BOUNDED INJECTIVITY AND HAAGERUP TENSOR PRODUCT

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ABSTRACT. In this paper, we prove that if $V \subseteq B(H)$ is an injective operator system on a separable Hilbert space H, then $V \otimes_h W$ is b-injective for any operator system W if and only if V is finite dimensional.

1. Introduction

Let $B(H)$ be the set of all bounded linear operators on a Hilbert space H. Operator spaces are the concrete closed subspaces of $B(H)$ as formulated in [3].

An operator space V is called *b-injective* if there is a $\lambda \geq 1$ such that for given operator spaces $W_1 \subseteq W_2$ any completely bounded map $\varphi_1 : W_1 \to V$ can be extended to a completely bounded map $\varphi_2 : W_2 \to V$ with $\|\varphi_2\|_{cb} \leq \lambda \|\varphi_1\|_{cb}$. An injective operator space V is a b-injective operator space with $\lambda = 1$. For more details see [6, 8].

An operator space $V \subseteq B(H)$ is called an operator system if V is unital and a self adjoint operator space. It is well known that every injective operator system is a unital C^*-algebra. In fact, if $V \subseteq B(H)$ is an injective operator system, then there is some completely contractive onto projection $\varphi : B(H) \to V$. Therefore, V equipped with the following multiplication

$$\circ : V \times V \to V \quad s.t \quad T \circ S := \varphi(TS)$$

is a C^*-algebra ([3, Theorem 6.1.3]). Therefore, every finite dimensional injective operator system V is in the form of $\oplus_{k=1}^n M_{m_k}$. Thus for any operator

space W,

$$V \check{\otimes} W \cong \oplus_{k=1}^{n} M_{m_k}(W).$$

Consequently, if W is an injective operator system then $V \check{\otimes} W$ is an injective operator system, too.

Furthermore, Takesaki in [11] shows that, for every two C^*-algebras A and B, the minimal C^*-tensor product $A \otimes B$ is injective if and only if A and B are injective and either A or B is finite-dimensional.

The above fact is not necessarily valid in the category of operator spaces, because there exist infinite dimensional injective operator spaces whose minimal tensor product is injective. In fact, let δ_{11} be the projection in M_∞ which is 1 in the first coordinate and zero elsewhere. Thus $K_{1 \times \infty} = M_{1 \times \infty} \cong \delta_{11} M_\infty$ is an injective operator space. By [3, Page 177],

$$K_{1 \times \infty} \check{\otimes} K_{1 \times \infty} \cong K_{1 \times \infty}(K_{1 \times \infty}) \cong K_{1, \infty \times \infty} = M_{1, \infty \times \infty}.$$

is again an injective operator space.

Now, in the operator space category the question naturally arises:

whether or not the above-mentioned fact is valid for another cross norm.

In this paper, we focus on the problem considering the Haagerup tensor product. In fact, we prove that if $V \subseteq B(H)$ is an injective operator system on a separable Hilbert space H, then $V \otimes_h W$ is b-injective for any operator system W if and only if V is finite dimensional.

2. The Main Theorem

In this paper, we use the notions of injective and Haagerup tensor products as well as infinite matrices of operator spaces; related to notations and theorems which can be found in [1, 3].

Given operator spaces V and W and a linear mapping $\varphi : V \to W$, for each $n \in \mathbb{N}$, there is a corresponding linear mapping $\varphi_n : M_n(V) \to M_n(W)$ defined by $\varphi_n(T) = [\varphi(T_{i,j})]$ for all $T = [T_{i,j}] \in M_n(V)$. The completely bounded norm of φ is defined by

$$\| \varphi \|_{cb} = \sup\{\| \varphi_n \| : n \in \mathbb{N}\}.$$

It is said that φ is completely bounded (respectively, completely contractive) if $\| \varphi \|_{cb} < \infty$ (respectively, $\|\varphi\|_{cb} \leq 1$). We say that the operator spaces V and W are completely isometrically isomorphic if there is an onto linear map $\varphi : V \to W$ such that each mapping $\varphi_n : M_n(V) \to M_n(W)$ is an isometry. This notion is indicated by $V \cong W$. If $\varphi : V \to W$ is a completely bounded linear bijection and its inverse is completely bounded, then we say φ is a completely isomorphism. In this case, we say that V and W are completely isomorphic and write $V \simeq W$. It is well known that the same dimensional operator spaces are completely isomorphic.

Let V and W be λ_V- and λ_W-injective operator spaces, respectively. Then $V \oplus W$ is a $\max\{\lambda_V, \lambda_W\}$-injective operator space. Also, if Z is an operator

subspace of V and there is a completely bounded onto projection $\varphi : V \to Z$, then Z is a $\lambda_V \|\varphi\|_{cb}$-injective operator space.

LEMMA 2.1. *Let V and W be completely isomorphic operator spaces. Then V is a b-injective operator space if and only if W is a b-injective operator space.*

PROOF. We assume that V is a λ-injective operator space for some $\lambda \geq 1$, and also $\varphi : W \to V$ is a completely isomorphic mapping. Let Z_1, Z_2 be two operator spaces satisfying $Z_1 \subseteq Z_2$ and $\phi : Z_1 \to W$ be a completely bounded map. Thus $\varphi \circ \phi : Z_1 \to V$ is a completely bounded map, and so there is a completely bounded map $\psi : Z_2 \to V$ extension for $\varphi \circ \phi$ with $\|\psi\|_{cb} \leq \lambda \|\varphi \circ \phi\|_{cb}$. Obviously $\varphi^{-1} \circ \psi : Z_2 \to W$ is a completely bounded extension map for ϕ such that

$$\|\varphi^{-1} \circ \psi\|_{cb} \leq \|\varphi^{-1}\|_{cb} \|\psi\|_{cb} \leq \lambda \|\varphi^{-1}\|_{cb} \|\varphi \circ \phi\|_{cb} \leq \lambda \|\varphi^{-1}\|_{cb} \|\varphi\|_{cb} \|\phi\|_{cb}.$$

Thus W is a $\lambda \|\varphi\|_{cb} \|\varphi^{-1}\|_{cb}$-injective operator space.   □

THEOREM 2.2. *$c_\circ$ is not a b-injective operator space.*

PROOF. Assume, to reach a contradiction, that $c_\circ$ is a λ-injective operator space for some $\lambda \geq 1$. This assumption implies that $c_\circ$ is a λ-injective Banach space, too. In fact, let E and F be Banach spaces, $E \subseteq F$ and $\varphi : E \to c_\circ$ be a bounded linear map. If the Banach spaces E and F endowed with the MIN operator space structure, respectively, then we have $\|\varphi\| = \|\varphi\|_{cb}$. And also, by the assumption, we can extend φ to a completely bounded map $\psi : \mathrm{MIN}\, F \to c_\circ$ such that $\|\psi\|_{cb} \leq \lambda \|\varphi\|_{cb}$, and so $\|\psi\| \leq \lambda \|\varphi\|$.

Therefore $c_\circ$ is a b-injective Banach space, and so $c_\circ$ has a subspace isomorphic to ℓ^∞ ([8, 9]). On the other hand, the Banach space $c_\circ$ is separable, but that ℓ^∞ is not separable. This is a contradiction.   □

THEOREM 2.3. *Let $V \subseteq B(H)$ be an injective operator system on a separable Hilbert space H. Then $V \otimes_h W$ is b-injective for all injective operator space W if and only if V is finite dimensional.*

PROOF. ($\Leftarrow$) Assume that V is a finite dimensional operator system with $dim V = n$. Then V is completely isomorphic to the injective column Hilbert space $M_{n,1}(\mathbb{C})$, ([3, Corollary 2.2.5]). Then by [3, Proposition 9.3.1], we have

$$V \otimes_h W \simeq M_{n,1}(\mathbb{C}) \otimes_h W \cong M_{n,1}(\mathbb{C}) \check{\otimes} W \cong M_{n,1}(W).$$

Now it is clear that, the injectivity of W implies the injectivity of $M_{n,1}(W)$. Hence $V \otimes_h W$ is b-injective.

($\Rightarrow$) Assume that V is an infinite dimensional injective operator system on a separable Hilbert space H. By [7], V is completely isomorphic to ℓ^∞ or M_∞.

CASE 1) $V \simeq \ell_\infty$: By the assumption of the theorem, $V \otimes_h M_\infty$ is a b-injective operator space. Thus, by Lemma 1, $\ell_\infty \otimes_h M_\infty \simeq V \otimes_h M_\infty$ is a

b-injective operator space, too. We can assume that the injective row Hilbert space $K_{1 \times \infty} = M_{1 \times \infty} (\cong \delta_{11} M_\infty)$ is an operator subspace of M_∞. Thus, there is some completely contractive onto projection $\varphi' : M_\infty \to K_{1 \times \infty}$. By the [3, Proposition 9.2.5],

$$I \otimes \varphi' : \ell^\infty \otimes_h M_\infty \to \ell^\infty \otimes_h K_{1 \times \infty}$$

is a completely contractive and onto projection. Therefore, $\ell^\infty \otimes_h K_{1 \times \infty}$ is a b-injective operator space. By [3, Page 177 and Proposition 9.3.1], we have

$$K_{1 \times \infty}(\ell^\infty) \cong \ell^\infty \check{\otimes} K_{1 \times \infty} \cong \ell^\infty \otimes_h K_{1 \times \infty}.$$

Therefore, by Lemma 1, there exists some $\lambda \geq 1$ such that $K_{1 \times \infty}(\ell^\infty)$ is a λ-injective operator space. Let $\delta_n \in \ell^\infty$ be the natural projection for each $n \in \mathbb{N}$, and $(\alpha_i)_i \in c_\circ$. We have $\sup_i |\alpha_i| < \infty$. Then, for each $n \in \mathbb{N}$

$$\|[\alpha_1 \delta_1 \quad \alpha_2 \delta_2 \quad \cdots \quad \alpha_n \delta_n]\| = \|\sum_{i=1}^n |\alpha_i|^2 \delta_i\|_\infty^{1/2} = \max_{1 \leq i \leq n} |\alpha_i| \leq \sup_i |\alpha_i| < \infty.$$

Thus, by definition of $M_{1 \times \infty}(\ell^\infty)$ (see [3], Section 10), we have

$$u = [\alpha_1 \delta_1 \quad \alpha_2 \delta_2 \quad \cdots] \in M_{1 \times \infty}(\ell^\infty).$$

For any $\varepsilon > 0$, there is some $n \in \mathbb{N}$ such that $|\alpha_i| \leq \varepsilon$ for each $i \geq n$. For each $m \geq n$, we define

$$u_m = [\alpha_1 \delta_1 \quad \cdots \quad \alpha_m \delta_m \quad 0 \quad \cdots].$$

We have

$$\|u - u_m\| = [0 \quad \cdots \quad 0 \quad \alpha_{m+1} \delta_{m+1} \quad \alpha_{m+2} \delta_{m+2} \quad \cdots]\|$$

$$= \sup_{p \geq m+1} \|\sum_{i=m+1}^p |\alpha_i|^2 \delta_i\|_\infty^{1/2}$$

$$= \sup_{p \geq m+1} \{\max_{m+1 \leq i \leq p} |\alpha_i|\} \leq \varepsilon.$$

Thus, by definition of $K_{1 \times \infty}(\ell^\infty)$, we have $u \in K_{1 \times \infty}(\ell^\infty)$. Therefore,

$$\varphi : c_\circ \to K_{1 \times \infty}(\ell^\infty) \ : \ (\alpha_i)_i \mapsto [\alpha_1 \delta_1 \quad \alpha_2 \delta_2 \quad \cdots]$$

is a completely isometric embedding. Now, we consider $[f_1 \quad f_2 \quad \cdots] \in K_{1 \times \infty}(\ell^\infty)$. Thus for any $\varepsilon > 0$ there is some $n \in \mathbb{N}$ such that

$$\|[0 \quad \cdots \quad 0 \quad f_n \quad f_{n+1} \quad \cdots]\| = \sup_{p \geq n} \|\sum_{i=n}^p |f_i|^2\|_\infty^{1/2} \leq \varepsilon.$$

Then for any $p \geq n$ we have $\|f_p\|_\infty \leq \varepsilon$. Hence

$$\psi : K_{1 \times \infty}(\ell^\infty) \to c_\circ \ : \ [f_1 \quad f_2 \quad \cdots] \mapsto (f_k(k))_k$$

is a completely contractive onto mapping such that $\psi \circ \varphi = id$.

Let W_1, W_2 be two operator spaces satisfying $W_1 \subseteq W_2$, and $\Phi_1 : W_1 \to c_\circ$ be a completely bounded mapping. Then $\varphi \circ \Phi_1 : W_1 \to K_{1 \times \infty}(\ell^\infty)$ is a

completely bounded map. Since $K_{1\times\infty}(\ell^\infty)$ is a λ-injective operator space, for $\varphi\circ\Phi_1$ there exists a completely bounded extension $\Phi_2 : W_2 \to K_{1\times\infty}(\ell^\infty)$, where $\|\Phi_2\|_{cb} \leq \lambda\|\varphi \circ \Phi_1\|_{cb}$. Obviously $\psi \circ \Phi_2 : W_2 \to c_\circ$ is a completely bounded extension of Φ_1 such that

$$\|\psi \circ \Phi_2\|_{cb} \leq \|\Phi_2\|_{cb} \leq \lambda\|\varphi \circ \Phi_1\|_{cb} \leq \lambda\|\Phi_1\|_{cb}.$$

Therefore, $c_\circ$ is b-injective, and this is a contradiction.

CASE 2) $V \simeq M_\infty$: Therefore $M_\infty \otimes_h M_\infty$ is a b-injective operator space. Also, ℓ^∞ and $K_{1\times\infty}$ are injective operator subspaces of M_∞. Thus, there are completely contractive onto projections

$$\varphi : M_\infty \to \ell^\infty \quad \text{and} \quad \psi : M_\infty \to K_{1\times\infty}.$$

Thus, by [3, Proposition 9.2.5],

$$\varphi \otimes \psi : M_\infty \otimes_h M_\infty \to \ell^\infty \otimes_h K_{1\times\infty}$$

is a completely contractive and onto projection. Therefore, $\ell^\infty \otimes_h K_{1\times\infty}$ is b-injective, too. This, again, leads to a contradiction. □

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