Observer analysis and synthesis for Lipschitz nonlinear systems under discrete time-varying measurements

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Abstract: Observer synthesis for nonlinear Lipschitz systems with time-varying sampling is studied. To establish the exponential convergence of the observer, in this paper, we model the impact of the sampling uncertainty by a reset integrator. First, generic conditions for stability of a sampled data system are recalled. Second it is shown how to derive tractable numerical conditions to analyze the robustness of a continuous-time Luenberger observer when the sampling is discrete and time-varying. Then it is demonstrated that this emulation approach can be passed over allowing for the direct computation of an observer gain. Simulations and comparisons with related articles show the efficiency of the proposed methodology.

Keywords: Continuous-discrete observers, Lipschitz Nonlinear systems, Sampled-data systems, LMIs

1. INTRODUCTION

In many control systems full state is not available for measurements. The main reasons for this fact can either be technical or economical and constitute the primary motivation for observer synthesis. This topic has numerous practical applications and has been the subject of an immense body of work (among which [Luenberger (1971), Besancon (2007), Ellis (2002)] . The classical way to deal with the observation when using digital device is to consider periodic sampling and to analyze the resulting discrete time system [Astrom (1997), Chen (1995)]. However, in recent years a growing amount of attention has been paid to networked control systems where information is gathered discretely and communicated through a network [Hespanha (2007)]. This network can introduce uncertainty into the sampling, and the classical assumption of periodic communication is no longer justified. This fact has motivated a growing amount of research in the field of sampled data systems [Antsaklis (2007), Hetel (2017)].

In this work we will consider observer synthesis for nonlinear Lipschitz systems with discrete time-varying measurements. In the past few years, this problem has received a lot of attention. Two main observation schemes are considered. The first one uses a reset on the observer state and is usually described as a continuous-discrete observer, where between sampling the model of the plant is integrated and at sampling time an instantaneous reset of the observer state occurs. In [Andrieu (2010), Chen (2013), Dinh (2015), Etienne (2016), Mazenc (2015), Raff (2007)] this structure is considered. In [Andrieu (2010), Dinh (2015), Mazenc (2015)] a reachable set between samplings is estimated, then a discrete time Lyapunov function is derived that establishes the observer convergence. In [Etienne (2016)] practical stability is considered, where an event triggering mechanism is introduced in order to reduce the communication burden. In [Raff (2007)] a set of LMIs is given that allows to show stability of the sampled system by using a time-varying periodic Lyapunov function. This approach was extended to the case of delayed systems and time varying sampling in [Chen (2013)]. The second observer structure the observer state trajectory is piece-wise continuous and a reset occurs on the derivative of the state.

This structure was considered in [Raff (2008)], where using results for time-delay systems, a Luenberger-like observer is synthesized. In [Farza (2014)] the gain of a Luenberger-like observer exponentially decreases as the time from the last sampled measurement increases. Here we will further investigate this second observer structure. To establish our results we will model the perturbation induced by sampling as a reset integrator. This approach was considered in [Mirkin (2007), Fujioka (2009), Omran (2014), Omran (2016)] and allows to use some structural properties of the reset integrator to derive sufficient conditions on the maximal allowable sampling interval. In this case the resulting approach is "emulation" based, i.e. it is assumed that a continuous-time stabilizing controller/observer is already known [Arcak (2004), Karafyllis (2009)].

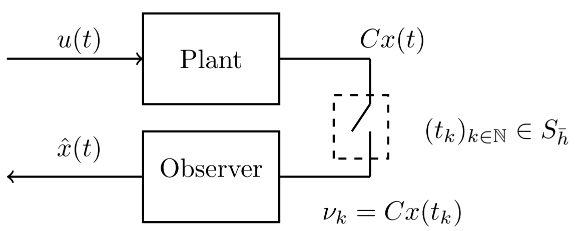

Fig. 1. Structure of the observation scheme

The contribution of our paper is twofold. First, new conditions for analysis of an *a priori known* observer gain under discrete time varying sampling will be provided. Then it will be shown how this emulation based analysis can be avoided to directly synthesize an observer for a nonlinear Lipschitz system with discrete time-varying measurements. With respect to other approaches the proposed method represents a trade-off between computational complexity and non conservatisme of the computed intersampling time. Note that with respect to continuous-discrete observer we have smoother trajectories due to conituous corrective action.

The remainder of this paper is organized as follows: first some notations are introduced, then in Section 2 the model of the plant, the observer and the concept of solution under consideration are given. In Section 3 we recall some notion allowing to give generic conditions for observer synthesis. In Section 4 numerically solvable conditions are derived for both analysis and synthesis of an observer gain. Last, in Section 5 we illustrate our approach on some examples and give elements of comparison with related articles.

Notation: v' denote the transpose of v for either a matrix or a vector. For a symmetric matrix the symbol $\star$ denotes the elements induced by symmetry: $\begin{pmatrix} A & B \\ B' & C \end{pmatrix}$ will be denoted $\begin{pmatrix} A & B \\ \star & C \end{pmatrix}$ and $He(A) := A + A'$. The number of elements of a set $\mathcal{P}$ will be denoted $Card(\mathcal{P})$. $\mathbb{R}_{\geq 0}$ corresponds to the positive real numbers. Given $p \in \mathbb{N}$, the set $\varDelta_p$ denotes the unit simplex,

$$\varDelta_p := \left\{ \lambda \in \mathbb{R}^p_{\geq 0} : \sum_{i=1}^{p} \lambda_i = 1 \right\}.$$

For a set of matrices $R_i \in \mathbb{R}^{n\times m}$, $i = 1, \dots p$, $Cov\{R_i\}_{i\in\{1\dots p\}}$ denotes its closed convex hull

$$Cov\{R_i\}_{i\in\{1\dots p\}} = \left\{ Z \in \mathbb{R}^{n\times m} : \exists \lambda \in \varDelta_p, \sum_{i=1}^{p} \lambda_i R_i = Z \right\}.$$

The Euclidean norm of $x \in \mathbb{R}^n$ is written $|x|$. In the following $\lambda_{\max}(Q)$ (resp $\lambda_{\min}(Q)$) denotes the biggest (resp smallest) eigenvalue of a symmetric matrix Q. For a square matrix $P > 0$ (resp $P < 0$) means that P is positive definite (resp negative definite). For $P > 0$, $||x||_P := \sqrt{x'Px}$. The space of functions $f : [t_0, t_1) \to \mathbb{R}^m$ which are quadratically integrable over the interval $[t_0, t_1)$ is denoted as $L_2^m[t_0, t_1)$. A function $\beta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class $\mathcal{K}$ function if $\beta(0) = 0$, α is continuous, strictly increasing, if furthermore $\lim_{s\to\infty} \beta(s) = \infty$, β is said to be $\mathcal{K}_\infty$.

2. PROBLEM STATEMENT

We consider a nonlinear Lipschitz system of the form

$$\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t) + G\phi\big(Hx(t)\big), t \geq 0, \\ &\nu_k = Cx(t_k), k \in \mathbb{N}, \\ &x(0) = x_0 \in \mathbb{R}^n, \end{aligned} \tag{1}$$

with $x(t) \in \mathbb{R}^n$ the state of the system at time t and $u(t) \in \mathbb{R}^l$ the input applied to the system at time t. Here $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, $C \in \mathbb{R}^{q\times n}$, $G \in \mathbb{R}^{n\times m}$, $H \in \mathbb{R}^{m\times n}$ and $\phi : \mathbb{R}^m \to \mathbb{R}^m$, where the nonlinear term satisfies the Lipschitz condition:

$$|\phi(a) - \phi(b)| \leq \gamma|a - b|, \quad \forall(a, b) \in \mathbb{R}^m \times \mathbb{R}^m, \tag{2}$$

for some $\gamma > 0$. Furthermore it is assumed that $u(t)$ is piece-wise continuous and bounded. As shown in Fig 1 the communication between the sensors and the observer is not continuous. We denote by $(t_k)_{k\in\mathbb{N}}$ the sequence of sampling times, $(t_k)_{k\in\mathbb{N}}$ is monotonously increasing. $\nu_k \in \mathbb{R}^p$ is the sampled output available at time t_k. In what follows we consider a class of sampling sequences of the form:

$$\mathcal{S}_{\bar{h}} = \{(t_k)_{k\in\mathbb{N}}\, s.t.\, t_0 = 0,\, 0 < t_{k+1} - t_k \leq \bar{h}\}. \tag{3}$$

with $\lim_{k\to\infty} t_k = \infty$. From (1) and the assumption in (2) for every initial condition $x_0 \in \mathbb{R}^n$ and every time $t \geq 0$ the solution of (1) exist and is unique [Khalil (1996)]. The observer will be defined as follows:

$$\begin{aligned} \dot{\hat{x}}(t) = &A\hat{x}(t) + Bu + G\phi\big(H\hat{x}(t)\big) \\ &+LC\big(\hat{x}(t_k) - x(t_k)\big), \forall\, t \in [t_k, t_{k+1}), \end{aligned} \tag{4}$$

where $\forall t \in \mathbb{R}_{\geq 0}$, $\hat{x}(t) \in \mathbb{R}^n$ is the state of the observer at time t, and $L \in \mathbb{R}^{n\times q}$ is the observer gain.

Defining $z(t) = x(t) - \hat{x}(t)$, the following observation error dynamics is obtained:

$$\begin{aligned} \dot{z}(t) = Az(t) + G\Big(\phi(Hx(t)) - \phi\Big(H\big(x(t) - z(t)\big)\Big)\Big) \\ +LCz(t_k), \forall t \in [t_k, t_{k+1}). \end{aligned} \tag{5}$$

Definition 1. The system (5) is globally exponentially stable with convergence rate $\alpha > 0$ if there exist $M > 0$ such that for any initial conditions $z_0 \in \mathbb{R}^n$, any solution $x(t)$ of (1) and any sampling sequence $(t_k)_{k\in\mathbb{N}} \in S_{\bar{h}}$ the solutions of (5) verify:

$$|z(t)| \leq Me^{-\alpha t}|z_0|.$$

We are now able to formulate the problems that we will consider in this work:

Analysis Problem (AP): Consider system (5) with a given gain L, a given set of admissible sampling sequence $S_{\bar{h}}$ and a convergence rate α. Give sufficient conditions to ensure that the system (5) is globally exponentially stable with convergence rate α for any sampling sequence $(t_k)_{k\in\mathbb{N}} \in S_{\bar{h}}$.

Synthesis Problem (SP): Consider system (5) with a given set of admissible sampling sequence $S_{\bar{h}}$ and a convergence rate α. Find a gain L such that the system (5) is globally exponentially stable with convergence rate α for any sampling sequence $(t_k)_{k\in\mathbb{N}} \in S_{\bar{h}}$.

3. PRELIMINARIES

In the case under consideration, system (5) can be described as an Linear Parameter Varying (LPV) system. This method is a classical one in the literature and can also be applied to some classes of non Lipschitz system. See for instance the work of [Zemouche (2008)]. In our setting it can be stated as follows:

Lemma 1. (Zemouche (2008)) Consider equations (1),(4), (5). Then, there exists a finite set of matrices R_i, $i \in \mathcal{P} := \{1, 2, \ldots, p\}$, such that for any $(x, z) \in \mathbb{R}^n \times \mathbb{R}^n$

$$Az + G\left[\phi\Big(Hx\Big) - \phi\Big(H(x-z)\Big)\right] \in Cov\left\{R_i z\right\}_{i\in\mathcal{P}}.$$

In practice, the set of matrices $R_i, i \in \mathcal{P}$ can be easily computed numerically. See [Zemouche (2008)] for further details. Note that the proposed methodology obviously can also be applied to linear systems as a special case.

To find a suitable L, in the case of continuous communication, numerous results are available. In particular, as stated previously, it is possible to recast (5) as an LPV system leading to a set of LMIs whose feasibility ensures asymptotic convergence of an observer (see [Zemouche (2008)]).

$$R_i'P + PR_i + C'Q + QC \leq 0, \forall\, i \in \mathcal{P} \tag{6}$$

with $P = P' > 0$, and where $\mathcal{P}$ is obtained using the differential mean value theorem. If LMIs (6) are satisfied, then the resulting gain is given by $L = P^{-1}Q$.

Using Lemma 1 and following the approach of [Fujioka (2009), Omran (2016)], we model the discrepancy between z and $z(t_k)$ in the following way:

$$\begin{aligned} \dot{z} &= \big(R(\lambda) + LC\big)z + LCw, \\ y &= \ \dot{z}, \quad \forall t \in [t_k, t_{k+1}), \end{aligned} \tag{7}$$

with $R(\lambda(t)) \in \overline{Cov}\{R_i\}_{i\in\mathcal{P}}$ and the reset integrator

$$\varDelta_{sh} : w(t) = z(t_k) - z(t) = -\int_{t_k}^{t} y(\rho)d\rho, \forall t \in [t_k, t_{k+1}). \tag{8}$$

Given the equations (5) and a sampling sequence $(t_k)_{k\in\mathbb{N}} \in \mathcal{S}_{\bar{h}}$, there is a unique solution that is defined recursively on each time interval of the form $[t_k, t_{k+1})$. Consider the sampled-data system (5) under the equivalent representation (7),(8). Assume that for every $(t_k)_{k\in\mathbb{N}} \in S_{\bar{h}}$:

H1. There exists a continuous function $s(y, w)$ which satisfies the following integral property
$$\int_{t_k}^{t} s\big(y(\rho), \omega(\rho)\big)d\rho \leq 0, \forall t \in\ [t_k, t_{k+1}),$$
along the trajectories of (7), (8).

H2. There exists a differentiable positive definite function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and $\alpha, c_1, c_2, p > 0$ such that
$$c_1|z|^p \leq V(z) \leq c_2|z|^p.$$
and
$$\dot{V}(z(t)) + \alpha V(z(t)) \leq e^{-\alpha(t-t_k)}s\big(y(t), w(t)\big), \qquad \forall t \in [t_k, t_{k+1}), \forall z(t) \in \mathbb{R}^n. \tag{9}$$

We particularize as follows the result from [Omran (2016)] to our case

Theorem 1. (Omran (2016)). Consider a sampling sequence in $S_{\bar{h}}$, and positive number $\alpha, p > 0$. Under the assumption H1-H2 the system (7) is globally exponentially stable with convergence rate $\dfrac{\alpha}{p}$.

Theorem 1 does not provide a constructive way to obtain V, or an observer gain to ensure convergence. Consider two technical Lemma's.

Lemma 2. [Omran (2014)] Consider the system (7) and the reset integrator in (8). Then, for any $y \in L_2^n[t_k, t_{k+1})$ and any $0 < X = X' \in \mathbb{R}^{n\times n}$ we have the following inequality:

$$\int_{t_k}^{t} w(\rho)'Xw(\rho) - \delta_0^2 y(\rho)'Xy(\rho)d\rho \leq 0$$

for all $t \in [t_k, t_{k+1})$, where $\delta_0 = \dfrac{2}{\pi}\bar{h}$.

Lemma 3. [Omran (2014)] Consider the system (7) and the reset integrator in (8). Then, for any $y \in L_2^n[t_k, t_{k+1})$ and any $0 < Y = Y' \in \mathbb{R}^{n\times n}$ we have the following inequality:

$$\int_{t_k}^{t} w(\rho)'Yy(\rho) + y(\rho)'Yw(\rho)d\rho \leq 0$$

for all $t \in [t_k, t_{k+1})$, where $\delta_0 = \dfrac{2}{\pi}\bar{h}$.

4. TRACTABLE CONDITIONS TO SOLVE (AP) AND (SP)

4.1 Analysis of convergence with known observation gain.

We will first assume that a suitable observer gain for system (4) is known when communication is continuous (i. e. in (5), $z(t_k) = z(t), \forall\, t \geq 0$) and give tractable sufficient conditions for analysis of a given observation gain and a given maximal inter sampling time $\bar{h}$. In other words, our approach gives sufficient conditions to test whether or not a couple $(L, \bar{h})$ (i.e. a given gain and a fixed maximal inter-sampling time) will guaranty a certain convergence rate.

In order to establish the convergence of the observer one defines the following LMIs:

Theorem 2. If the following set of LMIs is feasible:

$$W(\alpha)_{ij} := \left[\begin{array}{c|c|c} \begin{matrix} He(G_1'LC)+ \\ He(G_1'R_i)+\alpha P \end{matrix} & \begin{matrix} (R_i+LC)'G_2 \\ +P-G_1' \end{matrix} & \begin{matrix} (R_i+LC)'G_3 \\ +G_1'LC \end{matrix} \\ \hline \star & \begin{matrix} -He(G_2) \\ +\delta_0^2 Xl_j \end{matrix} & \begin{matrix} G_2'LC \\ -G_3 - Yl_j \end{matrix} \\ \hline \star & \star & \begin{matrix} He(G_3'LC) \\ -Xl_j \end{matrix} \end{array}\right] < 0, \tag{10}$$

$\forall i \in \mathcal{P}, \forall j = 1, 2$,with $l_1 = 1, l_2 = e^{-\alpha\bar{h}}$, $\alpha \geq 0, \delta_0 = \dfrac{2\bar{h}}{\pi}$, $G_1, G_2, G_3, P, X, Y \in \mathbb{R}^{n\times n}, P = P' > 0, X = X' > 0, Y = Y' > 0$, then system, (7) with observation gain L and a sampling period $(t_k)_{k\in\mathbb{N}} \in \mathcal{S}_{\bar{h}}$, is exponentially asymptotically stable with convergence rate $\dfrac{\alpha}{2}$

Proof 1. If the set of LMIs (10) is verified, then multiplying on the right by $\xi := (z', y', w')'$ and on the left by ξ' leads to

$$2z'P(R_i z + LCz + LCw) + \xi' He(\bar{G}' H_i)\xi$$
$$+ \alpha z' P z \leq -\delta_0^2 ||y||_X^2 + ||w||_X^2 + 2y'Yw, \forall i \in \mathcal{P}, \tag{11}$$

and

$$2z'P(R_i z + LCz + LCw) + \xi' He(\bar{G}' H_i)\xi$$
$$+\alpha z'Pz \leq e^{-\alpha \bar{h}} \left(-\delta_0^2 ||y||_X^2 + ||w||_X^2 + 2y'Yw, \right) \forall i \in \mathcal{P},$$

where $H_i = [R_i + LC, -I, LC]$,$\bar{G} = [G_1, G_2, G_3]$, and

$$He(\bar{G}'H_i) = \left[\begin{array}{c|c|c} He\Big(G_1'(R_i + LC)\Big) & \begin{array}{r} +(R_i + LC)'G_2 \\ -G_1' \end{array} & \begin{array}{r} (R_i + LC)'G_3 \\ +G_1'LC \end{array} \\ \hline \star & -He(G_2) & -G_3 + G_2'LC \\ \hline \star & \star & He(G_3'LC) \end{array}\right].$$

Recalling that

$$\begin{aligned} \dot{z} &= (R(\lambda(x, \hat{x})) + LC)z + LC(w) \\ y &= \dot{z}, \quad \forall t \in [t_k, t_{k+1}), \end{aligned} \tag{14}$$

Since $R\big(\lambda((x, \hat{x})\big) = \sum_{i\in\mathcal{P}} \lambda_i R_i$, for some $\lambda \in \Delta_{Card(\mathcal{P})}$, one has by convexity,

$$\sum_{i\in\mathcal{P}} \lambda_i \left[\xi'(\bar{G}'H_i + H_i'\bar{G})\xi\right] = 0.$$

Therefore by using (11) one obtains the following inequality:

$$\begin{aligned} &2z'P\Big(R(\lambda(t))z + LCz + LCw\Big) + \alpha z'Pz \\ &\leq -\delta_0^2 ||y||_X^2 + ||w||_X^2 + 2y'Yw. \end{aligned} \tag{15}$$

and

$$\begin{aligned} &2z'P\Big(R(\lambda(t))z + LCz + LCw\Big) + \alpha z'Pz \\ &\leq e^{-\alpha\bar{h}} \left(-\delta_0^2 ||y||_X^2 + ||w||_X^2 + 2y'Yw\right). \end{aligned} \tag{16}$$

Therefore $\dot{V} + \alpha V \leq e^{-\alpha(t-t_k)} s(y(t), w(t))$, where

$$s(y(t), w(t)) = \begin{pmatrix} y \\ w \end{pmatrix}' \begin{bmatrix} -\delta_0^2 X & Y \\ Y & X \end{bmatrix} \begin{pmatrix} y \\ w \end{pmatrix}.$$

By virtue of Lemma 2 and Lemma 3, $s(y(t), w(t))$ fulfils H1 of Theorem 1. Furthermore since

$$\lambda_{\min}(P)|z|^2 \leq V(z) := z'Pz \leq \lambda_{\max}(P)|z|^2.$$

One can apply Theorem 1:

$$|z(t)| \leq \sqrt{\frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}} e^{-\frac{\alpha t}{2}} |z_0|.$$

This concludes the proof. □

The proposed method consists in two steps. First a gain L is computed based on continuous-time LMIs given by (6), then the robustness of the proposed controller with respect to sampled measurements is assessed using (10). However since the gain is directly computed using continuous-time equation some conservatism may be introduced.

Remark 1. Until now no observability hypothesis were made. However it can be seen that a necessary condition for (10) to be feasible is that there exist $L, G_1, P > 0$ such that for all $i \in \mathcal{P}$, $G_1'(R_i + LC) + (R_i + LC)'G_1 + \alpha P < 0, G_1 + G_1' > 0$. This inequality requires detectability of each linear system of the form $\dot{x} = R_i x$, $y = Cx$.

4.2 Synthesis of an observation gain

In equation (10) the knowledge of the observation gain L is assumed, typically this L is computed using a continuous-time model. In order to find a L allowing (5) to converge for larger classes of sampling sequences, we show next how to circumvent the emulation based approach. Indeed modifying equation (10) one can deal simultaneously with both observer synthesis and convergence.

Corollary 1. If the following set of LMIs is feasible:

$$\Omega(\alpha)_{ij} = \left[\begin{array}{c|c|c} \begin{array}{r} \alpha P + He(R_i'G) \\ +He(QC) \end{array} & \begin{array}{c} P + R_i'G \\ -G' + C'Q' \end{array} & QC \\ \hline \star & -He(G) + \delta_0^2 X l_j & QC - Yl_j \\ \hline \star & \star & -Xl_j \end{array}\right] < 0, \tag{17}$$

$\forall i \in \mathcal{P}, \forall j = 1, 2$,with $l_1 = 1, l_2 = e^{-\alpha\bar{h}}$ $\alpha \geq 0, \delta_0 = \dfrac{2\bar{h}}{\pi}$, $P, G \in \mathbb{R}^{n\times n}, P = P' > 0,\ X = X' > 0, Y = Y' > 0 \in \mathbb{R}^{n\times n}$,

then system (7) with observation gain $L = (G')^{-1}Q$ is exponentially asymptotically stable with convergence rate $\dfrac{\alpha}{2}$.

Proof 2. If the set of LMIs (17) is feasible then $-G - G' + \delta_0^2 X < 0$ with $\delta_0^2 X > 0$. It follows that G is full rank and thus invertible.

Furthermore when (17) is feasible, (10) is feasible with a new gain $L = G'^{-1}Q$. where $G_1 = G_2 = G, G_3 = 0$.

Therefore one can apply Theorem 2. This concludes the proof. □

Some generality is lost when setting new constraints on G_2 and G_3. However the resulting gain may allow to establish convergence of the observer for larger values of $\bar{h}$ than by using the emulation based approach. Moreover the gain obtained using Corollary 1 can be analysed using Theorem 2 to maximize the value of $\bar{h}$.

It has always been assumed that the value $\bar{h}$ was given *a priori*. However if unknown the maximal admissible $\bar{h}$ can be found by performing a line search.

If one is only interested in exponential convergence without known guaranteed convergence rate (10) and (17) can be simplified, leading to less computation.

Corollary 2. If the following set of LMIs is feasible

$$\left[\begin{array}{c|c|c} \begin{array}{r} +He(R_i'G) \\ +He(QC) \end{array} & \begin{array}{c} P + R_i'G \\ -G' + C'Q' \end{array} & QC \\ \hline \star & -He(G) + \delta_0^2 X & QC - Y \\ \hline \star & \star & -X \end{array}\right] < 0, \tag{18}$$

$\forall i \in \mathcal{P}$, $\delta_0 = \dfrac{2\bar{h}}{\pi}$, $P, G \in \mathbb{R}^{n\times n}, P = P' > 0,\ X = X' > 0, Y = Y' > 0 \in \mathbb{R}^{n\times n}$, Then (17) is feasible for α sufficiently small.

Proof 3. By assumption on LMIs (18)

$$\Omega(0)_{ij} < 0, \forall i \in \mathcal{P}, j = \{1, 2\}.$$

Where $\Omega(0)_{ij}$ is defined in (17). Since $\forall i \in \mathcal{P}, j = \{1, 2\}$ $\Omega(\alpha)_{ij}$ is a continuous matrix function of α it follows that there exists $\tilde{\alpha}$ sufficiently small, such that $\forall \alpha \leq \tilde{\alpha}$ (17) is verified. □

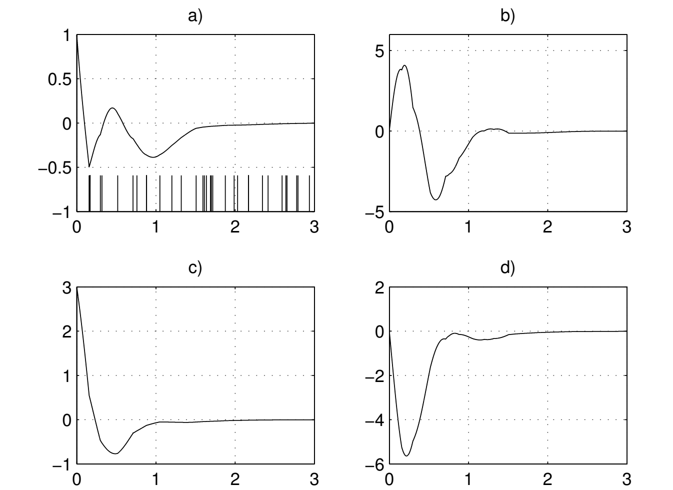

Fig. 2. Observation error of the 4 states for the flexible link. a)z_1, b) z_2, c)z_3, d)z_4. Sampling instants (plot a) bellow

This argument applies mutatis mutandis to the set of LMIs (10). As a consequences one can apply Theorem 1 to show exponential convergence of (7). From this continuity argument it follows that in (10), (17), one can choose $\alpha \geq 0$ while in Theorem 1 it is required that $\alpha > 0$.

5. EXAMPLES

5.1 Flexible Joint

Let us consider the model (1) of a flexible joint described by [Spong (1987)], and considered by [Raff (2007), Raff (2008), Chen (2013)]:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{pmatrix}, \quad \begin{array}{l} B = \begin{pmatrix} 0 & 21.6 & 0 & 0 \end{pmatrix}', \\ C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \end{array}$$

with $\phi = (0, 0, 0, 3.3 \sin x_3)'$. The input $u = sin(t)$ is applied to the system. One compute for $\mathcal{P} = \{1, 2\}$:

$$R_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -16.2 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -22.8 & 0 \end{pmatrix}.$$

Using the conditions of Corollary 1 for $\bar{h} = 0.18, \alpha = 1$ one obtain :

$$L = \begin{pmatrix} -4. & 19.35 & -1.52 & -10.12 \\ -0.68 & -4.41 & -0.27 & 1.19 \end{pmatrix}'.$$

The trajectory of the observation error along the sampling sequence is given in Fig. 2, where after a transient the observer converge to the true state. Note that this trajectory is not differentiable at sampling instants.

As expected simulation exhibits a trade-off between convergence rate and maximal allowable sampling interval. Using Corollary 1 the following parameters are given in Table 1

As shown in Table 2 it is possible to compare the maximal sampling time obtained with [Raff (2007), Raff (2008), Chen (2013)]. The class of sampling time considered in this articles is of the form $t_{k+1} - t_k \in [\underline{\tau}, \overline{\tau}]$.

In the case where measurement are continuous , this same example has also been considered in [Zemouche (2008)] with $C = (1, 0, 0, 0)$. Considering the gain obtained in this case, it is possible to apply Theorem 2 and use an emulation approach. One finds out that the LMIs (10) is verified until $\bar{h} = 0.15$. On the same system using Corollary 1 along with LMIs (17) one finds a new gain $\tilde{L}$ such that $\bar{h} = 0.38$. Using this new $\tilde{L}$ with the LMIs (10) $\bar{h}$ can be slightly increased to $\bar{h} = 0.4$. In order to compare our approach with related work, we have tested the feasibility of different sets of LMIs using the Sedumi solver ([Sturm (1999)]) in MATLAB, the results are listed in Table 2.

6. CONCLUSION

In this work the problem of observer synthesis of nonlinear Lipschitz systems subject to time-varying sampling has been studied. In order to conduct the analysis one defines a reset integrator, whose purpose is to model the impact of the time-varying sampling on the system. New conditions are proposed to analyse the robustness of an *a priori* known gain for a continuous-time Luenberger observer when the sensors communicate information discretely and the sampling is time-varying. The proposed framework also allows to give new conditions for observer gain synthesis. The effectiveness of this approach is illustrated on a robot flexible link and a single-link direct-drive manipulator actuated by a permanent magnet DC brush motor. It appears that the proposed methodology is competitive with respect to related approaches. Further study will investigate robustness with respect to noise as well as decentralized sensor communications.

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$\bar{h}$	0.38	0.31	0.2	0.18	0.15
α	0	0.2	0.6	1	1.5

Table 1. Sampling sequence and convergence rate

	Raff (2007)	Raff (2008)	Chen (2013)	Chen (2013)	Th 10
$\underline{\tau}$	0	0.3	0.45	0.1	0
$\overline{\tau}$	0.1	0.3	0.45	0.34	0.41

Table 2. Admissible sampling sequence for the flexible joint according to different approaches

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