# Nash Region of the Linear Deterministic Interference Channel with Noisy Output Feedback

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***Abstract*—In this paper, the $\eta$-Nash equilibrium ($\eta$-NE) region of the two-user linear deterministic interference channel (IC) with noisy channel-output feedback is characterized for all $\eta > 0$. The $\eta$-NE region, a subset of the capacity region, contains the set of all achievable information rate pairs that are stable in the sense of an $\eta$-NE. More specifically, given an $\eta$-NE coding scheme, there does not exist an alternative coding scheme for either transmitter-receiver pair that increases the individual rate by more than $\eta$ bits per channel use. Existing results such as the $\eta$-NE region of the linear deterministic IC without feedback and with perfect output feedback are obtained as particular cases of the result presented in this paper.**

***Index Terms*—Nash equilibrium, Linear Deterministic Interference Channel.**

## I. System Model

Consider the two-user decentralized linear deterministic interference channel with noisy channel-output feedback (D-LD-IC-NOF) depicted in Figure 1. For all $i \in \{1, 2\}$, with $j \in \{1, 2\} \setminus \{i\}$, the number of bit-pipes between transmitter $i$ and its intended receiver is denoted by $\overrightarrow{n}_{ii}$; the number of bit-pipes between transmitter $i$ and its non-intended receiver is denoted by $n_{ji}$; and the number of bit-pipes between receiver $i$ and its corresponding transmitter is denoted by $\overleftarrow{n}_{ii}$. These six non-negative integer parameters describe the D-LD-IC-NOF in Figure 1.

At transmitter $i$, the channel-input $\boldsymbol{X}_{i,n}$ at channel use $n$, with $n \in \{1, 2, \ldots, N_i\}$, is a $q$-dimensional binary vector $\boldsymbol{X}_{i,n} = \left(X_{i,n}^{(1)}, X_{i,n}^{(2)}, \ldots, X_{i,n}^{(q)}\right)^{\mathsf{T}} \in \mathcal{X}_i$, with $\mathcal{X}_i = \{0, 1\}^q$,

$$q = \max\left(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}\right), \tag{1}$$

and $N_i \in \mathbb{N}$ is the block-length of transmitter-receiver pair $i$. At receiver $i$, the channel-output $\overrightarrow{\boldsymbol{Y}}_{i,n}$ at channel use $n$, with $n \in \{1, 2, \ldots, \max(N_1, N_2)\}$, is also a $q$-dimensional binary vector $\overrightarrow{\boldsymbol{Y}}_{i,n} = \left(\overrightarrow{Y}_{i,n}^{(1)}, \overrightarrow{Y}_{i,n}^{(2)}, \ldots, \overrightarrow{Y}_{i,n}^{(q)}\right)^{\mathsf{T}}$. Let $\boldsymbol{S}$ be a $q \times q$ binary lower shift matrix. The input-output relation during channel use $n$ is given by

$$\overrightarrow{\boldsymbol{Y}}_{i,n} = \boldsymbol{S}^{q-\overrightarrow{n}_{ii}} \boldsymbol{X}_{i,n} + \boldsymbol{S}^{q-n_{ij}} \boldsymbol{X}_{j,n}, \tag{2}$$

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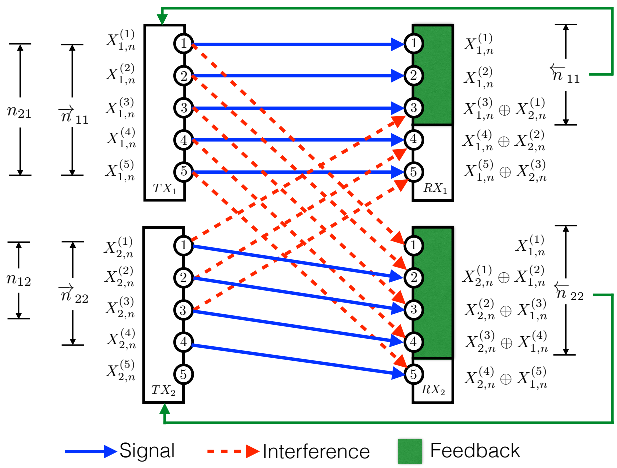


Fig. 1. Two-user linear deterministic interference channel with noisy channel-output feedback at channel use $n$.

where $\boldsymbol{X}_{i,n} = (0, 0, \ldots, 0)^{\mathsf{T}}$ for all $n > N_i$. The feedback signal $\overleftarrow{\boldsymbol{Y}}_{i,n}$ available at transmitter $i$ at the end of channel use $n$ is

$$\overleftarrow{\boldsymbol{Y}}_{i,n} = \boldsymbol{S}^{\left(\max\left(\overrightarrow{n}_{ii}, n_{ij}\right) - \overleftarrow{n}_{ii}\right)^{+}} \overrightarrow{\boldsymbol{Y}}_{i,n-d}, \tag{3}$$

where $d$ is a finite delay, additions and multiplications are defined over the binary field, and $(\cdot)^{+}$ is the positive part operator.

Without any loss of generality, the feedback delay is assumed to be equal to one channel use. Let $\mathcal{W}_i$ be the set of message indices of transmitter $i$. Transmitter $i$ sends the message index $W_i \in \mathcal{W}_i$ by transmitting the codeword $\boldsymbol{X}_i = (\boldsymbol{X}_{i,1}, \boldsymbol{X}_{i,2}, \ldots, \boldsymbol{X}_{i,N_i}) \in \mathcal{X}_i^{N_i}$, which is a binary $q \times N_i$ matrix. The encoder of transmitter $i$ can be modeled as a set of deterministic mappings $f_{i,1}^{(N)}, f_{i,2}^{(N)}, \ldots, f_{i,N_i}^{(N)}$, with $f_{i,1}^{(N)} : \mathcal{W}_i \times \mathbb{N} \rightarrow \{0, 1\}^q$ and for all $n \in \{2, 3, \ldots, N_i\}$, $f_{i,n}^{(N)} : \mathcal{W}_i \times \mathbb{N} \times \{0, 1\}^{q \times (n-1)} \rightarrow \{0, 1\}^q$, such that

$$\boldsymbol{X}_{i,1} = f_{i,1}^{(N)}\left(W_i, \Omega_i\right) \text{ and} \tag{4a}$$

$$\boldsymbol{X}_{i,n} = f_{i,n}^{(N)}\left(W_i, \Omega_i, \overleftarrow{\boldsymbol{Y}}_{i,1}, \overleftarrow{\boldsymbol{Y}}_{i,2}, \ldots, \overleftarrow{\boldsymbol{Y}}_{i,n-1}\right), \tag{4b}$$

where $\Omega_i$ is a randomly generated index known by both transmitter $i$ and receiver $i$, while unknown by transmitter $j$ and receiver $j$. The decoder of receiver $i$ is defined by a deterministic function $\psi_i^{(N)} : \{0, 1\}^{q \times N} \times \mathbb{N} \rightarrow \mathcal{W}_i$. At the end of the communication, receiver $i$ uses the $q \times N$ binary matrix $\left(\overrightarrow{\boldsymbol{Y}}_{i,1}, \overrightarrow{\boldsymbol{Y}}_{i,2}, \ldots, \overrightarrow{\boldsymbol{Y}}_{i,N}\right)$ and $\Omega_i$ to obtain an estimate $\widehat{W}_i \in \mathcal{W}_i$ of the message index $W_i$, i.e.,

$\widehat{W}_i = \psi_i^{(N)}\left(\overrightarrow{\boldsymbol{Y}}_{i,1}, \overrightarrow{\boldsymbol{Y}}_{i,2}, \ldots, \overrightarrow{\boldsymbol{Y}}_{i,N}, \Omega_i\right)$. Let $W_i$ be written as $c_{i,1}\, c_{i,2} \ldots c_{i,M_i}$ in binary form, with $M_i = \lceil \log_2 |\mathcal{W}_i| \rceil$. Let also $\widehat{W}_i$ be written as $\widehat{c}_{i,1}\, \widehat{c}_{i,2} \ldots \widehat{c}_{i,M_i}$ in binary form.

A transmit-receive configuration for transmitter-receiver pair $i$, denoted by $s_i$, can be described in terms of the block-length $N_i$, the number of bits per block $M_i$, the channel-input alphabet $\mathcal{X}_i$, the codebook, the encoding functions $f_{i,1}^{(N)}, f_{i,2}^{(N)}, \ldots, f_{i,N_i}^{(N)}$, the decoding function $\psi_i^{(N)}$, etc.

The average bit error probability at decoder $i$ given the configurations $s_1$ and $s_2$, denoted by $p_i(s_1, s_2)$, is given by

$$p_i(s_1, s_2) = \frac{1}{M_i} \sum_{\ell=1}^{M_i} \mathbb{1}_{\{\widehat{c}_{i,\ell} \neq c_{i,\ell}\}}. \tag{5}$$

Within this context, a rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ is said to be achievable if it complies with the following definition.

*Definition 1 (Achievable Rate Pairs): A rate pair $(R_1, R_2) \in \mathbb{R}_+^2$ is achievable if there exists at least one pair of configurations $(s_1, s_2)$ such that the decoding bit error probabilities $p_1(s_1, s_2)$ and $p_2(s_1, s_2)$ can be made arbitrarily small by letting the block-lengths $N_1$ and $N_2$ grow to infinity.*

The aim of transmitter $i$ is to autonomously choose its transmit-receive configuration $s_i$, in order to maximize its achievable rate $R_i$. Note that the rate achieved by transmitter-receiver $i$ depends on both configurations $s_1$ and $s_2$ due to mutual interference. This reveals the competitive interaction between both links in the decentralized interference channel. The following section models this interaction using tools from game theory.

## II. The Two-User Interference Channel as a Game

The competitive interaction between the two transmitter-receiver pairs in the decentralized interference channel can be modeled by the following game in normal-form:

$$\mathcal{G} = \left(\mathcal{K}, \{\mathcal{A}_k\}_{k\in\mathcal{K}}, \{u_k\}_{k\in\mathcal{K}}\right). \tag{6}$$

The set $\mathcal{K} = \{1, 2\}$ is the set of players, that is, the set of transmitter-receiver pairs. The sets $\mathcal{A}_1$ and $\mathcal{A}_2$ are the sets of actions of players 1 and 2, respectively. An action of a player $i \in \mathcal{K}$, which is denoted by $s_i \in \mathcal{A}_i$, is basically its transmit-receive configuration as described in Section I. The utility function of player $i$ is $u_i : \mathcal{A}_1 \times \mathcal{A}_2 \to \mathbb{R}_+$ and it is defined as the information rate of transmitter $i$,

$$u_i(s_1, s_2) = \begin{cases} R_i = \frac{M_i}{N_i}, & \text{if} \quad p_i(s_1, s_2) < \epsilon \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

where $\epsilon > 0$ is an arbitrarily small number.

This game formulation was first proposed in [1] and [2]. A class of transmit-receive configurations $\boldsymbol{s}^* = (s_1^*, s_2^*) \in \mathcal{A}_1 \times \mathcal{A}_2$ that are particularly important in the analysis of this game is referred to as the set of $\eta$-Nash equilibria ($\eta$-NE), with $\eta > 0$. This type of configuration satisfies the following definition.

*Definition 2 ($\eta$-Nash equilibrium): In the game $\mathcal{G} = \left(\mathcal{K}, \{\mathcal{A}_k\}_{k\in\mathcal{K}}, \{u_k\}_{k\in\mathcal{K}}\right)$, an action profile $(s_1^*, s_2^*)$ is an $\eta$-Nash equilibrium if for all $i \in \mathcal{K}$ and for all $s_i \in \mathcal{A}_i$, there exits an $\eta > 0$ such that*

$$u_i(s_i, s_j^*) \leqslant u_i(s_i^*, s_j^*) + \eta. \tag{8}$$

Let $(s_1^*, s_2^*)$ be an $\eta$-Nash equilibrium action profile of the game in (6). Then, none of the transmitters can increase its own information transmission rate more than $\eta$ bits per channel use by changing its own transmit-receive configuration and keeping the average bit error probability arbitrarily close to zero. Note that for $\eta$ sufficiently large, from Definition 2, any pair of configurations can be an $\eta$-NE. Alternatively, for $\eta = 0$, the classical definition of Nash equilibrium is obtained [3]. In this case, if a pair of configurations is a Nash equilibrium ($\eta = 0$), then each individual configuration is optimal with respect to each other. Hence, the interest is to describe the set of all possible $\eta$-NE rate pairs $(R_1, R_2)$ of the game in (6) with the smallest $\eta$ for which there exists at least one equilibrium configuration pair. The set of rate pairs that can be achieved at an $\eta$-NE is known as the $\eta$-Nash equilibrium region.

*Definition 3 ($\eta$-NE Region): Let $\eta > 0$ be fixed. An achievable rate pair $(R_1, R_2)$ is said to be in the $\eta$-NE region of the game $\mathcal{G} = \left(\mathcal{K}, \{\mathcal{A}_k\}_{k\in\mathcal{K}}, \{u_k\}_{k\in\mathcal{K}}\right)$ if there exists a pair $(s_1^*, s_2^*) \in \mathcal{A}_1 \times \mathcal{A}_2$ that is an $\eta$-NE and the following holds:*

$$u_1(s_1^*, s_2^*) = R_1 \quad \text{and} \quad u_2(s_1^*, s_2^*) = R_2. \tag{9}$$

The following section characterizes the $\eta$-NE region (Def. 3) of the two-user D-LD-IC-NOF in (6), denoted by $\mathcal{N}_\eta\left(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22}\right)$, for fixed parameters $\left(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22}\right) \in \mathbb{N}^6$ and for all $\eta > 0$.

## III. Main Results

The $\eta$-NE region is characterized in terms of two regions: the capacity region, denoted by $\mathcal{C}(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ and a convex region, denoted by $\mathcal{B}_\eta(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$. In the following, the tuple $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \overleftarrow{n}_{11}, \overleftarrow{n}_{22})$ is used only when needed.

The capacity region $\mathcal{C}$ of the two-user LD-IC-NOF is described in Theorem 1 in [4], which is a generalization of previous works in [5] and [6]. For all $\eta > 0$, the convex region $\mathcal{B}_\eta$ is defined as follows:

$$\mathcal{B}_\eta = \Big\{(R_1, R_2) : L_i \leqslant R_i \leqslant U_i, \text{ for all } i \in \{1, 2\}\Big\}, \tag{10}$$

where,

$$L_i = \left((\overrightarrow{n}_{ii} - n_{ij})^+ - \eta\right)^+ \text{ and} \tag{11a}$$

$$U_i = \max(\overrightarrow{n}_{ii}, n_{ij}) - \Bigg(\min\left((\overrightarrow{n}_{jj} - n_{ji})^+, n_{ij}\right) - \left(\min\left((\overrightarrow{n}_{jj} - n_{ij})^+, n_{ji}\right) - \left(\max(\overrightarrow{n}_{jj}, n_{ji}) - \overleftarrow{n}_{jj}\right)^+\right)^+\Bigg)^+ + \eta, \tag{11b}$$

with $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$. Theorem 1 uses the region $\mathcal{B}_\eta$ in (10) and the capacity region $\mathcal{C}$ to describe the $\eta$-NE region $\mathcal{N}_\eta$.

*Theorem 1: Let $\eta > 0$ be fixed. The $\eta$-NE region $\mathcal{N}_\eta$ of the two-user D-LD-IC-NOF with parameters $\overrightarrow{n}_{11}$, $\overrightarrow{n}_{22}$, $n_{12}$, $n_{21}$, $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$, is $\mathcal{N}_\eta = \mathcal{C} \cap \mathcal{B}_\eta$.*

Figure 2 shows the capacity region $\mathcal{C}$ and the $\eta$-NE region $\mathcal{N}_\eta$ of a channel with parameters $\overrightarrow{n}_{11} = 7$, $\overrightarrow{n}_{22} = 6$, $n_{12} = 4$, $n_{21} = 4$ and different values for $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$, with $\eta$ chosen arbitrarily small. Note that when $\overleftarrow{n}_{11} \in$

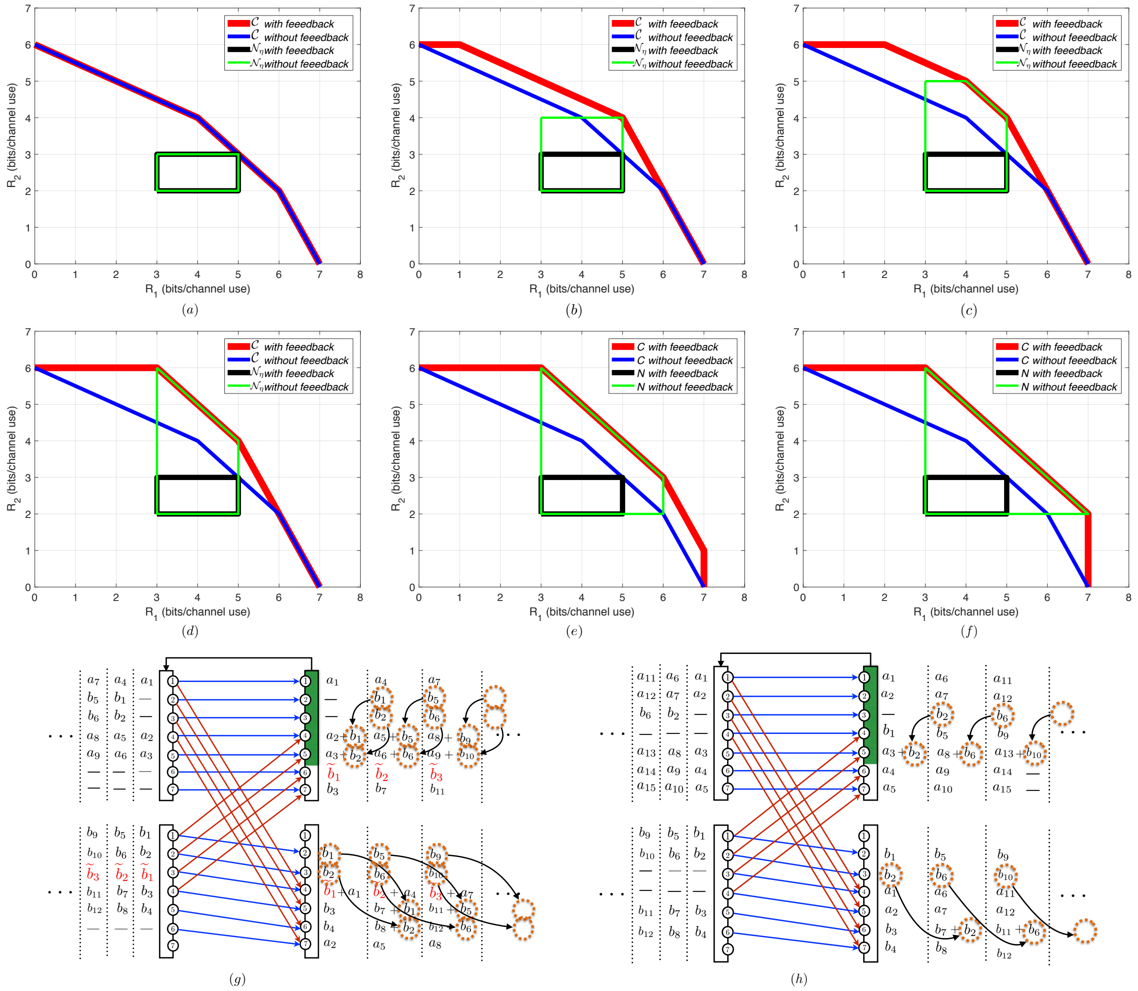


Fig. 2. Capacity region $\mathcal{C}(7,6,4,4,0,0)$ (thin blue line) and $\eta$-NE region $\mathcal{N}_\eta(7,6,4,4,0,0)$ (thick black line) with $\eta$ chosen arbitrarily small. Fig. 2a shows the capacity region $\mathcal{C}(7,6,4,4,\overleftarrow{n}_{11},\overleftarrow{n}_{22})$ (thick red line) and the $\eta$-NE region $\mathcal{N}_\eta(7,6,4,4,\overleftarrow{n}_{11},\overleftarrow{n}_{22})$ (thin green line), with $\overleftarrow{n}_{11}\in\{0,1,2,3,4\}$ and $\overleftarrow{n}_{22}\in\{0,1,2,3,4\}$. Fig. 2b shows the capacity region $\mathcal{C}(7,6,4,4,5,\overleftarrow{n}_{22})$ (thick red line) and the $\eta$-NE region $\mathcal{N}_\eta(7,6,4,4,5,\overleftarrow{n}_{22})$ (thin green line), with $\overleftarrow{n}_{22}\in\{0,1,2,3,4\}$. Fig. 2c shows the capacity region $\mathcal{C}(7,6,4,4,6,\overleftarrow{n}_{22})$ (thick red line) and the $\eta$-NE region $\mathcal{N}_\eta(7,6,4,4,6,\overleftarrow{n}_{22})$ (thin green line), with $\overleftarrow{n}_{22}\in\{0,1,2,3,4\}$. Fig. 2d shows the capacity region $\mathcal{C}(7,6,4,4,7,\overleftarrow{n}_{22})$ (thick red line) and the $\eta$-NE region $\mathcal{N}_\eta(7,6,4,4,7,\overleftarrow{n}_{22})$ (thin green line), with $\overleftarrow{n}_{22}\in\{0,1,2,3,4\}$. Fig. 2e shows the capacity region $\mathcal{C}(7,6,4,4,7,5)$ (thick red line) and the $\eta$-NE region $\mathcal{N}_\eta(7,6,4,4,7,5)$ (thin green line). Fig. 2f shows the capacity region $\mathcal{C}(7,6,4,4,7,6)$ (thick red line) and the $\eta$-NE region $\mathcal{N}_\eta(7,6,4,4,7,6)$ (thin green line). Fig. 2g and Fig. 2h illustrate the achievability scheme for the equilibrium rate pair $(3,4)$ and $(5,4)$ in $\mathcal{N}_\eta(7,6,4,4,5,0)$.

$\{0,1,2,3,4\}$ and $\overleftarrow{n}_{22}\in\{0,1,2,3,4\}$ (Figure 2a), it follows that $\mathcal{N}_\eta(7,6,4,4,\overleftarrow{n}_{11},\overleftarrow{n}_{22}) = \mathcal{N}_\eta(7,6,4,4,0,0)$. Thus, in this case the use of feedback in any of the transmitter-receiver pairs does not enlarge the $\eta$-Nash region. Alternatively, when $\overleftarrow{n}_{11} > 4$ and $\overleftarrow{n}_{22}\in\{0,1,2,3,4\}$ (Figures 2b, 2c and 2d), the resulting $\eta$-Nash region is strictly larger than in the previous case. A similar effect is observed in Figures 2e and 2f. This observation implies the existence of a threshold on each feedback parameter $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$ beyond which the $\eta$-Nash region is enlarged. The exact values of $\overleftarrow{n}_{11}$ and $\overleftarrow{n}_{22}$, given a fixed tuple $(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21})$, beyond which the $\eta$-Nash region can be enlarged is presented in [7]. Figure 2g and Figure 2h show the coding schemes to achieve the rate pairs $(3,4)$ and $(5,4)$, respectively, when $\overleftarrow{n}_{11}=5$ and $\overleftarrow{n}_{22}=0$. In Figure 2g, note that common randomness is used by transmitter-receiver pair 2 to prevent transmitter-receiver pair 1 from increasing its individual rate. More specifically, the bits $\tilde{b}_1$, $\tilde{b}_2$, $\tilde{b}_3$, ...are known by both transmitter 2 and receiver 2. The use of common randomness is also observed in [8], [9] and [10]. Common randomness reflects a competitive behavior between both transmitter-receiver pairs. In Figure 2g, common randomness is not used by transmitter-receiver pair 2 and thus, transmitter-receiver pair 1 achieves a higher rate at an $\eta$-NE with respect to the previous example. This suggests a more altruistic behavior.

The $\eta$-NE region $\mathcal{N}_\eta$ without feedback, i.e., when $\overleftarrow{n}_{11}=0$ and $\overleftarrow{n}_{22}=0$ (Theorem 1 in [8]), is $\mathcal{N}_\eta(\overrightarrow{n}_{11},\overrightarrow{n}_{22},n_{12},n_{21},0,0)$. The $\eta$-NE region with perfect feedback i.e., $\overleftarrow{n}_{11}=\max(\overrightarrow{n}_{11},n_{12})$ and $\overleftarrow{n}_{22}=\max(\overrightarrow{n}_{22},n_{21})$ (Theorem 1 in [9]), is $\mathcal{N}_\eta(\overrightarrow{n}_{11}, \overrightarrow{n}_{22}, n_{12}, n_{21}, \max(\overrightarrow{n}_{11},n_{12}), \max(\overrightarrow{n}_{22},n_{21}))$. From the comments above, it is interesting to highlight the following inclusions:

$$\begin{aligned}&\mathcal{N}_\eta\Big(\overrightarrow{n}_{11},\overrightarrow{n}_{22},n_{12},n_{21},0,0\Big)\subseteq\\&\mathcal{N}_\eta\Big(\overrightarrow{n}_{11},\overrightarrow{n}_{22},n_{12},n_{21},\overleftarrow{n}_{11},\overleftarrow{n}_{22}\Big)\subseteq\\&\mathcal{N}_\eta\Big(\overrightarrow{n}_{11},\overrightarrow{n}_{22},n_{12},n_{21},\max\left(\overrightarrow{n}_{11},n_{12}\right),\max\left(\overrightarrow{n}_{22},n_{21}\right)\Big),\end{aligned} \tag{12}$$

for all $\eta > 0$. The inclusions above might appear trivial, however, enlarging the set of actions often leads to paradoxes

(Braess Paradox [11]) in which the new game possesses equilibria at which players obtain smaller individual benefits and/or smaller total benefit. Nonetheless, letting both transmitter-receiver pairs to use feedback does not induce this type of paradoxes with respect to the case without feedback.

## IV. Proofs

To prove Theorem 1, the first step is to show that a rate pair $(R_1, R_2)$, with $R_i < L_i$ or $R_i > U_i$ for at least one $i \in \{1, 2\}$, is not achievable at an $\eta$-equilibrium for all $\eta > 0$. That is,

$$\mathcal{N}_\eta \subseteq \mathcal{C} \cap \mathcal{B}_\eta. \tag{13}$$

The second step is to show that, for all $\eta > 0$, any point in $\mathcal{C} \cap \mathcal{B}_\eta$ can be achievable at an $\eta$-equilibrium. That is,

$$\mathcal{N}_\eta \supseteq \mathcal{C} \cap \mathcal{B}_\eta, \tag{14}$$

which proves the equality $\mathcal{N}_\eta = \mathcal{C} \cap \mathcal{B}_\eta$.

*a) Proof of (13):* The proof of (13) is completed by the following lemmas.

*Lemma 1: A rate pair $(R_1, R_2) \in \mathcal{C}$, with either $R_1 < L_1$ or $R_2 < L_2$ is not achievable at an $\eta$-equilibrium for all $\eta > 0$.*

*Proof:* The proof of Lemma 1 is presented in [7]. ■

The intuition behind this proof is that the rate $R_i = (\overrightarrow{n}_{ii} - n_{ij})^+$ is always achievable independently of the coding scheme of transmitter-receiver pair $j$. To achieve $R_i = (\overrightarrow{n}_{ii} - n_{ij})^+$ transmitter $i$ uses the most significant bit-pipes, which are interference free, to transmit new bits at each channel use $n$.

*Lemma 2: A rate pair $(R_1, R_2) \in \mathcal{C}$, with either $R_1 > U_1$ or $R_2 > U_2$ is not achievable at an $\eta$-equilibrium for all $\eta > 0$.*

*Proof:* The proof of Lemma 2 is presented in [7]. ■

This proof is based on the fact that at an $\eta$-NE, transmitter $j$ might re-transmit some of the bits previously transmitted by transmitter $i$. The interference produced by those re-transmitted bits at receiver $i$ can be eliminated if they were received interference free during previous channel uses. This allows transmitter $i$ to use the bit-pipes interfered with by those re-transmitted bits to send new information bits at each channel use. The key point of this proof is to show that the maximum number of bits that can be re-transmitted at an $\eta$-NE is upper bounded.

*b) Proof of (14):* Consider a modification of the coding scheme with noisy feedback presented in [4], which combines rate splitting [12], block Markov superposition coding [13] and backward decoding [14]. The novelty with respect to [4] consists of allowing users to introduce common randomness as suggested in [8] and [9].

Consider without any loss of generality that $N = N_1 = N_2$. Let $W_i^{(t)} \in \{1, 2, \ldots, 2^{NR_i}\}$ and $\Omega_i^{(t)} \in \{1, 2, \ldots, 2^{NR_{i,R}}\}$ denote the message index and the random message index sent by transmitter $i$ during the $t$-th block, with $t \in \{1, 2, \ldots, T\}$, respectively. Following a rate-splitting argument, assume that $\left(W_i^{(t)}, \Omega_i^{(t)}\right)$ is represented by the indices $\left(W_{i,C1}^{(t)}, \Omega_{i,R1}^{(t)}, W_{i,C2}^{(t)}, \Omega_{i,R2}^{(t)}, W_{i,P}^{(t)}\right) \in \{1, 2, \ldots, 2^{NR_{i,C1}}\} \times \{1, 2, \ldots, 2^{NR_{i,R1}}\} \times \{1, 2, \ldots, 2^{NR_{i,C2}}\} \times \{1, 2, \ldots, 2^{NR_{i,R2}}\} \times \{1, 2, \ldots, 2^{NR_{i,P}}\}$, where $R_i = R_{i,C1} + R_{i,C2} + R_{i,P}$ and $R_{i,R} = R_{i,R1} + R_{i,R2}$. The rate $R_{i,R}$ is the number of transmitted bits that are known by both transmitter $i$ and receiver $i$ per channel use, and thus it does not have an impact on the information rate $R_i$.

The codeword generation follows a four-level superposition coding scheme. The indices $W_{i,C1}^{(t-1)}$ and $\Omega_{i,R1}^{(t-1)}$ are assumed to be decoded at transmitter $j$ via the feedback link of transmitter-receiver pair $j$ at the end of the transmission of block $t-1$. Therefore, at the beginning of block $t$, each transmitter possesses the knowledge of the indices $W_{1,C1}^{(t-1)}$, $\Omega_{1,R1}^{(t-1)}$, $W_{2,C1}^{(t-1)}$ and $\Omega_{2,R1}^{(t-1)}$. In the case of the first block $t = 1$, the indices $W_{1,C1}^{(0)}$, $\Omega_{1,R1}^{(0)}$, $W_{2,C1}^{(0)}$ and $\Omega_{1,R2}^{(0)}$ are assumed to be known by all transmitters and receivers. Using these indices both transmitters are able to identify the same codeword in the first code-layer. This first code-layer, which is common for both transmitter-receiver pairs, is a sub-codebook of $2^{N(R_{1,C1}+R_{2,C1}+R_{1,R1}+R_{2,R1})}$ codewords. Denote by $\boldsymbol{u}\left(W_{1,C1}^{(t-1)}, \Omega_{1,R1}^{(t-1)}, W_{2,C1}^{(t-1)}, \Omega_{2,R1}^{(t-1)}\right)$ the corresponding codeword in the first code-layer. The second codeword is chosen by transmitter $i$ using $\left(W_{i,C1}^{(t)}, \Omega_{i,R1}^{(t)}\right)$ from the second code-layer, which is a sub-codebook of $2^{N(R_{i,C1}+R_{i,R1})}$ codewords corresponding to the codeword $\boldsymbol{u}\left(W_{1,C1}^{(t-1)}, \Omega_{1,R1}^{(t-1)}, W_{2,C1}^{(t-1)}, \Omega_{2,R1}^{(t-1)}\right)$. Denote by $\boldsymbol{u}_i\left(W_{1,C1}^{(t-1)}, \Omega_{1,R1}^{(t-1)}, W_{2,C1}^{(t-1)}, \Omega_{2,R1}^{(t-1)}, W_{i,C1}^{(t)}, \Omega_{i,R1}^{(t)}\right)$ the corresponding codeword in the second code-layer. The third codeword is chosen by transmitter $i$ using $\left(W_{i,C2}^{(t)}, \Omega_{i,R2}^{(t)}\right)$ from the third code-layer, which is a sub-codebook of $2^{N(R_{i,C2}+R_{i,R2})}$ codewords corresponding to the codeword $\boldsymbol{u}_i\left(W_{1,C1}^{(t-1)}, \Omega_{1,R1}^{(t-1)}, W_{2,C1}^{(t-1)}, \Omega_{2,R1}^{(t-1)}, W_{i,C1}^{(t)}, \Omega_{i,R1}^{(t)}\right)$. Denote by $\boldsymbol{v}_i\left(W_{1,C1}^{(t-1)}, \Omega_{1,R1}^{(t-1)}, W_{2,C1}^{(t-1)}, \Omega_{2,R1}^{(t-1)}, W_{i,C1}^{(t)}, \Omega_{i,R1}^{(t)}, W_{i,C2}^{(t)}, \Omega_{i,R2}^{(t)}\right)$ the corresponding codeword in the third code-layer. The fourth codeword is chosen by transmitter $i$ using $W_{i,P}^{(t)}$ from the fourth code-layer, which is a sub-codebook of $2^{NR_{i,P}}$ codewords corresponding to the codeword $\boldsymbol{v}_i\left(W_{1,C1}^{(t-1)}, \Omega_{1,R1}^{(t-1)}, W_{2,C1}^{(t-1)}, \Omega_{2,R1}^{(t-1)}, W_{i,C1}^{(t)}, \Omega_{i,R1}^{(t)}, W_{i,C2}^{(t)}, \Omega_{i,R2}^{(t)}\right)$. Denote by $\boldsymbol{x}_{i,P}\left(W_{1,C1}^{(t-1)}, \Omega_{1,R1}^{(t-1)}, W_{2,C1}^{(t-1)}, \Omega_{2,R1}^{(t-1)}, W_{i,C1}^{(t)}, \Omega_{i,R1}^{(t)}, W_{i,C2}^{(t)}, \Omega_{i,R2}^{(t)}, W_{i,P}^{(t)}\right)$ the corresponding codeword in the fourth code-layer. Finally, the codeword $\boldsymbol{x}_i\left(W_{1,C1}^{(t-1)}, \Omega_{1,R1}^{(t-1)}, W_{2,C1}^{(t-1)}, \Omega_{2,R1}^{(t-1)}, W_{i,C1}^{(t)}, \Omega_{i,R1}^{(t)}, W_{i,C2}^{(t)}, \Omega_{i,R2}^{(t)}, W_{i,P}^{(t)}\right)$ to be sent during block $t \in \{1, 2, \ldots, T\}$ is a simple concatenation of the previous codewords, i.e., $\boldsymbol{x}_i = \left(\boldsymbol{u}_i^{\mathsf{T}}, \boldsymbol{v}_i^{\mathsf{T}}, \boldsymbol{x}_{i,P}^{\mathsf{T}}\right)^{\mathsf{T}} \in \{0, 1\}^{q \times N}$, where the message indices have been dropped for ease of notation.

The decoder follows a backward decoding scheme. In the following, this coding scheme is referred to as a randomized Han-Kobayashi coding scheme with noisy feedback (R-HK-NOF) and it is described in [7]. The rest of the proof consists of showing that the R-HK-NOF coding scheme is capable of achieving an $\eta$-NE with $(R_1, R_2) \in \mathcal{C} \cap \mathcal{B}_\eta$ for all $\eta > 0$, subject to a proper choice of the rates $R_{i,R1}$ and $R_{i,R2}$, for all $i \in \{1, 2\}$.

*Lemma 3: The achievable region of the randomized Han-Kobayashi coding scheme for the D-LD-IC-NOF is the set* *of non-negative rates* $\Big(R_{1,C1},\ R_{1,R1},\ R_{1,C2},\ R_{1,R2},\ R_{1,P},\ R_{2,C1},\ R_{2,R1},\ R_{2,C2},\ R_{2,R2},\ R_{2,P}\Big)$ *that satisfy the following conditions for all* $i \in \{1,2\}$ *and* $j \in \{1,2\} \setminus \{i\}$*:*

$$R_{j,C1} + R_{j,R1} \leqslant \theta_{1,i}, \tag{15a}$$
$$R_i + R_{j,C} + R_{j,R} \leqslant \theta_{2,i}, \tag{15b}$$
$$R_{j,C2} + R_{j,R2} \leqslant \theta_{3,i}, \tag{15c}$$
$$R_{i,P} \leqslant \theta_{4,i}, \tag{15d}$$
$$R_{i,P} + R_{j,C2} + R_{j,R2} \leqslant \theta_{5,i}, \tag{15e}$$
$$R_{i,C2} + R_{i,P} \leqslant \theta_{6,i}, \ \textit{and} \tag{15f}$$
$$R_{i,C2} + R_{i,P} + R_{j,C2} + R_{j,R2} \leqslant \theta_{7,i}, \tag{15g}$$

*where,*

$$\theta_{1,i} = \Big(n_{ij} - \left(\max\left(\overrightarrow{n}_{ii}, n_{ij}\right) - \overleftarrow{n}_{ii}\right)^{+}\Big)^{+}, \tag{16a}$$
$$\theta_{2,i} = \max\left(\overrightarrow{n}_{ii}, n_{ij}\right), \tag{16b}$$
$$\theta_{3,i} = \min\Big(n_{ij}, \left(\max\left(\overrightarrow{n}_{ii}, n_{ij}\right) - \overleftarrow{n}_{ii}\right)^{+}\Big), \tag{16c}$$
$$\theta_{4,i} = \left(\overrightarrow{n}_{ii} - n_{ji}\right)^{+}, \tag{16d}$$
$$\theta_{5,i} = \max\Big( \left(\overrightarrow{n}_{ii} - n_{ji}\right)^{+}, \min\Big(n_{ij}, \left(\max\left(\overrightarrow{n}_{ii}, n_{ij}\right) - \overleftarrow{n}_{ii}\right)^{+}\Big)\Big), \tag{16e}$$
$$\theta_{6,i} = \min\Big(n_{ji}, \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^{+}\Big) - \min\Big( \left(n_{ji} - \overrightarrow{n}_{ii}\right)^{+}, \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^{+} \Big) + \left(\overrightarrow{n}_{ii} - n_{ji}\right)^{+}, \ \textit{and} \tag{16f}$$
$$\theta_{7,i} = \max\Big( \min\Big(n_{ij}, \left(\max\left(\overrightarrow{n}_{ii}, n_{ij}\right) - \overleftarrow{n}_{ii}\right)^{+} \Big), \min\Big(n_{ji}, \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^{+} \Big) - \min\Big( \left(n_{ji} - \overrightarrow{n}_{ii}\right)^{+}, \left(\max\left(\overrightarrow{n}_{jj}, n_{ji}\right) - \overleftarrow{n}_{jj}\right)^{+} \Big) + \left(\overrightarrow{n}_{ii} - n_{ji}\right)^{+} \Big). \tag{16g}$$

*Proof:* The proof of Lemma 3 is presented in [7]. ■

The set of inequalities in (15) can be written in terms of the transmission rates $R_1 = R_{1,C1} + R_{1,C2} + R_{1,P}$ and $R_2 = R_{2,C1} + R_{2,C2} + R_{2,P}$ to observe that the R-HK-NOF achieves all the rates $(R_1, R_2) \in \mathcal{C}$, when $R_{1,R} = R_{2,R} = 0$.

The following lemma shows than when both transmitter-receiver links use the R-HK-NOF scheme and one of them unilaterally changes its coding scheme, it obtains a rate improvement that can be upper bounded.

*Lemma 4: Let* $\eta > 0$ *be fixed and let the rate tuple* $\boldsymbol{R} = (R_{1,C} - \frac{\eta}{6}, R_{1,R} - \frac{\eta}{6}, R_{1,P} - \frac{\eta}{6}, R_{2,C} - \frac{\eta}{6}, R_{2,R} - \frac{\eta}{6}, R_{2,P} - \frac{\eta}{6})$ *be achievable with the R-HK-NOF such that* $R_1 = R_{1,P} + R_{1,C} - \frac{1}{3}\eta$ *and* $R_2 = R_{2,P} + R_{2,C} - \frac{1}{3}\eta$*. Then, any unilateral deviation of transmitter-receiver pair* $i$ *by using any other coding scheme leads to a transmission rate* $R_i'$ *that satisfies* $R_i' \leqslant \max\left(\overrightarrow{n}_{ii}, n_{ij}\right) - (R_{j,C} + R_{j,R}) + \frac{2}{3}\eta$*.*

*Proof:* The proof of Lemma 4 is presented in [7]. ■

Lemma 4 reveals the relevance of the random symbols $\Omega_1$ and $\Omega_2$ used by the R-HK-NOF. Even though the random symbols used by transmitter $j$ do not increase the effective transmission rate of transmitter-receiver pair $j$, they strongly limit the rate improvement transmitter-receiver pair $i$ can obtain by deviating from the R-HK-NOF coding scheme. This observation can be used to show that the R-HK-NOF can be an $\eta$-NE, when both $R_{1,R}$ and $R_{2,R}$ are properly chosen. The following lemma formalizes this intuition.

*Lemma 5: Let* $\eta > 0$ *be fixed and let the rate tuple* $\boldsymbol{R} = (R_{1,C} - \frac{\eta}{6}, R_{1,R} - \frac{\eta}{6}, R_{1,P} - \frac{\eta}{6}, R_{2,C} - \frac{\eta}{6}, R_{2,R} - \frac{\eta}{6}, R_{2,P} - \frac{\eta}{6})$ *be achieved by using the R-HK-NOF, with*

$$R_{i,C} + R_{i,P} + R_{j,C} + R_{j,R} = \max(\overrightarrow{n}_{ii}, n_{ij}) + \frac{2}{3}\eta, \tag{17}$$

*for all* $i \in \{1,2\}$*. Then, the rate pair* $(R_1, R_2)$*, with* $R_i = R_{i,C} + R_{i,P} - \frac{1}{3}\eta$ *is achievable at an* $\eta$*-Nash equilibrium.*

*Proof:* The proof of Lemma 5 is presented in [7]. ■

The following lemma shows that all the rate pairs $(R_1, R_2) \in \mathcal{C} \cap \mathcal{B}_\eta$ are achievable by the R-HK-NOF coding scheme at an $\eta$-NE, for all $\eta > 0$.

*Lemma 6: Let* $\eta > 0$ *be fixed. Then, for all rate pairs* $(R_1, R_2) \in \mathcal{C} \cap \mathcal{B}_\eta$*, there always exists at least one* $\eta$*-NE transmit-receive configuration pair* $(s_1^*, s_2^*) \in \mathcal{A}_1 \times \mathcal{A}_2$*, such that* $u_1(s_1^*, s_2^*) = R_1$ *and* $u_2(s_1^*, s_2^*) = R_2$*.*

*Proof:* The proof of Lemma 6 is presented in [7]. ■

This proof consists of showing that the set of inequalities in (15) and (17) leads to a set of rate pairs identical to $\mathcal{C} \cap \mathcal{B}_\eta$. This concludes the proof of Theorem 1.

## V. Conclusions

In this paper, the $\eta$-NE region of the D-LD-IC-NOF has been characterized for all $\eta > 0$. This region contains the $\eta$-NE region without feedback studied in [8] and is contained within the $\eta$-NE region with perfect channel-output feedback studied in [9].

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