

Comparison Between Two Eccentricity-based Topological Indices of Graphs

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Abstract: For a connected graph G , the eccentric connectivity index (ECI) and the first Zagreb eccentricity index of G are defined as $\xi^c(G) = \sum_{v_i \in V(G)} \deg_G(v_i) \varepsilon_G(v_i)$ and $E_1(G) = \sum_{v_i \in V(G)} \varepsilon_G(v_i)^2$, respectively, where $\deg_G(v_i)$ is the degree of v_i in G and $\varepsilon_G(v_i)$ denotes the eccentricity of vertex v_i in G . In this paper we compare the eccentric connectivity index and the first Zagreb eccentricity index of graphs. It is proved that $E_1(T) > \xi^c(T)$ for any tree T . This improves a result by Das^[25] for the chemical trees. Moreover, we also show that there are infinite number of chemical graphs G with $E_1(G) > \xi^c(G)$. We also present an example in which infinite graphs G are constructed with $E_1(G) = \xi^c(G)$ and give some results on the graphs G with $E_1(G) < \xi^c(G)$. Finally, an effective construction is proposed for generating infinite graphs with each comparative inequality possibility between these two topological indices.

Keywords: Graph, First Zagreb eccentricity index, Eccentric connectivity index.

1. INTRODUCTION

WE only consider finite, undirected and simple graphs throughout this paper. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$. The *degree* of $v_i \in V(G)$, denoted by $\deg_G(v_i)$, is the number of vertices in G adjacent to v_i . For any two vertices v_i, v_j in a graph G , the distance between them, denoted by $d_G(v_i, v_j)$, is the length of a shortest path connecting them in G . Other undefined notations and terminology on the graph theory can be found in.^[1]

For any vertex of graph G , the eccentricity $\varepsilon_G(v_i)$ is the maximum distance from v_i to other vertices of G , i.e., $\varepsilon_G(v_i) = \max_{v_j \neq v_i} d_G(v_i, v_j)$. If $\varepsilon_G(v_i) = d_G(v_i, v_j)$, then v_j is an *eccentric vertex* of vertex v_i . For any graph G , we denote by \bar{G} the complement of G . As usual, let S_n , P_n , C_n , K_n be the star graph, path graph, cycle graph and complete graph, respectively, on n vertices. We denote by K_{n_1, n_2} the complete bipartite graph with bipartition of sizes n_1 and n_2 . The *Cartesian product* $G \square H$ of graphs G and H is the graph with $V(G \square H) = V(G) \times V(H)$ and (g, h) is adjacent to (g', h') if and only if $gg' \in E(G)$ and $h = h'$, or $g = g'$ and

$hh' \in E(H)$. If $G = H$, then $G \square H$ is denoted by $G^{(2)}$ for short. Moreover, $G^{(k)}$ can be similarly defined.

A graphical invariant is a number related to a graph which is a structural invariant, in other words, it is a fixed number under graph automorphisms. In chemical graph theory, these invariants are also known as the topological indices. Two of the oldest graph invariants are the well-known Zagreb indices first introduced in Ref. [2] where Gutman and Trinajstić examined the dependence of total π -electron energy on molecular structure and elaborated in Ref. [3] For a (molecular) graph G , the first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ are, respectively, defined as follows:

$$M_1 = M_1(G) = \sum_{v_i \in V(G)} \deg_G(v_i)^2,$$

$$M_2 = M_2(G) = \sum_{v_i, v_j \in E(G)} \deg_G(v_i) \deg_G(v_j).$$

These two classical topological indices reflect the extent of branching of the molecular carbon-atom skeleton.^[4] The main properties of M_1 and M_2 were summarized in Refs. [5,6]. Other recent results on Zagreb indices can be found in Ref. [7] and the references cited therein.

In analogy with the first and second Zagreb indices of graphs, some variants of them are invented, such as multiplicative Zagreb indices,^[8,9] multiplicative sum Zagreb index,^[10,11] Zagreb coindex^[12,13] and so on. In particular, Vukičević and Graovac^[14] defined the first and second Zagreb eccentricity indices as follows:

$$E_1(G) = \sum_{v_i \in V(G)} \varepsilon_G(v_i)^2, E_2(G) = \sum_{v_i, v_j \in E(G)} \varepsilon_G(v_i) \varepsilon_G(v_j).$$

Some mathematical properties of E_1 and E_2 can be found in Refs. [15,16].

In 1997, Sharma, Goswami and Madan^[17] introduced a distance-based molecular structure descriptor, which is named as "eccentric connectivity index" and defined as

$$\xi^c(G) = \sum_{v_i \in V(G)} \deg_G(v_i) \varepsilon_G(v_i).$$

The eccentric connectivity index (ECI) has been employed successfully for the development of numerous mathematical models for the prediction of biological activities of diverse nature.^[18–20] The ECI also can be written as follows:

$$\xi^c(G) = \sum_{v_i, v_j \in E(G)} (\varepsilon(v_i) + \varepsilon(v_j)).$$

Some properties of ECI have been reported in Refs. [21–23].

A tree with maximum degree at most 4 is called *chemical tree*, which provides the graph representation of alkanes.^[24] In particular, a graph with maximum degree at most 4 is called *chemical graph*. Denote by $\mathcal{T}_n(d)$ the set of trees of order n and with diameter d .

From definition, we have $E_1(S_n) > \xi^c(S_n)$ and $E_1(K_n) < \xi^c(K_n)$ for $n \geq 3$. Therefore these two topological indices E_1 and ξ^c are incomparable. In Ref. [25], Das proved that $E_1(T) > \xi^c(T)$ for any chemical tree T . The paper is organized as follows. In Section 2, we show that $E_1(T) > \xi^c(T)$ for any tree T . And we present that there are infinite number of chemical graphs G with $E_1(G) > \xi^c(G)$. In Section 3, we give an example in which infinite number of graphs G are constructed with $E_1(G) = \xi^c(G)$. Also several sufficient conditions are proved for graphs G with $E_1(G) < \xi^c(G)$. In Section 4, an effective construction is presented for generating infinite graphs with each comparison possibility between these two topological indices.

2. THE GRAPHS WITH $E_1(G) > \xi^c(G)$

In this section we characterize some graphs G with $E_1(G) > \xi^c(G)$. Clearly, $E_1(P_2) = \xi^c(P_2)$. Then we deal with the case when T is a tree of order $n \geq 3$. Next we will prove that there are infinite number of chemical graphs G with $E_1(G) > \xi^c(G)$.

Before presenting the main results, we need to introduce some notations. A *caterpillar*,^[26] denoted by $P_{k+1}^n(a_2, a_3, \dots, a_k)$ with $\sum_{i=2}^k a_i = n - k - 1$, is a tree of order n with diameter k obtained from a path $P_{k+1} = v_1 v_2 \dots v_{k+1}$ by attaching $a_i \geq 0$ pendant vertices to the vertex v_i for $i = 2, 3, \dots, k$. If k is even, then P_{k+1} has a unique central vertex $v_{\frac{k+1}{2}}$. Otherwise, P_{k+1} has two adjacent central vertices $v_{\frac{k-1}{2}}$ and $v_{\frac{k+1}{2}}$. If $k = 3$ in $P_{k+1}^n(a_2, a_3, \dots, a_k)$ with $a_2 = p - 2 \geq 0$, $a_3 = q - 1 \geq 0$ and $p + q = n - 2$, then $P_{k+1}^n(a_2, a_3, \dots, a_k)$ is a *double star* and denoted by $DS_n(p, q)$ for short. And $P_{k+1}^n(a_2, a_3, \dots, a_k)$ is a *dumbbell* and denoted by $DB_n(a_2, a_k)$ if $a_2 \geq 0$, $a_k \geq 0$ and $a_t = 0$ for $3 \leq t \leq k - 1$. Moreover, $P_{k+1}^n(a_2, a_3, \dots, a_k)$ is called a *volcano tree* and denoted by $V_n(n - k - 1)$ if $a_t = n - k - 1$ when v_t is a central vertex of P_{k+1} and $a_j = 0$ for any $j \neq t$ for even k , and in the set $\mathcal{V}_n(n - k - 1)$ if $a_t + a_{t+1} = n - k - 1$ when v_t, v_{t+1} are two central vertices of P_{k+1} and $a_j = 0$ for any $j \neq t, t + 1$ for odd k . If $d = 2$, $\mathcal{T}_n(d)$ contains a single tree S_n . The case is same when $n = 3$. For $n = 4$, there are exactly two trees S_n and P_n with $E_1(P_n) > \xi^c(P_n)$ and $E_1(S_n) > \xi^c(S_n)$. So in the following we always assume that $d \geq 3$ and $n \geq 5$. For convenience, here we set $\xi^A(G) = E_1(G) - \xi^c(G)$ for any connected graph G .

Lemma 2.1. Suppose that $T \in \mathcal{T}_n(d)$ with $n \geq 5$ and $3 \leq d < n - 1$ minimizes the value of ξ^A . Then T must be a caterpillar.

Proof. We choose an arbitrary tree $T \in \mathcal{T}_n(d)$ with $\xi^A(T)$ as small as possible. If $T \cong P_{d+1}^n(a_2, a_3, \dots, a_d)$, then our result holds immediately. Otherwise, we can assume that $P_{d+1} = v_1 v_2 \dots v_d v_{d+1}$ is a diametral path in T . Then T can be viewed as a tree obtained by attaching a subtree T_i to each of vertex v_i with $i \in \{2, 3, \dots, d\}$ such that $\text{diam}(T_i) \leq d - \varepsilon_T(v_i)$. There must be a pendant vertex, say v_k , from $V(T_m)$ with $d_T(v_k, v_m) \geq 2$ where $m \in \{3, 4, \dots, d - 1\}$ and $v_k v_j \in E(T)$ ($j \notin \{1, 2, \dots, d, d + 1\}$). Without loss of generality, we assume that $\varepsilon_T(v_m) = d_T(v_m, v_{d+1}) = d + 1 - m$ and $\varepsilon_T(v_k) = t$ with $t > d + 1 - m + 1$.

Now we construct a new tree T' obtained from T by deleting the edge $v_k v_j$ and adding a new edge $v_k v_m$. Then T' still belongs to $\mathcal{T}_n(d)$ with $\varepsilon_{T'}(v_k) = d + 1 - m + 1$ with $\varepsilon_{T'}(v_m) = d + 1 - m$ and $\varepsilon_{T'}(v_x) = \varepsilon_T(v_x)$ where $x \neq k$. Note that $t > d + 1 - m + 1$. Therefore, only considering the contribution of the vertices v_j, v_k and v_m to ξ^A , we have

$$\begin{aligned} \xi^A(T) - \xi^A(T') &= [t - 1 - \deg_T(v_j)](t - 1) + t(t - 1) \\ &\quad + (d + 1 - m)[d + 1 - m - \deg_T(v_m)] \\ &\quad - [t - 1 - (\deg_T(v_j) - 1)](t - 1) \\ &\quad - (d + 2 - m - 1)(d + 2 - m) \\ &\quad - (d + 1 - m)[d + 1 - m - (\deg_T(v_m) + 1)] \end{aligned}$$

$$\begin{aligned} &= -(t-1) + (d+1-m) + t(t-1) \\ &\quad - (d+1-m)(d+2-m) \\ &= (t-1)^2 - (d+1-m)^2 > 0, \end{aligned}$$

that is, $\xi^A(T') < \xi^A(T)$. If T' is a caterpillar, our result follows. If not, we can continue the above construction process until we obtain a caterpillar $P_{d+1}^n(a_2, a_3, \dots, a_d) \in \mathcal{T}_n(d)$. Then our result holds from the fact that ξ^A strictly decreases in the above construction process. \square

Lemma 2.2. Let $T \cong P_{d+1}^n(a_2, a_3, \dots, a_d)$ be a non-volcano tree. Then there is another caterpillar $T' \in \mathcal{T}_n(d)$ with $\xi^A(T') < \xi^A(T)$.

Proof. By assumption, there is a non-central vertex v_i in the diametral path of $T \cong P_{d+1}^n(a_2, a_3, \dots, a_d)$ such that $a_i > 0$. First, let d be even. When d is odd, our proof can be similarly completed and so omitted here. Without loss of generality, assume that $i < \frac{d}{2}$. Now we construct a new tree T' from T by deleting all the pendant edges incident with v_i and joining them with the vertex v_{i+1} . Note that $\deg_T(v_i) = a_i + 2$ and T' is still a caterpillar with diameter d . Moreover, we have

$$\begin{aligned} &\xi^A(T) - \xi^A(T') \\ &= a_i [(d-i+2)(d-i+2-1) - (d-i+1)(d-i+1-1)] \\ &\quad + (d-i+1) [(d-i+1-a_i-2) - (d-i+1-2)] \\ &\quad + (d-i) [(d-i-\deg_T(v_{i+1})) - (d-i-\deg_T(v_{i+1})-a_i)] \\ &= a_i [(d-i+2)^2 - (d-i+1)^2 - 1] - a_i(d-i+1) + a_i(d-i) \\ &= a_i [2(d-i)+1] \\ &> 0, \end{aligned}$$

which finishes the proof of this lemma. \square

Now we define the a function as follows:

$$h(n, d) = \begin{cases} 2 \sum_{i=\frac{d+1}{2}}^d (i^2 - 2i) + \frac{(d+1)}{4} [(n-d-2)d + n] + \frac{7d-1}{4} & \text{if } d \text{ is odd;} \\ 2 \sum_{i=\frac{d}{2}}^d (i^2 - 2i) + \frac{d^2(n-d-2)}{4} + 3d & \text{if } d \text{ is even.} \end{cases}$$

By some calculations, we have $\xi^A(V_n(n-d-1)) = h(n, d)$ for even d and $\xi^A(T) = h(n, d)$ for any $T \in \mathcal{V}_n(n-d-1)$ when d is odd.

Theorem 2.3. Let T be a tree of order $n \geq 4$. Then

$$\xi^A(T) \geq n$$

with equality holding if and only if $T \cong S_n$.

Proof. Note that the set of all trees of order n can be partitioned into the union of the sets $\mathcal{T}_n(d)$ with

$d \in \{2, 3, \dots, n-1\}$. Moreover, it can be easily checked that $h(n, d) > h(n, d-2)$ for any polarity of d .

Assume that T is a tree of order $n \geq 3$ with ξ^A as small as possible. By the above argument, we conclude that T is a tree in $\mathcal{T}_n(2)$ or in $\mathcal{T}_n(3)$, that is, $T \cong S_n$ or $T \cong DS_n(n_1, n_2)$ with $n_1 \geq 1$, $n_2 \geq 1$ and $n_1 + n_2 = n-2$. By definition, we have $\xi^A(S_n) = n < 4n-4 = \xi^A(DS_n(n_1, n_2))$. Thus $T \cong S_n$ from the choice of T .

Conversely, if $T \cong S_n$, we have $\xi^A(T) = n$, finishing the proof of this theorem. \square

From Theorem 2.3, the following corollary can be easily obtained.

Corollary 2.4. Let T be a tree of order $n \geq 3$. Then $E_1(T) > \xi^c(T)$.

From Corollary 2.4, any chemical tree T fulfills the property that $E_1(T) > \xi^c(T)$, which is also recently proved by Das.^[25]

Note that any vertex in the cycle C_n has the same eccentricity $\lfloor \frac{n}{2} \rfloor$. Therefore we have $E_1(C_n) > \xi^c(C_n)$ for $n > 4$. In the following theorem we prove the existence of chemical graphs G with $E_1(G) > \xi^c(G)$.

Theorem 2.5. There are infinite number of chemical graphs G such that $E_1(G) > \xi^c(G)$.

Proof. Now we consider the graph $G = C_n \square K_2$ with $n \geq 6$. Note that G is a 3-regular graph of order $2n$. Assume that $V(G) = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n\}$ where $v_1 v_2 \dots v_n$ and $u_1 u_2 \dots u_n$ with their natural adjacency relation form two induced cycle C_n in G .

For any positive integer k with $1 \leq k \leq n$, we set $|k|_n = \min\{k, n-k\}$. It can be verified that $\varepsilon_G(v_i) = \lfloor \frac{n}{2} \rfloor + 1 > 3$ for any vertex v_i with u_i as its eccentric vertex in G where $|t|_n = \lfloor \frac{n}{2} \rfloor$. By symmetry, we have $\varepsilon_G(u_j) = \lfloor \frac{n}{2} \rfloor + 1$ for any vertex u_j . Thus $\xi^A(G) > 0$, that is, $E_1(G) > \xi^c(G)$ as desired. \square

3. THE GRAPHS WITH $E_1(G) \leq \xi^c(G)$

In this section we prove several results on the graphs G with $E_1(G) \leq \xi^c(G)$.

Recall that $E_1(K_2) = \xi^c(K_2)$. Also $E_1(C_4) = \xi^c(C_4)$ with $C_4 = K_2^{(2)}$. Note that, for $k \geq 2$, $K_2^{(k)}$ is just the k -cube which is a k -regular graph with each vertex with eccentricity k . Now we give a more general result.

Example 3.1. $E_1(K_2^{(k)}) = \xi^c(K_2^{(k)})$ for any $k \geq 1$.

Next we turn to the results for the graphs G with $E_1(G) < \xi^c(G)$. Although in Section 2 we prove that $E_1(T) > \xi^c(T)$ for any tree of order $n > 2$, we have the opposite result for the complements of all trees of order $n > 2$. Below we first list an essential lemma for the complement of a graph.

Lemma 3.2.^[27] Let G be a connected graph with the connected complement.

- (i) If $d > 3$, then \bar{G} has diameter $\bar{d} = 2$.
- (ii) If $d = 3$, then \bar{G} has a spanning subgraph which is a double star.

Lemma 3.3. Let G be a self-centered graph of order $n > 2$ with $v_i \in V(G)$. Then $\deg_G(v_i) \geq 2$.

Proof. To the contrary, we assume that $\deg_G(v_i) = 1$ with $v_i v_j \in E(G)$. Note that $n > 2$. Then $\varepsilon_G(v_i) = \varepsilon_G(v_j) + 1$. This is a contradiction since G is self-centered. \square

Note that \bar{P}_3 is disconnected and $\bar{P}_4 = P_4$. Moreover, it can be verified that $E_1(\bar{T}) < \xi^c(\bar{T})$ for the trees of order 5 except $DS_5(1,2)$ with $E_1(DS_5(1,2)) > \xi^c(DS_5(1,2))$. So we assume that $n > 5$ in the following theorem.

Theorem 3.4. Let T be a tree of order $n > 5$ with diameter $d > 2$. Then $E_1(\bar{T}) < \xi^c(\bar{T})$.

Proof: If $d = 3$, then $T \cong DS_n(n_1, n_2)$ with $n_1 + n_2 = n - 2 > 2$, $n_1 \geq 1$ and $n_2 \geq 1$. Assume that the only two vertices with eccentricity 2 in T are v_1 and v_2 with $\deg_T(v_1) = n_1 + 1 \leq n_2 + 1 = \deg_T(v_2)$. Then $\varepsilon_{\bar{T}}(v_1) = \varepsilon_{\bar{T}}(v_2) = 3$, and all other vertices have the same eccentricity 2 in \bar{T} . Moreover, all vertices other than v_1, v_2 in \bar{T} have degrees $n - 2$. If $n_1 = 1$, then $n_2 = n - 3$ and $\deg_{\bar{T}}(v_1) = n - 3$, $\deg_{\bar{T}}(v_2) = 1$. Note that $n > 5$. Then we have

$$\begin{aligned} \xi^c(\bar{T}) - E_1(\bar{T}) &= 3(n - 3 - 3) + 3(1 - 3) + 2(n - 2)(n - 2 - 2) \\ &= 2n^2 - 9n - 8 > 0. \end{aligned}$$

For $n_1 \geq 2$, similarly as above, we get

$$\xi^c(\bar{T}) - E_1(\bar{T}) = 2n^2 - 9n - 8 > 0.$$

If $d > 3$, then, by Lemma 3.2(i), \bar{T} has diameter $\bar{d} = 2$. Now we claim that \bar{T} is a 2-self-centered graph. If not, \bar{T} has a vertex v_i with $\varepsilon_{\bar{T}}(v_i) = 1$, i.e., $\deg_{\bar{T}}(v_i) = n - 1$. Thus v_i is an isolated vertex in T , which is a contradiction since T is a tree. Observe that a connected 2-regular graph is just a cycle C_n whose complement is not a tree for $n > 5$, in view of Lemma 3.3, we find that \bar{T} is a 2-self-centered graph with $\deg_{\bar{T}}(v_i) \geq 2$ for any vertex v_i and there is at least one vertex v_j with $\deg_{\bar{T}}(v_j) > 2$. Thus $\xi^c(\bar{T}) - E_1(\bar{T}) > 0$, finishing the proof of the theorem. \square

In the following theorem we give a sufficient condition for the graphs G of order n and with $E_1(G) < \xi^c(G)$.

Theorem 3.5. Let G be a connected graph of order $n \geq 5$ with $K_{2,n-2}$ as its subgraph. Then $E_1(G) < \xi^c(G)$.

Proof. If $G \cong K_{2,n-2}$, then we have $\xi^c(G) = 8(n - 2)$ and $E_1(G) = 4n$. It follows that $\xi^c(G) > E_1(G)$ from $n \geq 5$. Therefore in the following we assume that G contains $K_{2,n-2}$ as a proper spanning subgraph.

Since G contains $K_{2,n-2}$ as a proper spanning subgraph, the eccentricity of any vertex in G is 1 or 2. Moreover, $\deg_G(v_k) \geq 2$ for any vertex $v_k \in V(G)$ with $\varepsilon_G(v_k) = 2$, and $d_G(v_i) = n - 1$ for any vertex $v_i \in V(G)$ with $\varepsilon_G(v_i) = 1$. If there is a vertex $v_k \in V(G)$ with degree $n - 1$, then $\xi^c(G) > E_1(G)$ immediately. Otherwise, G is 2-self-centered graph. Considering that $K_{2,n-2}$ is a proper spanning subgraph of G , there are at least two vertices v_i, v_j with $\deg_G(v_i) > \varepsilon_G(v_i)$ and $\deg_G(v_j) > \varepsilon_G(v_j)$. So $\xi^c(G) > E_1(G)$, finishing the proof of the theorem. \square

We can easily observe that the graph G in Theorem 3.5 has minimum degree at least 2 and $\deg_G(v_i) \geq \varepsilon_G(v_i)$ for any vertex $v_i \in V(G)$ with at least one strict inequality. But these conditions are not necessary for $\xi^c(G) > E_1(G)$. Next we will give an example with $\xi^c(G) > E_1(G)$ but not satisfying these above conditions. Denote by G^* the graph obtained by attaching a pendant vertex to each of the vertices in a connected graph G . Clearly, for $n \geq 3$, any pendant vertex in K_n^* has eccentricity 3, and any other vertex has a same eccentricity 2 in it. Thus $\xi^c(K_n^*) - E_1(K_n^*) = 2n(n - 2) + 3n(1 - 3) = 2n(n - 5) > 0$ if $n > 5$.

Example 3.6. $E_1(K_n^*) < \xi^c(K_n^*)$ for $n > 5$.

Note that all the graphs described in Theorems 3.4 and 3.5 have diameter at most 3. In Ref. [28], a class of graphs are constructed with exactly two distinct eccentricities (see Figure 1). Here G_0 is an arbitrary graph, each of whose vertices is adjacent to any vertex from $\{x_1, y_1, z_1, w_1\}$. The black vertices have eccentricity $r - 1$ and the white vertices have eccentricity r in G . The graph schematically shown in Figure 1 is denoted by $G = S(G_0; r - 1, r)$ with diameter r . In the following theorem we prove the existence of graphs G with diameter more than 2 fulfilling $E_1(G) < \xi^c(G)$.

Theorem 3.7. For any integer $r \geq 2$, there is a graph G with diameter r and fulfilling $E_1(G) < \xi^c(G)$.

Proof. For any integer $r \geq 2$, we choose a graph $G = S(K_n; r - 1, r)$ with $n \geq 3r$. From the structure of $S(K_n; r - 1, r)$, we find that there are n vertices in K_n of $S(K_n; r - 1, r)$ with eccentricity $r - 1$ and degree $n + 3$, other vertices have a

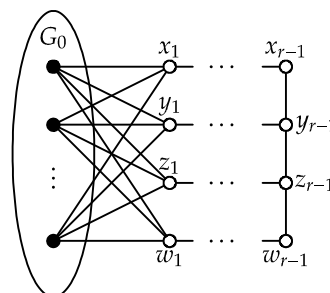


Figure 1. Graph G with only two eccentricities $r - 1$ and r .

same eccentricity r in $S(K_n, r-1, r)$. Note that $n \geq 3r$ and $r \geq 2$. Then

$$\begin{aligned} \xi^c(G) - E_1(G) &= [n+3-(r-1)]n(r-1) + (3-r)2r \\ &\quad + (2-r)[4(r-3)+2]r + (n+1-r)4r \\ &= (n+4-r)n(r-1) + (n+1-r)4r \\ &\quad - 2r(2r^2-8r+7) \\ &\geq 3r(2r+4)(r-1) + (2r+1)4r \\ &\quad - 2r(2r^2-8r+7) \\ &= 2r(r^2+15r-11) > 0. \end{aligned}$$

This completes the proof of the theorem. \square

4. AN EFFECTIVE CONSTRUCTION

From Example 3.1, we would like to give a more general result for generating infinite graphs with different comparative relations. To do it, we first prove a useful lemma as follows.

Lemma 4.1. *Let G be a connected graph of order $n > 2$ and $H = G \square K_2$. Then $\varepsilon_H(w) = \varepsilon_G(w) + 1$ for any vertex $w \in V(H)$.*

Proof. Assume that $V(G) = \{v_1, v_2, \dots, v_n\}$ and $V(H) = \{v_1, v_2, \dots, v_n, v'_1, v'_2, \dots, v'_n\}$ with $H[\{v_1, v_2, \dots, v_n\}] = H[\{v'_1, v'_2, \dots, v'_n\}] \cong G$ where v'_i is the copy of vertex v_i in H for $i = 1, 2, \dots, n$. By symmetry, it suffices to prove that $\varepsilon_H(v_i) = \varepsilon_G(v_i) + 1$ for $i = 1, 2, \dots, n$.

Assume that $\varepsilon_G(v_i) = k$ for an arbitrary vertex $v_i \in V(G) \subseteq V(H)$. Then there exists a vertex v_j as an eccentric vertex of v_i in G . Therefore $\varepsilon_H(v_i) \geq d_H(v_i, v'_j) = k+1$ from the structure of H . Next we prove that $\varepsilon_H(v_i) \leq k+1$. Otherwise, we have $\varepsilon_H(v_i) \geq k+2$. Then there is a vertex $v'_m \in V(H) \setminus V(G)$ with $d_H(v_i, v'_m) \geq k+2$. By the structure of H , again, we have $d_G(v_i, v_m) \geq k+1$, contradicting the fact that $\varepsilon_G(v_i) = k$. So $\varepsilon_H(v_i) = k+1 = \varepsilon_G(v_i) + 1$, finishing the proof of the lemma. \square

Theorem 4.2. Assume that G is a connected graph of order $n \geq 2$ with $\varepsilon_G(v_i) \geq \deg_G(v_i)$ for any vertex $v_i \in V(G)$. Then $E_1(G \square K_2) \geq \xi^c(G \square K_2)$.

Proof. Let $H = G \square K_2$. From the structure of H , we have $\deg_H(w) = \deg_G(w) + 1$ for any vertex $w \in V(G)$. Thus $E_1(G) \geq \xi^c(G)$. By Lemma 4.1, we have

$$\begin{aligned} E_1(G \square K_2) - \xi^c(G \square K_2) &= 2 \sum_{v_i \in V(G)} (\varepsilon_G(v_i) + 1) [(\varepsilon_G(v_i) + 1) - (\deg_G(v_i) + 1)] \\ &\geq 4(E_1(G) - \xi^c(G)) \geq 0. \end{aligned}$$

Therefore, our result holds immediately. \square

Similarly as above, we can easily obtain the corollary below.

Corollary 4.3. *Assume that G is a connected graph of order $n > 2$ with $\varepsilon_G(v_i) < \deg_G(v_i)$ for any vertex $v_i \in V(G)$. Then $E_1(G \square K_2) < \xi^c(G \square K_2)$.*

By Theorem 4.2 and Corollary 4.3, we can get infinite graphs with each possibility for comparison between the first Zagreb eccentricity index and eccentric connectivity index.

Recall that the graph G^* is defined in Section 3. In addition to Example 3.1 for $E_1 = \xi^c$, by Theorem 4.2 and Corollary 4.3, we can give the following examples for other comparative inequalities between E_1 and ξ^c .

Example 4.4. $E_1(C_n^* \square K_2) > \xi^c(C_n^* \square K_2)$ for any $n \geq 4$, $E_1(K_{n_1, n_2} \square K_2) < \xi^c(K_{n_1, n_2} \square K_2)$ for $n_2 \geq n_1 > 2$.

5. CONCLUSION

In this paper we present some results on the comparison between $E_1(G)$ and $\xi^c(G)$ on the graphs G including some chemical graphs. In particular, we show that $E_1(T) > \xi^c(T)$ for any trees T including chemical trees. Some sufficient conditions are obtained on the graphs G with $E_1(G) \leq \xi^c(G)$. Moreover, we also give a construction method for generating infinite graphs G for each comparative inequality between $E_1(G)$ and $\xi^c(G)$. Now it seems to be an open and attractive problem to characterize completely chemical graphs with some comparative inequality between E_1 and ξ^c .

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