=1}^{\infty}$ is a constant sequence Definition 2.1 coincide with definition of α-admissible and α_*-admissible given in [21, Page 4] and [7, Page 1] respectively. Furthermore, if T is a singlevalued mapping then these definition 2.1 coincide with [26, Definition 2.2].

DEFINITION 2.3. *Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a function. A sequence of mappings $\{T_i : X \to C(X)\}_{i=1}^{\infty}$ is an F_α-contraction of Hardy-Rogers-type, if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that for each $i, j \in \mathbb{N}$, we have*

$$\tau + F(\alpha(x, y) H(T_i x, T_j y)) \leq F(N(x, y)), \tag{2.1}$$

for each $x, y \in X$, whenever $\min\{\alpha(x, y) H(T_i x, T_j y), N(x, y)\} > 0$, where

$$N(x, y) = a_1 d(x, y) + a_2 d(x, T_i x) + a_3 d(y, T_j y) + a_4 d(x, T_j y) + L d(y, T_i x),$$

with $a_1, a_2, a_3, a_4, L \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

THEOREM 2.4. *Let (X, d) be a complete metric space and let $\{T_i : X \to C(X)\}_{i=1}^{\infty}$ be an F_α-contraction of Hardy-Rogers-type satisfying the following conditions:*

(i) $\{T_i\}_{i=1}^{\infty}$ is strictly α-admissible sequence;
(ii) there exist $x_0 \in X$ and $x_1 \in T_i x_0$ for some $i \in \mathbb{N}$ with $\alpha(x_0, x_1) > 1$;

(iii) for any sequence $\{x_n\} \subseteq X$ *such that* $x_n \to x$ *as* $n \to \infty$ *and* $\alpha(x_n, x_{n+1}) > 1$ *for each* $n \in \mathbb{N}$*, we have* $\alpha(x_n, x) > 1$ *for each* $n \in \mathbb{N}$*.*

Then the mappings in the sequence $\{T_i\}_{i=1}^{\infty}$ *have a common fixed point.*

Proof. By hypothesis (ii), we assume without loss of generality that there exist $x_0 \in X$ and $x_1 \in T_1 x_0$ with $\alpha(x_0, x_1) > 1$. If $x_1 \in T_i x_1$ $\forall i \in \mathbb{N}$, then x_1 is a common fixed point. Let $x_1 \notin T_2 x_1$, as $\alpha(x_0, x_1) > 1$ there exists $x_2 \in T_2 x_1$ such that

$$d(x_1, x_2) \leq \alpha(x_0, x_1) H(T_1 x_0, T_2 x_1). \tag{2.2}$$

Since F is increasing, we have

$$F(d(x_1, x_2)) \leq F(\alpha(x_0, x_1) H(T_1 x_0, T_2 x_1)). \tag{2.3}$$

From (2.1) we have

$$\begin{aligned} \tau + F(d(x_1, x_2)) &\leq \tau + F(\alpha(x_0, x_1) H(T_1 x_0, T_2 x_1)) \\ &\leq F\big(a_1 d(x_0, x_1) + a_2 d(x_0, T_1 x_0) + a_3 d(x_1, T_2 x_1) \\ &\quad + a_4 d(x_0, T_2 x_1) + L d(x_1, T_1 x_0)\big) \\ &\leq F\big(a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_3 d(x_1, x_2) \\ &\quad + a_4 d(x_0, x_2) + L.0\big) \\ &\leq F\big(a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_3 d(x_1, x_2) \\ &\quad + a_4 (d(x_0, x_1) + d(x_1, x_2))\big) \\ &= F\big((a_1 + a_2 + a_4) d(x_0, x_1) + (a_3 + a_4) d(x_1, x_2)\big). \end{aligned} \tag{2.4}$$

Since F is increasing, we get from above that

$$d(x_1, x_2) < (a_1 + a_2 + a_4) d(x_0, x_1) + (a_3 + a_4) d(x_1, x_2).$$

That is,

$$(1 - a_3 - a_4) d(x_1, x_2) < (a_1 + a_2 + a_4) d(x_0, x_1).$$

As $a_1 + a_2 + a_3 + 2a_4 = 1$, thus we have

$$d(x_1, x_2) < d(x_0, x_1).$$

From (2.4), we have

$$\tau + F(d(x_1, x_2)) \leq F(d(x_0, x_1)).$$

If $x_2 \in T_i x_2$ $\forall i \in \mathbb{N}$ then x_2 is a common fixed point. Let $x_2 \notin T_3 x_2$. Since $\{T_i\}_{i=1}^{\infty}$ is strictly α-admissible, we have $\alpha(x_1, x_2) > 1$. There exists $x_3 \in T_3 x_2$ such that

$$d(x_2, x_3) \leq \alpha(x_1, x_2) H(T_2 x_1, T_3 x_2). \tag{2.5}$$

Since F is increasing, we have

$$F(d(x_2, x_3)) \leq F(\alpha(x_1, x_2) H(T_2 x_1, T_3 x_2)). \tag{2.6}$$

From (2.1) we have

$$\begin{aligned}\tau + F(d(x_2, x_3)) &\leq \tau + F(\alpha(x_1, x_2)H(T_2x_1, T_3x_2))\\ &\leq F\big(a_1d(x_1, x_2) + a_2d(x_1, T_2x_1) + a_3d(x_2, T_3x_2)\\ &\quad + a_4d(x_1, T_3x_2) + Ld(x_2, T_2x_1)\big)\\ &\leq F\big(a_1d(x_1, x_2) + a_2d(x_1, x_2) + a_3d(x_2, x_3)\\ &\quad + a_4d(x_1, x_3) + L.0\big)\\ &\leq F\big(a_1d(x_1, x_2) + a_2d(x_1, x_2) + a_3d(x_2, x_3)\\ &\quad + a_4(d(x_1, x_2) + d(x_2, x_3))\big)\\ &= F\big((a_1 + a_2 + a_4)d(x_1, x_2) + (a_3 + a_4)d(x_2, x_3)\big).\end{aligned} \tag{2.7}$$

Since F is increasing, we get from above that

$$d(x_2, x_3) < (a_1 + a_2 + a_4)d(x_1, x_2) + (a_3 + a_4)d(x_2, x_3).$$

That is,

$$(1 - a_3 - a_4)d(x_2, x_3) < (a_1 + a_2 + a_4)d(x_1, x_2).$$

As $a_1 + a_2 + a_3 + 2a_4 = 1$, thus we have

$$d(x_2, x_3) < d(x_1, x_2).$$

Now from (2.7) we have

$$\tau + F(d(x_2, x_3)) \leq F(d(x_1, x_2)).$$

So we have

$$F(d(x_2, x_3)) \leq F(d(x_1, x_2)) - \tau \leq F(d(x_0, x_1)) - 2\tau.$$

Continuing in the same way we get a sequence $\{x_n\} \subset X$ such that

$$x_n \in T_nx_{n-1},\ x_{n-1} \neq x_n \text{ and } \alpha(x_{n-1}, x_n) > 1 \text{ for each } n \in \mathbb{N}.$$

Furthermore,

$$F(d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\tau \text{ for each } n \in \mathbb{N}. \tag{2.8}$$

Letting $n \to \infty$ in (2.8) we get $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$. Thus by property (F_2), we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Let $d_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N}$. From (F_3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} d_n^k F(d_n) = 0.$$

From (2.8) we have

$$d_n^k F(d_n) - d_n^k F(d_0) \leq -d_n^k n\tau \leq 0 \text{ for each } n \in \mathbb{N}. \tag{2.9}$$

Letting $n \to \infty$ in (2.9) we get,

$$\lim_{n\to\infty} nd_n^k = 0. \tag{2.10}$$

This implies that there exists $n_1 \in \mathbb{N}$ such that $nd_n^k \leq 1$ for each $n \geq n_1$. Thus we have

$$d_n \leq \frac{1}{n^{1/k}}, \quad \text{for each } n \geq n_1. \tag{2.11}$$

To prove that $\{x_n\}$ is a Cauchy sequence. Consider $m, n \in \mathbb{N}$ with $m > n > n_1$. By using the triangular inequality and (2.11), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d_i \leq \sum_{i=n}^{\infty} d_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent series. Thus, $\lim_{n\to\infty} d(x_n, x_m) = 0$. Which implies that $\{x_n\}$ is a Cauchy sequence. As (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. By condition (iii) we have $\alpha(x_n, x^*) > 1$ for each $n \in \mathbb{N}$. We claim that $d(x^*, T_i x^*) = 0$ $\forall i \in \mathbb{N}$. On contrary suppose that $d(x^*, T_{i_0} x^*) > 0$ for some $i_0 \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, T_{i_0} x^*) > 0$ for each $n \geq n_0$. For each $n \geq n_0$ and for above i_0 we have

$$\begin{aligned} d(x^*, T_{i_0} x^*) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, T_{i_0} x^*) \\ &< d(x^*, x_{n+1}) + \alpha(x_n, x^*) H(T_{n+1} x_n, T_{i_0} x^*) \\ &< d(x^*, x_{n+1}) + a_1 d(x_n, x^*) + a_2 d(x_n, x_{n+1}) \\ &\quad + a_3 d(x^*, T_{i_0} x^*) + a_4 d(x_n, T_i x^*) + L d(x^*, x_{n+1}). \end{aligned} \tag{2.12}$$

Letting $n \to \infty$ in (2.12) we have

$$d(x^*, T_{i_0} x^*) \leq (a_3 + a_4) d(x^*, T_{i_0} x^*) < d(x^*, T_{i_0} x^*).$$

Which is a contradiction. Thus $d(x^*, T_i x^*) = 0$ $\forall i \in \mathbb{N}$. ∎

EXAMPLE 2.5. Let $X = \mathbb{N}$ be endowed with the usual metric $d(x, y) = |x - y|$ for each $x, y \in X$. Define $\{T_i : X \to C(X)\}_{i=1}^{\infty}$ by

$$T_i x = \begin{cases} \{0, 1\} \text{ if } x = 0, 1, \\ \{2x - 2, 2x\} \text{ if } x > 1 \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 2 \text{ if } x, y \in \{0, 1\}, \\ \frac{1}{4} \text{ if } x, y > 1, \\ 0 \text{ otherwise.} \end{cases}$$

Take $F(x) = x + \ln x$ for each $x \in (0, \infty)$. Under this F condition (2.1) reduces to

$$\frac{\alpha(x, y) H(T_i x, T_j y)}{N(x, y)} e^{\alpha(x,y) H(T_i x, T_j y) - N(x,y)} \leq e^{-\tau} \tag{2.13}$$

for each $x, y \in X$ with $\min\{\alpha(x,y)H(T_ix, T_jy), N(x,y)\} > 0$. Assume that $a_1 = 1$, $a_2 = a_3 = a_4 = L = 0$ and $\tau = \frac{1}{2}$. Clearly,

$$\min\{\alpha(x,y)H(T_ix, T_jy), d(x,y)\} > 0$$

for each $x, y > 1$ with $x \neq y$. From (2.13) for each $x, y > 1$ with $x \neq y$ we have

$$\frac{1}{4}e^{-\frac{1}{2}|x-y|} < e^{-\frac{1}{2}}.$$

Thus $\{T_i\}_{i=1}^{\infty}$ is an α-F-contraction of Hardy-Rogers-type with $F(x) = x + \ln x$. For $x_0 = 1$ we have $x_1 = 0 \in T_1x_0$ such that $\alpha(x_0, x_1) > 1$. Moreover, it is easy to see that $\{T_i\}_{i=1}^{\infty}$ is strictly α-admissible sequence and for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$. Therefore, by Theorem 2.4 $\{T_i\}_{i=1}^{\infty}$ has a common fixed point in X.

Definition 2.6. *Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a function. A sequence of mappings $\{T_i : X \to C(X)\}_{i=1}^{\infty}$ is an $F_\alpha*$-contraction of Hardy-Rogers-type, if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that for each $i, j \in \mathbb{N}$, we have*

$$\tau + F(\alpha_*(T_ix, T_jy)H(T_ix, T_jy)) \leq F(N(x,y)), \tag{2.14}$$

for each $x, y \in X$, whenever

$$\min\{\alpha_*(T_ix, T_jy)H(T_ix, T_jy), N(x,y)\} > 0,$$

where

$$N(x,y) = a_1d(x,y) + a_2d(x,T_ix) + a_3d(y,T_jy) + a_4d(x,T_jy) + Ld(y,T_ix),$$

with $a_1, a_2, a_3, a_4, L \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

Theorem 2.7. *Let (X, d) be a complete metric space and let $\{T_i : X \to C(X)\}_{i=1}^{\infty}$ be an α_*-F-contraction of Hardy-Rogers-type satisfying the following conditions:*

(i) $\{T_i\}_{i=1}^{\infty}$ is strictly α_*-admissible sequence;

(ii) there exist $x_0 \in X$ and $x_1 \in T_ix_0$ for some $i \in \mathbb{N}$ with $\alpha(x_0, x_1) > 1$;

(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) > 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) > 1$ for each $n \in \mathbb{N}$.

Then the mappings in a sequence $\{T_i\}_{i=1}^{n}$ have a common fixed point.

Proof. The proof of this theorem runs along the same lines as the proof of Theorem 2.9. ∎

Definition 2.8. *Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a function. A sequence of mappings $\{T_i : X \to B(X)\}_{i=1}^{\infty}$ is an F_α-contraction of Hardy-Rogers-type, if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that for each $i, j \in \mathbb{N}$, we have*

$$\tau + F(\alpha(x,y)\delta(T_ix, T_jy)) \leq F(N(x,y)), \tag{2.15}$$

for each $x, y \in X$*, whenever* $\min\{\alpha(x,y)\delta(T_i x, T_j y), N(x,y)\} > 0$*, where*

$$\begin{aligned} N(x,y) = a_1 d(x,y) + a_2 d(x, T_i x) + a_3 d(y, T_j y) \\ + a_4 d(x, T_j y) + L d(y, T_i x), \end{aligned}$$

with $a_1, a_2, a_3, a_4, L \geq 0$ *satisfying* $a_1 + a_2 + a_3 + 2a_4 = 1$ *and* $a_3 \neq 1$*.*

Note that H is not a metric on the set of bounded subsets of X, as the following example shows.

Let $X = \mathbb{R}$, endowed with usual metric then $H(A,B) = 0$ but $A \neq B$ for $A = [0,1)$ and $B = [0,1]$. This implies that H is not a metric on Bounded subsets of $\mathbb{R}$. It would be interesting to see whether the conclusions of Theorem 2.4 hold for bounded subsets of X. We will show that the conclusions of Theorem 2.4 still hold for bounded subsets of X provided that the Housdorff distance $H(A,B)$ in definition 2.3 is replaced with $\delta(A,B)$ and the strict inequality in (ii) of Theorem 2.4 is replaced by the soft inequality. More precisely we have the following result.

THEOREM 2.9. *Let* (X,d) *be a complete metric space and let* $\{T_i : X \to B(X)\}_{i=1}^{\infty}$ *be an* F_α*-contraction of Hardy-Rogers-type satisfying the following conditions:*

(i) $\{T_i\}_{i=1}^{\infty}$ *is* α*-admissible sequence;*
(ii) there exist $x_0 \in X$ *and* $x_1 \in T_i x_0$ *for some* $i \in \mathbb{N}$ *with* $\alpha(x_0, x_1) \geq 1$*;*
(iii) for any sequence $\{x_n\} \subseteq X$ *such that* $x_n \to x$ *as* $n \to \infty$ *and* $\alpha(x_n, x_{n+1}) \geq 1$ *for each* $n \in \mathbb{N}$*, we have* $\alpha(x_n, x) \geq 1$ *for each* $n \in \mathbb{N}$*.*

Then the mappings in the sequence $\{T_i\}_{i=1}^{\infty}$ *have a common fixed point.*

PROOF. By hypothesis (ii), we assume without loss of generality that there exist $x_0 \in X$ and $x_1 \in T_1 x_0$ with $\alpha(x_0, x_1) \geq 1$. If $x_1 \in T_i x_1$ $\forall i \in \mathbb{N}$, then x_1 is a common fixed point. Let $x_1 \notin T_2 x_1$. As $\alpha(x_0, x_1) \geq 1$, there exists $x_2 \in T_2 x_1$ such that

$$d(x_1, x_2) \leq \alpha(x_0, x_1)\delta(T_1 x_0, T_2 x_1). \tag{2.16}$$

Since F is increasing, we have

$$F(d(x_1, x_2)) \leq F(\alpha(x_0, x_1)\delta(T_1 x_0, T_2 x_1)). \tag{2.17}$$

From (2.15) we have

$$\begin{aligned}\tau + F(d(x_1,x_2)) &\le \tau + F(\alpha(x_0,x_1)\delta(T_1x_0,T_2x_1))\\ &\le F\big(a_1d(x_0,x_1)+a_2d(x_0,T_1x_0)+a_3d(x_1,T_2x_1)\\ &\quad + a_4d(x_0,T_2x_1)+Ld(x_1,T_1x_0)\big)\\ &\le F\big(a_1d(x_0,x_1)+a_2d(x_0,x_1)+a_3d(x_1,x_2)\\ &\quad + a_4d(x_0,x_2)+L.0\big)\\ &\le F\big(a_1d(x_0,x_1)+a_2d(x_0,x_1)+a_3d(x_1,x_2)\\ &\quad + a_4(d(x_0,x_1)+d(x_1,x_2))\big)\\ &= F\big((a_1+a_2+a_4)d(x_0,x_1)+(a_3+a_4)d(x_1,x_2)\big).\end{aligned} \tag{2.18}$$

Since F is increasing, we get from above that

$$d(x_1,x_2) < (a_1+a_2+a_4)d(x_0,x_1)+(a_3+a_4)d(x_1,x_2).$$

That is,

$$(1-a_3-a_4)d(x_1,x_2) < (a_1+a_2+a_4)d(x_0,x_1).$$

As $a_1+a_2+a_3+2a_4 = 1$, thus we have

$$d(x_1,x_2) < d(x_0,x_1).$$

Now from (2.18), we have

$$\tau + F(d(x_1,x_2)) \le F(d(x_0,x_1)).$$

If $x_2 \in T_ix_2$ $\forall i \in \mathbb{N}$ then x_2 is a common fixed point. Let $x_2 \notin T_3x_2$, since $\{T_i\}_{i=1}^{\infty}$ is α-admissible, we have $\alpha(x_1,x_2) \ge 1$. There exists $x_3 \in T_3x_2$ such that

$$d(x_2,x_3) \le \alpha(x_1,x_2)\delta(T_2x_1,T_3x_2). \tag{2.19}$$

Since F is increasing, we have

$$F(d(x_2,x_3)) \le F(\alpha(x_1,x_2)\delta(T_2x_1,T_3x_2)). \tag{2.20}$$

From (2.15) we have

$$\begin{aligned}\tau + F(d(x_2,x_3)) &\le \tau + F(\alpha(x_1,x_2)\delta(T_2x_1,T_3x_2))\\ &\le F\big(a_1d(x_1,x_2)+a_2d(x_1,T_2x_1)+a_3d(x_2,T_3x_2)\\ &\quad + a_4d(x_1,T_3x_2)+Ld(x_2,T_2x_1)\big)\\ &\le F\big(a_1d(x_1,x_2)+a_2d(x_1,x_2)+a_3d(x_2,x_3)\\ &\quad + a_4d(x_1,x_3)+L.0\big)\\ &\le F\big(a_1d(x_1,x_2)+a_2d(x_1,x_2)+a_3d(x_2,x_3)\\ &\quad + a_4(d(x_1,x_2)+d(x_2,x_3))\big)\\ &= F\big((a_1+a_2+a_4)d(x_1,x_2)+(a_3+a_4)d(x_2,x_3)\big).\end{aligned} \tag{2.21}$$

Since F is increasing, we get from above that

$$d(x_2, x_3) < (a_1 + a_2 + a_4)d(x_1, x_2) + (a_3 + a_4)d(x_2, x_3).$$

That is,

$$(1 - a_3 - a_4)d(x_2, x_3) < (a_1 + a_2 + a_4)d(x_1, x_2).$$

As $a_1 + a_2 + a_3 + 2a_4 = 1$, thus we have

$$d(x_2, x_3) < d(x_1, x_2).$$

Now from (2.21) we have

$$\tau + F(d(x_2, x_3)) \le F(d(x_1, x_2)).$$

So we have

$$F(d(x_2, x_3)) \le F(d(x_1, x_2)) - \tau \le F(d(x_0, x_1)) - 2\tau.$$

Continuing in the same way we get a sequence $\{x_n\} \subset X$ such that

$$x_n \in T_n x_{n-1},\ x_{n-1} \neq x_n \text{ and } \alpha(x_{n-1}, x_n) \ge 1 \text{ for each } n \in \mathbb{N}.$$

Furthermore,

$$F(d(x_n, x_{n+1})) \le F(d(x_0, x_1)) - n\tau \text{ for each } n \in \mathbb{N}. \tag{2.22}$$

Letting $n \to \infty$ in (2.22) we get $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$. Thus, by property (F_2), we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Let $d_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N}$. From (F_3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} d_n^k F(d_n) = 0.$$

From (2.22) we have

$$d_n^k F(d_n) - d_n^k F(d_0) \le -d_n^k n\tau \le 0 \text{ for each } n \in \mathbb{N}. \tag{2.23}$$

Letting $n \to \infty$ in (2.23) we get

$$\lim_{n\to\infty} n d_n^k = 0. \tag{2.24}$$

This implies that there exists $n_1 \in \mathbb{N}$ such that $n d_n^k \le 1$ for each $n \ge n_1$. Thus we have

$$d_n \le \frac{1}{n^{1/k}}, \quad \text{for each } n \ge n_1. \tag{2.25}$$

To prove that $\{x_n\}$ is a Cauchy sequence. Consider $m, n \in \mathbb{N}$ with $m > n > n_1$. By using the triangular inequality and (2.25) we have

$$\begin{aligned} d(x_n, x_m) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d_i \le \sum_{i=n}^{\infty} d_i \le \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent series. Thus $\lim_{n\to\infty} d(x_n, x_m) = 0$. Which implies that $\{x_n\}$ is a Cauchy sequence. As (X, d) is complete so there exists

$x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. By condition (iii) we have $\alpha(x_n, x^*) \geq 1$ for each $n \in \mathbb{N}$. We claim that $d(x^*, T_i x^*) = 0$ $\forall i \in \mathbb{N}$. On contrary suppose that $d(x^*, T_{i_0} x^*) > 0$ for some $i_0 \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, T_{i_0} x^*) > 0$ for each $n \geq n_0$. For each $n \geq n_0$ and for above i_0, we have

$$\begin{aligned} d(x^*, T_{i_0}x^*) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, T_{i_0}x^*) \\ &< d(x^*, x_{n+1}) + \alpha(x_n, x^*)\delta(T_{n+1}x_n, T_{i_0}x^*) \\ &< d(x^*, x_{n+1}) + a_1 d(x_n, x^*) + a_2 d(x_n, x_{n+1}) \\ &\quad + a_3 d(x^*, T_{i_0}x^*) + a_4 d(x_n, T_i x^*) + L d(x^*, x_{n+1}). \end{aligned} \tag{2.26}$$

Letting $n \to \infty$ in (2.26) we have

$$d(x^*, T_{i_0}x^*) \leq (a_3 + a_4) d(x^*, T_{i_0}x^*) < d(x^*, T_{i_0}x^*).$$

Which is a contradiction. Thus $d(x^*, T_i x^*) = 0$ for all $i \in \mathbb{N}$. ∎

EXAMPLE 2.10. Let $X = \{0, 1, 2, 3, ...\}$ and

$$d(x,y) = \begin{cases} 0 \text{ if } x = y, \\ x + y \text{ if } x \neq y. \end{cases}$$

Define $\{T_i : X \to B(X)\}_{i=1}^{\infty}$ by

$$T_i x = \begin{cases} \{0\} \text{ if } x = 0, \\ \{0, 1, 2, 3, ..., x\} \text{ if } x \neq 0 \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1 \text{ if } x = y = 0, \\ \frac{1}{2} \text{ if } x, y > 1, \\ 0 \text{ otherwise.} \end{cases}$$

Take $F(x) = x + \ln(x)$ for each $x \in (0, \infty)$. Under this F condition (2.15) reduces to

$$\frac{\alpha(x,y)\delta(T_i x, T_j y)}{N(x,y)} e^{\alpha(x,y)\delta(T_i x, T_j y) - N(x,y)} \leq e^{-\tau} \tag{2.27}$$

for each $x, y \in X$ with $\min\{\alpha(x,y)\delta(T_i x, T_j y), N(x,y)\} > 0$. Assume that $a_1 = 1$, $a_2 = a_3 = a_4 = L = 0$ and $\tau = \frac{1}{2}$. Clearly

$$\min\{\alpha(x,y)\delta(T_i x, T_j y), d(x,y)\} > 0$$

for each $x, y > 1$ with $x \neq y$. From (2.15) for each $x, y > 1$ with $x \neq y$, we have

$$\frac{1}{2} e^{-\frac{1}{2}(x+y)} < e^{-\frac{1}{2}}.$$

Thus $\{T_i\}_{i=1}^{\infty}$ is an F_α-contraction of Hardy-Roger-type with $F(x) = x + \ln x$. For $x_0 = 1$, we have $x_1 = 0 \in T_1 x_0$ such that $\alpha(x_0, x_1) \geq 1$. Moreover, it is easy to see that $\{T_i\}_{i=1}^{\infty}$ is α-admissible sequence and for any sequence

$\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$. Therefore by Theorem 2.9 $\{T_i\}_{i=1}^{\infty}$ has a common fixed point in X.

DEFINITION 2.11. *Let (X, d) be a metric space and $\alpha : X \times X \to [0, \infty)$ be a function. A sequence of mappings $\{T_i : X \to B(X)\}_{i=1}^{\infty}$ is an $F_\alpha*$-contraction of Hardy-Rogers-type, if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that for each $i, j \in \mathbb{N}$, we have*

$$\tau + F(\alpha_*(T_i x, T_j y)\delta(T_i x, T_j y)) \leq F(N(x, y)), \tag{2.28}$$

for each $x, y \in X$, whenever $\min\{\alpha_*(T_i x, T_j y)\delta(T_i x, T_j y), N(x, y)\} > 0$, where

$$N(x, y) = a_1 d(x, y) + a_2 d(x, T_i x) + a_3 d(y, T_j y) + a_4 d(x, T_j y) + L d(y, T_i x),$$

with $a_1, a_2, a_3, a_4, L \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$.

THEOREM 2.12. *Let (X, d) be a complete metric space and let $\{T_i : X \to B(X)\}_{i=1}^{\infty}$ be an $F_\alpha*$-contraction of Hardy-Rogers-type satisfying the following conditions:*

(i) $\{T_i\}_{i=1}^{\infty}$ is α_*-admissible sequence;

(ii) there exist $x_0 \in X$ and $x_1 \in T_i x_0$ for some $i \in \mathbb{N}$ with $\alpha(x_0, x_1) \geq 1$;

(iii) for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geq 1$ for each $n \in \mathbb{N}$.

Then the mappings in a sequence $\{T_i\}_{i=1}^{n}$ have a common fixed point.

PROOF. The proof of this theorem runs along the same lines as the proof of Theorem 2.9. ∎

3. APPLICATION

In this section, as a consequence of our result we establish an existence theorem for a system of integral equations. Let $X = (C[a, b], \mathbb{R})$ be the space of all real valued continuous functions defined on $[a, b]$. Note that X is complete ([25]) with respect to the metric $d_\tau(x, y) = \sup_{t \in [a,b]}\{|x(t) - y(t)|e^{-|\tau t|}\}$.

Consider the system of integral equations of the form

$$x(t) = f(t) + \int_a^b K_i(t, s, x(s))ds, \tag{3.1}$$

for $t, s \in [a, b]$ and $i \in \{1, 2, 3, \cdots, N\}$ with $N \in \mathbb{N}$. Where $K_i : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ are continuous functions.

THEOREM 3.1. *Let $X = (C[a, b], \mathbb{R})$ and let $\{T_i : X \to X\}_{i=1}^{N}$ be the operators defined as*

$$T_i x(t) = f(t) + \int_a^b K_i(t, s, x(s))ds, \tag{3.2}$$

for $t, s \in [a, b]$. *Where* $K_i : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ *and* $f : [a, b] \to \mathbb{R}$ *are continuous functions. Assume that there exist* $\gamma : X \to (0, \infty)$, $\alpha : X \times X \to (0, \infty)$ *and the following conditions hold:*

(i) for each $i, j \in \{1, 2, 3, \cdots, N\}$ *there exists* $\tau > 0$ *such that*

$$|K_i(t, s, x) - K_j(t, s, y)| \leq \frac{e^{-\tau}}{\gamma(x + y)}|x - y|$$

for each $t, s \in [a, b]$ *and* $x, y \in X$. *Moreover,*

$$\Big|\int_a^b \frac{e^{|\tau s|}}{\gamma(x + y)} ds\Big| \leq \frac{e^{|\tau t|}}{\alpha(x, y)}$$

for each $t \in [a, b]$;

(ii) for $x, y \in X$, $\alpha(x, y) \geq 1$ *implies* $\alpha(T_i x, T_j y) \geq 1$ *for each* $i, j \in \{1, 2, 3, \cdots, N\}$;

(iii) there exist $x_0 \in X$ *such that* $\alpha(x_0, T_i x_0) \geq 1$ *for some* $i \in \{1, 2, 3, \cdots, N\}$;

(iv) for any sequence $\{x_n\} \subseteq X$ *such that* $x_n \to x$ *as* $n \to \infty$ *and* $\alpha(x_n, x_{n+1}) \geq 1$ *for each* $n \in \mathbb{N}$, *we have* $\alpha(x_n, x) \geq 1$ *for each* $n \in \mathbb{N}$.

Then the system of integral equations (3.1) *has a solution in* X.

Proof. First we show that $\{T_i\}$ is an F_α-contraction of Hardy-Rogers-type. For each $i, j \in \{1, 2, 3, \cdots, N\}$, we have

$$\begin{aligned} |T_i x(t) - T_j y(t)| &\leq \int_a^b |K_i(t, s, x(s)) - K_j(t, s, y(s))| ds \\ &\leq \int_a^b \frac{e^{-\tau}}{\gamma(x(s) + y(s))}|x(s) - y(s)| ds \\ &= \int_a^b \frac{e^{-\tau} e^{|\tau s|}}{\gamma(x(s) + y(s))}|x(s) - y(s)| e^{-|\tau s|} ds \\ &\leq e^{-\tau} d_\tau(x, y) \int_a^b \frac{e^{|\tau s|}}{\gamma(x(s) + y(s))} ds \leq \frac{e^{|\tau t|}}{\alpha(x, y)} e^{-\tau} d_\tau(x, y). \end{aligned}$$

Thus we have

$$\alpha(x, y)|T_i x(t) - T_j y(t)| e^{-|\tau t|} \leq e^{-\tau} d_\tau(x, y).$$

Equivalently,

$$\alpha(x, y) d_\tau(T_i x, T_j y) \leq e^{-\tau} d_\tau(x, y).$$

Clearly natural logarithm belongs to $\mathfrak{F}$. Applying it on above inequality we get

$$\ln(\alpha(x, y) d_\tau(T_i x, T_j y)) \leq \ln(e^{-\tau} d_\tau(x, y)),$$

after some simplification we get

$$\tau + \ln(\alpha(x, y) d_\tau(T_i x, T_j y)) \leq \ln(d_\tau(x, y)).$$

Thus $\{T_i\}_{i=1}^N$ is an F_α-contraction of Hardy-Rogers-type with $a_1 = 1$, $a_2 = a_3 = a_4 = L = 0$ and $F(x) = \ln x$. Therefore by 2.9 it follows that the system of operators (3.2) have a common fixed point, that is, the system of integral equations (3.1) has a solution in X. ∎

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