ON THE IRREDUCIBLE ACTION OF PSL(2,R) ON THE 3-DIMENSIONAL EINSTEIN UNIVERSE

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Abstract. We describe the orbits of the irreducible action of PSL$(2,\mathbb{R})$ on the 3-dimensional Einstein universe $\mathbb{E}\mathrm{in}^{1,2}$. This work completes the study in [2], and is one element of the classification of cohomogeneity one actions on $\mathbb{E}\mathrm{in}^{1,2}$ ([5]).

1. Introduction

1.1. **Einstein universe.** Let $\mathbb{R}^{2,n+1}$ denote a $(n+3)$-dimensional real vector space equipped with a non-degenerate symmetric bilinear form $\mathfrak{q}$ with signature $(2, n+1)$. The nullcone of $\mathbb{R}^{2,n+1}$ is

$$\mathfrak{N}^{2,n+1} = \{v \in \mathbb{R}^{2,n+1} \setminus \{0\} : \mathfrak{q}(v) = 0\}.$$

The $(n+1)$-dimensional **Einstein universe** $\mathbb{E}\mathrm{in}^{1,n}$ is the image of the nullcone $\mathfrak{N}^{2,n+1}$ under the projectivization:

$$\mathbb{P} : \mathbb{R}^{2,n+1} \setminus \{0\} \longrightarrow \mathbb{RP}^{n+2}.$$

The degenerate metric on $\mathfrak{N}^{2,n+1}$ induces a $O(2, n+1)$-invariant conformal Lorentzian structure on Einstein universe. The group of conformal transformations on $\mathbb{E}\mathrm{in}^{1,n}$ is $O(2, n+1)$ [4].

A lightlike geodesic in Einstein universe is a **photon**. A photon is the projectivisation of an isotropic 2-plane in $\mathbb{R}^{2,n+1}$. The set of photons through a point $p \in \mathbb{E}\mathrm{in}^{1,n}$ denoted by $L(p)$ is the **lightcone** at p. The complement of a lightcone $L(p)$ in Einstein universe is the **Minkowski patch** at p and we denote it by $Mink(p)$. A Minkowski patch is conformally equivalent to the $(n+1)$-dimensional Minkoski space $\mathbb{E}^{1,n}$ [1].

The complement of the Einstein universe in $\mathbb{RP}^{n+2}$ has two connected components: the $(n+2)$-dimensional Anti de-Sitter space $\mathrm{AdS}^{1,n+1}$ and the generalized hyperbolic space $\mathbb{H}^{2,n}$: the first (respectively the second) is the projection of the domain $\mathbb{R}^{2,n+1}$ defined by $\{\mathfrak{q} < 0\}$ (respectively $\{\mathfrak{q} > 0\}$).

An immersed submanifold S of $\mathrm{AdS}^{1,n+1}$ or $\mathbb{H}^{2,n}$ is of **signature** (p, q, r) (respectively $\mathbb{E}\mathrm{in}^{1,n}$) if the restriction of the ambient pseudo-Riemmanian metric (respectively the conformal Lorentzian metric) is of signature (p, q, r), meaning that the radical has dimension r, and that maximal definite negative and positive subspaces have dimensions p and q, respectively. If S is nondegenerate, we forgot r and simply denote its signature by (p, q).

1.2. **The irreducible representation of PSL$(2,\mathbb{R})$.** A subgroup of $O(2, n+1)$ is **irreducible** if it preserves no proper subspace of $\mathbb{R}^{2,n+1}$. By [3, Theorem 1], up to conjugacy, $SO_\circ(1,2) \simeq \mathrm{PSL}(2,\mathbb{R})$ is the only irreducible connected Lie subgroup of $O(2,3)$.

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On the other hand, for every integer n, it is well known that, up to isomorphism, there is only one n-dimensional irreducible representation of $\mathrm{PSL}(2,\mathbb{R})$. For $n=5$, this representation is the natural action of $\mathrm{PSL}(2,\mathbb{R})$ on the vector space $\mathbb{V}=\mathbb{R}_4[X,Y]$ of homogeneous polynomials of degree 4 in two variables X and Y. This action preserves the following quadratic form

$$\mathfrak{q}(a_4X^4+a_3X^3Y+a_2X^2Y^2+a_1XY^3+a_0Y^4)=2a_4a_0-\frac{1}{2}a_1a_3+\frac{1}{6}a_2^2.$$

The quadratic form $\mathfrak{q}$ is nondegenerate and has signature $(2,3)$. This induces an irreducible representation $\mathrm{PSL}(2,\mathbb{R})\to O(2,3)$ [2].

Theorem 1.1. *The irreducible action of $PSL(2,\mathbb{R})$ on the 3-dimensional Einstein universe $\mathbb{E}\mathrm{in}^{1,2}$ admits three orbits:*

- *An 1-dimensional lightlike orbit, i.e. of signature $(0,0,1)$*
- *A 2-dimensional orbit of signature $(0,1,1)$,*
- *An open orbit (hence of signature $(1,2)$) on which the action is free.*

The 1-dimensional orbit is lightlike, homeomorphic to $\mathbb{RP}^1$, but not a photon. The union of the 1-dimensional orbit and the 2-dimensional orbit is an algebraic surface, whose singular locus is precisely the 1-dimensional orbit. It is the union of all projective lines tangent to the 1-dimensional orbit. Figure 1 describes a part of the 1 and 2-dimensional orbits in the Minkowski patch $Mink(Y^4)$.

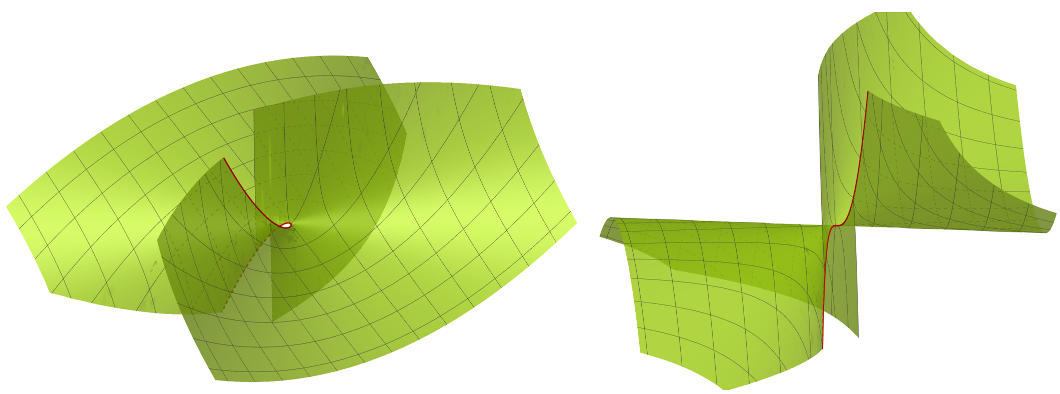

FIGURE 1. Two partial views of the intersection of the 1 and 2-dimensional orbits in Einstein universe with $\mathrm{Mink}(Y^4)$. **Red:** Part of the 1-dimensional orbit in Minkowski patch. **Green:** Part of the 2-dimensional orbit in Minkowski patch.

We will also describe the actions on Anti de-Sitter space and the generalized hyperbolic space $\mathbb{H}^{2,2}$:

Theorem 1.2. *The orbits of $PSL(2,\mathbb{R})$ in the Anti de-sitter component $\mathrm{AdS}^{1,3}$ are Lorentzian, i.e. of signature $(1,2)$. They are the leaves of a codimension 1 foliation. In addition, $PSL(2,\mathbb{R})$ induces three types of orbits in $\mathbb{H}^{2,2}$: a 2-dimensional spacelike orbit (of signature $(2,0)$) homeomorphic to the hyperbolic plane $\mathbb{H}^2$, a 2-dimensional Lorentzian orbit (i.e., of signature $(1,1)$) homeomorphic to the de-Sitter space $\mathrm{dS}^{1,1}$, and four kinds of 3-dimensional orbits where the action is free:*

- *one-parameter family of orbits of signature $(2,1)$ consisting of elements with four distinct non-real roots,*

- *one-parameter family of Lorentzian (i.e. of signature* $(1, 2)$*) orbits consisting of elements with four distinct real roots,*
- *two orbits of signature* $(1, 1, 1)$*,*
- *one-parameter family of Lorentzian (i.e. of signature* $(1, 2)$*) orbits consisting of elements with two distinct real roots, and a complex root* z *in* $\mathbb{H}^2$ *making an angle* θ *smaller than* $5\pi/6$ *with the two real roots.*

Remark 1.3. F. Fillastre indicated to us an alternative description for the last case stated in Theorem 1.2: these orbits correspond to polynomials whose roots in $\mathbb{CP}^1$ are ideal vertexes of regular ideal tetraedra in $\mathbb{H}^3$.

2. Proofs of the Theorems

Let f be an element in $\mathbb{V}$. We consider it as a polynomial function from $\mathbb{C}^2$ into $\mathbb{C}$. Actually, by specifying $Y = 1$, we consider f as a polynomial of degree at most 4. Such a polynomial is determined, up to a scalar, by its roots z_1, z_2, z_3, z_4 in $\mathbb{CP}^1$ (some of these roots can be ∞ if f can be divided by Y). It provides a natural identification between $\mathbb{P}(\mathbb{V})$ and the set $\widehat{\mathbb{CP}}^1_4$ made of 4-tuples (up to permutation) (z_1, z_2, z_3, z_4) of $\mathbb{CP}^1$ such that if some z_i is not in $\mathbb{RP}^1$, then its conjugate $\bar{z}_i$ is one of the z_j's. This identification is PSL$(2, \mathbb{R})$-equivariant, where the action of PSL$(2, \mathbb{R})$ on $\widehat{\mathbb{CP}}^1_4$ is simply the one induced by the diagonal action on $(\mathbb{CP}^1)^4$.

Actually, the complement of $\mathbb{RP}^1$ in $\mathbb{CP}^1$ is the union of the upper half-plane model $\mathbb{H}^2$ of the hyperbolic plane, and the lower half-plane. We can represent every element of $\widehat{\mathbb{CP}}^1_4$ by a 4-tuple (up to permutation) (z_1, z_2, z_3, z_4) such that:

– either every z_i lies in $\mathbb{RP}^1$,
– or z_1, z_2 lies in $\mathbb{RP}^1$, z_3 lies in $\mathbb{H}^2$ and $z_4 = \bar{z}_3$,
– or z_1, z_2 lies in $\mathbb{H}^2$ and $z_3 = \bar{z}_1$, $z_4 = \bar{z}_2$.

Theorems 1.1 and 1.2 will follow from the following proposition:

Proposition 2.1. *Let* $[f]$ *be an element of* $\mathbb{P}(\mathbb{V})$*. Then:*

- *it lies in* $\mathrm{Ein}^{1,2}$ *if and only if it has a root of multiplicity at least* 3*, or two distinct real roots* z_1*,* z_2*, and two complex roots* z_3*,* $z_4 = \bar{z}_3$*, with* z_3 *in* $\mathbb{H}^2$ *and such that the angle at* z_3 *between the hyperbolic geodesic rays* $[z_3, z_1)$ *and* $[z_3, z_2)$ *is* $5\pi/6$*,*
- *it lies in* $\mathrm{AdS}^{1,3}$ *if and only it has two distinct real roots* z_1*,* z_2*, and two complex roots* z_3*,* $z_4 = \bar{z}_3$*, with* z_3 *in* $\mathbb{H}^2$ *and such that the angle at* z_3 *between the hyperbolic geodesic rays* $[z_3, z_1)$ *and* $[z_3, z_2)$ *is* $> 5\pi/6$*,*
- *it lies in* $\mathbb{H}^{2,2}$ *if and only if it has no real roots, or four distinct real roots, or a root of multiplicity exactly* 2*, or it has two distinct real roots* z_1*,* z_2*, and two complex roots* z_3*,* $z_4 = \bar{z}_3$*, with* z_3 *in* $\mathbb{H}^2$ *and such that the angle at* z_3 *between the hyperbolic geodesic rays* $[z_3, z_1)$ *and* $[z_3, z_2)$ *is* $< 5\pi/6$*.*

Proof of Proposition 2.1. Assume that f has no real root. Hence we are in the situation where z_1, z_2 lie in $\mathbb{H}^2$ and $z_3 = \bar{z}_1$, $z_4 = \bar{z}_2$. By applying a suitable element of PSL$(2, \mathbb{R})$, we can assume $z_1 = i$, and $z_2 = ri$ for some $r > 0$. In other words, f is in the PSL$(2, \mathbb{R})$-orbit of $(X^2 + Y^2)(X^2 + r^2Y^2)$. The value of $\mathfrak{q}$ on this polynomial is $2 \times 1 \times r^2 + \frac{1}{6}(1 + r^2)^2 > 0$, hence $[f]$ lies in $\mathbb{H}^{2,2}$.

Hence we can assume that f admits at least one root in $\mathbb{RP}^1$, and by applying a suitable element of PSL$(2, \mathbb{R})$, one can assume that this root is ∞, i.e. that f is a multiple of Y.

We first consider the case where this real root has multiplicity at least 2:

$$f = Y^2(aX^2 + bXY + cY^2)$$

Then, $\mathfrak{q}(f) = \frac{1}{6}a^2$: it follows that if f has a root of multiplicity at least 3, it lies in $\mathbb{E}\mathrm{in}^{1,2}$, and if it has a real root of mulitplicity 2, it lies in $\mathbb{H}^{2,2}$.

We assume from now that the real roots of f have multiplicity 1. Assume that all roots are real. Up to $\mathrm{PSL}(2,\mathbb{R})$, one can assume that these roots are 0, 1, r and ∞ with $0 < r < 1$.

$$f(X,Y) = XY(X-Y)(X-rY) = X^3Y - (r+1)X^2Y^2 + rXY^3$$

Then, $\mathfrak{q}(f) = -\frac{1}{2}r + \frac{1}{6}(r+1)^2 = \frac{1}{6}(r^2 - r + 1) > 0$. Therefore f lies in $\mathbb{H}^{2,2}$ once more.

The only remaining case is the case where f has two distinct real roots, and two complex conjugate roots z, $\bar{z}$ with $z \in \mathbb{H}^2$. Up to $\mathrm{PSL}(2,\mathbb{R})$, one can assume that the real roots are 0, ∞, hence:

$$f(X,Y) = XY(X-zY)(X-\bar{z}Y) = XY(X^2 - 2|z|\cos\theta XY + |z|^2Y^2)$$

where $z = |z|e^{i\theta}$. Then:

$$\mathfrak{q}(f) = \frac{2|z|^2}{3}(\cos^2\theta - \frac{3}{4})$$

Hence f lies in $\mathbb{E}\mathrm{in}^{1,2}$ if and only if $\theta = \pi/6$ or $5\pi/6$. The proposition follows easily. □

Remark 2.2. In order to determine the signature of the orbits induced by $\mathrm{PSL}(2,\mathbb{R})$ in $\mathbb{P}(\mathbb{V})$, we consider the tangent vectors induced by the action of 1-parameter subgroups of $\mathrm{PSL}(2,\mathbb{R})$. We denote by E, P and H, the 1-parameter elliptic, parabolic and hyperbolic subgroups stabilizing i, ∞ and $\{0,\infty\}$, respectively.

Proof of Theorem 1.1. It follows from Proposition 2.1 that there are precisely three $\mathrm{PSL}(2,\mathbb{R})$-orbits in $\mathbb{E}\mathrm{in}^{1,2}$:

– one orbit $\mathcal{N}$ comprising polynomials with a root of multiplicity 4, i.e. of the form $[(sY - tX)^4]$ with $s,t \in \mathbb{R}$. It is clearly 1-dimensional, and equivariantly homeomorphic to $\mathbb{RP}^1$ with the usual projective action of $\mathrm{PSL}(2,\mathbb{R})$. Since $\frac{d}{dt}|_{t=0}(Y-tX)^4 = -4XY^3$ is a $\mathfrak{q}$-null vector, this orbit is lightlike (but cannot be a photon since the action is irreducible),

– one orbit $\mathcal{L}$ comprising polynomials with a real root of multiplicity 3, and another real root. These are the polynomials of the form $[(sY - tX)^3(s'Y - t'X)]$ with $s,t,s',t' \in \mathbb{R}$. It is 2-dimensional, and it is easy to see that it is the union of the projective lines tangent to $\mathcal{N}$. The vectors tangent to $\mathcal{L}$ induced by the 1-parameter subgroups P and E at $[XY^3] \in \mathcal{L}$ are $v_P = -Y^4$ and $v_E = 3X^2Y^2 + Y^4$. Obviously, v_P is orthogonal to v_E and $v_E + v_P$ is spacelike. Hence $\mathcal{L}$ is of signature $(0,1,1)$.

– one open orbit comprising polynomials admitting two distinct real roots and a root z in $\mathbb{H}^2$ making an angle $5\pi/6$ with the two real roots in $\partial\mathbb{H}^2$. The stabilizers of points in this orbit are trivial since an isometry of $\mathbb{H}^2$ preserving a point in $\mathbb{H}^2$ and one point in $\partial\mathbb{H}^2$ is necessarily the identity. □

Proof of Theorem 1.2. According to Proposition 2.1, the polynomials in $\mathrm{AdS}^{1,3}$ have two distinct real roots, and a complex root z in $\mathbb{H}^2$ making an angle θ greater than $5\pi/6$ with the two real roots. It follows that the action in $\mathrm{AdS}^{1,3}$ is free, and that the orbits are the level sets of the function θ. Suppose that M is a $\mathrm{PSL}(2,\mathbb{R})$-orbit in $\mathrm{AdS}^{1,3}$. There exists $r \in \mathbb{R}$ such that $[f] = [Y(X^2+Y^2)(X-rY)] \in M$. The orbit induced by the 1-parameter elliptic subgroup E at $[f]$ is

$$\gamma(t) = \left[(X^2+Y^2)\big((\sin t\cos t - r\sin^2 t)X^2 - (\sin t\cos t + r\cos^2 t)Y^2 + (\cos^2 t - \sin^2 t + 2r\sin t\cos t)XY\big)\right].$$

Then $\mathfrak{q}(\frac{d\gamma}{dt}|_{t=0}) = -2 - 2r^2 < 0$. This implies, as for any submanifold of a Lorentzian manifold admitting a timelike vector, that M is Lorentzian, i.e., of signature $(1,2)$.

The case of $\mathbb{H}^{2,2}$ is the richest one. According to Proposition 2.1 there are four cases to consider:

- *No real roots.* Then f has two complex roots z_1, z_2 in $\mathbb{H}^2$ (and their conjugates). It corresponds to two orbits: one orbit corresponding to the case $z_1 = z_2$: it is spacelike and has dimension 2. It is the only maximal $\mathrm{PSL}(2,\mathbb{R})$-invariant surface in $\mathbb{H}^{2,2}$ described in [2, Section 5.3]. The case $z_1 \neq z_2$ provides a one-parameter family of 3-dimensional orbits on which the action is free (the parameter being the hyperbolic distance between z_1 and z_2). One may assume that $z_1 = i$ and $z_2 = ri$ for some $r > 0$. Denote by M the orbit induced by $\mathrm{PSL}(2,\mathbb{R})$ at $[f] = [(X^2+Y^2)(X^2+r^2Y^2)]$. The vectors tangent to M at $[f]$ induced by the 1-parameter subgroups H, P and E are:

$$v_H = -4X^4+4r^2Y^4, \qquad v_P = -4X^3Y - 2(r^2+1)XY^3,$$
$$v_E = 2(r^2-1)X^3Y + 2(r^2-1)XY^3,$$

  respectively. The timelike vector v_H is orthogonal to both v_P and v_E. It is easy to see that the 2-plane generated by $\{v_P, v_E\}$ is of signature $(1,1)$. Therefore, the tangent space $T_{[f]}M$ is of signature $(2,1)$.
- *Four distinct real roots.* This case provides a one-parameter family of 3-dimensional orbits on which the action is free - the parameter being the cross-ratio between the roots in $\mathbb{RP}^1$. Denote by M the $\mathrm{PSL}(2,\mathbb{R})$-orbit at $[f] = [XY(X-Y)(X-rY)]$ (here as explained in the proof of Proposition 2.1, we can restrict ourselves to the case $0 < r < 1$). The vectors tangent to M at $[f]$ induced by the 1-parameter subgroups H, P, and E are:

$$v_H = -rY^4 + 2(r+1)XY^3 - 3X^2Y^2, \qquad v_P = -2X^3Y + 2rXY^3,$$
$$v_E = X^4 - rY^4 + 3(r-1)X^2Y^2 + 2(r+1)XY^3 - 2(r+1)X^3Y,$$

  respectively. A vector $x = av_H + bv_P + cv_E$ is orthogonal to v_P if and only if $2ra + b(r+1) + c(r+1)^2 = 0$. Set $a = \big(b(r+1) + c(r+1)^2\big)/-2r$ in

$$\mathfrak{q}(x) = 2ra^2 + \frac{3}{2}b^2 + \big(\frac{7}{2}(r^2+1) - r\big)c^2 + 2(r+1)ab + 2(r+1)^2 + ac(2r^2 - r + 5).$$

  Consider $\mathfrak{q}(x) = 0$ as a quadratic polynomial F in b. Since $0 < r < 1$, the discriminant of F is non-negative and it is positive when $c \neq 0$. Thus, the intersection of the orthogonal complement of the spacelike vector v_P with the tangent space $T_{[f]}M$ is a 2-plane of signature $(1,1)$. This implies that M is Lorentzian, i.e., of signature $(1,2)$.
- *A root of multiplicity* 2. Observe that if there is a non-real root of multiplicity 2, when we are in the first "no real root" case. Hence we consider here only the case where the root of multiplicity 2 lies in $\mathbb{RP}^1$. Then, we have three subcases to consider:
  - two distinct real roots of multiplicity 2: The orbit induced at X^2Y^2 is the image of the $\mathrm{PSL}(2,\mathbb{R})$-equivariant map

$$\mathrm{dS}^{1,1} \subset \mathbb{P}(\mathbb{R}_2[X,Y]) \longrightarrow \mathbb{H}^{2,2}, \qquad [L] \mapsto [L^2],$$

    where $\mathbb{R}_2[X,Y]$ is the vector space of homogeneous polynomials of degree 2 in two variables X and Y, endowed with discriminant as a $\mathrm{PSL}(2,\mathbb{R})$-invariant bilinear form of signature $(1,2)$ [2, Section 5.3]. (Here, L is the projective class of a Lorentzian bilinear form on $\mathbb{R}^2$). The vectors tangent to the orbit at X^2Y^2 induced by the 1-parameter subgroups P and E are $v_P = -2XY^3$ and $v_E = 2X^3Y - 2XY^3$, respectively. It is easy to see that the 2-plane generated by $\{v_p, v_E\}$ is of signature $(1,1)$. Hence, the orbit induced at X^2Y^2 is Lorentzian.

– three distinct real roots, one of them being of multiplicity 2: Denote by M the orbit induced by $\mathrm{PSL}(2,\mathbb{R})$ at $[f]=[XY^2(X-Y)]$. The vectors tangent to M at $[f]$ induced by the 1-parameter subgroups H, P and E are:

$$v_H=-2XY^3,\quad v_P=Y^4-2XY^3,\quad v_E=Y^4-X^4-2X^2Y^2+X^3Y-XY^3,$$

respectively. Obviously, the lightlike vector v_H+v_P is orthogonal to $T_{[f]}M$. Therefore, the restriction of the metric on $T_{[f]}M$ is degenerate. It is easy to see that the quotient of $T_{[f]}M$ by the action of the isotropic line $\mathbb{R}(v_H+v_P)$ is of signature $(1,1)$. Thus, M is of signature $(1,1,1)$.

– one real root of multiplicity 2, and one root in $\mathbb{H}^2$: Denote by M the orbit induced by $\mathrm{PSL}(2,\mathbb{R})$ at $[f]=[Y^2(X^2+Y^2)]$. The vectors tangent to M at $[f]$ induced by the 1-parameter subgroups H, P and E are

$$v_H=4Y^4,\qquad v_P=-2XY^3,\qquad v_E=2X^3Y+2XY^3,$$

respectively. Obviously, the lightlike vector v_H is orthogonal $T_{[f]}M$. Therefore, the restriction of the metric on $T_{[f]}M$ is degenerate. It is easy to see that the quotient of $T_{[f]}M$ by the action of the isotropic line $\mathbb{R}(v_H)$ is of signature $(1,1)$. Thus M is of signature $(1,1,1)$.

- *Two distinct real roots, and a complex root z in $\mathbb{H}^2$ making an angle θ smaller than $5\pi/6$ with the two real roots.* Denote by M the orbit induced by $\mathrm{PSL}(2,\mathbb{R})$ at $[f]=[Y(X^2+Y^2)(X-rY)]$. The vectors tangent to M at $[f]$ induced by the 1-parameter subgroups H, P and E are:

$$v_H=-4rY^4-2X^3Y+2XY^3,\qquad v_P=-3X^2Y^2+2rXY^3-Y^4,$$
$$v_E=X^4-Y^4-2rX^3Y-2rXY^3,$$

respectively. The following set of vectors is an orthogonal basis for $T_{[f]}M$ where the first vector is timelike and the two others are spacelike.

$$\{(7r+3r^3)v_H+(6-2r^2)v_P+(5+r^2)v_E,4v_P+v_E,v_H\}.$$

Therefore, M is Lorentzian, i.e., of signature $(1,2)$. □

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