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Periodic solutions of abstract neutral functional differential equations and applications

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Problème

In this work, we study the existence of periodic solutions for the following neutral partial functional differential equations of the following form

$$\frac{d}{dt}[x(t) - L(x_t)] = A[x(t) - L(x_t)] + G(x_t) + f(t), \qquad (1)$$

where $A : D(A) \subseteq X \to X$ is a linear closed operator on Banach space $(X, \|.\|)$ and $f \in L^p(\mathbb{T}, X)$ for all $p \geq 1$. For $r_{2\pi} := 2\pi N$ (some $N \in \mathbb{N}$) L and G are in $B(L^p([-r_{2\pi}, 0], \ X); \ X)$ is the space of all bounded linear operators and x_t is an element of $L^p([-r_{2\pi} \ , 0], \ X)$ which is defined as follows $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$.

Definition

For $1 \leq p < \infty$, a sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathbf{B}(X, Y)$ is said to be an L^p-multiplier if for each $f \in L^p(\mathbb{T}, X)$, there exists $u \in L^p(\mathbb{T}, Y)$ such that $\hat{u}(k) = M_k\hat{f}(k)$ for all $k \in \mathbb{Z}$.

Theorem (Neumann Expansion)

Let $A \in B(X, X)$, where X is a Banach space. If $\| A \| < 1$ then $I - A$ is invertible, moreover

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

Definition

Assume that A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X. A function x is called a mild solution of Eq. (1) if :

$$Dx_t = T(t)D\varphi + \int_0^t T(t-s)(Gx_s + f(s))ds \quad \text{for} \quad 0 \leq t \leq 2\pi.$$

Remark

Let $(T(t))_{t\geq 0}$ be the C_0-semigroup generated by A. If $g : [0, a] \to X$ is a continuous function, then $\int_0^t \int_0^s T(t-s)g(\xi)d\xi ds \in D(A)$ and

$$A\int_0^t \int_0^s T(t-s)g(\xi)d\xi ds = \int_0^t (T(t-s) - I)g(s)ds; 0 \leq t \leq a.$$

Lemma

Assume that A generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on X, if x is a mild solution then

$$Dx_t = D\varphi + A\int_0^t Dx_s ds + \int_0^t (Gx_s + f(s))ds \text{ for } 0 \leq t \leq 2\pi.$$

Theorem

Assume that A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X and $f \in L^p(\mathbb{T}; X)$
For some $1 \leq p < \infty$; if x is a mild solution of Eq. (1). Then

$$(ikD_k - AD_k - G_k)\hat{x}(k) = \hat{f}(k) \quad \text{for all} \quad k \in \mathbb{Z}.$$

Theorem

Let $1 \leq p < \infty$. Assume that A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X. If $\sigma_Z(\Delta) = \phi$ and $(ikD_k - AD_k - G_k)^{-1}$ is an L^p-multiplier Then there exists a unique mild periodic solution of Eq. (1).

Application

Let A be a closed **linear** operator on a Hilbert space H and suppose that $i\mathbb{Z} \subset \rho(A)$ and $\sup_{k\in\mathbb{Z}} \left\| k(ikD_k - AD_k)^{-1} \right\| =: M < \infty$.

If $\|G\| < \dfrac{1}{(2r_{2\pi})^{1/p} M}$ then for every $f \in L^p(\mathbb{T}; X)$, there exists a unique strong L^p-solution of Eq. (1).

From the identity

$$ikD_k - AD_k - G_k = (ikD_k - AD_k)(I - G_k(ikD_k - AD_k)^{-1})$$

it follows that

$ikD_k - AD_k - G_k$ is invertible whenever $\left\| G_k(ikD_k - AD_k)^{-1} \right\| < 1.$,

we observe that $\|G_k\| \leq (2r_{2\pi})^{1/p} \|G\|$.

Hence,

$\left\| G_k(ikD_k - AD_k)^{-1} \right\| = \left\| G_k(ikD_k - AD_k)^{-1} \right\| \leq (2r_{2\pi})^{1/p} \|G\| M := \alpha < 1.$

Then $\sigma_Z(\Delta) = \phi$ and we deduce that

$$(ikD_k - AD_k - G_k)^{-1} = (ikD_k - AD_k)^{-1}(I - G_k(ikD_k - AD_k)^{-1})^{-1}$$
$$= (ikD_k - AD_k)^{-1} \sum_{n=0}^{\infty} [G_k(ikD_k - AD_k)^{-1}]^n$$

Moreovery

$$\left\| ik(ikD_k - AD_k - G_k)^{-1} \right\| \leq \| ik(D_k - AD_k\| \sum_{n=0}^{\infty} [G_k(ikD_k - AD_k)^{-1}]^n$$
$$\leq \frac{1+M}{1-\alpha}$$

and

$$\sup_{k\in\mathbb{Z}} \left\| ik(ikD_k - AD_k - G_k)^{-1} \right\| < \infty.$$

we conclude that there exists a unique strong L^p-solution of Eq. (1).

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Merci pour votre attention