学校编码：**10384** 分类号______密级______

学号：**19020120154098** **UDC**______

厦门大学

博 士 学 位 论 文

与M-矩阵相关的一类二次矩阵方程的数值解

Numerical methods for a quadratic matrix equation associated with an M-matrix

Zubair Ahmed

指导教师姓名： 卢 琳 璋 教授

专 业 名 称： 计 算 数 学

论文提交日期： 2016 年 4 月

论文答辩时间： 2016 年 5 月

学位授予日期： 2016 年 月

答辩委员会主席：__________

评 阅 人：__________

2016 年 5 月

厦门大学学位论文原创性声明

本人呈交的学位论文是本人在导师指导下,独立完成的研究成果。 本人在论文写作中参考其他个人或集体已经发表的研究成果，均在文中以适当方式明确标明，并符合法律规范和《厦门大学研究生学术活动规范（试行）》。

另外，该学位论文为（　　　　　　　　　　　　）课题（组）的研究成果， 获得（　　　　　　　　　　）课题（组）经费或实验室的资助，在 （　　　　　　　　）实验室完成。（请在以上括号内填写课题或课题组负责人或实验室名称，未有此项声明内容的，可以不作特别声明。）

声明人（签名）：

年　　月　　日

厦门大学学位论文著作权使用声明

本人同意厦门大学根据《中华人民共和国学位条例暂行实施办法》等规定保留和使用此学位论文，并向主管部门或其指定机构送交学位论文（包括纸质版和电子版），允许学位论文进入厦门大学图书馆及其数据库被查阅、借阅。本人同意厦门大学将学位论文加入全国博士、硕士学位论文共建单位数据库进行检索，将学位论文的标题和摘要汇编出版，采用影印、缩印或者其它方式合理复制学位论文。

本学位论文属于：

（　　　）1.经厦门大学保密委员会审查核定的保密学位论文，于　　年　　月　　日解密，解密后适用上述授权。

（　　　）2.不保密，适用上述授权。

（请在以上相应括号内打"√"或填上相应内容。保密学位论文应是已经厦门大学保密委员会审定过的学位论文，未经厦门大学保密委员会审定的学位论文均为公开学位论文。此声明栏不填写的，默认为公开学位论文，均适用上述授权。）

声明人（签名）：

年　　月　　日

摘要

在科学与工程计算中常常遇到非线性矩阵方程, 尤其是二次矩阵方程. 本文研究了一类特殊的二次矩阵方程 $X^2 - EX - F = 0$, 其中 $E, F \in \mathbb{R}^{n\times n}$, E 是正对角矩阵, 而 F 是 M-矩阵. 这种类型的二次矩阵方程在 Markov 链的含噪 Wiener-Hopf 问题中有着一定的应用. 本文主要内容包括:

第一章, 简单介绍了二次矩阵方程的相关理论与算法, 并对相关的预备知识也作了简要说明. 主要内容包括非负矩阵, M-矩阵与一般矩阵理论的基本知识.

第二章, 简单介绍了非对称代数 Riccati 方程的相关理论与数值算法.

第三章, 在相关文献的基础上利用新的变换将这种特殊的二次矩阵方程转化为非对称代数 Riccati 方程, 并进而利用非对称代数 Riccati 方程的相关数值算法去求解. 理论分析和数值实验表明新的变换具有更好的性质.

第四章, 将上述结果推广到 F 是不可约奇异 M-矩阵的情形, 并给出了相应的理论分析和数值实验.

第五章, 研究了这类二次矩阵方程的特殊情况, M-矩阵的平方根. 理论分析和数值实验表明新的方法相比之前的已有方法具有一定的优势.

第六章, 总结和展望了本文中的方法和相关的课题.

关键词： 二次矩阵方程; 非对称代数 Riccati 方程; 最小非负解; M-矩阵; 不动点迭代法; Newton 迭代法; 保结构加倍算法; 矩阵平方根.

Abstract

Nonlinear matrix equation often arises in areas of science and engineering computation. This thesis, discusses numerical solution for a quadratic matrix equation associated with an M-matrix (M-QME) i.e. $X^2 - EX - F = 0$, where $E, F \in \mathbb{R}^{n\times n}$, E is a diagonal matrix and F is an M-matrix. This type of quadratic matrix equation arises in the study of noisy Wiener-Hopf problems for the Markov chains. We first transform the M-QME equation to a nonsymmetric algebraic Riccati equation (NARE) of special form and then considers some numerical methods on this special NARE to show how to compute the desired M-matrix solution, since the solution of practical interest is the M-matrix solution.

It is argued by both theoretical analysis and numerical experiments that the new transformation used in some numerical methods are effective and efficient than the existing transformation in Chun-Hua Guo [IMA journal of Numerical Analysis. 2003].

Since square roots of a matrix play an important role in many applications of matrix theory. Therefore, we generalize the above numerical method for finding the square root of a nonsingular M-matrix. Theoretical analysis and numerical experiments show that the new method is more effective and efficient.

Key Words: Quadratic matrix equation; Nonsymmetric algebraic Riccati equations; Minimal nonnegative solution; M-matrix; Fixed point iteration; Newton method; Alternating-directional doubling algorithm; Matrix square root

CONTENTS

List of figures

Chapter 1 Introduction

1.1 Background

Quadratic matrix equations arise in a number of areas of engineering applications and scientific computing. The quadratic matrix equation is just one kind of many non-linear matrix equations that need further research. The well known quadratic matrix equation is the algebraic Riccati equation. Numerical methods and theory of algebraic Riccati equations are well developed, which occur in a variety of applications[26, 47, 52].

The quadratic matrix equation (QME)

$$X^2 + EX + F = 0; \quad E, F \in \mathbb{R}^{n\times n} \tag{1.1}$$

occurs in many applications. Motivation for studying the quadratic equation comes either from the quadratic eigen value problems (see [24, 29, 35, 44, 61])

$$(\lambda^2 + \lambda E + F)x = 0, \quad E, F \in \mathbb{R}^{n\times n}, \tag{1.2}$$

or from the noisy Wiener-Hopf problems for markov chains [19, 42, 57, 58].

The answer to the question, whether the normal formula of the roots of the scalar quadratic can be generalized to (1.1), would be no unless it is specified for the special case, where E and F would not only commute but also E^2-4F would has a square root. Obviously, such solutions of (1.1) would be complicated and their characterization and computing would be interesting and challenging.

Newton's method is applied by Higham and Kim [35] with and without exact line searches to solve QME (1.1). Besides this, Bai et al. [63] prove local convergence and developed a Newton method and successive approximation method by converting the QME into an equivalent fixed point iteration. More specifically, if E is diagonal and F is an M-matrix Guo[20] developed an iterative method and prove existence and uniqueness of M-matrix solution of QME (1.1),. He achieved this by transforming QME (1.1) into a special NARE, and proved that the monotonic convergence can be obtained by iterative methods.

This thesis mainly focuses on the numerical solution of a quadratic matrix equation of the form

$$X^2 - EX - F = 0; \quad E, F \in \mathbb{R}^{n\times n} \tag{1.3}$$

where E is a diagonal matrix and F is an M-matrix. The M-QME (1.3) has application in Wiener-Hopf problems in Markov chains, (see [20, 57]). It is proved in [20] that the M-QME (1.3) has a unique non-singular M-matrix solution, which is of practical importance. Some numerical methods have been developed for solving the M-QME by transforming it into an equivalent

NARE to solve (see [20, 25, 46]). This study, mainly focus on further development of the transformation methods.

To compute the M-matrix solution of the M-QME (1.3), Guo in [20] turned it into a special NARE to solve by the transformation

$$X = \alpha I - Y$$

The motivational thoughts behind this are that many effective numerical methods for the NARE can be utilized to solve M-QME (1.3). This research uses a more general transformation to expect to achieve better effectiveness.

Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$, a matrix $X \in \mathbb{C}^{n\times n}$ is called a square root of A if

$$X^2 = A. \tag{1.4}$$

The study of square root of a general (real or complex) matrix can be traced back to the early works of Sylvester, Cayley, Frobenius in the 19th century. These researchers were followed by the works of Cecioni and Kreis in the early 20th century. The square roots of a matrix play significant role in many applications of matrix theory, for example computation of the matrix logarithm, the boundary value problems, etc. For the background of the square root of matrix, we refer to [34].

The computation of square root of real matrix A has been studied for many years by several authors, see [9, 14, 32, 34, 45] and references therein. If A is real, it may or may not have a real square root. A sufficient condition for one to exist is that A has no real negative eigenvalue see [32, 37]. Besides then, any matrix, with no non-positive real eigen values, has a unique square root. The eigen value for this class of matrices lies in the open right plane and is at times named as the principal square root, which is of usual interest. A class of matrices with no non-positive real eigen values is a highly studied class of non-singular M-matrices. The methods for finding square root of a matrix can be divided into two category, direct methods and iterative methods. In the class of direct methods, for example Schur method (see [14]) and the iterative methods depends on later class (see [30, 31, 33, 56, 62]). This work considers the computation of square root of the non-singular M-matrix of the