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博　士　学　位　论　文

几类偏微分方程的数值解法及其分析

Numerical Methods and Analysis for Several Kinds of Partial Differential Equations

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摘要

本篇论文我们探讨几类偏微分方程的数值方法，针对这些方程，我们提出一些数值格式，严格给出该格式的稳定性结果和收敛性误差估计，并给出一些例子验证我们的结论。我们主要讨论的是分数阶型微分方程。近几年，此类方程在数学建模中的应用越来越广泛，由不同分数阶微分方程导出的模型被很多领 域被提出，如材料、机械和生物系统，并发现针对一些具有记忆，不均匀或遗传性质的材料，分数阶微分方程相对于整数阶更有优势。随 着分数阶微分方程在建模领域的发展，其数值格式的构造也引起了越来越多人的兴趣。

论文主要内容如下：第一章，介绍分数阶微积分的历史和背景，并回顾一些现有的工作，基本定义和预备知识，这些都会在后面章节所用到。后我们总结本篇论文的主要结果。

第二章，致力于分数阶粘弹性模型方程MHD流的数值逼近。我们提出一种在时间方向采用有限差分，空间方向采用谱方法的离散 格式，并给出了其稳定性分析和收敛估计。若$0<\beta<1$为时间方向阶数，文中证明了在时间方向上有$(2-\beta)$阶，空间方向有谱收 敛精度。我们列出一些数值例子验证我们的理论估计，也给出不同参数β对流体的影响。

第三章，考虑广义二级流体通过多孔介质反常扩散问题的数值方法。我们给出基于时间方向有限差分，空间方向Legendre谱方法的数值格式，严格推导分析了其稳定性和收敛估计，证明了格式是无条件稳定的，收敛阶为$\mathcal{O}(\Delta t^{\min(2-\alpha,2-\beta)}+N^{1-m})$, 其中$\Delta t$, N 和 m分别为时间步长，多项式次数及精确解在空间方向的正则度。α 和 $\beta, 0<\alpha,\beta<1$ 是所涉及的两个不同的分数阶导数阶数。一些数值试验验证该方法的有效性，和所得的理论结果。

第四章，讨论Rosenau-Burgers 方程的有限差分/傅里叶谱方法，一个要特别注意的是非线性对流项的处理。对该格式我们给出了详细的分析，并证明是无条件稳定的。误差估计表明数值有$\mathcal{O}(\Delta t^2+N^{1-m})$的收敛阶数，其中$\Delta t$, N 和 m分别是时间步长，多项式次数及精确解在空间方向的正则度。同样一些数值例子验证我们的结论。

第五章，提出了一种有效的数值格式，用于求解具有非局部粘性项的水波模型，该模型描述了表面水波的传播。利用Caputo型的分数阶导数定义逼近非局部分数阶算子， 导出一种时间空间格式。该格式对分数阶导数采用已知的5/2阶格式，并

用混合线性化处理非线性项。分析表明格式是无条件稳定的。误差估计给出结合二阶向后差 分和5/2阶格式，在时间方向为二阶收敛，空间方向有谱收敛。一些数值例子验证该格式的有效性和精度。最后本方法用于考察非局部粘性波方程的解的渐近衰减率 ，同时也考虑了不同参数对衰减率的影响。

关键词：数值方法、稳定性、收敛性、分数阶粘弹性，广义二级流体，Rosenau Burgers，水波模型。

Abstract

In this dissertation, we investigate numerical methods for several partial differential equations. A number of numerical schemes are proposed and analyzed for numerical solutions to these equations. Some stability and error estimates are vigorously derived and numerically tested.

Our work is focused on fractional differential equations. The use of fractional differential equations in mathematical models has become increasingly popular in recent years. Different models using fractional differential equations have been proposed in more and more fields, covering materials, mechanical, and biological systems, and it's found that fractional differential equations gain some advantage over the classical one in modeling materials with memory, heterogeneity or inheritable character. The modeling progress on using fractional differential equations has led to increasing interest in developing numerical schemes for their solutions. The outline of this thesis is as follows:

In Chapter 1, we first discuss a brief history of fractional calculus, background, and recall some existing work, together with some basic definitions and preliminary properties which will be used in the paper. Then we summarize the main results obtained in this dissertation.

Chapter 2 is devoted to the numerical approximations for MHD flows of fractional viscoelastic model equation. A schema combining a finite difference approach in time direction and a spectral method for the space discretization is proposed. We give a detailed analysis for the proposed schema by providing stability and error estimates. We prove that convergence order of the schema is $(2-\beta)$ in time and spectral accuracy in space, with $0<\beta<1$ being the order of the fractional derivative in time. Some numerical examples are carried out to support the theoretical predictions. The impact of β on the fluid flow is also shown.

In Chapter 3, some numerical methods for generalized second grade fluid through porous media with anomalous diffusion are considered. Some schemes using finite difference approach in time direction and Legendre spectral approximations in the space direction are proposed and analyzed. The stability and error estimates are rigorously established showing that the proposed schemes are unconditionally stable, and that the convergent order is $\mathcal{O}(\Delta t^{\min(2-\alpha,2-\beta)}+N^{1-m})$, where Δt , N and m are respectively time step size, polynomial degree, and regularity in the space variable of the exact solution. α and β, $0<\alpha,\beta<1$, are two different orders involved in the fractional derivatives. Numerical computations are presented to demonstrate the effectiveness of the method and confirm the theoretical claims.

In Chapter 4, we consider finite difference/ Fourier spectral methods for the numerical solution of the Rosenau-Burgers equation. A particular attention is paid to the treatment of the nonlinear convection term. A detailed analysis is carried out for the proposed schemes. We prove that the overall schemes are unconditionally stable. The error estimation shows that the numerical solutions converge with the order $\mathcal{O}(\Delta t^2 + N^{1-m})$, where Δt , N and m are respectively time step size, polynomial degree, and regularity in the space variable of the exact solution. Numerical tests are conducted to support the theoretical results.

In Chapter 5, efficient numerical schemes are proposed for solving the water wave model with nonlocal viscous term that describes the propagation of surface water wave. By using the Caputo fractional derivative definition to approximate the nonlocal fractional operators, a time stepping scheme and spectral method in space are constructed for the considered model. The proposed method employs a known 5/2 order scheme for the fractional derivative and a mixed linearization for the nonlinear term. The analysis shows that the proposed numerical schemes are unconditionally stable. Some error estimates are provided to show that the second order backward differentiation plus 5/2 order scheme converges with order 2 in time, and exponentially in space. Several numerical examples are provided to verify the efficiency and accuracy of our method. Finally, the proposed method is used to investigate the asymptotic decay rate of the solutions of the nonlocal viscous wave equation, as well as the effect of different parameters on this decay rates.

Key words: Numerical method, stability, convergence, fractional viscoelastic, generalized second grade, Rosenau-Burgers, water wave model.

Contents

Chapter 1 Preface

1.1 Background

This thesis is devoted to numerical methods and numerical investigation for solutions of some kinds of partial differential equations. This study focuses on a number of time fractional partial differential equations, to which we construct and analyze efficient finite difference schemes in time direction and spectral methods in space. A series of numerical examples are provided to validate the proposed methods.

Given our approach to the solution of differential equations, we need to also consider the meaning of concepts such as differentiation and integration. The derivative is perhaps one of the most fundamental concept in the field of mathematical sciences. This concept has physical meaning and interpretation is direct and apparent; a rate of change, the relationship of distance and velocity or velocity and acceleration. Seemingly a completely different concept is integration, given the physical meaning of the area under a curve. Obviously there are many different geometric and physical interpretations for the concept of differentiation and integration, however that discussion is not really relevant to our current work. Rather we consider the slightly more uncommon concept of fractional differentiation and fractional integration. Fractional calculus is a mathematical branch investigating the properties of derivatives and integrals of non-integer orders (called fractional derivatives and integrals). In particular, this discipline involves the notion and methods of solving differential equations involving fractional derivatives of the unknown function (called fractional differential equations).

The idea of fractional differentiation is attributed first to a series of letters written between Leibnitz and L'Hopital in 1695, in which L'Hopital asks the pertinent question when referring to $\frac{d^n y}{dx^n} \equiv D^n y$:"What if $n = \frac{1}{2}$?", to which Leibnitz wrote prophetically,"...Thus it follows that $d^{\frac{1}{2}}x$ will be equal to $x\sqrt[2]{dx : x}$, an apparent paradox, from which one day useful consequences will be drawn."[1, 2]. A simple enough question to ask is does the operator $D^{\frac{1}{2}}$ exist, such that when applied twice we get $(D^{\frac{1}{2}})^2 = D$? The answer to this is yes. Another question that is concisely answered is, given D^n what happens if n is negative? The operator described is nothing other than a fractional integral similar to the duality of integer differentiation and integration, where integration is the inverse operation to differentiation.

This leads us immediately to the term ‘differintegration’ or ‘differintegral’, as dubbed by Oldham and Spanier in [2], notated by D^n where n is no longer constrained to be a positive integer.

A lot of contributions to the theory of fractional calculus up to the middle of the 20th century, of famous mathematicians are known: Laplace (1812), Fourier (1822), Abel (1823-1826), Liouville (1832-1837), Riemann (1847), Grünwald (1867-1872), Letnikov (1868-1872), Heaviside (1892-1912), Weyl (1917), Erdélyi (1939-1965), and many others (see Gorenflo and Mainardi [3]). Nowadays, not only fractions but also arbitrary real and even complex numbers are considered as order of differentiation. Nevertheless, the name “fractional calculus” is kept for the general theory.

There are three main approaches to fractional derivatives; the Riemann-Liouville derivative, the Caputo derivative and more recently the modified Riemann-Liouville derivative introduced by Jumarie in [4]. Jumarie has made extensive use of his modfied Riemann-Louiville derivative and accompanying generalized Taylor Series for investigations of Brownian motion and Poisson processes to a fractional order, for solving stochastic differential equations governed by fractional Brownian motion, for solving fractional partial differential equations and developing a Fourier transform of fractional order [4–9]. With this in mind the focus of our work is based primarily on the Caputo derivative due to its popularity among researchers particularly in the field of partial differential equations. Moreover the use of the Caputo derivative allows one to formulate problems with integer order boundary conditions where the Riemann-Liouville derivative requires fractional order boundary conditions which are difficult to interpret physically.

A perfect elastic material does not exists since in reality: inelasticity is always present. This inelasticity leads to energy dissipation or damping. Therefore, for a wide class of materials it is not sufficient to use an elastic constitutive model to capture the mechanical behaviour. In order to replace expensive experimental tests by numerical simulations there is a need for an accurate material model. Therefore viscoelastic constitutive models have frequently been used to simulate the time dependent behaviour of polymeric materials. The classical linear viscoelastic models that use integer order time derivatives in the constitutive laws, requires an excessive number of parameters to accurately predict observed material behaviour.

Linear viscoelasticity in combination with fractional order operators , i.e., the fractional order viscoelasticity model, have attracted considerable attention in the last decades. The fractional order viscoelasticity model is capable of describing the behaviour of many non-