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STRONG SIZE PROPERTIES

Sergio Macías and César Piceno
Universidad Nacional Autónoma de México, Mexico

Abstract. We prove that countable aposyndesis, finite-aposyndesis, continuum chainability, acyclicity (for $n \geq 3$), and acyclicity for locally connected continua are strong size properties. As a consequence of our results we obtain that arcwise connectedness is a strong size property which is originally proved by Hosokawa.

1. Introduction

Hosokawa defines strong size maps on the n-fold hyperspace of a continuum in [5] as a generalization of Whitney maps for the hyperspace of subcontinua of a continuum and proves the existence of such maps ([5, Theorem 2.2]). He also proves that local connectedness ([5, Theorem 3.1]), arcwise connectedness ([5, Theorem 3.3]) and aposyndesis ([5, Theorem 3.4]) are strong size properties. It is natural to ask what other topological properties are strong size properties. We prove that countable aposyndesis (Theorem 4.1), finite-aposyndesis (Corollary 4.2), continuum chainability (Corollary 4.4), acyclicity for $n \geq 3$ (Corollary 4.17), and acyclicity for locally connected continua (Corollary 4.18) are strong size properties. We note that Corollary 4.2 answers one of the questions asked by Hosokawa ([5, Question, p. 964]). As a consequence of Theorem 4.3 we obtain that arcwise connectedness is a strong size property (Corollary 4.5) which is originally proved by Hosokawa. We end the paper with a theorem about extending strong size map defined on a closed subset of the n-fold hyperspace of a continuum to the complete n-fold hyperspace (Theorem 5.4).

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Strong size properties are the natural generalization of Whitney properties which have been study extensively in [15, Chapter XIV] and [8, Chapter VIII].

2. Definitions and notation

Given a subset A of a metric space Z with metric d, the closure, the boundary and the interior of A are denoted by $Cl_Z(A)$, $Bd_Z(A)$ and $Int_Z(A)$, respectively. Also, $\mathcal{V}_r(A)$ denotes the open ball of radius r about A.

Let Z be a metric space. By a *deformation* we mean a map $H\colon Z\times[0,1]\to Z$ such that $H((z,0)) = z$ for each $z \in Z$. Let $A = \{H((z,1)) \mid z \in Z\}$. If the map $r\colon Z \to A$ given by $r(z) = H((z,1))$ is a retraction from Z onto A, then H is a *deformation retraction from Z onto A*. If H is a deformation retraction from Z onto A and for each $a \in A$ and each $t \in [0,1]$, $H((a,t)) = a$, then H is a *strong deformation retraction from Z onto A*. The set A is called a *deformation retract of Z* (*strong deformation retract of Z*, respectively). A metric space Z is an *absolute retract* provided that for each embedding $e\colon Z \to X$ of Z into a metric space X such that $e(Z)$ is closed in X, $e(Z)$ is a retract of X.

A *map* is a continuous function. The symbol $\twoheadrightarrow$ denotes a surjective map.

A *continuum* is a nonempty compact connected metric space. A continuum X is *aposyndetic* provided that for each pair of points x_1 and x_2 of X, there exists a subcontinuum W of X such that $x_1 \in Int_X(W) \subset W \subset X \setminus \{x_2\}$. The continuum X is *finitely aposyndetic* (*countable aposyndetic*) if for each finite (countable closed) subset F of X and each point $x \in X \setminus F$, there exists a subcontinuum W of X such that $x \in Int_X(W) \subset W \subset X \setminus F$.

Let p and q be two points of the continuum X. A finite collection $\{L_1, \ldots, L_m\}$ of sets is called a *chain from p to q* provided that $p \in L_1$, $q \in L_m$ and $L_j \cap L_k \neq \emptyset$ if and only if $|j-k| \leq 1$. A chain is called a *continuum chain* if each of its elements is a continuum. A continuum chain is an *ε-continuum chain* if the diameter of each of its elements is less than ε. The continuum X is said to be *continuum chainable* provided that for each pair of points p and q of X and each $\varepsilon > 0$, there exists an ε-continuum chain from p to q in X.

Remark 2.1. Observe that any arcwise connected continuum is a continuum chainable continuum.

If X is a continuum and n is a positive integer, then $\check{H}^n(X)$ denotes the reduced nth Čech cohomology group of X with integer coefficients. A continuum X is said to be *acyclic* if $\check{H}^1(X)$ is trivial.

Given a continuum X, we consider the following *hyperspaces* of X:

$$2^X = \{A \subset X \mid A \text{ is nonempty and closed}\}$$

and

$$\mathcal{C}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ components}\},$$

where n is a positive integer. $\mathcal{C}_n(X)$ is called the *n-fold hyperspace* of X. These spaces are topologized with the Hausdorff metric defined as follows:

$$\mathcal{H}(A,B) = \inf\{\varepsilon > 0 \mid A \subset \mathcal{V}_\varepsilon(B) \text{ and } B \subset \mathcal{V}_\varepsilon(A)\},$$

$\mathcal{H}$ always denotes the Hausdorff metric on 2^X. When $n = 1$, we write $\mathcal{C}(X)$ instead of $\mathcal{C}_1(X)$. Given an element B of $\mathcal{C}_n(X)$, the *mesh of* B, denoted by mesh(B), is

$$\text{mesh}(B) = \max\{\text{diam}(K) \mid K \text{ is a component of } B\}.$$

The symbol $\mathcal{F}_n(X)$ denotes the *n-fold symmetric product* of X; that is:

$$\mathcal{F}_n(X) = \{A \in \mathcal{C}_n(X) \mid A \text{ has at most } n \text{ points}\}.$$

Note that, by definition, $\mathcal{F}_n(X) \subset \mathcal{C}_n(X)$. It is known that if X is a continuum, then 2^X and $\mathcal{C}_n(X)$ are arcwise connected continua (for 2^X and $\mathcal{C}(X)$ see [15, (1.13)]; for $\mathcal{C}_n(X)$ and $n \geq 2$, see [11, 1.8.12]). Also, $\mathcal{F}_n(X)$ is a continuum for all positive integers n ([1, p. 877]).

Let X be a continuum and let n be a positive integer. An *order arc* in $\mathcal{C}_n(X)$ is an arc $\alpha\colon [0,1] \to \mathcal{C}_n(X)$ such that if $0 \leq s < t \leq 1$, then $\alpha(s) \subset \alpha(t)$ and $\alpha(s) \neq \alpha(t)$.

Let B and A be two elements of $\mathcal{C}_n(X)$. We say that the pair (B, A) satisfies *property* (OA) provided that $B \subset A$ and each component of A intersects B. Let us note that this condition guaratees the existence of an order arc, in $\mathcal{C}_n(X)$, from B to A when $B \subset A$ and $B \neq A$ [15, (1.8)].

Let X be a continuum and let n be a positive integer. If $B \in \mathcal{C}_n(X)$, define:

$$\mathcal{C}_n(B,X) = \{A \in \mathcal{C}_n(X) \mid B \subset A\};$$

$$\mathcal{OA}_n(B,X) = \{A \in \mathcal{C}_n(X) \mid (B,A) \text{ satisfies property } (OA)\}.$$

If $A \in \mathcal{OA}_n(B,X)$, then

$$\mathcal{OA}_n(B,A) = \{D \in \mathcal{OA}_n(B,X) \mid D \subset A\}.$$

A map $\mu\colon \mathcal{C}_n(X) \to [0,\infty)$ is said to be a *strong size map* provided that:

(1) $\mu(A) = 0$ for every $A \in \mathcal{F}_n(X)$;

(2) if $A \subset B$, $A \neq B$ and $B \notin \mathcal{F}_n(X)$, then $\mu(A) < \mu(B)$.

Since X is nondegenerate, we may assume that $\mu(X) = 1$. By [5, Theorem 2.2], strong size maps exist for each continuum X and each positive integer n. Note that for $n = 1$ a strong size map is just a Whitney map.

Each set of the form $\mu^{-1}(t)$ for any strong size map μ for $\mathcal{C}_n(X)$ and any $t \in [0,1]$ is called a *strong size level of* $\mathcal{C}_n(X)$.

A topological property P is called a *strong size property* if whenever X has property P, so does every strong size level of $\mathcal{C}_n(X)$ for each positive integer n.

3. Preliminary Results

Lemma 3.1. *Let X be a continuum, let n be a positive integer and let $\mu\colon \mathcal{C}_n(X)\twoheadrightarrow[0,1]$ be a strong size map. If $t>0$ and $\mathcal{S}=\mu^{-1}(t)$ is a strong size level for $\mathcal{C}_n(X)$, $\mathcal{A}$ is a closed subset of $\mathcal{S}$ and $B\in\mathcal{S}\setminus\mathcal{A}$, then there exists a $\varepsilon>0$ such that $A\not\subset\mathcal{V}_\varepsilon(B)$ for any $A\in\mathcal{A}$.*

Proof. Suppose the result is not true. Then for each positive integer m, there exists $A_m\in\mathcal{A}$ such that $A_m\subset\mathcal{V}_{\frac{1}{m}}(B)$. Since $\mathcal{A}$ is closed, without loss of generality, we assume that the sequence $\{A_m\}_{m=1}^\infty$ converges to an element A of $\mathcal{A}$. Note that $A\subset B$. Since $\mathcal{S}$ a strong size level, we have that $A=B$. Hence, $B\in\mathcal{A}$, a contradiction to the election of B. Therefore, the lemma is true. □

The proof of the following lemma is similar to the one done for a similar result for Whitney levels ([15, (14.8.1)]); we include the details for completeness.

Lemma 3.2. *Let X be a continuum, let n be a positive integer, let $\mu\colon \mathcal{C}_n(X)\twoheadrightarrow[0,1]$ be a strong size map, and let $\mathcal{S}=\mu^{-1}(t)$ be a strong size level for $\mathcal{C}_n(X)$. Let $A,B\in\mathcal{S}$, let $A_1,\dots,A_\ell$ be the components of A and let $B_1,\dots,B_m$ be the components of B. If $P\in\mathcal{F}_n(X)$ is such that $P\subset A\cap B$, $P\cap A_j\neq\emptyset$ for each $j\in\{1,\dots,\ell\}$ and $P\cap B_k\neq\emptyset$ for all $k\in\{1,\dots,m\}$, then there exists an arc in $\mathcal{S}$ joining A and B.*

Proof. Let $\alpha,\beta\colon[0,1]\to\mathcal{C}_n(X)$ be two order arcs such that $\alpha(0)=P$, $\alpha(1)=A$, $\beta(0)=P$ and $\beta(1)=B$ ([2, Proposition 2.6]). Given $s\in[0,1]$, define $f_s\colon[0,1]\to\mathcal{C}_n(X)$ by $f_s(r)=\alpha(s)\cup\beta(r)$. Then f_s is well defined and continuous. Since $\mu(f_s(0))=\mu(\alpha(s)\cup\beta(0))=\mu(\alpha(s))\leq t$ and $\mu(f_s(1))=\mu(\alpha(s)\cup\beta(1))=\mu(\alpha(s)\cup B)\geq t$, there exists $r_s\in[0,1]$ such that $\mu(f_s(r_s))=t$.

Let $\gamma\colon[0,1]\to\mathcal{S}$ be given by $\gamma(s)=\alpha(s)\cup\beta(r_s)$. We show γ is well defined. To this end, let $s\in[0,1]$ and suppose there exists $r\in[0,1]$ such that $\alpha(s)\cup\beta(r)\in\mathcal{S}$. Since β is an order arc, we have that either $\beta(r)\subset\beta(r_s)$ or $\beta(r_s)\subset\beta(r)$. Without loss of generality we assume that $\beta(r)\subset\beta(r_s)$. Then $\alpha(s)\cup\beta(r)\subset\alpha(s)\cup\beta(r_s)$. Since μ is a strong size map, we obtain that $\alpha(s)\cup\beta(r)=\alpha(s)\cup\beta(r_s)$. Thus, γ is well defined.

To see that γ is continuous, let $\{s_m\}_{m=1}^\infty$ be sequence of elements of $[0,1]$ converging to an element s of $[0,1]$. Then the corresponding sequence $\{r_{s_m}\}_{m=1}^\infty$ has a convergent subsequence $\{r_{s_{m_k}}\}_{k=1}^\infty$. Let r be the limit of the sequence $\{r_{s_{m_k}}\}_{k=1}^\infty$. Since α and β are continuous, we have that $\lim_{k\to\infty}\gamma(s_{m_k})=\lim_{k\to\infty}(\alpha(s_{m_k})\cup\beta(r_{s_{m_k}}))=\alpha(s)\cup\beta(r)$. By definition of γ, $\gamma(s)=\alpha(s)\cup\beta(r_s)$. Since both $\alpha(s)\cup\beta(r)$ and $\alpha(s)\cup\beta(r_s)$ belong to $\mathcal{S}$ and either $\alpha(s)\cup\beta(r)\subset\alpha(s)\cup\beta(r_s)$ or $\alpha(s)\cup\beta(r_s)\subset\alpha(s)\cup\beta(r)$, we have that $\alpha(s)\cup\beta(r)=\alpha(s)\cup\beta(r_s)$. Therefore, γ is continuous. □

It seems that the following lemma is well known but we cannot find a reference for it.

Lemma 3.3. *If X is a continuum, then $\check{H}^0(X)$ is trivial.*

Proof. The result follows form three the facts: (1) each continuum is an inverse limit of connected polyhedra ([13, Theorem 2]), (2) the 0th reduced cohomology group of a connected polyhedron is trivial ([14, 42.2]), and (3) the continuity theorem for Čech cohomology ([18, Theorem 7–7]). ∎

4. Strong Size Properties

A topological property P is called a *strong size property* if whenever X has property P, so does every strong size level of $\mathcal{C}_n(X)$ for each positive integer n.

H. Hosokawa proves in [5, Theorem 3.4] that aposyndesis is a strong size property, we extend this result to countable aposyndesis.

Theorem 4.1. *Countable aposyndesis is a strong size property.*

Proof. Let X be a countable aposyndetic continuum and let $\mu\colon \mathcal{C}_n(X) \twoheadrightarrow [0,1]$ be a strong size map, let $t \in [0,1]$ and let $\mathcal{S} = \mu^{-1}(t)$ be a strong size level for $\mathcal{C}_n(X)$. It is known that if $n \geq 2$, then $\mathcal{F}_n(X)$ is countable aposyndetic [10, Theorem 8]. Hence, the case $t = 0$ follows. Suppose $t > 0$ and let $\mathcal{A}$ be a countable closed subset of $\mathcal{S}$. Let $B \in \mathcal{S} \setminus \mathcal{A}$. By Lemma 3.1, there exists $\varepsilon > 0$ such that $A \not\subset \mathcal{V}_\varepsilon(B)$ for any $A \in \mathcal{A}$. Let $U = X \setminus Cl_X(\mathcal{V}_{\frac{\varepsilon}{2}}(B))$. By [7, Theorem 2.1], there exists a map $s\colon \mathcal{A} \to X$ such that $s(A) \in A \cap U$ for each $A \in \mathcal{A}$. Hence, $s(\mathcal{A})$ is a countable closed subset of X such that $s(\mathcal{A}) \cap B = \emptyset$. Then for each $b \in B$, there exists a subcontinuum K_b of X such that $b \in Int_X(K_b) \subset K_b \subset X \setminus s(\mathcal{A})$. Thus, $\{Int_X(K_b) \mid b \in B\}$ is an open cover of B. Since B is compact, there exist $b_1, \ldots, b_\ell \in B$ such that $B \subset \cup_{j=1}^{\ell} Int_X(K_{b_j}) \subset \cup_{j=1}^{\ell} K_{b_j}$. Without loss of generality, we assume that the family $\{K_{b_1}, \ldots, K_{b_\ell}\}$ consists of pairwise disjoint continua. Hence, by [5, Theorem 2.14], $\langle K_{b_1}, \ldots, K_{b_\ell}\rangle \cap \mathcal{S}$ is a subcontinuum of $\mathcal{S}$. Note that $B \in Int_{\mathcal{C}_n(X)}(\langle K_{b_1}, \ldots, K_{b_\ell}\rangle) \cap \mathcal{S}$. Since for each $j \in \{1, \ldots, \ell\}$, $s(\mathcal{A}) \cap K_{b_j} = \emptyset$, we obtain that $A \not\subset \cup_{j=1}^{\ell} K_{b_j}$ for any $A \in \mathcal{A}$. Hence, $(\langle K_{b_1}, \ldots, K_{b_\ell}\rangle \cap \mathcal{S}) \cap \mathcal{A} = \emptyset$. Therefore, $\mathcal{S}$ is countable aposyndetic. ∎

The following corollary answers one of the questions of Hosokawa [5, Question, p. 964].

Corollary 4.2. *Finite aposyndesis is a strong size property.*

Theorem 4.3. *Let X be a continuum chainable continuum, let n be a positive integer and let $\mu\colon \mathcal{C}_n(X) \twoheadrightarrow [0,1]$ be a strong size map. If $t \in (0,1)$ and $\mathcal{S} = \mu^{-1}(t)$, then $\mathcal{S}$ is arcwise connected.*

PROOF. Let A and B be two elements of $\mathcal{S}$. Since $\mathcal{C}_n(X)$ is compact and μ is continuous, there exists $\varepsilon > 0$ such that if $D \in \mathcal{C}_n(X)$ and $\text{mesh}(D) < \varepsilon$, then $\mu(D) < t$. Since $t > 0$ and $A, B \in \mathcal{S}$, at least one of the components of A and B is nondegenerate. Hence, A and B have uncountably many points. Let $\{a_1, \ldots, a_n\}$ be a subset of A such that it intersects each component of A. Similarly, let $\{b_1, \ldots, b_n\}$ be a subset of B such that it intersects each component of B.

Since X is a continuum chainable continuum, for each $i \in \{1, \ldots, n\}$, there exist subcontinua $D_1^i, \ldots, D_{k_i}^i$ of X such that $a_i \in D_1^i$, $b_i \in D_{k_i}^i$, $D_j^i \cap D_\ell^i \neq \emptyset$ if and only if $|j - \ell| \leq 1$, and $\text{diam}(D_j^i) < \frac{\varepsilon}{n}$ for $j \in \{1, \ldots, k_i\}$. Let $k = \max\{k_1, \ldots, k_n\}$. For each $j \in \{1, \ldots, k\}$, let $D_j = \cup_{i=1}^n D_j^i$, where $D_j^i = D_{k_i}^i$ if $j \geq k_i$. Note that for every $j \in \{1, \ldots, k\}$, $D_j \in \mathcal{C}_n(X)$ and $\text{mesh}(D_j) < \varepsilon$. Hence, $\mu(D_j) < t$ for all $j \in \{1, \ldots, k\}$. For each $j \in \{1, \ldots, k-1\}$, let $p_j^i \in D_j^i \cap D_{j+1}^i$ and let $P_j = \{p_j^1, \ldots, p_j^n\}$.

For each $j \in \{1, \ldots, k\}$, let α_j be an order arc from D_j to X and let $D'_j \in \mathcal{S}$ be such that $\{D'_j\} = \alpha_j \cap \mathcal{S}$. Note that A, D'_1 and $\{a_1, \ldots, a_n\}$ satisfy the hypothesis of Lemma 3.2. Then there exists an arc β_1 in $\mathcal{S}$ from A to D'_1. Also note that if $j \in \{1, \ldots, k-1\}$, then D'_j, D'_{j+1} and P_j satisfy the hypothesis of Lemma 3.2. Thus, there exists an arc β_{j+1} in $\mathcal{S}$ from D'_j to D'_{j+1}. Similarly, by Lemma 3.2, there exists an arc β_{k+1} in $\mathcal{S}$ from D'_k to B. Hence, $\cup_{j=1}^{k+1} \beta_j$ contains an arc from A to B. Therefore, $\mathcal{S}$ is arcwise connected. □

We have the following:

COROLLARY 4.4. *Being a continuum chainable continuum is a strong size property.*

PROOF. Let X be a continuum chainable continuum and let $\mu\colon \mathcal{C}_n(X) \twoheadrightarrow [0,1]$ be a strong size map, let $t \in [0,1]$ and let $\mathcal{S} = \mu^{-1}(t)$ be a strong size level for $\mathcal{C}_n(X)$. It is known that X is continuum chainable if and only if $\mathcal{F}_n(X)$ is continuum chainable for each positive integer n ([2, Theorem 2.9]). Hence, the case $t = 0$ follows. For $t > 0$, the result follows from Remark 2.1 and Theorem 4.3. □

As a consequence of Remark 2.1, Theorem 4.3 and [2, Proposition 2.7], we obtain the following result of Hosokawa ([5, Theorem 3.3]):

COROLLARY 4.5. *Being an arcwise connected continuum is a strong size property.*

Our next goal is to prove that for an integer $n \geq 3$, the strong size levels of $\mathcal{C}_n(X)$ are acyclic (Corollary 4.17). To this end, we follow [17]. We include all the details for the convenience of the reader.

Let us mention that it is known that acyclicity is not a Whitney property ([16, Example 2]) and it is for 1-dimensional continua ([17, Corollary 7]). Since we do not ask any additional properties to the continuum X, Theorem 4.16 says that, for $n \geq 3$, the levels of strong size maps are much nicer than the ones of Whitney maps. In particular, Corollary 4.17, tells us the acyclicity is a strong size property.

A nonempty collection Σ of closed subsets of a continuum X is called a *structure* if Σ is closed with respect to finite unions, finite intersections, and intersections of towers ordered by inclusion. If Σ is a structure on X, then an element P of Σ is called an *indecomposable set* provided that whenever $P = A \cup B$, for some elements A and B of Σ, we have that $P = A$ or $P = B$.

Given a continuum X, we consider two structures in $\mathcal{C}_n(X)$. If $\mathcal{B}$ is a closed subset of $\mathcal{C}_n(X)$, let

$$\mathcal{M}(\mathcal{B}) = \cup\{\mathcal{OA}_n(B, X) \mid B \in \mathcal{B}\}.$$

REMARK 4.6. Note that if $\mathcal{B}$ and $\mathcal{D}$ are two closed subsets of $\mathcal{C}_n(X)$, then $\mathcal{M}(\mathcal{B} \cup \mathcal{D}) = \mathcal{M}(\mathcal{B}) \cup \mathcal{M}(\mathcal{D})$ and $\mathcal{M}(\mathcal{B}) \cap \mathcal{M}(\mathcal{D}) = \mathcal{M}(\mathcal{M}(\mathcal{B}) \cap \mathcal{M}(\mathcal{D}))$. Also, if $\{\mathcal{B}_\lambda\}_{\lambda\in\Lambda}$ is a tower ordered by inclusion, then $\mathcal{M}\left(\cap_{\lambda\in\Lambda}\mathcal{M}(\mathcal{B}_\lambda)\right) = \cap_{\lambda\in\Lambda}\mathcal{M}(\mathcal{B}_\lambda)$.

LEMMA 4.7. *Let X be a continuum and let n be a positive integer. If $\mathcal{B}$ is a closed subset of $\mathcal{C}_n(X)$, then $\mathcal{M}(\mathcal{B})$ is closed in $\mathcal{C}_n(X)$.*

PROOF. Let $D \in Cl_{\mathcal{C}_n(X)}(\mathcal{M}(\mathcal{B}))$. Then there exists a sequence $\{D_m\}_{m=1}^{\infty}$ of elements of $\mathcal{M}(\mathcal{B})$ converging to D. For each m, there exists $B_m \in \mathcal{B}$ such that $D_m \in \mathcal{OA}_n(B_m, X)$. Since $\mathcal{B}$ is compact, there exists a subsequence $\{B_{m_k}\}_{k=1}^{\infty}$ of the sequence $\{B_m\}_{m=1}^{\infty}$ that converges to an element B of $\mathcal{B}$. Since $D_{m_k} \in \mathcal{OA}_n(B_{m_k}, X)$, we have that $D \in \mathcal{OA}_n(B, X)$. Therefore, $\mathcal{M}(\mathcal{B})$ is closed in $\mathcal{C}_n(X)$. ☐

For a continuum X and a positive integer n, let

$$\Sigma_1 = \{\mathcal{M}(\mathcal{B}) \mid \mathcal{B} \text{ is a closed subset of } \mathcal{C}_n(X)\}.$$

By Remark 4.6 and Lemma 4.7, Σ_1 is a structure.

REMARK 4.8. Note that the indecomposable sets of Σ_1 are the sets of the form $\mathcal{M}(\{B\})$ where $B \in \mathcal{C}_n(X)$. Since $\mathcal{M}(\{B\})$ is homeomorphic to $\mathcal{OA}_n(B, X)$ and this set is an absolute retract ([12, 4.3]), the indecomposable sets of Σ_1 have all its reduced Čech cohomology groups trivial. Hence, all the reduced Čech cohomology groups of each member of Σ_1 are trivial ([17, Theorem 2]).

The second structure is found in $\mathcal{OA}_n(Z, X)$, where Z is an arbitrary point of $\mathcal{C}_n(X)$. If $\mathcal{B}$ is a closed subset of $\mathcal{OA}_n(Z, X)$, let

$$\mathcal{L}(\mathcal{B}) = \cup\{\mathcal{OA}_n(Z, B) \mid B \in \mathcal{B}\}.$$

REMARK 4.9. Observe that if $\mathcal{B}$ and $\mathcal{D}$ are two closed subsets of $\mathcal{C}_n(X)$, then $\mathcal{L}(\mathcal{B} \cup \mathcal{D}) = \mathcal{L}(\mathcal{B}) \cup \mathcal{L}(\mathcal{D})$ and $\mathcal{L}(\mathcal{B}) \cap \mathcal{L}(\mathcal{D}) = \mathcal{L}(\mathcal{L}(\mathcal{B}) \cap \mathcal{L}(\mathcal{D}))$. Also, if $\{\mathcal{B}_\lambda\}_{\lambda\in\Lambda}$ is a tower ordered by inclusion, then $\mathcal{L}\left(\cap_{\lambda\in\Lambda}\mathcal{L}(\mathcal{B}_\lambda)\right) = \cap_{\lambda\in\Lambda}\mathcal{L}(\mathcal{B}_\lambda)$.

LEMMA 4.10. *Let X be a continuum, let n be a positive integer and let $Z \in \mathcal{C}_n(X)$. If $\mathcal{B}$ is a closed subset of $\mathcal{OA}_n(Z,X)$, then $\mathcal{L}(\mathcal{B})$ is closed in $\mathcal{C}_n(X)$.*

PROOF. Let $D \in Cl_{\mathcal{C}_n(X)}(\mathcal{L}(\mathcal{B}))$. Then there exists a sequence $\{D_m\}_{m=1}^{\infty}$ of elements of $\mathcal{L}(\mathcal{B})$ converging to D. For each m, there exists $B_m \in \mathcal{B}$ such that $D_m \in \mathcal{OA}_n(Z, B_m)$. Since $\mathcal{OA}_n(Z,X)$ is a continuum ([12, 4.3]), $\mathcal{B}$ is compact. Then there exists a subsequence $\{B_{m_k}\}_{k=1}^{\infty}$ of the sequence $\{B_m\}_{m=1}^{\infty}$ that converges to an element B of $\mathcal{B}$. Since $D_{m_k} \in \mathcal{OA}_n(Z, B_{m_k})$, we obtain that $D \in \mathcal{OA}_n(Z,B)$. Therefore, $\mathcal{L}(\mathcal{B})$ is closed in $\mathcal{C}_n(X)$. □

Let X be a continuum and let n be a positive integer. For an element Z of $\mathcal{C}_n(X)$, let

$$\Sigma_2 = \{\mathcal{L}(\mathcal{B}) \mid \mathcal{B} \text{ is a closed subset of } \mathcal{OA}_n(Z,X)\}.$$

By Remark 4.9 and Lemma 4.10, Σ_2 is a structure.

REMARK 4.11. Note that the indecomposable sets of Σ_2 are the sets of the form $\mathcal{L}(\{B\})$ where $B \in \mathcal{OA}_n(Z,X)$. Since $\mathcal{L}(\{B\})$ is homeomorphic to $\mathcal{OA}_n(Z,B)$ and this set is an absolute retract ([12, 4.3]), all the reduced Čech cohomology groups of the indecomposable sets of Σ_2 are trivial. Thus, all the reduced Čech cohomology groups of each member of Σ_2 are trivial ([17, Theorem 2]).

Let X be a continuum, let n be a positive integer and let $\mu\colon \mathcal{C}_n(X) \twoheadrightarrow [0,1]$ be a strong size map. For an element Z of $\mathcal{C}_n(X)$ and an element $t \in [\mu(Z),1]$, let

$$\mathcal{D}_n(Z,t) = \mathcal{M}(\{Z\}) \cap \mu^{-1}(t).$$

As a consequence of Lemma 3.2, $\mathcal{D}_n(Z,t)$ is an arcwise connected continuum.

The proof of the following theorem is similar to the one given in [17, Theorem 4].

THEOREM 4.12. *Let X be a continuum, let n be a positive integer, let $\mu\colon \mathcal{C}_n(X) \twoheadrightarrow [0,1]$ be a strong size map and let $Z \in \mathcal{C}_n(X)$. If $t \in [\mu(Z),1]$, then all the reduced Čech cohomology groups of $\mathcal{D}_n(Z,t)$ are trivial.*

PROOF. Consider the pair $\{\mathcal{M}(\mathcal{D}_n(Z,t)), \mathcal{L}(\mathcal{D}_n(Z,t))\}$ of subsets of $\mathcal{OA}_n(Z,X)$. For an integer $m \geq 0$, consider the following part of the reduced Mayer-Vietoris sequence:

$$\check{H}^m(\mathcal{M}(\mathcal{D}_n(Z,t))) \oplus \check{H}^m(\mathcal{L}(\mathcal{D}_n(Z,t))) \to \check{H}^m(\mathcal{D}_n(Z,t)) \to \check{H}^{m+1}(\mathcal{OA}_n(Z,X))$$

for this pair. By Remarks 4.8 and 4.11, we have that $\check{H}^m(\mathcal{M}(\mathcal{D}_n(Z,t)))$ and $\check{H}^m(\mathcal{L}(\mathcal{D}_n(Z,t)))$ are trivial. Since $\mathcal{OA}_n(Z,X)$ is an absolute retract ([12, 4.3]), $H^{m+1}(\mathcal{OA}_n(Z,X))$ is trivial too. Hence, $\check{H}^m(\mathcal{D}_n(Z,t))$ is trivial. Therefore, all the reduced Čech cohomology groups of $\mathcal{D}_n(Z,t)$ are trivial. □

Let X be a continuum, let n be a positive integer and let $\mu\colon \mathcal{C}_n(X)\twoheadrightarrow[0,1]$ be a strong size map. Let $s,t\in[0,1]$ be such that $s\le t$. Define ${}_n\gamma_s^t\colon \mu^{-1}(s)\to\mu^{-1}(t)$ by ${}_n\gamma_s^t(Z)=\mathcal{D}_n(Z,t)$. The next lemma shows that ${}_n\gamma_s^t$ is upper semicontinuous.

Lemma 4.13. *Let X be a continuum, let n be a positive integer and let $\mu\colon \mathcal{C}_n(X)\twoheadrightarrow[0,1]$ be a strong size map. If $s,t\in[0,1]$ are such that $s\le t$, then ${}_n\gamma_s^t$ is upper semicontinuous.*

Proof. Let $\{Z_m\}_{m=1}^{\infty}$ be a sequence of elements of $\mu^{-1}(s)$ that converges to an element Z of $\mu^{-1}(s)$. Let $Y\in\limsup {}_n\gamma_s^t(Z_m)$. Then there exists a subsequence $\{m_k\}_{k=1}^{\infty}$ of the natural sequence such that for each positive integer k, there exists $Y_{m_k}\in{}_n\gamma_s^t(Z_{m_k})$ such that the sequence $\{Y_{m_k}\}_{k=1}^{\infty}$ converges to Y. Since for all k $Z_{m_k}\subset Y_{m_k}$ and $\{Z_{m_k}\}_{k=1}^{\infty}$ converges to Z, we have that $Z\subset Y$. It is easy to see that $Y\in\mathcal{OA}_n(Z,X)$. Hence, $Y\in{}_n\gamma_s^t(Z)$. Therefore, ${}_n\gamma_s^t$ is upper semicontinuous. ☐

Theorem 4.14. *Let X be a continuum, let n be a positive integer and let $\mu\colon \mathcal{C}_n(X)\twoheadrightarrow[0,1]$ be a strong size map. If $s,t\in[0,1]$ are such that $s\le t$, then ${}_n\gamma_s^t$ induces a monomorphism $({}_n\gamma_s^t)^*\colon \check{H}^1(\mu^{-1}(t))\to\check{H}^1(\mu^{-1}(s))$.*

Proof. By Theorem 4.12, all the reduced Čech cohomology groups of ${}_n\gamma_s^t(Z)$ are trivial. Suppose $t\ne 1$, the result is clear for $t=1$. Let $B\in\mu^{-1}(t)$ and suppose that $B_1,\dots,B_m$ are the components of B. Note that $({}_n\gamma_s^t)^{-1}(B)=\langle B_1,\dots,B_m\rangle_n\cap\mu^{-1}(s)$ and this set is a proper continuum of $\mu^{-1}(s)$ by [5, Theorem 2.14]. The result now follows from Lemmas 4.13, 3.3 and [17, Theorem 3]. ☐

Corollary 4.15. *Let X be a continuum, let n be a positive integer and let $\mu\colon \mathcal{C}_n(X)\twoheadrightarrow[0,1]$ be a strong size map. If $t\in[0,1]$, then ${}_n\gamma_0^t$ induces a monomorphism $({}_n\gamma_0^t)^*\colon \check{H}^1(\mu^{-1}(t))\to\check{H}^1(\mu^{-1}(0))$.*

Theorem 4.16. *Let X be a continuum, let $n\ge 3$ be an integer and let $\mu\colon \mathcal{C}_n(X)\twoheadrightarrow[0,1]$ be a strong size map. If $\mathcal{S}=\mu^{-1}(t)$ is a strong size level, then $\mathcal{S}$ is acyclic.*

Proof. By [9, Theorem 8], each map from $\mathcal{F}_n(X)$ into the unit circle in the plane is homotopic to a constant map. This implies, by [3, 8.1], that $\check{H}^1(\mathcal{F}_n(X))$ is trivial; i.e., $\mathcal{F}_n(X)$ is acyclic. The theorem now follows from the fact that $\mu^{-1}(0)=\mathcal{F}_n(X)$ and Corollary 4.15. ☐

Corollary 4.17. *The property of being acyclic is a strong size property for each integer $n\ge 3$.*

Corollary 4.18. *The property of being acyclic is a strong size property for locally connected continua.*

Proof. Let X be a locally connected continuum. For $n \geq 3$, the corollary follows from Corollary 4.17. Suppose $n = 2$. By [4, Satz 1], [20, (7.4)] and [3, 8.1], we have that $\mathcal{F}_2(X)$ is acyclic. Hence, since $\mu^{-1}(0) = \mathcal{F}_2(X)$, by Corollary 4.15, $\mu^{-1}(t)$ is acyclic for all $t \in (0, 1]$. If $n = 1$, the corollary follows from [6, p. 253], [20, (7.4)] and [3, 8.1]. Therefore, the property of being acyclic is a strong size property for locally connected continua. □

5. Extending Strong Size Maps

Let X be a continuum and let n be a positive integer. We show that if $\mathfrak{C}$ is a nonempty closed subset of $\mathcal{C}_n(X)$ and $\mu \colon \mathfrak{C} \to [0, 1]$ is a strong size map, then μ can be extended to a strong size map defined on $\mathcal{C}_n(X)$. To this end, we follow [19].

If P is a partially ordered space and $x \in P$, we write $L(x) = \{p \in P \mid p \leq x\}$ and $M(x) = \{p \in P \mid x \leq p\}$, and if $A \subset P$ then $L(A) = \cup\{L(a) \mid a \in A\}$ and $M(A) = \cup\{M(a) \mid a \in A\}$. An element m of a partially ordered space P is *minimal* (*maximal*) if, whenever $x \in P$ and $x \leq m$ ($m \leq x$), it follows that $m = x$. The set of minimal elements of P is denoted by $\min(P)$ and the set of maximal elements of P is denoted by $\max(P)$.

Recall that given a nondegenerate continuum X, H. Hosokawa [5] defined the following order on $\mathcal{C}_n(X)$: For $A, B \in \mathcal{C}_n(X)$, define $A < B$ if $A \subset B$, $A \neq B$ and $B \notin \mathcal{F}_n(X)$. We denote $A \leq B$ if $A < B$ or $A = B$. Then $\mathcal{C}_n(X)$ is a partially ordered space with respect to this order. Clearly $\min(\mathcal{C}_n(X)) = \mathcal{F}_n(X)$; $\max(\mathcal{C}_n(X)) = \{X\}$ and these sets are closed and since X is a nondegenerate continuum, they are disjoint.

The following three theorems are Theorems 2.2, 2.3 and Lemma 3.2 of [19]:

Theorem 5.1. *If K is a compact subset of a partially ordered space, then $L(K)$ and $M(K)$ are closed sets.*

Theorem 5.2. *If x and y are elements of a compact partially ordered space and if $M(x) \cap L(y) = \emptyset$, then there are disjoint open sets U and V such that $x \in U = M(U)$ and $y \in V = L(V)$.*

Theorem 5.3. *Suppose P is a compact partially ordered space such that $\min(P)$ and $\max(P)$ are disjoint closed sets, Q is a closed subset containing $(\min(P)) \cup (\max(P))$, and suppose A and B are disjoint nonempty closed subsets such that $A = M(A)$ and $B = L(B)$. If $f \colon Q \to [0, 1]$ is a continuous order-preserving function such that $f(\min(P)) = \{0\}$ and $f(\max(P)) = \{1\}$, then f admits a continuous order-preserving extension $\hat{f} \colon P \to [0, 1]$ such that $\hat{f}(a) \geq \inf f(A \cap Q)$ for each $a \in A$ and $\hat{f}(b) \leq \sup f(B \cap Q)$ for each $b \in B$.*

The proof of the following theorem is similar to the one given for [19, Theorem 3.1]; we include the appropriate changes for the convenience of the reader.

THEOREM 5.4. *Let X be a continuum and let n be a positive integer. If $\mathfrak{C}$ is a nonempty closed subset of $\mathcal{C}_n(X)$ and $\mu\colon \mathfrak{C} \to [0,1]$ is a strong size map, then μ can be extended to a strong size map μ_n defined on $\mathcal{C}_n(X)$.*

PROOF. Let $\mathfrak{C}$ be a nonempty closed subset of $\mathcal{C}_n(X)$ and let $\mu\colon \mathfrak{C} \to [0,1]$ be a strong size map. Without loss of generality we assume that $\mathcal{F}_n(X) \cup \{X\} \subset \mathfrak{C}$ (if this is not true, let $\mathfrak{K} = \mathfrak{C} \cup \mathcal{F}_n(X) \cup \{X\}$ and note that μ can be extended to a strong size map μ' on $\mathfrak{K}$ by defining $\mu'(X) = 1$ and $\mu'(A) = 0$ for each $A \in \mathcal{F}_n(X)$).

Let $\mathfrak{U}$ be a countable base for $\mathcal{C}_n(X)$ and let

$$\mathfrak{B} = \{(\mathcal{U}, \mathcal{V}) \mid M(Cl(\mathcal{U})) \cap L(Cl(\mathcal{V})) = \emptyset \text{ and } \mathcal{U}, \mathcal{V} \in \mathfrak{U}\}$$

Then $\mathfrak{B}$ is countable and we may enumerate its elements $\mathfrak{B} = \{(\mathcal{U}_k, \mathcal{V}_k) \mid k \text{ is a positive integer}\}$. By Theorem 5.1 the sets $M(Cl(\mathcal{U}))$ and $L(Cl(\mathcal{V}))$ are closed. Hence, by Theorem 5.3, for each positive integer k, there exists a continuous order-preserving function $\omega_k\colon \mathcal{C}_n(X) \to [0,1]$ such that $\omega_k|_{\mathfrak{C}} = \mu$ and:

$$\begin{aligned} &\omega_k(A) \geq \inf \mu(M(Cl(\mathcal{U}_k)) \cap \mathfrak{C}) \text{ if } A \in M(Cl(\mathcal{U}_k)),\\ &\omega_k(B) \leq \max \mu(L(Cl(\mathcal{V}_k)) \cap \mathfrak{C}) \text{ if } B \in L(Cl(\mathcal{V}_k)). \end{aligned}$$

Define $\mu_n\colon \mathcal{C}_n(X) \to [0,1]$ by $\mu_n(A) = \sum_{k=1}^{\infty} \frac{1}{2^k}\omega_k(A)$ for all $A \in \mathcal{C}_n(X)$. Observe that μ_n is a continuous extension of μ. Since each ω_k is order-preserving, μ_n is also order preserving.

We need to show that if $A, B \in \mathcal{C}_n(X)$ and $A < B$ (in Hosokawa's sense) then $\mu_n(A) < \mu_n(B)$. It suffices to prove that there exists a positive integer k such that $\omega_k(A) < \omega_k(B)$.

Let $t_A = \sup \mu(L(A) \cap \mathfrak{C})$ and let $t_B = \inf \mu(M(B) \cap \mathfrak{C})$. Since μ is a strong size map, $t_A < t_B$. Let $\varepsilon > 0$ be such that $\varepsilon < \frac{1}{2}(t_B - t_A)$. By Theorem 5.2, there exist two disjoint open subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{C}_n(X)$ such that $A \in \mathcal{V} = L(\mathcal{V})$ and $B \in \mathcal{U} = M(\mathcal{U})$ and, by compactness, we may assume that $\mu(\mathcal{V} \cap \mathfrak{C}) \subset [0, t_A + \varepsilon)$ and $\mu(\mathcal{U} \cap \mathfrak{C}) \subset (t_B - \varepsilon, 1]$. It follows that there is a positive integer k such that $A \in \mathcal{V}_k \subset Cl(\mathcal{V}_k) \subset \mathcal{V}$ and $B \in \mathcal{U}_k \subset Cl(\mathcal{U}_k) \subset \mathcal{U}$, from here we obtain:

$$\omega_k(A) \leq t_A + \varepsilon < t_B - \varepsilon \leq \omega_k(B).$$

□

COROLLARY 5.5. *Let X be a continuum. If $\mu\colon \mathcal{C}(X) \to [0,1]$ is a Whitney map, μ can be extended to a strong size map μ_n defined on $\mathcal{C}_n(X)$.*

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S. Macías
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Circuito Exterior, Ciudad Universitaria
México D. F., C. P. 04510
México
E-mail: `sergiom@matem.unam.mx`

C. Piceno
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Circuito Exterior, Ciudad Universitaria
México D. F., C. P. 04510
México
E-mail: `cesarpicman@hotmail.com`

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