SURFACES OF REVOLUTION IN THE THREE DIMENSIONAL PSEUDO-GALILEAN SPACE

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Abstract. In the present paper, we study surfaces of revolution in the three dimensional pseudo-Galilean space G_3^1. Also, we characterize surfaces of revolution in G_3^1 in terms of the position vector field and Gauss map.

1. Introduction

Let $\mathbf{x} : M \longrightarrow \mathbb{E}^m$ be an isometric immersion of a connected n-dimensional manifold in the m-dimensional Euclidean space $\mathbb{E}^m$. Denote by H and Δ the mean curvature and the Laplacian of M with respect to the Riemannian metric on M induced from that of $\mathbb{E}^m$, respectively. Takahashi ([16]) proved that the submanifolds in $\mathbb{E}^m$ satisfying $\Delta \mathbf{x} = \lambda \mathbf{x}$, that is, all coordinate functions are eigenfunctions of the Laplacian with the same eigenvalue $\lambda \in \mathbb{R}$ are either the minimal submanifolds of $\mathbb{E}^m$ or the minimal submanifolds of hypersphere $\mathbb{S}^{m-1}$ in $\mathbb{E}^m$.

As an extension of Takahashi theorem, Garay studied in [12] hypersurfaces in $\mathbb{E}^m$ whose coordinate functions are eigenfunctions of the Laplacian, but not necessarily associated to the same eigenvalue. He considered hypersurfaces in $\mathbb{E}^m$ satisfying the condition

$$(1.1) \qquad \Delta \mathbf{x} = A\mathbf{x},$$

where $A \in \mathrm{Mat}(m, \mathbb{R})$ is an $m \times m$- diagonal matrix, and proved that such hypersurfaces are minimal ($H = 0$) in $\mathbb{E}^m$ and open pieces of either round

hyperspheres or generalized right spherical cylinders. Related to this, Dillen, Pas and Verstraelen ([9]) investigated surfaces in $\mathbb{E}^3$ whose immersions satisfy the condition

$$(1.2) \qquad \Delta\mathbf{x} = A\mathbf{x} + B,$$

where $A \in \mathrm{Mat}(3,\mathbb{R})$ is a 3×3-real matrix and $B \in \mathbb{R}^3$. In other words, each coordinate function is of 1-type in the sense of Chen ([6]). For the Lorentzian version of surfaces satisfying (1.2), Alías, Ferrández and Lucas ([1]) proved that the only such surfaces are minimal surfaces and open pieces of Lorentz circular cylinders, hyperbolic cylinders, Lorentz hyperbolic cylinders, hyperbolic spaces or pseudo-spheres.

The notion of an isometric immersion $\mathbf{x}$ is naturally extended to smooth functions on submanifolds of Euclidean space or pseudo-Euclidean space. The most natural one of them is the Gauss map of the submanifold. In particular, if the submanifold is a hypersurface, the Gauss map can be identified with the unit normal vector field to it. Dillen, Pas and Verstraelen ([10]) studied surfaces of revolution in the three dimensional Euclidean space $\mathbb{E}^3$ such that its Gauss map G satisfies the condition

$$(1.3) \qquad \Delta G = AG, \qquad A \in \mathrm{Mat}(3,\mathbb{R}).$$

Also, Baikoussis and Blair ([3]) investigated the ruled surfaces in $\mathbb{E}^3$ satisfying the condition (1.3), and Baikoussis and Verstraelen ([4, 5]) studied the helicoidal surfaces and the spiral surfaces in $\mathbb{E}^3$. Choi ([7, 8]) completely classified the surfaces of revolution and the ruled surfaces with non-null base curve satisfying the condition (1.3) in the three dimensional Minkowski space $\mathbb{E}^3_1$. Furthermore, Alías, Ferrández, Lucas and Meroño ([2]) studied the ruled surfaces with null ruling in $\mathbb{E}^3_1$, and Yoon ([17]) classified translation surfaces in $\mathbb{E}^3_1$.

The main purpose of this paper is to complete classification of surfaces of revolution in the three dimensional pseudo-Galilean space G^1_3 satisfying (1.1) and (1.3).

The study for some surfaces in Galilean or pseudo-Galilean space can find in [11, 13–15, 18].

2. Preliminaries

The pseudo-Galilean space G^1_3 is a Cayley-Klein space with absolute figure consisting of the ordered triple $\{\omega, f, I\}$, where ω is the ideal (absolute) plane in the three dimensional real projective space RP_3, f the line (the absolute line) in ω and L the fixed hyperbolic involution of points of f.

Homogenous coordinates in G^1_3 are introduced in such a way that the absolute plane ω is given by $x_0 = 0$, the absolute line f by $x_0 = x_1 = 0$ and the hyperbolic involution η by $\eta : (x_0 : x_1 : x_2 : x_3) \to (0 : 0 : x_3 : x_2)$. The last condition is equivalent to the requirement that the conic $x_2^2 - x_3^2 = 0$ is the

absolute conic. Metric relations are introduced with respect to the absolute figure. In affine coordinates defined by $(x_0 : x_1 : x_2 : x_3) = (1 : x : y : z)$, the distance between the points $P_i = (x_i, y_i, z_i)(i = 1, 2)$ is defined by (see [11])

$$d(P_1, P_2) = \begin{cases} |x_2 - x_1|, & \text{if } x_1 \neq x_2, \\ \sqrt{|(y_2 - y_1)^2 - (z_2 - z_1)^2|}, & \text{if } x_1 = x_2. \end{cases}$$

The group motions of G_3^1 is a six-parameter group given (in affine coordinates) by

$$\bar{x} = a + x,$$
$$\bar{y} = b + cx + y \cosh \varphi + z \sinh \varphi,$$
$$\bar{z} = d + ex + y \sinh \varphi + z \cosh \varphi.$$

Let $\mathbf{x} = (x_1, y_1, z_1)$ and $\mathbf{y} = (x_2, y_2, z_2)$ be vectors in G_3^1. A vector $\mathbf{x}$ is called isotropic if $x_1 = 0$, otherwise it is called non-isotropic. The pseudo-Galilean scalar product of $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$(2.1) \qquad \mathbf{x} \cdot \mathbf{y} = \begin{cases} x_1 x_2, & \text{if } x_1 \neq 0 \quad \text{or} \quad x_2 \neq 0, \\ y_1 y_2 - z_1 z_2, & \text{if } \quad x_1 = 0 \quad \text{and} \quad x_2 = 0. \end{cases}$$

From this, the pseudo-Galilean norm of a vector $\mathbf{x}$ in G_3^1 is given by $||\mathbf{x}|| = \sqrt{|\mathbf{x} \cdot \mathbf{x}|}$ and all unit non-isotropic vectors are the form $(1, y_1, z_1)$. For isotropic vector $x_1 = 0$ holds. There are four types of isotropic vectors: spacelike $(y_1^2 - z_1^2 > 0)$, timelike $(y_1^2 - z_1^2 < 0)$ and the two types of lightlike $(y_1 = \pm z_1)$ vectors. A non-lightlike isotropic vector is a unit vector if $y_1^2 - z_1^2 = \pm 1$.

The pseudo-Galilean cross product of $\mathbf{x}$ and $\mathbf{y}$ on G_3^1 is defined by

$$(2.2) \qquad \mathbf{x} \times \mathbf{y} = \begin{vmatrix} 0 & -e_2 & e_3 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix},$$

where $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

Consider a C^r-surface M, $r \geq 1$, in G_3^1 parametrized by

$$\mathbf{x}(u, v) = (x(u_1, u_2), y(u_1, u_2), z(u_1, u_2)).$$

Let us denote $g_i = \frac{\partial x}{\partial u_i}$, $h_{ij} = \frac{\partial \tilde{\mathbf{x}}}{\partial u_i} \cdot \frac{\partial \tilde{\mathbf{x}}}{\partial u_j} (i, j = 1, 2)$, where $\sim$ stands for the projection of a vector on the pseudo-Euclidean yz-plane. Then, the matrix of the first fundamental form ds^2 of a surface M in G_3^1 is given by

$$(2.3) \qquad ds^2 = \begin{pmatrix} ds_1^2 & 0 \\ 0 & ds_2^2 \end{pmatrix},$$

where $ds_1^2 = (g_1 du_1 + g_2 du_2)^2$ and $ds_2^2 = h_{11} du_1^2 + 2h_{12} du_1 du_2 + h_{22} du_2^2$ (cf. [14]). In such case, we denote the components of ds^2 by g_{ij}^*.

It is well known that in terms of local coordinates $\{u_1, u_2\}$ of M the Laplacian operator Δ of the first fundamental form on M is defined by

$$(2.4) \qquad \Delta = \frac{1}{\sqrt{|\mathfrak{g}|}} \sum_{i,j=1}^{2} \frac{\partial}{\partial u_i} (\sqrt{|\mathfrak{g}|} g^{*ij} \frac{\partial}{\partial u_j}),$$

where $\mathfrak{g} = \det(g_{ij}^*)$ and $(g^{*ij}) = (g_{ij}^*)^{-1}$.

3. SURFACES OF REVOLUTION IN G_3^1

In the pseudo-Galilean space G_3^1, there are two types of rotations: pseudo-Euclidean rotations given the normal form

$$(3.1) \qquad \begin{aligned} \bar{x} &= x, \\ \bar{y} &= y \cosh t + z \sinh t, \\ \bar{z} &= y \sinh t + z \cosh t \end{aligned}$$

and isotropic rotations with the normal form

$$(3.2) \qquad \begin{aligned} \bar{x} &= x + bt, \\ \bar{y} &= y + xt + b\frac{t^2}{2}, \\ \bar{z} &= z, \end{aligned}$$

where $t \in \mathbb{R}$ and $b = $ constant > 0.

The trajectory of a single point under a pseudo-Euclidean rotation is a pseudo-Euclidean circle(i.e., a rectangular hyperbolic)

$$x = \text{constant}, \quad y^2 - z^2 = r^2, \quad r \in \mathbb{R}.$$

The invariant r is the radius of the circle. Pseudo-Euclidean circles intersect the absolute line f in the fixed points of the hyperbolic involution (F_1, F_2). There are three kinds of pseudo-Euclidean circles: circles of real radius, of imaginary radius, and of radius zero. Circles of real radius are timelike curves (having timelike tangent vectors) and imaginary radius spacelike curves (having spacelike tangent vectors).

The trajectory of a point under isotropic rotation is an isotropic circle whose normal form is

$$z = \text{constant}, \quad y = \frac{x^2}{2b}.$$

The invariant b is the radius of the circle. The fixed line of the isotropic rotation (3.2) is the absolute line f.

First of all, we consider a nonisotropic curve α parametrized by

$$\alpha(u) = (f(u), g(u), 0) \quad \text{or} \quad \alpha(u) = (f(u), 0, g(u))$$

around the x-axis by pseudo-Euclidean rotation (3.1), where g is a positive function and f is a smooth function on an open interval I. Then the surface M of revolution can be written as

(3.3) $$\mathbf{x}(u, v) = (f(u), g(u) \cosh v, g(u) \sinh v),$$

or

(3.4) $$\mathbf{x}(u, v) = (f(u), g(u) \sinh v, g(u) \cosh v),$$

for any $v \in \mathbb{R}$.

Next, we consider the isotropic rotations. By isotropic curve $\alpha(u) = (0, f(u), g(u))$ about the z-axis by isotropic rotation (3.2), we obtain a surface

(3.5) $$\mathbf{x}(u, v) = \left(v, f(u) + \frac{v^2}{2b}, g(u) \right),$$

where f and g are smooth functions and $b \neq 0$ ([14]).

4. Surfaces of revolution satisfying $\Delta \mathbf{x} = A\mathbf{x}$

In this section, we classify surfaces of revolution in G_3^1 satisfying the equation

(4.1) $$\Delta \mathbf{x} = A\mathbf{x},$$

where $A = (a_{ij}), i, j = 1, 2, 3$.

First of all, let M be a surface of revolution in G_3^1 defined by (3.3). Assume that the rotated curve α is parametrized by arc-length. Then it is rewritten as the form:

$$\alpha(u) = (u, g(u), 0).$$

In this case, the parametrization of M is given by

(4.2) $$\mathbf{x}(u, v) = (u, g(u) \cosh v, g(u) \sinh v),$$

where g is a positive function. From (2.3) we have

$$g_1 = 1, \ g_2 = 0, \ h_{11} = h_{12} = 0, \ h_{22} = -g^2(u),$$

which imply the components of the first fundamental form ds^2 on M are given by

(4.3) $$g_{11}^* = 1, \ g_{12}^* = 0, \ g_{22}^* = -g^2(u).$$

So, M is a timelike surface. By a straightforward computation, the Laplacian operator Δ on M with the help of (4.3) and (2.4) turns out to be

(4.4) $$\Delta = -\frac{g'(u)}{g(u)} \frac{\partial}{\partial u} - \frac{\partial^2}{\partial u^2} + \frac{1}{g^2(u)} \frac{\partial^2}{\partial v^2}.$$

Suppose that M satisfies (4.1). Then from (4.2) and (4.4), we have
(4.5)

$$a_{11}u + a_{12}g(u)\cosh v + a_{13}g(u)\sinh v = -\frac{g'(u)}{g(u)},$$

$$a_{21}u + a_{22}g(u)\cosh v + a_{23}g(u)\sinh v = \left(-\frac{g'(u)^2}{g(u)} - g''(u) + \frac{1}{g(u)}\right)\cosh v,$$

$$a_{31}u + a_{32}g(u)\cosh v + a_{33}g(u)\sinh v = \left(-\frac{g'(u)^2}{g(u)} - g''(u) + \frac{1}{g(u)}\right)\sinh v.$$

Since the functions $\sinh v, \cosh v$ and the constant function are linearly independent, by (4.5) we get $a_{12} = a_{13} = a_{21} = a_{23} = a_{31} = a_{32} = 0$ and $a_{22} = a_{33} = \mu$. Consequently the matrix A satisfies

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{pmatrix},$$

where $a_{11} = \lambda$ and (4.5) is rewritten as the following:

(4.6) $\qquad \lambda u = -\frac{g'(u)}{g(u)}, \qquad \mu g(u) = -\frac{g'(u)^2}{g(u)} - g''(u) + \frac{1}{g(u)}.$

Combining the first and the second equation of (4.6), we obtain

(4.7) $\qquad\qquad\qquad g^2(2\lambda^2 u^2 - \lambda + \mu) = 1.$

THEOREM 4.1. *There is no harmonic surface of revolution given by (3.3) in the three dimensional pseudo-Galilean space G_3^1.*

PROOF. Suppose that M satisfies $\Delta \mathbf{x} = 0$. Then from (4.6) we obtain $g'(u) = 0$ and $\frac{1}{g(u)} = 0$, it is impossible. $\qquad\qquad\qquad\qquad\qquad\square$

THEOREM 4.2. *Let M be a non-harmonic surface of revolution given by (3.3) in the three dimensional pseudo-Galilean space G_3^1. Then M satisfies (4.1) if and only if it is an open part of a Lorentz hyperbolic cylinder.*

PROOF. If $2\lambda^2 u^2 - \lambda + \mu = 0$ in (4.7), then we get $\lambda = 0$ and $\mu = 0$, that is, $A = 0$. It is contradiction because M is a non-harmonic surface. So we assume $2\lambda^2 u^2 - \lambda + \mu \neq 0$. Then from (4.7) we get

(4.8) $\qquad\qquad\qquad g(u) = \pm\frac{1}{\sqrt{|2\lambda^2 u^2 - \lambda + \mu|}}.$

If we substitute (4.8) in the first equation of (4.6), one find $\lambda = 0$ and $g(u) = \pm\frac{1}{\sqrt{|\mu|}}$, $\mu \neq 0$. It follows that M is a Lorentz hyperbolic cylinder $y^2 - z^2 = \frac{1}{\mu}$ (see Figure 1). It can be easily shown that the converse assertion is also true. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$
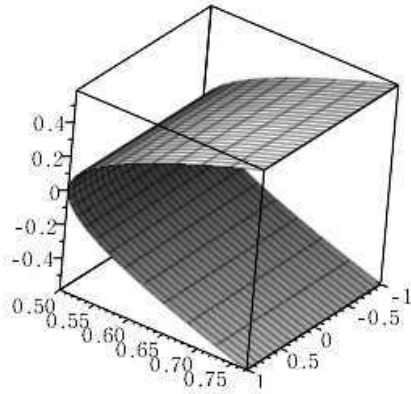
FIGURE 1. A Lorentz hyperbolic cylinder

Let M be a surface of revolution given by (3.4) in the three dimensional pseudo-Galilean space G_3^1. Assume that a nonisotropic curve α is a unit speed curve. Then it is given by $\alpha(u) = (u, 0, g(u))$. In this case, the spacelike surface M of revolution is parametrized by

(4.9) $$\mathbf{x}(u,v) = (u, g(u)\sinh v, g(u)\cosh v).$$

Suppose that the surface of revolution M given by (4.9) satisfies (4.1). Then, by using similar method of Theorem 4.2, we have the following result:

THEOREM 4.3. *Let M be a non-harmonic surface of revolution given by (4.9) in the three dimensional pseudo-Galilean space G_3^1. Then M satisfies (4.1) if and only if it is an open part of a hyperbolic cylinder $z^2 - y^2 = r^2$.*

Last of all, let M be a surface of revolution in G_3^1 generated by the rotated curve $\alpha(u) = (0, f(u), g(u))$. Assume that the curve α is parametrized by arclength, that is,

$$f'(u)^2 - g'(u)^2 = -\epsilon(=\pm 1).$$

Then the parametrization of M is given by

(4.10) $$\mathbf{x}(u,v) = \left(v, f(u) + \frac{v^2}{2b}, g(u)\right),$$

where f and g are smooth functions and $b \neq 0$. From (2.3) we have

$$g_1 = 0, \ g_2 = 1, \ h_{11} = -\epsilon, \ h_{12} = h_{22} = 0,$$

which imply the components of the first fundamental form ds^2 on M are given by

(4.11) $$g_{11}^* = 1, \ g_{12}^* = 0, \ g_{22}^* = -\epsilon.$$

By (4.11) and (2.4) the Laplacian operator Δ of M can be expressed as

$$(4.12) \qquad \Delta = \epsilon \frac{\partial^2}{\partial u^2} + \epsilon \frac{\partial^2}{\partial v^2}.$$

Suppose that M satisfies (4.1). Then from (4.10) and (4.12), we have

$$a_{11}v + a_{12}\left(f(u) + \frac{v^2}{2b}\right) + a_{13}g(u) = 0,$$

$$(4.13) \qquad a_{21}v + a_{22}\left(f(u) + \frac{v^2}{2b}\right) + a_{23}g(u) = \epsilon f''(u) + \frac{\epsilon}{b},$$

$$a_{31}v + a_{32}\left(f(u) + \frac{v^2}{2b}\right) + a_{33}g(u) = \epsilon g''(u).$$

It imply we can easily find $a_{11} = a_{12} = a_{13} = a_{21} = a_{22} = a_{31} = a_{32} = 0$. In this case, (4.13) is rewritten as the following:

$$(4.14) \qquad \lambda g(u) = \epsilon f''(u) + \frac{\epsilon}{b}, \qquad \mu g(u) = \epsilon g''(u),$$

where $\lambda = a_{23}$ and $\mu = a_{33}$. The general solution of the second equation in (4.14) is given by

$$(4.15) \qquad g(u) = a_1 u + a_2$$

if $-\epsilon\mu = 0$, where a_1, a_2 are constants,

$$(4.16) \qquad g(u) = b_1 \cos(ku + b_2)$$

if $-\epsilon\mu = k^2$, where b_1, b_2 and k are constants, or

$$(4.17) \qquad g(u) = c_1 \cosh(ku + c_2)$$

if $\epsilon\mu = k^2$, where c_1, c_2 are constants. Thus from (4.15), (4.16), (4.17) and the first equation in (4.14) the function $f(u)$ is given by

$$(4.18) \qquad f(u) = \frac{1}{6}\epsilon\lambda a_1 u^3 + \frac{1}{2}\left(\epsilon\lambda a_2 - \frac{1}{b}\right)u^2 + a_3 u + a_4,$$

$$(4.19) \qquad f(u) = -\frac{\epsilon b_1 \lambda}{k^2}\cos(ku + b_2) - \frac{1}{2b}u^2 + b_3 u + b_4,$$

or

$$(4.20) \qquad f(u) = \frac{\epsilon c_1 \lambda}{k^2}\cosh(ku + c_2) - \frac{1}{2b}u^2 + c_3 u + c_4,$$

where $a_i, b_i, c_i (i = 1, 2, 3, 4)$ and k are constants.

Consequently, the following theorem holds.

THEOREM 4.4. *A surface of revolution generated by the isotropic curve in the three dimensional pseudo-Galilean space G_3^1 satisfies the equation $\Delta\mathbf{x} = A\mathbf{x}, A \in Mat(3,\mathbb{R})$ if and only if the surface can be parametrized as*

$$\mathbf{x}(u, v) = \left(v, f(u) + \frac{v^2}{2b}, g(u)\right),$$

where

(i) *either* $f(u) = \frac{1}{6}\epsilon\lambda a_1 u^3 + \frac{1}{2}(\epsilon\lambda a_2 - \frac{1}{b})u^2 + a_3 u + a_4$ *and* $g(u) = a_1 u + a_2$,

(ii) $f(u) = -\frac{\epsilon b_1 \lambda}{k^2}\cos(ku + b_2) - \frac{1}{2b}u^2 + b_3 u + b_4$ *and* $g(u) = b_1\cos(ku + b_2)$,

(iii) *or* $f(u) = \frac{\epsilon c_1 \lambda}{k^2}\cosh(ku + c_2) - \frac{1}{2b}u^2 + c_3 u + c_4$ *and* $g(u) = c_1\cosh(ku + c_2)$

with $a_i, b_i, c_i, k \in \mathbb{R}$ $(i = 1, 2, 3, 4)$.

REMARK 4.5. For specific functions $f(u)$ and $g(u)$ appropriate intervals of u and v in (i) and (ii), we have the graphs shown in Figure 2 and Figure 3.
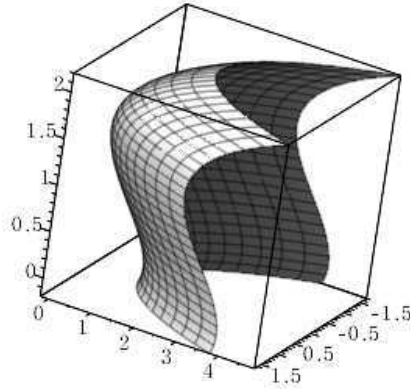

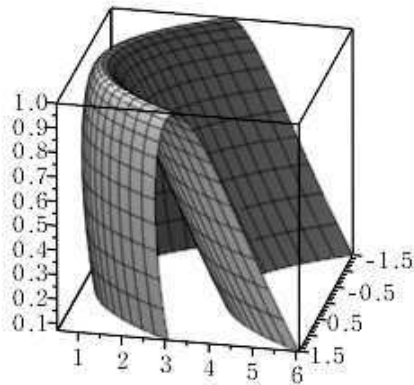FIGURE 2. A timelike surface with $f(u) = u^3 + \frac{3}{2}u^2 - u$ and $g(u) = u + 1$

FIGURE 3. A spacelike surface with $f(u) = \cos(u) + u^2 + u$ and $g(u) = \cos(u)$

5. SURFACES OF REVOLUTION SATISFYING $\Delta G = AG$

In this section, we classify surfaces of revolution in G_3^1 satisfying the following equation

$$(5.1) \qquad \Delta G = AG,$$

where $A = (a_{ij}), i, j = 1, 2, 3$.

THEOREM 5.1. *Let M be a surface of revolution given by (4.2) in the three dimensional pseudo-Galilean space G_3^1. Then M satisfies (5.1) if and only if it is an open part of a Lorentz hyperbolic cylinder.*

PROOF. Let M be a surface of revolution generated by a unit speed non-isotropic curve in G_3^1. Then M is parametrized by

$$(5.2) \qquad \mathbf{x}(u, v) = (u, g(u) \cosh v, g(u) \sinh v),$$

where g is a positive function. From the natural frame $\{\mathbf{x}_u, \mathbf{x}_v\}$ the Gauss map G of M is obtained by

$$(5.3) \qquad G = \left(\frac{1}{||\mathbf{x}_u \times \mathbf{x}_v||} \right) \mathbf{x}_u \times \mathbf{x}_v = (0, \cosh v, \sinh v).$$

Suppose that M satisfies (5.1). Then from (4.4) and (5.3) we get the following system of differential equations:

$$a_{12} \cosh v + a_{13} \sinh v = 0,$$

$$(5.4) \qquad a_{22} \cosh v + a_{23} \sinh v = \frac{1}{g^2(u)} \cosh v,$$

$$a_{32} \cosh v + a_{33} \sinh v = \frac{1}{g^2(u)} \sinh v.$$

In order to prove the theorem we have to solve the above system of ordinary differential equations. From (5.4) we easily deduce that

$$a_{12} = a_{13} = a_{23} = a_{32} = 0, a_{22} = a_{33}$$

and

$$(5.5) \qquad \frac{1}{g^2(u)} = a_{22}.$$

From this $g(u)$ is a constant function. Consequently, M is an open part of a Lorentz hyperbolic cylinder. It can be easily shown that the converse assertion is also true. $\square$

THEOREM 5.2. *Let M be a surface of revolution given by (4.9) in the three dimensional pseudo-Galilean space G_3^1. Then M satisfies (5.1) if and only if it is an open part of a hyperbolic cylinder.*

Let M be a surface of revolution generated by a unit speed isotropic curve in G_3^1. Then M is parametrized by

$$(5.6) \qquad \mathbf{x}(u,v) = \left(v, f(u) + \frac{v^2}{2b}, g(u) \right).$$

From this the Gauss map G of M is given by

$$(5.7) \qquad G = (0, -g'(u), -f'(u)).$$

Suppose that M satisfies (5.1). Then from (4.12) and (5.7) we get the following system of differential equations:

$$(5.8) \qquad a_{12}g'(u) + a_{13}f'(u) = 0,$$

$$(5.9) \qquad a_{22}g'(u) + a_{23}f'(u) = \epsilon g'''(u),$$

$$(5.10) \qquad a_{32}g'(u) + a_{33}f'(u) = \epsilon f'''(u).$$

If $f'(u) = 0$, then $g(u) = \pm u + c_1$ and $\epsilon = 1$. Also, we find $a_{12} = a_{22} = a_{32} = 0$ and the matrix A satisfies

$$A = \begin{pmatrix} a_{11} & 0 & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{pmatrix}.$$

Thus, M is parametrized by

$$\mathbf{x}(u,v) = \left(v, c_2 + \frac{v^2}{2b}, \pm u + c_1 \right),$$

where c_1, c_2 are constants, and M is timelike parabolic cylinder.

If $g'(u) = 0$, then $f(u) = \pm u + d_1$ and $\epsilon = -1$. From (5.8), (5.9) and (5.10) we get $a_{13} = a_{23} = a_{33} = 0$ and the matrix A satisfies

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix}.$$

Thus, M is parametrized by

$$\mathbf{x}(u,v) = \left(v, \pm u + \frac{v^2}{2b} + d_1, d_2 \right),$$

where d_1, d_2 are constants, and M is spacelike plane.

We suppose that $f'(u) \neq 0$ and $g'(u) \neq 0$. If $a_{12} \neq 0$, then from (5.8) we get

$$(5.11) \qquad g'(u) = -\frac{a_{13}}{a_{12}} f'(u).$$

Substituting $g'''(u)$ in (5.9) and comparing it and (5.10), we have

$$a_{22} = a_{32}, a_{23} = a_{33}, a_{13} = -a_{12}.$$

It follows that we can obtain the following ODEs

(5.12) $f'(u) = g'(u),$

(5.13) $f'''(u) = \epsilon(a_{32} + a_{33})f'(u)$

and the relation

$$a_{32} + a_{33} = a_{22} + a_{23}.$$

Thus, we have the general solution of (5.13) as follows

(5.14) $f(u) = \dfrac{a_1}{k}\sin(ku + a_2) + a_3$

if $\epsilon(a_{32} + a_{33}) = -k^2$,

(5.15) $f(u) = \dfrac{b_1}{k}\sinh(ku + b_2) + b_3$

if $\epsilon(a_{32} + a_{33}) = k^2$ or

(5.16) $f(u) = \dfrac{1}{2}c_1 u^2 + c_2 u + c_3$

if $\epsilon(a_{32} + a_{33}) = 0$, where $a_i, b_i, c_i (i = 1, 2, 3)$ are constants.

THEOREM 5.3. *A surface of revolution generated by the isotropic curve in the three dimensional pseudo-Galilean space G_3^1 satisfies the equation $\Delta G = AG, A \in Mat(3, \mathbb{R})$ if and only if the surface can be parametrized as*

$$x(u, v) = \left(v, f(u) + \frac{v^2}{2b}, g(u) \right),$$

where

(i) *either $f(u) = d_1$ and $g(u) = \pm u + d_2$,*
(ii) *$f(u) = \pm u + d_3$ and $g(u) = d_4$,*
(iii) *$f(u) = \frac{1}{2}c_1 u^2 + c_2 u + c_3$ and $g(u) = \frac{1}{2}c_1 u^2 + c_2 u + c_3$,*
(iv *$f(u) = \frac{a_1}{k}\sin(ku + a_2) + a_3$ and $g(u) = \frac{a_1}{k}\sin(ku + a_2) + a_3$*
(v) *or $f(u) = \frac{b_1}{k}\sinh(ku + b_2) + b_3$ and $g(u) = \frac{b_1}{k}\sinh(ku + b_2) + b_3$.*

REMARK 5.4. For specific functions $f(u)$ and $g(u)$ appropriate intervals of u and v in (iii), we have the graph shown in Figure 4.

THEOREM 5.5. (CLASSIFICATION). *Let M be a surface of revolution given by (4.2) in the three dimensional pseudo-Galilean space G_3^1. Then, for some matrix $A \in Mat(3, \mathbb{R})$ the following are equivalent:*

(1) $\Delta \mathbf{x} = A\mathbf{x}$.
(2) $\Delta G = AG$.
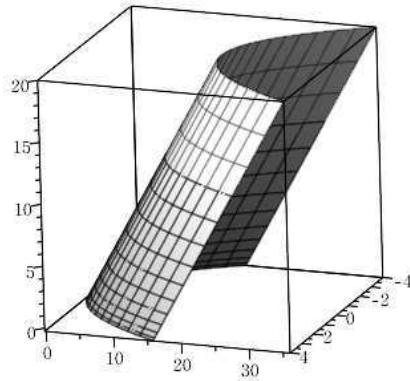(3) *M is an open part of a Lorentz hyperbolic cylinder.*

Figure 4. A surface of revolution with $f(u) = u^2 + u + 1$ and $g(u) = u^2 + u + 1$.

Theorem 5.6. (Classification). *Let M be a surface of revolution given by (4.9) in the three dimensional pseudo-Galilean space G_3^1. Then, for some matrix $A \in Mat(3, \mathbb{R})$ the following are equivalent:*

(1) $\Delta \mathbf{x} = A\mathbf{x}$.

(2) $\Delta G = AG$.

(3) *M is an open part of a hyperbolic cylinder.*

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