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A NOTE ON REPRESENTATIONS OF SOME AFFINE VERTEX ALGEBRAS OF TYPE D

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Abstract. In this note we construct a series of singular vectors in universal affine vertex operator algebras associated to $D_\ell^{(1)}$ of levels $n-\ell+1$, for $n \in \mathbb{Z}_{>0}$. For $n=1$, we study the representation theory of the quotient vertex operator algebra modulo the ideal generated by that singular vector. In the case $\ell = 4$, we show that the adjoint module is the unique irreducible ordinary module for simple vertex operator algebra $L_{D_4}(-2,0)$. We also show that the maximal ideal in associated universal affine vertex algebra is generated by three singular vectors.

1. Introduction

The classification of irreducible modules for simple vertex operator algebra $L_{\mathfrak{g}}(k,0)$ associated to affine Lie algebra $\hat{\mathfrak{g}}$ of level k is still an open problem for general $k \in \mathbb{C}$ ($k \neq -h^\vee$). This problem is connected with the description of the maximal ideal in the universal affine vertex algebra $N_{\mathfrak{g}}(k,0)$. One approach to this classification problem is through construction of singular vectors in $N_{\mathfrak{g}}(k,0)$.

The known (non-generic) cases include positive integer levels (cf. [14, 19, 20]) and some special cases of rational admissible levels, in the sense of Kac and Wakimoto ([17]) (cf. [1, 3, 4, 6, 9, 21, 22]). It turns out that negative integer levels also have some interesting properties. They appeared in bosonic realizations in [10], and also recently in the context of conformal embeddings (cf. [5]).

In this note we study a vertex operator algebra associated to affine Lie algebra of type $D_\ell^{(1)}$ and negative integer level $-\ell+2$. This level appeared in

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[5] in the context of conformal embedding of $L_{B_{\ell-1}}(-\ell+2,0)$ into $L_{D_\ell}(-\ell+2,0)$. This conformal embedding is in some sense similar to the conformal embedding of $L_{D_\ell}(-\ell+\frac{3}{2},0)$ into $L_{B_\ell}(-\ell+\frac{3}{2},0)$.

We will show that there are also similarities in singular vectors in universal affine vertex algebras $N_{B_\ell}(-\ell+\frac{3}{2},0)$ (studied in [21]) and $N_{D_\ell}(-\ell+2,0)$. More generally, we construct a series of singular vectors

$$v_n=\Big(\sum_{i=2}^{\ell}e_{\epsilon_1-\epsilon_i}(-1)e_{\epsilon_1+\epsilon_i}(-1)\Big)^n\mathbf{1}$$

in $N_{D_\ell}(n-\ell+1,0)$, for any $n\in\mathbb{Z}_{>0}$. For $n=1$, we study the representation theory of the quotient $N_{D_\ell}(-\ell+2,0)$ modulo the ideal generated by v_1. Using the methods from [1, 2, 4, 20], we obtain the classification of irreducible weak modules in the category $\mathcal{O}$ for that vertex algebra. It turns out that there are infinitely many of these modules.

In the special case $\ell=4$, we obtain the classification of irreducible weak modules from the category $\mathcal{O}$ for simple vertex operator algebra $L_{D_4}(-2,0)$. This vertex algebra also appeared in [5] in the context of conformal embedding of $L_{G_2}(-2,0)$ into $L_{D_4}(-2,0)$. It follows that there are finitely many irreducible weak $L_{D_4}(-2,0)$–modules from the category $\mathcal{O}$, that the adjoint module is the unique irreducible ordinary $L_{D_4}(-2,0)$–module, and that every ordinary $L_{D_4}(-2,0)$–module is completely reducible. We also show that the maximal ideal in $N_{D_4}(-2,0)$ is generated by three singular vectors.

2. Preliminaries

We assume that the reader is familiar with the notion of vertex operator algebra (cf. [7, 11–14, 16, 18, 19]) and Kac-Moody algebra (cf. [15]).

Let V be a vertex operator algebra. Denote by $A(V)$ the associative algebra introduced in [23], called the Zhu's algebra of V. As a vector space, $A(V)$ is a quotient of V, and we denote by $[a]$ the image of $a\in V$ under the projection of V onto $A(V)$. We recall the following fundamental result from [23]:

Proposition 2.1. *The equivalence classes of the irreducible $A(V)$–modules and the equivalence classes of the irreducible $\mathbb{Z}_+$–graded weak V–modules are in one-to-one correspondence.*

Let $\mathfrak{g}$ be a simple Lie algebra with a triangular decomposition $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$, and $\hat{\mathfrak{g}}$ the (untwisted) affine Lie algebra associated to $\mathfrak{g}$. Denote by $V(\mu)$ the irreducible highest weight $\mathfrak{g}$–module with highest weight μ, and by $L(k,\mu)$ the irreducible highest weight $\hat{\mathfrak{g}}$–module with highest weight $k\Lambda_0+\mu$.

Furthermore, denote by $N(k,0)$ (or $N_{\mathfrak{g}}(k,0)$) the universal affine vertex algebra associated to $\hat{\mathfrak{g}}$ of level $k\in\mathbb{C}$. For $k\neq -h^\vee$, $N(k,0)$ is a vertex operator algebra with Segal-Sugawara conformal vector, and $L(k,0)$ is a simple vertex operator algebra. The Zhu's algebra of $N(k,0)$ was determined in [14]:

PROPOSITION 2.2. *The associative algebra $A(N(k,0))$ is canonically isomorphic to $U(\mathfrak{g})$. The isomorphism is given by $F : A(N(k,0)) \to U(\mathfrak{g})$*

$$F([x_1(-n_1-1)\cdots x_m(-n_m-1)\mathbf{1}]) = (-1)^{n_1+\cdots+n_m} x_m \cdots x_1,$$

for any $x_1,\ldots,x_m \in \mathfrak{g}$ and any $n_1,\ldots,n_m \in \mathbb{Z}_+$.

We have:

PROPOSITION 2.3. *Assume that a $\hat{\mathfrak{g}}$–submodule J of $N(k,0)$ is generated by m singular vectors ($m \in \mathbb{Z}_{>0}$), i.e. $J = U(\hat{\mathfrak{g}})v^{(1)} + \ldots + U(\hat{\mathfrak{g}})v^{(m)}$. Then*

$$A(N(k,0)/J) \cong U(\mathfrak{g})/I,$$

where I is the two-sided ideal of $U(\mathfrak{g})$ generated by $u^{(1)} = F([v^{(1)}]),\ldots,u^{(m)} = F([v^{(m)}])$.

Let $J = U(\hat{\mathfrak{g}})v^{(1)} + \ldots + U(\hat{\mathfrak{g}})v^{(m)}$ be a $\hat{\mathfrak{g}}$–submodule of $N(k,0)$ generated by singular vectors $v^{(1)},\ldots,v^{(m)}$. Now we recall the method from [1, 2, 4, 20] for the classification of irreducible $A(N(k,0)/J)$–modules from the category $\mathcal{O}$ by solving certain systems of polynomial equations.

Denote by $_L$ the adjoint action of $U(\mathfrak{g})$ on $U(\mathfrak{g})$ defined by $X_L f = [X,f]$ for $X \in \mathfrak{g}$ and $f \in U(\mathfrak{g})$. Let $R^{(j)}$ be a $U(\mathfrak{g})$–submodule of $U(\mathfrak{g})$ generated by the vector $u^{(j)} = F([v^{(j)}])$ under the adjoint action, for $j = 1,\ldots,m$. Clearly, $R^{(j)}$ is an irreducible highest weight $U(\mathfrak{g})$–module. Let $R_0^{(j)}$ be the zero-weight subspace of $R^{(j)}$.

The next proposition follows from [1, 4, 20]:

PROPOSITION 2.4. *Let $V(\mu)$ be an irreducible highest weight $U(\mathfrak{g})$–module with the highest weight vector v_μ, for $\mu \in \mathfrak{h}^*$. The following statements are equivalent:*

(1) *$V(\mu)$ is an $A(N(k,0)/J)$–module,*
(2) *$R^{(j)}V(\mu) = 0$, for every $j = 1,\ldots,m$,*
(3) *$R_0^{(j)}v_\mu = 0$, for every $j = 1,\ldots,m$.*

Let $r \in R_0^{(j)}$. Clearly there exists the unique polynomial $p_r \in S(\mathfrak{h})$ such that

$$rv_\mu = p_r(\mu)v_\mu.$$

Set $\mathcal{P}_0^{(j)} = \{\, p_r \mid r \in R_0^{(j)}\}$, for $j = 1,\ldots,m$. We have:

COROLLARY 2.5. *There is one-to-one correspondence between*

(1) *irreducible $A(N(k,0)/J)$–modules from the category $\mathcal{O}$,*
(2) *weights $\mu \in \mathfrak{h}^*$ such that $p(\mu) = 0$ for all $p \in \mathcal{P}_0^{(j)}$, for every $j = 1,\ldots,m$.*

In the case $m = 1$, we use the notation R, R_0 and $\mathcal{P}_0$ for $R^{(1)}$, $R_0^{(1)}$ and $\mathcal{P}_0^{(1)}$, respectively.

3. Vertex operator algebra associated to $D_\ell^{(1)}$ of level $-\ell + 2$

In this section we study the representation theory of the quotient of universal affine vertex operator algebra associated to $D_\ell^{(1)}$ of level $-\ell + 2$, modulo the ideal generated by a singular vector of conformal weight two.

Denote by $\mathfrak{g}$ the simple Lie algebra of type D_ℓ. We fix the root vectors for $\mathfrak{g}$ as in [8, 10]. We have:

Theorem 3.1. *Vector*

$$v_n = \Big(\sum_{i=2}^{\ell} e_{\epsilon_1 - \epsilon_i}(-1) e_{\epsilon_1 + \epsilon_i}(-1) \Big)^n \mathbf{1}$$

is a singular vector in $N_{D_\ell}(n - \ell + 1, 0)$, for any $n \in \mathbb{Z}_{>0}$.

Proof. Direct verification of relations $e_{\epsilon_k - \epsilon_{k+1}}(0) v_n = 0$, for $k = 1, \ldots, \ell - 1$, $e_{\epsilon_{\ell-1} + \epsilon_\ell}(0) v_n = 0$ and $f_{\epsilon_1 + \epsilon_2}(1) v_n = 0$. □

In the case $n = 1$, we obtain the singular vector

$$v = \sum_{i=2}^{\ell} e_{\epsilon_1 - \epsilon_i}(-1) e_{\epsilon_1 + \epsilon_i}(-1) \mathbf{1} \tag{3.1}$$

in $N_{D_\ell}(-\ell + 2, 0)$.

Remark 3.2. Vector v from relation (3.1) has a similar formula as singular vector

$$-\frac{1}{4} e_{\epsilon_1}(-1)^2 \mathbf{1} + \sum_{i=2}^{\ell} e_{\epsilon_1 - \epsilon_i}(-1) e_{\epsilon_1 + \epsilon_i}(-1) \mathbf{1}$$

for $B_\ell^{(1)}$ in $N_{B_\ell}(-\ell + \frac{3}{2}, 0)$. The representation theory of the quotient of $N_{B_\ell}(-\ell + \frac{3}{2}, 0)$ modulo the ideal generated by that vector was studied in [21].

We will consider representations of the vertex operator algebra

$$\mathcal{V}_{D_\ell}(-\ell + 2, 0) = \frac{N_{D_\ell}(-\ell + 2, 0)}{U(\hat{\mathfrak{g}}) v}.$$

Proposition 2.3 gives:

Proposition 3.3. *The associative algebra $A(\mathcal{V}_{D_\ell}(-\ell + 2, 0))$ is isomorphic to the algebra $U(\mathfrak{g})/I$, where I is the two-sided ideal of $U(\mathfrak{g})$ generated by*

$$u = \sum_{i=2}^{\ell} e_{\epsilon_1 - \epsilon_i} e_{\epsilon_1 + \epsilon_i}.$$

We have the following classification:

THEOREM 3.4. *For any subset* $S = \{i_1, \dots, i_k\} \subseteq \{1, 2, \dots, \ell - 2\}$, $i_1 < \dots < i_k$, *and* $t \in \mathbb{C}$, *we define weights*

$$\mu_{S,t} = \sum_{j=1}^{k} \left(i_j + 2 \sum_{s=j+1}^{k} (-1)^{s-j} i_s + (-1)^{k-j+1}(t + \ell - 1) \right) \omega_{i_j} + t\omega_{\ell-1},$$

$$\mu'_{S,t} = \sum_{j=1}^{k} \left(i_j + 2 \sum_{s=j+1}^{k} (-1)^{s-j} i_s + (-1)^{k-j+1}(t + \ell - 1) \right) \omega_{i_j} + t\omega_{\ell},$$

where $\omega_1, \dots, \omega_\ell$ *are fundamental weights for* $\mathfrak{g}$*. Then the set*

$$\{L_{D_\ell}(-\ell + 2, \mu_{S,t}), L_{D_\ell}(-\ell + 2, \mu'_{S,t}) \mid S \subseteq \{1, 2, \dots, \ell - 2\}, t \in \mathbb{C}\}$$

provides the complete list of irreducible weak $\mathcal{V}_{D_\ell}(-\ell+2, 0)$*–modules from the category* $\mathcal{O}$*.*

PROOF. We use the method for classification of irreducible $A(\mathcal{V}_{D_\ell}(-\ell + 2, 0))$–modules in the category $\mathcal{O}$ from Corollary 2.5. In this case $R \cong V_{D_\ell}(2\omega_1)$, and similarly as in [21, Lemma 28] one obtains that

$$\dim R_0 = \ell - 1.$$

Furthermore, one obtains by direct calculation that

$$(f_{\epsilon_1-\epsilon_2} f_{\epsilon_1+\epsilon_2})_L u \in p_1(h) + U(\mathfrak{g})\mathfrak{n}_+,$$
$$(f_{\epsilon_1-\epsilon_{i+1}} f_{\epsilon_1+\epsilon_{i+1}} - f_{\epsilon_1-\epsilon_i} f_{\epsilon_1+\epsilon_i})_L u \in p_i(h) + U(\mathfrak{g})\mathfrak{n}_+, \; i = 2, \dots, \ell - 1,$$

where

$$p_i(h) = h_i(h_{\epsilon_i+\epsilon_{i+1}} + \ell - i - 1), \quad \text{for } i = 1, \dots, \ell - 1 \tag{3.2}$$

are linearly independent polynomials in $\mathcal{P}_0$. Here h_i $(i = 1, \dots, \ell)$ denote the simple coroots for $\mathfrak{g}$ and

$$h_{\epsilon_i+\epsilon_{i+1}} = h_i + 2h_{i+1} + \ldots + 2h_{\ell-2} + h_{\ell-1} + h_\ell, \quad \text{for } i < \ell - 1.$$

Corollary 2.5 now implies that the highest weights of irreducible $A(\mathcal{V}_{D_\ell}(-\ell + 2, 0))$–modules from the category $\mathcal{O}$ are given as solutions of polynomial equations

$$p_i(h) = 0, \; i = 1, \dots, \ell - 1. \tag{3.3}$$

First we note that for $i = \ell - 1$, we obtain the equation

$$h_{\ell-1} h_\ell = 0.$$

Thus, either $h_{\ell-1} = 0$ or $h_\ell = 0$. Assume first that $h_{\ell-1} = 0$, and let $S = \{i_1, \dots, i_k\}$, $i_1 < \ldots < i_k$ be the subset of $\{1, 2, \dots, \ell - 2\}$ such that

$h_i = 0$ for $i \notin S$ and $h_i \neq 0$ for $i \in S$. Then we have the system

$$
\begin{aligned}
&h_{i_1} + 2h_{i_2} + \ldots + 2h_{i_k} + h_\ell + \ell - i_1 - 1 = 0,\\
&h_{i_2} + 2h_{i_3} + \ldots + 2h_{i_k} + h_\ell + \ell - i_2 - 1 = 0,\\
&\qquad\vdots\\
&h_{i_{k-1}} + 2h_{i_k} + h_\ell + \ell - i_{k-1} - 1 = 0,\\
&h_{i_k} + h_\ell + \ell - i_k - 1 = 0.
\end{aligned}
\tag{3.4}
$$

The solution of this system is given by

$$
\begin{aligned}
&h_{i_j} = i_j + 2\sum_{s=j+1}^{k} (-1)^{s-j} i_s + (-1)^{k-j+1}(t + \ell - 1), \quad \text{for } j = 1, \ldots, k;\\
&h_\ell = t \quad (t \in \mathbb{C}).
\end{aligned}
$$

It follows that $V_{D_\ell}(\mu'_{S,t})$ is an irreducible $A(\mathcal{V}_{D_\ell}(-\ell+2,0))$–module. Similarly, the case $h_\ell = 0$ gives that $V_{D_\ell}(\mu_{S,t})$ is irreducible $A(\mathcal{V}_{D_\ell}(-\ell+2,0))$–module. We conclude that the set

$$\{V_{D_\ell}(\mu_{S,t}), V_{D_\ell}(\mu'_{S,t}) \mid S \subseteq \{1, 2, \ldots, \ell-2\}, t \in \mathbb{C}\}$$

provides the complete list of irreducible $A(\mathcal{V}_{D_\ell}(-\ell+2,0))$-modules from the category $\mathcal{O}$. The claim of theorem now follows from Zhu's theory. □

Example 3.5. For $\ell = 4$, we have subsets $S = \emptyset, \{1\}, \{2\}, \{1,2\}$ of the set $\{1,2\}$, so we obtain that the set

$$
\begin{aligned}
&\{L_{D_\ell}(-\ell+2, t\omega_3), L_{D_\ell}(-\ell+2, t\omega_4), L_{D_\ell}(-\ell+2, (-2-t)\omega_1 + t\omega_3),\\
&L_{D_\ell}(-\ell+2, (-2-t)\omega_1 + t\omega_4), L_{D_\ell}(-\ell+2, (-1-t)\omega_2 + t\omega_3),\\
&L_{D_\ell}(-\ell+2, (-1-t)\omega_2 + t\omega_4), L_{D_\ell}(-\ell+2, t\omega_1 + (-1-t)\omega_2 + t\omega_3),\\
&L_{D_\ell}(-\ell+2, t\omega_1 + (-1-t)\omega_2 + t\omega_4) \mid t \in \mathbb{C}\}
\end{aligned}
\tag{3.5}
$$

provides the complete list of irreducible weak $\mathcal{V}_{D_4}(-2,0)$–modules from the category $\mathcal{O}$.

Recall that a module for vertex operator algebra is called ordinary if $L(0)$ acts semisimply with finite-dimensional weight spaces. We have:

Corollary 3.6. *The set*

$$\{L_{D_\ell}(-\ell+2, t\omega_{\ell-1}), L_{D_\ell}(-\ell+2, t\omega_\ell) \mid t \in \mathbb{Z}_{\geq 0}\}$$

provides the complete list of irreducible ordinary $\mathcal{V}_{D_\ell}(-\ell+2,0)$–modules.

Proof. If $L_{D_\ell}(-\ell+2, \mu)$ is an ordinary $\mathcal{V}_{D_\ell}(-\ell+2,0)$–module, then μ is a dominant integral weight. Then $\mu(h_{\epsilon_i+\epsilon_{i+1}}) \in \mathbb{Z}_{\geq 0}$, for $i = 1, \ldots, \ell-1$. Relations (3.2) and (3.3) then give that

$$\mu(h_i) = 0, \quad \text{for } i = 1, \ldots, \ell-2,$$

and $\mu(h_{\ell-1}) = 0$ or $\mu(h_\ell) = 0$. Thus, $\mu = t\omega_{\ell-1}$ or $\mu = t\omega_\ell$, and $t \in \mathbb{Z}_{\geq 0}$ since μ is a dominant integral weight. □

It follows that:

Corollary 3.7. *The set of irreducible ordinary $L_{D_\ell}(-\ell+2, 0)$–modules is a subset of the set*

$$\{L_{D_\ell}(-\ell+2, t\omega_{\ell-1}), L_{D_\ell}(-\ell+2, t\omega_\ell) \mid t \in \mathbb{Z}_{\geq 0}\}.$$

4. Case $\ell = 4$

In this section we study the case $\ell = 4$. We determine the classification of irreducible weak $L_{D_4}(-2, 0)$–modules from the category $\mathcal{O}$. It turns out that there are finitely many of these modules and that the adjoint module is the unique irreducible ordinary $L_{D_4}(-2, 0)$–module. We also show that the maximal ideal in $N_{D_4}(-2, 0)$ is generated by three singular vectors.

Denote by θ the automorphism of $N_{D_4}(-2, 0)$ induced by the automorphism of the Dynkin diagram of D_4 of order three such that

$$\theta(\epsilon_1 - \epsilon_2) = \epsilon_3 - \epsilon_4, \qquad \theta(\epsilon_2 - \epsilon_3) = \epsilon_2 - \epsilon_3,$$
$$\theta(\epsilon_3 - \epsilon_4) = \epsilon_3 + \epsilon_4, \qquad \theta(\epsilon_3 + \epsilon_4) = \epsilon_1 - \epsilon_2.$$

Relation (3.1) implies that

$$v = (e_{\epsilon_1-\epsilon_2}(-1)e_{\epsilon_1+\epsilon_2}(-1) + e_{\epsilon_1-\epsilon_3}(-1)e_{\epsilon_1+\epsilon_3}(-1) + e_{\epsilon_1-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_4}(-1))\mathbf{1}$$

is a singular vector in $N_{D_4}(-2, 0)$. Furthermore,

$$\begin{aligned}\theta(v) = (&e_{\epsilon_3-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_2}(-1) - e_{\epsilon_2-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_3}(-1) \\ &+ e_{\epsilon_2+\epsilon_3}(-1)e_{\epsilon_1-\epsilon_4}(-1))\mathbf{1},\end{aligned}$$

and

$$\begin{aligned}\theta^2(v) = (&e_{\epsilon_3+\epsilon_4}(-1)e_{\epsilon_1+\epsilon_2}(-1) - e_{\epsilon_2+\epsilon_4}(-1)e_{\epsilon_1+\epsilon_3}(-1) \\ &+ e_{\epsilon_1+\epsilon_4}(-1)e_{\epsilon_2+\epsilon_3}(-1))\mathbf{1}\end{aligned}$$

are also singular vectors in $N_{D_4}(-2, 0)$. We consider the vertex operator algebra

$$\widetilde{L}_{D_4}(-2, 0) = \frac{N_{D_4}(-2, 0)}{J},$$

where J is the ideal in $N_{D_4}(-2, 0)$ generated by vectors v, $\theta(v)$ and $\theta^2(v)$.

Proposition 2.3 gives that the associative algebra $A(\widetilde{L}_{D_4}(-2, 0))$ is isomorphic to the algebra $U(\mathfrak{g})/I$, where I is the two-sided ideal of $U(\mathfrak{g})$ generated by u, $\theta(u)$ and $\theta^2(u)$, and

$$u = e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2} + e_{\epsilon_1-\epsilon_3}e_{\epsilon_1+\epsilon_3} + e_{\epsilon_1-\epsilon_4}e_{\epsilon_1+\epsilon_4}.$$

Proposition 4.1. *We have:*

(i) *The set*

$$\{L_{D_4}(-2,0), L_{D_4}(-2,-2\omega_1), L_{D_4}(-2,-2\omega_3), L_{D_4}(-2,-2\omega_4), L_{D_4}(-2,-\omega_2)\}$$

provides a complete list of irreducible weak $\widetilde{L}_{D_4}(-2,0)$–modules from the category $\mathcal{O}$.

(ii) $L_{D_4}(-2,0)$ *is the unique irreducible ordinary module for $\widetilde{L}_{D_4}(-2,0)$.*

PROOF. (i) We use the method for classification from Corollary 2.5. In this case $R^{(1)} \cong V_{D_4}(2\omega_1)$, $R^{(2)} \cong V_{D_4}(2\omega_3)$, $R^{(3)} \cong V_{D_4}(2\omega_4)$ and

$$\dim R_0^{(1)} = \dim R_0^{(2)} = \dim R_0^{(3)} = 3.$$

Using polynomials from relation (3.2) and automorphisms θ and θ^2, one obtains that the highest weights μ of $A(\widetilde{L}_{D_4}(-2,0))$–modules $V_{D_4}(\mu)$ are obtained as solutions of these 9 polynomial equations:

$$\begin{aligned}
&h_{\epsilon_1-\epsilon_2}(h_{\epsilon_1+\epsilon_2}+2)=0,\\
&h_{\epsilon_2-\epsilon_3}(h_{\epsilon_2+\epsilon_3}+1)=0,\\
&h_{\epsilon_3-\epsilon_4}h_{\epsilon_3+\epsilon_4}=0,\\
&h_{\epsilon_3-\epsilon_4}(h_{\epsilon_1+\epsilon_2}+2)=0,\\
&h_{\epsilon_2-\epsilon_3}(h_{\epsilon_1+\epsilon_4}+1)=0,\\
&h_{\epsilon_3+\epsilon_4}h_{\epsilon_1-\epsilon_2}=0,\\
&h_{\epsilon_3+\epsilon_4}(h_{\epsilon_1+\epsilon_2}+2)=0,\\
&h_{\epsilon_2-\epsilon_3}(h_{\epsilon_1-\epsilon_4}+1)=0,\\
&h_{\epsilon_1-\epsilon_2}h_{\epsilon_3-\epsilon_4}=0.
\end{aligned}$$

This easily gives that $\mu = 0$, $-2\omega_1$, $-2\omega_3$, $-2\omega_4$ or $-\omega_2$, and the claim follows from Zhu's theory.

Claim (ii) follows from the fact that $\mu = 0$ is the only dominant integral weight such that $L_{D_4}(-2,\mu)$ is in the set given in the claim (i). □

We have:

THEOREM 4.2. *Vertex operator algebra $\widetilde{L}_{D_4}(-2,0)$ is simple, i.e.,*

$$L_{D_4}(-2,0) = \frac{N_{D_4}(-2,0)}{U(\hat{\mathfrak{g}}).v + U(\hat{\mathfrak{g}}).\theta(v) + U(\hat{\mathfrak{g}}).\theta^2(v)}.$$

PROOF. Let w be a singular vector for $\hat{\mathfrak{g}}$ in $\widetilde{L}_{D_4}(-2,0)$. The classification result from Proposition 4.1 (ii) implies that $U(\hat{\mathfrak{g}}).w$ is a highest weight $\hat{\mathfrak{g}}$–module with highest weight $-2\Lambda_0$ and that w is proportional to $\mathbf{1}$. The claim follows. □

We conclude:

THEOREM 4.3. (i) *The set*

$$\{L_{D_4}(-2,0), L_{D_4}(-2,-2\omega_1), L_{D_4}(-2,-2\omega_3), L_{D_4}(-2,-2\omega_4), L_{D_4}(-2,-\omega_2)\}$$

provides a complete list of irreducible weak $L_{D_4}(-2,0)$–modules from the category $\mathcal{O}$.

(ii) $L_{D_4}(-2,0)$ *is the unique irreducible ordinary module for* $L_{D_4}(-2,0)$.

(iii) *Every ordinary $L_{D_4}(-2,0)$–module is completely reducible.*

PROOF. Proposition 4.1 and Theorem 4.2 imply claims (i) and (ii).

(iii) Let M be an ordinary $L_{D_4}(-2,0)$–module, and let w be a singular vector for $\hat{\mathfrak{g}}$ in M. The classification result from (ii) implies that $U(\hat{\mathfrak{g}}).w$ is a highest weight $\hat{\mathfrak{g}}$–module with highest weight $-2\Lambda_0$. Claim (ii) also implies that any singular vector in $U(\hat{\mathfrak{g}}).w$ has highest weight $-2\Lambda_0$ and it is proportional to w. Thus, $U(\hat{\mathfrak{g}}).w$ is an irreducible $\hat{\mathfrak{g}}$–module and the claim follows. $\square$

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