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EXISTENCE OF LOCAL STRONG SOLUTIONS TO FLUID-BEAM AND FLUID-ROD INTERACTION SYSTEMS

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Abstract. We study an unsteady nonlinear fluid–structure interaction problem. We consider a Newtonian incompressible two-dimensional flow described by the Navier-Stokes equations set in an unknown domain depending on the displacement of a structure, which itself satisfies a linear wave equation or a linear beam equation. The fluid and the structure systems are coupled *via* interface conditions prescribing the continuity of the velocities at the fluid–structure interface and the action-reaction principle. We prove existence of a unique local-in-time strong solution. In the case of a damped beam this is an alternative proof (and a generalization) of the result that can be found in [19]. In the case of the wave equation or a beam equation with inertia of rotation, this is, to our knowledge the first result of existence of strong solutions for which no viscosity is added. One key point, is to use the fluid dissipation to control, in appropriate function spaces, the structure velocity.

Keywords: Fluid-structure interaction, strong solution, existence and uniqueness.

1. Introduction

In this paper, we focus on the interactions between a viscous incompressible Newtonian fluid and a moving elastic structure located on one part of the fluid domain boundary. Precisely, we consider a $2D$ fluid container whose top boundary is made of a $1D$ elastic rod or beam. The fluid domain, denoted by $\mathcal{F}(t) \subset \mathbb{R}^2$, depends on time since it depends on the structure displacement. It reads

$$\mathcal{F}(t) := \{(x,y) \in \mathbb{R}^2 \,,\; x \in (0,L)\,,\; y \in (0, 1+\eta(x,t))\}\,.$$

where $(x,t) \mapsto \eta(x,t)$ stands for the displacement of the structure.

We assume that the fluid is two dimensional, homogeneous, viscous, incompressible and Newtonian. Its velocity-field u and internal pressure p satisfy the incompressible Navier–Stokes equations in $\mathcal{F}(t)$:

$$\rho_f(\partial_t u + (u\cdot\nabla)u) - \operatorname{div}\sigma(u,p) \;=\; 0\,, \tag{1.1}$$

$$\operatorname{div} u \;=\; 0\,. \tag{1.2}$$

The fluid stress tensor $\sigma(u,p)$ is given by the Newton law:

$$\sigma(u,p) = \mu(\nabla u + \nabla u^\top) - p\mathrm{I}_2\,.$$

Here μ denotes the viscosity of the fluid and ρ_f its density, and are both positive constants.

The structure displacement η satisfies a linear, possibly damped, beam or wave equation:

$$\rho_s \partial_{tt}\eta - \delta\partial_{xxtt}\eta + \alpha\,\partial_{xxxx}\eta - \beta\partial_{xx}\eta - \gamma\,\partial_{xxt}\eta \;=\; \phi(u,p,\eta), \quad \text{on } (0,L)\,, \tag{1.3}$$

where $\alpha, \beta, \gamma, \delta$ are non-negative given constants and $\rho_s > 0$ denotes the constant structure density. Three cases are studied depending on the possibly vanishing parameters among $\alpha, \beta, \gamma, \delta$. We name the cases by the symbol $\mathbf{C}$ with the non-vanishing parameters as indices. Precisely, we denote:

- $(\mathbf{C}_\beta)$ the case for which $\beta > 0$, $\gamma = \delta = \alpha = 0$; this case corresponds to a rod equation with no additional damping (*i.e.* a wave equation):

$$\rho_s\partial_{tt}\eta - \beta\partial_{xx}\eta = \phi(u,p,\eta), \quad \text{on } (0,L)\,, \tag{$\mathbf{C}_\beta$}$$

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- $(\mathbf{C}_{\alpha,\delta})$ the case for which $\alpha > 0, \delta > 0$ and $\beta = \gamma = 0$; this one models a beam in flexion where the term $\delta \partial_{xxtt}\eta$ accounts for the inertia of rotation [23]:

$$(\mathbf{C}_{\alpha,\delta}) \qquad \rho_s \partial_{tt}\eta - \delta\partial_{xxtt}\eta + \alpha\,\partial_{xxxx}\eta = \phi(u,p,\eta), \quad \text{on } (0,L)\,,$$

- $(\mathbf{C}_{\alpha,\gamma})$ the case for which $\alpha > 0, \gamma > 0$ and $\beta = \delta = 0$; this last one models again a beam in flexion equation but with additional viscosity (already considered in [19, 15]):

$$(\mathbf{C}_{\alpha,\gamma}) \qquad \rho_s \partial_{tt}\eta + \alpha\,\partial_{xxxx}\eta - \gamma\,\partial_{xxt}\eta = \phi(u,p,\eta), \quad \text{on } (0,L)\,.$$

We emphasize that the structure equation is set in a reference configuration whereas the fluid equations are written in Eulerian coordinates and consequently in an unknown domain.

The fluid and structure equations are coupled through the source term $\phi(u,p,\eta)$ in (1.3), which corresponds to the trace of the second component of $\sigma(u,p)ndl$ transported in the structure reference configuration. The coupling term writes:

$$(1.4) \quad \phi(u,p,\eta)(x,t) = -e_2 \cdot \sigma(u,p)(x, 1+\eta(x,t),t)(-\partial_x \eta(x,t)\,e_1 + e_2)\,, \quad (x,t) \in (0,L)\times(0,T),$$

where (e_1, e_2) denotes the canonical basis of $\mathbb{R}^2$. The fluid and the structure are coupled also through the kinematic condition, which corresponds to a no-slip boundary condition at the interface:

$$(1.5) \qquad u(x, 1+\eta(x,t),t) = \partial_t \eta(x,t) e_2, \quad (x,t) \in (0,L)\times(0,T).$$

We complement our system with the following conditions on the remaining boundaries of the container:

- L-periodicity w.r.t. x for the fluid and the structure;
- no-slip boundary conditions on the bottom of the fluid container:

$$(1.6) \qquad u(x,0,t) = 0\,.$$

In what follows, we call (FS) the fluid–structure system (1.1)-(1.2)-(1.3)-(1.4)-(1.5)-(1.6)-(2.2)-(2.3)-(2.4). We study herein the (FS) system, completed with initial conditions:

$$\begin{aligned}
(1.7) \qquad \eta(x,0) &= \eta^0(x)\,, && x \in (0,L),\\
(1.8) \qquad \partial_t \eta(x,0) &= \dot{\eta}^0(x)\,, && x \in (0,L)\,,\\
(1.9) \qquad u(x,y,0) &= u^0(x,y)\,, && (x,y) \in \{x \in (0,L)\,, y \in (0, 1+\eta^0(x))\} =: \mathcal{F}^0\,.
\end{aligned}$$

The construction of a reasonable Cauchy-theory for free-boundary problems such as (FS) is a long-standing issue in the mathematical analysis of fluid-structure problems. Studies have been developed along two lines depending on whether the structure is immersed or on some part of the container boundary.

In the case of a 3D elastic structure evolving in a 3D viscous incompressible Newtonian flow, we refer the reader to [9] and [4] where the structure is described by a finite number of eigenmodes or to [2] for an artificially damped elastic structure. For the case of the full system describing the motion of a three-dimensional elastic structure interacting with a three-dimensional fluid, we mention [12, 10] in the steady state case and [7, 8, 17, 24] for the full unsteady case. In [7, 8], the authors consider the existence of strong solutions for small enough data locally in time, whereas, in [17, 24], the existence of local-in-time strong solutions is proven in the case where the fluid structure interface is flat and for a zero initial displacement field.

Concerning the fluid-beam – or more generally fluid-shell – coupled systems, that we consider herein, the 2D/1D steady state case is considered in [11] for homogeneous Dirichlet boundary conditions on the fluid boundaries (that are not the fluid–structure interface). Existence of a unique strong enough solution is obtained for small enough applied forces. In the unsteady framework, we refer to [5] where a 3D/2D fluid-plate coupled system is studied and where the structure is a damped plate in flexion. The case of an undamped plate is studied in [14]. The previous results deal with the existence of weak solutions, *i.e.* in the energy spaces, and rely on the only tranversal motion of the elastic beam that enables to circumvent the lack of regularity of the fluid domain boundary (that is not even Lipschitz). These results also apply to a 2D/1D fluid-shell coupled problem which is considered in [22]. In this reference, the authors give an alternative proof of existence of weak solutions based on ideas coming from numerical schemes [16]. The existence of strong solutions for 3D/2D, or 2D/1D coupled problem involving a damped elastic structure is studied in [1, 19, 20]. The proofs of [19, 20] are based on a splitting strategy for the Stokes system and on an implicit treatment of the *so called* fluid added mass effect. Moreover, they are valid for a zero (or small) initial displacement field. The coupling of a 3D Newtonian fluid and a linearly elastic Koiter shell is recently studied in [18]. In this study, the mid-surface of the structure is not flat anymore and existence of weak solutions is obtained. More recently, existence of a unique global-in-time solution for a 2D/1D coupling with a damped beam has been proven in [15]. This result includes that there is no contact between the structure and the bottom boundary and the additional viscosity of the beam is a key ingredient of the proof.

The results in the references above apply to the system under consideration here as follows. Existence of weak solutions as long as the structure does not touch the bottom of the fluid cavity is obtained in [14, 22] and is valid for $\beta > 0$ or $\alpha > 0$ without any additional damping or inertia of rotation terms. The existence of strong solution is proven only in the case where some viscosity is added to the structure equation. The case for which $\alpha = \delta = 0$, $\beta > 0$, $\gamma > 0$ is studied in [20], whereas the third case ($\mathbf{C}_{\alpha,\gamma}$) is studied in [19, 15]. In [19] local existence and uniqueness of a strong is obtained and, in [15], the solution is proven to be a global one and, in particular, no collision occur between the elastic struture and the bottom of the fluid cavity. Nevertheless [19, 20] seem to require the initial displacement to be equal to zero (or small enough).

A critical issue raised by the above references is the possibility of constructing a strong solution theory, for coupled systems describing the interactions of an elastic structure with a viscous fluid, with no regularity loss (*i.e.* a solution such that the fluid velocity remains in the same Sobolev spaces as the initial data, at least locally in time) and with no additional damping term on the structure. In the present paper, we tackle this issue in the case where the structure occupies a part of the container boundaries (corresponding to cases ($\mathbf{C}_{\beta}$) and ($\mathbf{C}_{\alpha,\delta}$)). But, we consider also the case where the structure displacement satisfies a damped beam equation as in [19, 15]. In this latter case, we complement the proof of the result of [19] which is developed only when the initial displacement field is equal to zero (or small enough).

The outline of the paper is as follows. In next section, we present the functional framework for our study and state our main result. In the rest of the section, we focus on the change of variables turning the system of equations into a system written in the reference configuration. We recall an elliptic regularity result for steady state Stokes-like equations obtained in [15] and other technical lemmas. The section after is devoted to the study of a linear system, for which we prove existence of a unique strong solution on any time interval $(0, T)$ and derive energy estimates or equalities uniform in T for any bounded T (we will consider $T < 1$). For the cases ($\mathbf{C}_{\beta}$) of a wave equation and ($\mathbf{C}_{\alpha,\delta}$) of a beam equation with inertia of rotation the key point is to obtain a regularity estimate taking advantage of the dissipation coming from the fluid. These estimates rely strongly on the previous elliptic results and on the fact that the system is studied without decoupling the fluid and structure. The decoupling allows to take advantage of the specificities of each sub problems [3, 19, 20, 24]. Nevertheless, this method enhances the gap of regularities between each sub-problem, leading to the need of adding some viscosity [19, 20] or deriving additionnal hidden regularity [24]. In this paper, we derive regularity estimates directly on a coupled linear system as for instance in [7, 17]. This enables us to obtain no gap between the regularities of initial data and of the solution. In the last section, we prove the existence of a local-in-time solution for the full nonlinear system by applying a classical Picard fixed point Theorem. We write the full system as a perturbation of the previous linear system. By doing so, a non homogenous divergence condition appears that we first lift in an appropriate way. Note also that, in estimating the nonlinear terms, a special attention is paid on the dependency of the various constants with respect to time. Eventually, we extend to the full nonlinear problem the existence result with no mismatch between the regularities of the initial data and of the solution.

2. General setting, main result

Below, we apply the same conventions and function spaces as in [15]. Time is the last variable of a function. This enables to write a unified definition for periodic functions whether they depend on one space variable only (such as the displacement η) or two space variables (such as the velocity-field u). In particular, we denote with sharped notations the periodic version in the first variable of a function space ($C_\sharp$, $L^p_\sharp$, $H^m_\sharp$ *etc*). We refer the reader to [15] for more details. Then, for any given function $b \in C_\sharp(0,L)$, *i.e.* the set of continuous and L-periodic functions on $\mathbb{R}$, satisfying $\min(1+b) > 0$, we define

$$\Omega_b := \{(x,y) \in \mathbb{R}^2 \,, \text{such that } x \in (0,L), y \in (0, 1+b(x))\} .$$

With this definition, the unknown fluid domain $\mathcal{F}(t)$ appearing in (FS) is related to the displacement η of the fluid-container top-boundary *via* $\mathcal{F}(t) = \Omega_{\eta(\cdot,t)}$. Finally, zero-average functions play a central role in our construction (as explained below). So, we denote:

$$L^2_{\sharp,0}(\Omega_b) := \left\{ f \in L^2_\sharp(\Omega_b) \text{ s.t.} \int_{\Omega_b} f(x)\mathrm{d}x = 0 \right\} ,$$

and, in the same way,

$$L^2_{\sharp,0}(0,L) := \left\{ f \in L^2_\sharp(0,L) \text{ s.t.} \int_0^L f(x)\mathrm{d}x = 0 \right\} .$$

We denote by $H^{-1}_\sharp(0,L)$ the dual of $H^1_\sharp(0,L) \cap L^2_{\sharp,0}(0,L)$. We emphasize that this choice is consistent with the L-periodic case that we consider herein.

2.1. **Main result.** An important remark on (FS) system is that the incompressibility condition together with boundary conditions imply:

$$\int_0^L \partial_t \eta = 0 \,, \quad \forall\, t > 0 \,. \tag{2.1}$$

Consequently, for any classical solution (u,p,η) to this system, the right-hand side of (1.3) must have zero mean:

$$\int_0^L \phi(u,p,\eta) = 0.$$

This property is achieved thanks to a good choice of the constant normalizing the pressure which is consequently uniquely defined. More precisely, we split the pressure into:

$$p = p_0 + c, \tag{2.2}$$

where, one imposes

$$\int_{\mathcal{F}(t)} p_0 = 0, \tag{2.3}$$

and c satisfies then

$$c(t) = \frac{1}{L}\int_0^L e_2 \cdot (\sigma(u,p_0))(x, 1+\eta(x,t), t)(-\partial_x \eta(x,t)\, e_1 + e_2). \tag{2.4}$$

This constant c is the Lagrange multiplier associated with the constraint (2.1). Note that, since the displacement of the structure is tranverse, the condition (2.1) is linear with respect to η. It is not the case when considering also longitudinal displacement.

Note that condition (2.1) imposes for compatibility reason that $\dot\eta^0$ satisfies also $\int_0^L \dot\eta^0 = 0$ and thus that $\int_0^L \eta^0$ is a constant that we choose to fix to equal to zero in the following, without any loss of generality.

We proceed with the definition of strong solution to (FS). We choose to define such solutions with respect to the classical strong solution theory for Navier Stokes equations. We remind that, on a fixed domain $\mathcal{F}$, a strong solution (u,p) to the incompressible Navier Stokes equations on $(0,T)$ would satisfy:

$$u \in H^1(0,T;L^2(\mathcal{F})) \cap C([0,T];H^1(\mathcal{F})) \cap L^2(0,T;H^2(\mathcal{F})), \quad p \in L^2(0,T;H^1(\mathcal{F})).$$

In full generality, it is required that $\partial\mathcal{F} \in C^{1,1}$ to obtain such a solution (in order to apply elliptic regularity results for the stationary Stokes problem). It is proven in [15] that, in the subgraph case that we consider herein, it is sufficient that the top boundary of the fluid domain is $H^2_\sharp(0,L)$ to obtain this class of solution (more generally it is sufficient to have a $H^{\frac{3}{2}+\varepsilon_0}$ boundary for $\varepsilon_0 > 0$). Hence, we consider in what follows that this H^2 subgraph property is at least satisfied initially. So, we consider initial conditions for which:

$$\eta^0 \in H^2_\sharp(0,L) \cap L^2_{\sharp,0}(0,L), \qquad \min(1+\eta^0) > 0, \tag{2.5}$$

and correspondingly:

$$\dot\eta^0 \in H^1_\sharp(0,L) \cap L^2_{\sharp,0}(0,L), \quad u^0 \in H^1_\sharp(\mathcal{F}^0) \tag{2.6}$$

where $\mathcal{F}^0 = \mathcal{F}_{\eta^0}$. Note that $\int_0^L \eta^0 = 0$ is not essential in all that follows and one could have considered only $\eta^0 \in H^2_\sharp(0,L)$.

Given $T > 0$, one may then look for a solution η to (1.3) with the minimal regularity for a standard wave equation:

$$\eta \in H^2(0,T;L^2_\sharp(0,L)) \cap W^{1,\infty}(0,T;H^1_\sharp(0,L)) \cap L^\infty(0,T;H^2_\sharp(0,L)), \tag{2.7}$$

When higher-order derivatives are involved, assuming further regularity of the initial data (see hypothesis $(\mathbf{H}_{\alpha,\delta})$–$(\mathbf{H}_{\alpha,\gamma})$ depending on the case $(\mathbf{C}_{\alpha,\delta})$–$(\mathbf{C}_{\alpha,\gamma})$ respectively, see Remark 2.3), we obtain additional regularity of the solution:

$$\left\{\begin{array}{l} \sqrt{\delta}\,\eta \in H^2(0,T;H^1_\sharp(0,L)) \cap W^{1,\infty}(0,T;H^2_\sharp(0,L)), \\ \sqrt{\alpha}\,\eta \in L^\infty(0,T;H^3_\sharp(0,L)), \\ \sqrt{\gamma}\,\eta \in H^1(0,T;H^2_\sharp(0,L)). \end{array}\right. \tag{2.8}$$

In order for the fluid domain to remain connected in time, we also require that:

$$\min_{t\in[0,T]} \min(1+\eta(\cdot,t)) > 0. \tag{2.9}$$

This yields a well-defined open space-time fluid domain

$$\mathcal{Q}_T := \{(x,y,t) \in (0,L) \times \mathbb{R} \times (0,T) \text{ s.t. } y \in (0,\eta(x,t))\},$$

on which we may require that:

$$\left\{\begin{array}{l} \partial_t u \in L^2_\sharp(\mathcal{Q}_T)\,, \quad \nabla^2 u \in L^2_\sharp(\mathcal{Q}_T)\,, \\ \nabla p \in L^2_\sharp(\mathcal{Q}_T)\,. \end{array}\right. \tag{2.10}$$

We emphasize that we do not ask for a regularity statement such as $u \in C([0,T];H^1_\sharp(\mathcal{F}))$. Indeed, here, as the fluid domain $\mathcal{F}$ moves with time, such a regularity statement can only be stated through a change of variables. Here, we choose to work with an intrinsic formulation. Our definition of strong solution reads:

Definition 2.1. *Given $(\alpha,\beta,\gamma,\delta) \in [0,\infty)^4$ satisfying one of the three assumptions $(\mathbf{C}_{\alpha,\gamma})$, $(\mathbf{C}_{\alpha,\delta})$ or $(\mathbf{C}_\beta)$, and $\rho_s > 0$, let us consider $(\eta^0,\dot\eta^0,u^0)$ satisfying* (2.5)-(2.6) *and $T > 0$. A strong solution to (FS) on $(0,T)$, associated with the initial data $(\eta^0,\dot\eta^0,u^0)$, is a triplet (η,u,p) satisfying* (2.7)-(2.8)-(2.9), (2.10) *and such that*

- *equations* (1.1)-(1.2) *are satisfied a.e. in $\mathcal{Q}_T$,*
- *equations* (2.2)-(2.3)-(2.4) *are satisfied a.e. in* $(0,T)$,
- *equation* (1.3) *is satisfied in* $L^2(0,T;H^{-1}_\sharp(0,L))$,
- *equations* (1.5)-(1.6) *are satisfied a.e. in* $(0,T)\times(0,L)$,
- *equations* (1.7)-(1.8)-(1.9) *are satisfied a.e. in* $(0,L)$ *and* $\mathcal{F}^0$.

In all cases but $(\mathbf{C}_{\alpha,\delta})$, our definition yields that equation (1.3) contains only terms (except possibly one) in $L^2(0,T;L^2_\sharp(0,L))$. Consequently, in both cases $(\mathbf{C}_\beta)$ and $(\mathbf{C}_{\alpha,\gamma})$, equation (1.3) actually holds a.e. and helps to gain regularity on the only term which does not belong to $L^2(0,T;L^2_\sharp(0,L))$. This remark yields that $\eta \in L^2(0,T;H^4_\sharp(0,L))$ in the case $(\mathbf{C}_{\alpha,\gamma})$. In the case $(\mathbf{C}_{\alpha,\delta})$, two terms in (1.3) are only in $L^2(0,T;H^{-1}_\sharp(0,L))$ (namely $\partial_{ttxx}\eta$ and $\partial_{xxxx}\eta$) so that no better regularity can be gained from the equation. To summarize, the definition above yields the following regularity of displacement fields:

- in the case $(\mathbf{C}_\beta)$ of a wave equation
$$\eta \in H^2(0,T;L^2_\sharp(0,L)) \cap L^\infty(0,T;H^2_\sharp(0,L)) \cap W^{1,\infty}(0,T;H^1_\sharp(0,L)),$$
- in the case $(\mathbf{C}_{\alpha,\delta})$ of a beam equation with inertia of rotation
$$\eta \in H^2(0,T;H^1_\sharp(0,L)) \cap L^\infty(0,T;H^3_\sharp(0,L)) \cap W^{1,\infty}(0,T;H^2_\sharp(0,L)),$$
- in the case $(\mathbf{C}_{\alpha,\gamma})$ of a damped beam
$$\eta \in H^2(0,T;L^2_\sharp(0,L)) \cap H^1(0,T;H^2_\sharp(0,L)) \cap L^\infty(0,T;H^3_\sharp(0,L)).$$

Nevertheless, from the kinematic condition (1.5) together with the fluid velocity regularity, we have moreover that $\partial_t\eta \in L^2(0,T;H^{3/2}_\sharp(0,L))$ in all cases.

We emphasize also that it is legitimate for the initial fluid velocity-field condition to verify (1.9). Indeed, for arbitrary $\Omega \Subset \mathcal{F}^0$, we have that, for small time, $\Omega \subset \mathcal{F}(t)$. Then, by restriction and interpolation $u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)) \subset C([0,T];H^1(\Omega))$.

In the strong solution framework that we depicted above, solutions are "so continuous" that initial data must keep track of some properties that are required in the equations. For instance, as classical in Navier Stokes equation, we have to require that the initial velocity-field u^0 satisfies:

$$\operatorname{div} u^0 = 0 \quad \text{on } \mathcal{F}^0\,, \tag{2.11}$$

and corresponding to no-slip conditions, we also have to require that:

$$u^0(x,0) = 0\,, \quad u^0(x,1+\eta^0(x)) = \dot\eta^0(x)e_2\,, \quad \forall\, x \in (0,L)\,. \tag{2.12}$$

Finally, the initial no-flux condition is to be satisfied by the initial structure velocity/displacement:

$$\int_0^L \dot\eta^0 = 0. \tag{2.13}$$

With these remarks, our main result reads as follows

Theorem 2.2. *Given $(\alpha,\beta,\gamma,\delta) \in [0,\infty)^4$ satisfying one of the three assumptions $(\mathbf{C}_{\alpha,\gamma})$, $(\mathbf{C}_{\alpha,\delta})$ or $(\mathbf{C}_\beta)$, and $\rho_s > 0$, let us consider initial data $(\eta^0,\dot\eta^0,u^0)$ satisfying (2.5)-(2.6) and compatibility conditions (2.11)-(2.12)-(2.13). Assume further that $(\eta^0,\dot\eta^0)$ satisfy*

$$\sqrt{\alpha}\,\eta^0 \in H^3_\sharp(0,L), \qquad \sqrt{\delta}\,\dot\eta^0 \in H^2_\sharp(0,L).$$

Then there exists a time T_0 depending decreasingly on:

$$\|u^0\|_{H^1_\sharp(\mathcal{F}^0)} + \|\eta^0\|_{H^2_\sharp(0,L)} + \|\sqrt{\alpha}\,\eta^0\|_{H^3_\sharp(0,L)} + \|\dot\eta^0\|_{H^1_\sharp(0,L)} + \|\sqrt{\delta}\,\dot\eta^0\|_{H^2_\sharp(0,L)} + \|(1+\eta^0)^{-1}\|_{L^\infty_\sharp(0,L)},$$

such that (FS) admits a unique strong solution on $(0,T_0)$.

Remark 2.3. *Several comments are in order:*

1. *The further assumptions on initial displacement and structure velocities read:*

$(\mathbf{H}_\beta)$ $\quad(\eta^0,\dot\eta^0) \in H^2_\sharp(0,L) \times H^1_\sharp(0,L)$, *for the wave equation* $(\mathbf{C}_\beta)$,

$(\mathbf{H}_{\alpha,\delta})$ $\quad(\eta^0,\dot\eta^0) \in H^3_\sharp(0,L) \times H^2_\sharp(0,L)$, *for the beam equation with inertia of rotation* $(\mathbf{C}_{\alpha,\delta})$,

$(\mathbf{H}_{\alpha,\gamma})$ $\quad(\eta^0,\dot\eta^0) \in H^3_\sharp(0,L) \times H^1_\sharp(0,L)$, *for the beam equation with additional viscosity* $(\mathbf{C}_{\alpha,\gamma})$.

Note that with the regularities (2.8) of the structure displacement, $\eta(\cdot,0)$ and $\partial_t\eta(\cdot,0)$ make sense for each considered cases in the above spaces. For instance in the case $(\mathbf{C}_\beta)$,

$\eta(\cdot,0) \in H^2_\sharp(0,L)$ and $\partial_t\eta(\cdot,0) \in H^1_\sharp(0,L)$, thanks, respectivelly, to the embedding of $L^\infty(0,T;H^2_\sharp(0,L))\cap W^{1,\infty}(0,T;H^1_\sharp(0,L))$ in $C([0,T];H^2_\sharp(0,L)_w)$ and of $H^1(0,T;L^2_\sharp(0,L))\cap L^\infty(0,T;H^1_\sharp(0,L))$ in $C([0,T];H^1_\sharp(0,L)_w)$, where the subscript w denotes the weak topology (see [21]*).*

2. *In the case $(\mathbf{C}_\beta)$, the dissipation of the fluid induces a dissipation on the structure that is sufficient to regularize the solution to the wave equation. Indeed, the dissipation from the fluid comes from the Dirichlet-to-Neumann type stationary Stokes system which is roughly speaking equivalent to $(-\partial_x^2)^{\frac12}$ and therefore, applying a result from Chen & Triggiani* [6] *in the space $L^2(0,T;H^{\frac12}_\sharp(0,L))$, we get from the previous regularity obtained in the Theorem above (see Definition 2.1) that $\partial_{tt}\eta$ and $\partial_{xx}\eta$ both belong to $L^2(0,T;H^{\frac12}_\sharp(0,L))$, that is $\eta\in H^2(0,T;H^{\frac12}_\sharp(0,L))\cap L^2(0,T;H^{\frac52}_\sharp(0,L))$.*

One important consequence of this local-in-time existence result is that it enables extension of solutions. In particular, classical dynamical-system methods enable to derive the following corollary from Theorem 2.2:

Corollary 2.4. *Given $(\alpha,\beta,\gamma,\delta)\in[0,\infty)^4$ satisfying $(\mathbf{C}_{\alpha,\gamma})$, or $(\mathbf{C}_{\alpha,\delta})$ or $(\mathbf{C}_\beta)$, and $\rho_s>0$, let us consider initial data $(\eta^0,\dot\eta^0,u^0)$ satisfying* (2.5)-(2.6) *and compatibility conditions* (2.11)-(2.12)-(2.13)*. Assume further that $(\eta^0,\dot\eta^0)$ satisfy*

$$\sqrt{\alpha}\,\eta^0\in H^3_\sharp(0,L),\qquad \sqrt{\delta}\,\dot\eta^0\in H^2_\sharp(0,L). \tag{2.14}$$

Then there exists a unique non-extendable strong solution to (FS) with initial data $(\eta^0,\dot\eta^0,u^0)$. Furthermore, this solution is defined on $(0,T_*)$ with the alternative:

- *either $T_*=+\infty$*
- *or $T_*<\infty$ and*

$$\limsup_{t\to T_*}\Big(\|u(\cdot,t)\|_{H^1_\sharp(\mathcal{F}(t))}+\|\eta(\cdot,t)\|_{H^2_\sharp(0,L)}+$$
$$\|\sqrt{\alpha}\,\eta(\cdot,t)\|_{H^3_\sharp(0,L)}+\|\dot\eta(\cdot,t)\|_{H^1_\sharp(0,L)}+\|\sqrt{\delta}\,\dot\eta(\cdot,t)\|_{H^2_\sharp(0,L)}+\|(1+\eta(\cdot,t))^{-1}\|_{L^\infty_\sharp(0,L)}\Big)=+\infty.$$

2.2. **Change of variables.** The strategy of proof for Theorem 2.2 is standard and consists first to rewrite the fluid equation in the reference configuration, namely Ω_0 and then perform a perturbation analysis on this quasilinear system. We note that Ω_0 is related to a displacement $\eta=0$ and is not necessary equal to $\mathcal{F}^0$ the initial configuration since $\eta^0\neq 0$. In this section, we introduce a change of variables that maps the reference configuration Ω_0 onto $\mathcal{F}(t)$. Our aim is to construct a unique change of variables which is properly defined for all cases $(\mathbf{C}_\beta)$, $(\mathbf{C}_{\alpha,\delta})$, $(\mathbf{C}_{\alpha,\gamma})$. Consequently, we want our choice to be valid for a minimal regularity of the displacement η, that is $L^\infty(0,T;H^2_\sharp(0,L))\cap W^{1,\infty}(0,T;H^1_\sharp(0,L))$. This minimal regularity corresponds to the worst case $(\mathbf{C}_\beta)$ of the wave equation.

Let fix time at first and denote with symbol b any displacement-field $\eta(\cdot,t)$. To tranform Ω_0 into Ω_b, the first choice that we can think about is to set simply:

$$\chi^1_b(x,y)=(x,y(1+b(x)))=(x,y+\mathcal{R}^1 b(x,y,t)).$$

This choice is made in [15, 19]. In particular we recall the following proposition that can be found in [15]:

Proposition 2.5. *Let us consider $b\in H^2_\sharp(0,L)$ satisfying $\min 1+b>0$. Then for any given $m\le 2$,*

- *the mapping $f\mapsto f\circ\chi^1_b$ realizes a linear homeomorphism from $H^m_\sharp(\Omega_b)$ onto $H^m_\sharp(\Omega_0)$,*

- *there exists a non-decreasing function* $K^1_m : [0,\infty) \to (0,\infty)$ *such that, if we assume* $\|b\|_{H^2_\sharp(0,L)} + \|(1+b)^{-1}\|_{L^\infty_\sharp(0,L)} \leq R_1$ *then*
$$\|f\circ\chi^1_b\|_{H^m_\sharp(\Omega_0)} \leq K^1_m(R_1)\|f\|_{H^m_\sharp(\Omega_b)}\,, \quad \|f\|_{H^m_\sharp(\Omega_b)} \leq K^1_m(R_1)\|f\circ\chi^1_b\|_{H^m_\sharp(\Omega_0)}\,.$$
- *there exists a universal constant* C *for which:*
$$\|(\nabla\chi^1_b)\,v\|_{H^1_\sharp(\Omega_0)} \leq C\|b\|_{H^2_\sharp(0,L)}\|v\|_{H^1_\sharp(\Omega_0)}, \qquad \forall v\in H^1_\sharp(\Omega_0).$$

The last item of this proposition is a consequence of the fact that $\nabla\chi^1_b$ belongs to the space $H^1(0,L;H^s(0,1))$, for all $s\geq 0$, which is a multiplier space of $H^1_\sharp(\Omega_0)$, whenever $s\geq 1$.

We emphasize that the change of variable χ^1_b is well-defined for arbitrary $b\in H^2_\sharp(0,L)$ under the sole condition that $1+b$ remains non-negative, and, in particular, without any restriction on $\|b\|_{H^2_\sharp(0,L)}$. However, one shortcoming of this first choice is that, when considering a displacement field
$$\eta\in L^\infty(0,T;H^2_\sharp(0,L))\cap W^{1,\infty}(0,T;H^1_\sharp(0,L))$$
nothing ensures that $\eta(\cdot,t)-\eta(\cdot,0)$ remains small for small times in $L^\infty(0,T;H^2_\sharp(0,L))$. And thus nothing ensures that, for instance, $\nabla\chi^1_{\eta(\cdot,t)} - \nabla\chi^1_{\eta(\cdot,0)}$ is small for a small time in the multiplier space $L^\infty(0,T;H^1(0,L;H^s(0,1))$ whereas this property is critical in our perturbation analysis. To overcome this difficulty, we note that
$$L^\infty(0,T;H^2_\sharp(0,L))\cap W^{1,\infty}(0,T;H^1_\sharp(0,L)) \hookrightarrow C^{0,\theta}([0,T];H^{2-\theta}_\sharp(0,L)), \quad \forall\,\theta\in(0,1),$$
with an embedding constant that does not depend on T. Consequently, another possible choice is to consider
$$\chi^2_b(x,y) = (x, y+\mathcal{R}^2 b(x,y)),$$
where $\mathcal{R}^2$ is a continuous lifting $H^s_\sharp(0,L)\to\{b\in H^{s+1/2}_\sharp(\Omega_0), \text{s.t. } b|_{y=0}=0\}$. Indeed, such an operator exists for $s>1/2$ so that, for $\varepsilon_0>0$ and $b\in H^{3/2+\varepsilon_0}$, we have that $\mathcal{R}^2 b$ and χ^2_b satisfy:

- $\mathcal{R}^2 b\in C^1(\overline{\Omega}_0)$ with
$$\|\mathcal{R}^2 b\|_{C^1(\overline{\Omega}_0)} \leq C\|b\|_{H^{\frac{3}{2}+\varepsilon_0}(0,L)},$$
for a universal constant C,
- χ^2_b maps $\partial\Omega_0$ into $\partial\Omega_b$.

Consequently, for small b (in the $H^{3/2+\varepsilon_0}_\sharp(0,L)$ norm), the mapping χ^2_b realizes a C^1-diffeomorphism from Ω_0 onto Ω_b. In the following proposition, we complement this remark with quantitative statement on the subsequent change of variables:

Proposition 2.6. *Let us consider* $b\in H^{3/2+\varepsilon_0}_\sharp(0,L)$, $0<\varepsilon_0$. *There exists a constant* $\mathcal{M}>0$ *such that, for every* $0<R_2\leq\mathcal{M}$, *if* $\|b\|_{H^{3/2+\varepsilon_0}_\sharp(0,L)}\leq R_2$, *then for any given* $m\leq 2$,

- *the mapping* $f\mapsto f\circ\chi^2_b$ *realizes a linear homeomorphism from* $H^m_\sharp(\Omega_b)$ *onto* $H^m_\sharp(\Omega_0)$,
- *there exists a non-decreasing function* $K^2_m:[0,\infty)\to(0,\infty)$ *such that:*
$$\|f\circ\chi^2_b\|_{H^m_\sharp(\Omega_0)} \leq K^2_m(R_2)\|f\|_{H^m_\sharp(\Omega_b)}\,, \quad \|f\|_{H^m_\sharp(\Omega_b)} \leq K^2_m(R_2)\|f\circ\chi^2_b\|_{H^m_\sharp(\Omega_0)}\,.$$
- *there exists an absolute constant* $C>0$ *for which we have:*
$$\|(\nabla\chi^2_b)\,v\|_{H^1_\sharp(\Omega_0)} \leq C\|b\|_{H^{3/2+\varepsilon_0}_\sharp(0,L)}\|v\|_{H^1_\sharp(\Omega_0)}, \quad \forall\,v\in H^1_\sharp(\Omega_0).$$

The proof of this proposition is straightforward and is left to the reader. With this second choice, we obtain that, given η in $L^\infty(0,T;H^2_\sharp(0,L))\cap W^{1,\infty}(0,T;H^1_\sharp(0,L))$ such that $\eta(\cdot,0)=\eta^0$, the functions $\eta(\cdot,t)-\eta^0$ and $\nabla\chi^2_{\eta(\cdot,t)}-\nabla\chi^2_{\eta^0}$ are small for small times in $L^\infty(0,T;H^{3/2+\varepsilon_0}_\sharp(0,L))$ and $L^\infty(0,T;H^{1+\varepsilon_0}_\sharp(\Omega_0))$ respectively. However, this second choice is restricted to small displacements.

For the time evolution problem, we choose a change of variables that take advantage of both constructions above. Given $\eta:(0,t)\times(0,L)\to\mathbb{R}$, we introduce a mapping χ_η which writes

$$\chi_\eta(x,y,t) = \big(x, y+\mathcal{R}^1(\eta^0)(x,y)+\mathcal{R}^2(\eta-\eta^0))(x,y,t)\big) \tag{2.15}$$

Thus $\chi_\eta = \chi^1_{\eta^0} + (0, \mathcal{R}^2(\eta - \eta^0))$ where $(0, \mathcal{R}^2(\eta-\eta^0))$ is a small-in-time perturbation of the mapping $\chi^1_{\eta^0}$ in $L^\infty(0,T;H^{2+\varepsilon_0}_\sharp(\Omega_0))$. Note that $\chi^1_{\eta^0}$ does not depend on time and is a C^1 diffeomorphism if $\min 1 + \eta^0 > 0$. For this "hybrid" change of variables, we can state

Proposition 2.7. *Let us consider $\eta \in L^\infty(0,T;H^2_\sharp(0,L)) \cap W^{1,\infty}(0,T;H^1_\sharp(0,L))$ and denote $\eta^0 := \eta(\cdot,0)$. If η^0 satisfies $\min 1+\eta^0 > 0$ with $\|\eta^0\|_{H^2_\sharp(0,L)} + \|(1+\eta^0)^{-1}\|_{L^\infty_\sharp(0,L)} \leq R_1$ then there exists a constant $\mathcal{K}$ such that if $\|\eta - \eta^0\|_{L^\infty(0,T;H^{3/2+\varepsilon_0}_\sharp(0,L))} \leq \mathcal{K}$ for some $\varepsilon_0 > 0$, then*

- *the mapping $f \mapsto f \circ \chi_\eta$ realizes a linear homeomorphism from $L^2(\mathcal{Q}_T)$ onto $L^2(0,T;L^2_\sharp(\Omega_0))$,*
- *there exists increasing functions K_0, K_1, K_2 such that, for arbitrary $f \in C^\infty(\overline{\mathcal{Q}_T})$ there holds:*

$$\|\partial_t [f\circ\chi_\eta]\|_{L^2(Q_T)} \leq K_1(R_1,\mathcal{K})\|f\|_{H^1(\mathcal{Q}_T)},$$
$$\|\nabla [f\circ\chi_\eta]\|_{L^2(Q_T)} \leq K_1(R_1,\mathcal{K})\|\nabla f\|_{L^2(\mathcal{Q}_T)},$$
$$\|\nabla^2 [f\circ\chi_\eta]\|_{L^2(Q_T)} \leq K_2(R_1,\mathcal{K})\left[\|\nabla f\|_{L^2(\mathcal{Q}_T)} + \|\nabla^2 f\|_{L^2_\sharp(\mathcal{Q}_T)}\right],$$
$$\|f\circ\chi_\eta\|_{C([0,T];H^1_\sharp(\Omega_0))} \leq K_0(R_1,\mathcal{K})\left[\|f\|_{H^1(\mathcal{Q}_T)} + \|\nabla^2 f\|_{L^2_\sharp(\mathcal{Q}_T)}\right],$$

and conversely:

$$\|\partial_t f\|_{L^2(\mathcal{Q}_T)} \leq K_1(R_1,\mathcal{K})\|f\circ\chi_\eta]\|_{H^1(Q_T)}$$
$$\|\nabla f\|_{L^2(\mathcal{Q}_T)} \leq K_1(R_1,\mathcal{K})\|\nabla[f\circ\chi_\eta]\|_{L^2(Q_T)}$$
$$\|\nabla^2 f\|_{L^2_\sharp(\mathcal{Q}_T)} \leq K_2(R_1,\mathcal{K})[\|\nabla^2[f\circ\chi_\eta]\|_{L^2(Q_T)} + \|\nabla[f\circ\chi_\eta]\|_{L^2(Q_T)}].$$

We do not give an exhaustive proof for this proposition, as it mostly follows the previous propositions in this section straightforwardly. The main point requiring new informations is the last item of the first list:

$$\|f\circ\chi_\eta\|_{C([0,T];H^1_\sharp(\Omega_0))} \leq K_0(R_1,\mathcal{K})\left[\|f\|_{H^1(\mathcal{Q}_T)} + \|\nabla^2 f\|_{L^2_\sharp(\mathcal{Q}_T)}\right].$$

This one is recovered by applying [21, Theorem 3.1], see the proof of [15, Theorem 2] for more details in a similar context.

Remark 2.8. *1. Note that, with our definition of χ_η, when η does not depend on time we have that $\chi_\eta = \chi^1_\eta$. Consequently we shall omit the superscript 1 when no confusion can be made.*

2. In Appendix A, we give further informations on this last change of variables in order to study the nonlinearities that are involved in the quasilinear introduced below.

3. Although Propositions the results of 2.6 and 2.7 depend on the parameter $\varepsilon_0 > 0$, we will make the explicit choice $\varepsilon_0 = \frac{1}{4}$ in the differents proofs, for simplicity (see Appendix A for details).

2.3. **Equivalent system in a fixed domain.** Now we rewrite the fluid–structure system in the reference configuration Ω_0. With the definition of the previous subsection, we set

$$v(x,y,t) = u(\chi_\eta(x,y,t),t) \quad \text{and} \quad q(x,y,t) = p(\chi_\eta(x,y,t),t).$$

These pseudo-Lagrangian quantities satisfy in $Q_T := \Omega_0 \times (0,T)$:

$$(\rho_f \det\nabla\chi_\eta)\partial_t v + \rho_f((v - \partial_t\chi_\eta)\cdot(B_\eta\nabla))v - \mu\mathrm{div}\,((A_\eta\nabla)v) + (B_\eta\nabla)q = 0, \tag{2.16}$$
$$\mathrm{div}\,(B_\eta^\top v) = 0, \tag{2.17}$$

where the matrices A_η and B_η are defined by

$$B_\eta = \mathrm{cof}\ \nabla\chi_\eta, \quad A_\eta = \frac{1}{\det\nabla\chi_\eta}B_\eta^\top B_\eta. \tag{2.18}$$

To rewrite the structure equation, we remark that, by using the fact that $u_1(x, 1+\eta(x,t),t) = 0$, on $(0,L)$ and that $\mathrm{div}\, u = 0$, we can show that (see for instance [5]):

$$((\nabla u)^\top(x, 1+\eta(x,t))(-\partial_x\eta(x,t)\,e_1 + e_2)\cdot e_2 = 0.$$

Thus the forcing term applied by the fluid on the structure can be simplified into:

$$\phi(u,p,\eta)(x,t) = p(x,1+\eta(x,t),t) - \mu e_2 \cdot \nabla u(x,1+\eta(x,t),t)(-\partial_x\eta(x,t)\, e_1 + e_2)\,,$$

and, with the unknowns (v,q) defined in the fixed domain, the structure equation reads:

$$\rho_s\partial_{tt}\eta - \delta\partial_{xxtt}\eta + \alpha\,\partial_{xxxx}\eta - \beta\partial_{xx}\eta - \gamma\,\partial_{xxt}\eta = -\left[\mu((A_\eta\nabla)v - qB_\eta\right]\, e_2\cdot e_2, \quad \text{on } (0,L). \tag{2.19}$$

Finally, the kinematic condition at the fluid–structure interface reads

$$v(x,1,t) = \partial_t\eta(x,t)e_2, \quad \text{on } (0,L)\times(0,T). \tag{2.20}$$

The other boundary conditions are preserved:

- L-periodicity w.r.t. x for the fluid and the structure;
- no-slip boundary conditions on the bottom of the fluid container:

$$v(x,0,t) = 0\,. \tag{2.21}$$

The initial conditions for the structure displacement and velocity are still (1.7), (1.8). We remind that we require the initial data of the structure equations to satisfy:

$$\min(1+\eta^0) > 0, \qquad \int_0^L \dot\eta^0 = 0. \tag{2.22}$$

The initial condition for the fluid velocity reads:

$$v(x,y,0) = u^0(x, y(1+\eta^0(x)) = v^0(x,y)\,,\ \ (x,y)\in\Omega_0. \tag{2.23}$$

The initial data v^0 has to satisfy also compatibility conditions set in the reference domain:

$$v^0(x,0) = 0\,, \quad v^0(x,1) = \dot\eta^0(x)e_2\,, \quad \forall\, x\in(0,L)\,, \tag{2.24}$$

$$\operatorname{div}(B_{\eta^0}^\top v^0) = 0 \quad \text{on } \Omega_0\,. \tag{2.25}$$

Note that, with our decomposition (2.2) of the pressure p, the image q of p by the mapping χ_η satisfies

$$q(x,y,t) = p(\chi_\eta(x,y,t),t) = p_0(\chi_\eta(x,y,t),t) + c(t) = q_0(x,y,t) + c(t),$$

and q_0 verifies the following constraint

$$\int_{\Omega_0} q_0 \det\nabla\chi_\eta = 0, \tag{2.26}$$

which corresponds to (2.4) in the reference configuration. Moreover, with the new unknowns, the constant c writes

$$c(t) = \frac{1}{L}\int_0^L -\mu((A_\eta\nabla)v\, e_2)\cdot e_2 + q_0(B_\eta\, e_2)\cdot e_2.$$

Nevertheless, working with (2.26) is not easy since it is a non linear combination of unknowns q_0 and η. So, we split the pressure q differently by setting:

$$q(x,y,t) = r_0(x,y,t) + d(t), \tag{2.27}$$

with

$$\int_{\Omega_0} r_0 = 0, \tag{2.28}$$

Then,

$$d(t) = \frac{1}{L}\int_0^L -\mu((A_\eta\nabla)v\, e_2)\cdot e_2 + r_0(B_\eta\, e_2)\cdot e_2. \tag{2.29}$$

From the knowledge of (r_0,d) we can recover (q_0,c) with exactly the same regularities. Indeed, we have

$$c(t) = \frac{1}{|\mathcal{F}(t)|}\left(d(t) + \int_{\Omega_0} r_0 \det\nabla\chi_\eta\right) \quad \text{and} \quad q_0 = r_0 + d(t) - \frac{1}{|\mathcal{F}(t)|}\left(d(t) + \int_{\Omega_0} r_0\det\nabla\chi_\eta\right).$$

Due to Proposition 2.7, Theorem 2.2 is equivalent to the existence of a strong solution (η, v, q) of the system written in the reference configuration, denoted $(\mathrm{FS})_{ref}$, stated in the following theorem:

Theorem 2.9. *Given $(\alpha,\beta,\gamma,\delta)\in[0,\infty)^4$ satisfying one of the three assumptions $(\mathbf{C}_{\alpha,\gamma})$, $(\mathbf{C}_{\alpha,\delta})$ or $(\mathbf{C}_{\beta})$, and $\rho_s>0$, let us consider initial data*

$$(\eta^0,\dot\eta^0,v^0)\in H^2_\sharp(0,L)\times H^1_\sharp(0,L)\times H^1_\sharp(\Omega_0),$$

satisfying compatibility conditions (2.22)-(2.24)-(2.25) and $\mathcal{K}>0$. Assume further that:

$$\sqrt{\alpha}\,\eta^0\in H^3_\sharp(0,L),\qquad \sqrt{\delta}\,\dot\eta^0\in H^2_\sharp(0,L).$$

Then, there exists $T>0$ depending decreasingly on

$$\|v^0\|_{H^1_\sharp(\mathcal{F}^0)}+\|\eta^0\|_{H^2_\sharp(0,L)}+\|\sqrt{\alpha}\,\eta^0\|_{H^3_\sharp(0,L)}+\|\dot\eta^0\|_{H^1_\sharp(0,L)}+\|\sqrt{\delta}\,\dot\eta^0\|_{H^2_\sharp(0,L)}+\|(1+\eta^0)^{-1}\|_{L^\infty_\sharp(0,L)},$$

such that there exists a unique strong solution to $(FS)_{ref}$ on $(0,T)$ i.e. *a triplet (η,v,q) satisfying:*

- *the following regularity statement for the fluid unknowns (v,q,d):*

$$\begin{aligned}&v\in L^2(0,T;H^2_\sharp(\Omega_0))\cap H^1(0,T;L^2_\sharp(\Omega_0))\cap C([0,T];H^1_\sharp(\Omega_0)),\\ &q\in L^2(0,T;H^1_\sharp(\Omega_0)),\quad d\in L^2(0,T)\,,\end{aligned}$$

- *the regularity statement (2.7) for the structure unknown η,*
- *equations (2.16), (2.17), (2.18), (2.15) a.e. in $\Omega_0\times(0,T)$,*
- *equations (2.27), (2.28), (2.29) a.e. in $(0,T)$,*
- *equation (2.19) in $L^2(0,T;H^{-1}_\sharp(0,L))$,*
- *equations (2.20), (2.21) a.e. in $(0,T)\times(0,L)$,*
- *equations (1.7), (1.8), (2.23) a.e. in $(0,L)$ and Ω_0.*

Furthermore, there exists a positive time (still denoted T) depending on $\mathcal{K}$ (defined in Proposition 2.7 with $\varepsilon_0=\frac14$ here) for which we have:

$$\|\eta-\eta^0\|_{L^\infty(0,T;H^{7/4}_\sharp(0,L))}\le\mathcal{K}.$$

The two next sections are devoted to the proof of this theorem. We apply a standard perturbation method. We write $(\mathrm{FS})_{ref}$ as follows:

$$\rho_{f,\eta^0}\partial_t v-\mu\mathrm{div}\,((A_{\eta^0}\nabla)v)+(B_{\eta^0}\nabla)q=f_1[v,\eta]+f_2[v,\eta]+\mathrm{div}\,h[v,q,\eta],\qquad\text{in }\Omega_0\,,$$

$$\mathrm{div}\,(B^\top_{\eta^0}v)=g[v,\eta],\qquad\text{in }\Omega_0\,,$$

$$\begin{aligned}\rho_s\partial_{tt}\eta-\delta\partial_{xxtt}\eta+\alpha\,\partial_{xxxx}\eta-\beta\partial_{xx}\eta-\gamma\,\partial_{xxt}\eta=&\\ -\mu((A_{\eta^0}\nabla)v\,e_2)\cdot e_2+q(B_{\eta^0}\,e_2)\cdot e_2-(h[v,q,\eta]e_2)\cdot e_2,&\quad\text{on }(0,L)\,,\end{aligned}$$

$$v=\partial_t\eta e_2\quad\text{on }(0,L)\times\{1\},$$

with $\rho_{f,\eta^0}=\rho_f\det\nabla\chi_{\eta^0}$, and

$$f_1[v,\eta]=\rho_f(\det\nabla\chi_{\eta^0}-\det\nabla\chi_\eta)\partial_t v \tag{2.30}$$

$$f_2[v,\eta]=-\rho_f((v-\partial_t\chi_\eta)\cdot(B_\eta\nabla))v \tag{2.31}$$

$$g[v,\eta]=\mathrm{div}\,((B^\top_{\eta^0}-B^\top_\eta)v) \tag{2.32}$$

$$h[v,q,\eta]=-\mu((A_{\eta^0}-A_\eta)\nabla)v+q(B_{\eta^0}-B_\eta). \tag{2.33}$$

Consequenlty we consider at first f_1,f_2,g,h as data for the above linear coupled problem in the unknowns (η,v,q). The dependance of f_1,f_2,g,h on (η,v,q) is then anlayzed in order to apply a classical fixed-point theorem. In the linear problem, the fluid unknowns solve an unsteady Stokes–like problem with a non homogeneous divergence constraint. We underline that the matrices B_{η^0} and A_{η^0} appearing in this problem do not depend on time. We note moreover the peculiar form of the right-hand side of the structure equation where the perturbation term $-(he_2)\cdot e_2$ is the counter part of the one in the fluid $\mathrm{div}\,h$.

2.4. **Preliminary results.** To derive the desired regularity on the fluid velocity and pressure solution to $(\mathrm{FS})_{ref}$, we need an elliptic regularity result for a steady state Stokes–like problem. We use results derived in [15] that we recall here for the sake of completeness. In the cases $(\mathbf{C}_{\alpha,\delta})$, $(\mathbf{C}_{\alpha,\gamma})$ of beam equations, the regularity of η^0 (namely $H^3_\sharp(0,L)\subset C^{1,1}_\sharp(0,L)$, which corresponds to the one coming from the fact that $\alpha>0$) is sufficient to apply standard elliptic results for the Stokes–like equation. Nevertheless in the case $(\mathbf{C}_\beta)$ of the wave equation the initial displacement is only $H^2_\sharp(0,L)$ (which does not embed in $C^{1,1}$) so that finer estimates are required.

Let us consider the following Stokes–like problem, for $b\in H^2_\sharp(0,L)$

$$-\mathrm{div}[(A_b\nabla)z]+(B_b\nabla)r_0 = f\,,\quad \text{in } \Omega_0\,, \tag{2.34}$$

$$\mathrm{div}(B_b^\top z) = g\,,\quad \text{in } \Omega_0\,, \tag{2.35}$$

with $f\in L^2_\sharp(\Omega_0)$, $g\in H^1_\sharp(\Omega_0)$. Here we underline that, since b does not depend on time the deformation mapping χ_b that defines A_b and B_b is equal to χ^1_b. The system is completed with the following boundary conditions:

$$z(x,1) = \dot\eta(x)e_2\,,\qquad \forall\, x\in(0,L)\,, \tag{2.36}$$

$$z(x,0) = 0\,,\qquad \forall\, x\in(0,L)\,, \tag{2.37}$$

with $\dot\eta\in H^{3/2}_\sharp(0,L)$. Due to the boundary conditions and to the incompressibility constraint the data should satisfy

$$\int_0^L \dot\eta = \int_{\Omega_0} g. \tag{2.38}$$

Following the lines of the proof of [15, Lemma 1] we obtain

Lemma 2.10. *For any $b\in H^2_\sharp(0,L)$ such that $(1+b)^{-1}\in L^\infty_\sharp(0,L)$, source terms and boundary condition*

$$(f,g)\in L^2_\sharp(\Omega_h)\times(H^1_\sharp(\Omega_h)\cap L^2_{\sharp,0}(\Omega_h)),\quad \dot\eta\in H^{\frac32}_\sharp(0,L),$$

satisfying (2.38), there exists a unique solution $(z,r_0)\in H^2_\sharp(\Omega_0)\times(H^1_\sharp(\Omega_0)\cap L^2_{\sharp,0}(\Omega_0))$ to the Stokes system (2.34)–(2.37). Moreover, there exists a non-decreasing function $K^{\mathrm s}:[0,\infty)\to(0,\infty)$ such that, if we assume $\|b\|_{H^2_\sharp(0,L)}+\|(1+b)^{-1}\|_{L^\infty_\sharp(0,L)}\le R_1$ then, this solution satisfies:

$$\|z\|_{H^2_\sharp(\Omega_0)}+\|r_0\|_{H^1_\sharp(\Omega_0)}\le K^{\mathrm s}(R_1)\left(\|f\|_{L^2_\sharp(\Omega_0)}+\|g\|_{H^1_\sharp(\Omega_0)}+\|\dot\eta\|_{H^{\frac32}_\sharp(0,L)}\right).$$

Remark 2.11. *The proof of the previous lemma relies mainly on the fact that the matrices B_b and A_b, that are well-defined and invertible for b satisfying $\min\limits_{x\in(0,L)}(1+b(x))>0$, belong to $H^1_\sharp((0,L);H^s(0,1))$, for any $s\ge0$, which is a multiplier of $H^1_\sharp(\Omega_0)$ for any $s\ge1$. The very same elliptic regularity result would hold true if:*

$$B_b=\mathrm{cof}\,\nabla(\chi^2_b),\qquad A_b=\frac{1}{\det\nabla(\chi^2_b)}B_b^\top B_b$$

whenever $b\in H^{\frac74}_\sharp(0,L)$ is such that $\|b\|_{H^{\frac74}_\sharp(0,L)}\le R_2$, where R_2 is defined in Proposition 2.6 (with $\varepsilon_0=\frac14$ here).

We end up this section with an approximation lemma that we shall use without mention later on. This lemma is in particular used tacitly many times in energy estimates to justify integration by parts.

Lemma 2.12. *Let $m\in\mathbb N\setminus\{0\}$ and $u\in H^1_\sharp(\Omega_0)$. Assume that there exists $\dot\eta\in H^m_\sharp(0,L)$ such that:*

$$u(x,1)=\dot\eta(x)e_2,\qquad u(x,0)=0,\qquad \text{on } (0,L).$$

There exists a sequence $(u_n)_{n\in\mathbb N}\in C^\infty_\sharp(\Omega_0)$ such that:

- *$u_n(x,0)=0$ for all $n\in\mathbb N$*

- *$u_n(\cdot,1) = \dot\eta_n e_2$ for all $n\in\mathbb{N}$, with $\dot\eta_n$ converging toward $\dot\eta$ in $H^m_\sharp(0,L)$ when $n\to\infty$.*
- *u_n converges towards u in $H^1_\sharp(\Omega_0)$.*

Proof. For any $\xi\in H^1_\sharp(0,L)$ let denote:

$$U[\xi](x,y) = \xi(x) y e_2 \quad \forall\,(x,y)\in\Omega_0.$$

It is straightforward that $U[\xi]\in H^1_\sharp(\Omega_0)$ with

- $\|U[\xi]; H^1_\sharp(\Omega_0)\| \le K\|\xi\|_{H^1_\sharp(0,L)}$, for some universal constant K,
- $U[\xi](x,1) = \xi(x)e_2$ on $(0,L)$,
- $U[\xi](x,0) = 0$ on $(0,L)$.

We have also that, if $\xi\in C^\infty_\sharp(0,L)$, then $U[\xi]\in C^\infty_\sharp(\Omega_0)$.

Let us now consider $(u,\dot\eta)$ satisfying the assumptions of our lemma. We denote $v := u - U[\dot\eta]$ so that $v\in H^1_\sharp(\Omega_0)$ vanishes on $y=0$ and $y=1$. We can easily construct (by dilation in the y-variable and convolution) a sequence $(v_n)_{n\in\mathbb{N}}\in C^\infty_\sharp(\Omega_0)$ converging towards v in $H^1_\sharp(\Omega_0)$ such that

$$v_n(x,1) = v_n(x,0) = 0, \qquad \forall\, x\in(0,L), \quad \forall\, n\in\mathbb{N}.$$

Let then construct a sequence $(\dot\eta_n)_{n\in\mathbb{N}}$ converging to $\dot\eta$ in $H^m_\sharp(0,L)$ (by projecting $\dot\eta$ onto a finite number of Fourier modes for instance). The candidates

$$u_n = v_n + U[\dot\eta_n], \quad \forall\, n\in\mathbb{N},$$

satisfy all the requirements of the lemma. □

3. Study of a linear system

In this section, we fix parameters $(\alpha,\beta,\gamma,\delta)\in[0,\infty)^4$ satisfying one of the three assumptions $(\mathbf{C}_{\alpha,\gamma})$, $(\mathbf{C}_{\alpha,\delta})$ or $(\mathbf{C}_{\beta})$. Given $b\in H^2_\sharp(0,L)$, s.t. $\min(1+b)>0$, we study the associated linear problem that we introduced in the previous section:

$$\rho_{f,b}\partial_t v - \mu \mathrm{div}\,((A_b\nabla)v) + (B_b\nabla)q = f + \mathrm{div}\, h, \quad \text{in } \Omega_0\,, \tag{3.1}$$

$$\mathrm{div}\,(B_b^\top v) = g, \quad \text{in } \Omega_0\,, \tag{3.2}$$

$$\rho_s\partial_{tt}\eta - \delta\partial_{xxtt}\eta + \alpha\,\partial_{xxxx}\eta - \beta\partial_{xx}\eta - \gamma\,\partial_{xxt}\eta = - \mu((A_b\nabla)v\, e_2)\cdot e_2 + q(B_b\, e_2)\cdot e_2 - (he_2)\cdot e_2, \quad \text{on } (0,L)\,, \tag{3.3}$$

with the coupling conditions

$$v(x,1,t) = \partial_t\eta(x,t)e_2, \quad \text{on } (0,L)\times(0,T). \tag{3.4}$$

and the boundary conditions

- L-periodicity w.r.t. x for the fluid and the structure;
- no-slip boundary conditions on the bottom of the fluid container:

$$v(x,0,t) = 0\,. \tag{3.5}$$

This system is completed with initial boundary conditions

$$\eta(x,0) = \eta^0(x)\,, \qquad x\in(0,L), \tag{3.6}$$
$$\partial_t\eta(x,0) = \dot\eta^0(x)\,, \qquad x\in(0,L)\,, \tag{3.7}$$
$$v(x,y,0) = v^0(x,y)\,, \quad (x,y)\in\Omega_0, \tag{3.8}$$

satisfying the compatibility conditions:

$$v^0(x,0) = 0\,, \quad v^0(x,1) = \dot\eta^0(x)e_2\,, \quad \forall\, x\in(0,L)\,, \tag{3.9}$$

$$\mathrm{div}\,(B_b^\top v^0)(x,y) = 0\,, \;\; (x,y)\in\Omega_0 \qquad \text{and} \qquad \int_0^L \dot\eta^0 = 0. \tag{3.10}$$

We recall that, as mentioned in the previous section, the pressure of this system is uniquely fixed by the condition that the volume of the bulk is conserved with time. Hence, the solution to our linear system is a triplet (η, v, q) where q splits into:

$$q(x,y,t) = r_0(x,y,t) + d(t), \tag{3.11}$$

with

$$\int_{\Omega_0} r_0 = 0, \qquad d(t) = \frac{1}{L}\int_0^L -\mu((A_\eta \nabla)v\, e_2)\cdot e_2 + r_0(B_\eta\, e_2)\cdot e_2. \tag{3.12}$$

First we study the linear system with a divergence free constraint, then we consider the case of a non homogeneous divergence.

3.1. **Function spaces.** Our purpose is to see the resolution of the linear system above as a linear mapping involved in a fixed point argument. With this purpose in mind, we fix function spaces for the initial data/solution/source terms in which we solve this problem.

Concerning initial data, in consistency with the assumptions of Theorem 2.9, we set:

$$X_s^0 := \{(\eta^0,\dot\eta^0) \in H^2_\sharp(0,L)\times H^1_\sharp(0,L) \text{ s.t. } \sqrt{\alpha}\eta^0 \in H^3_\sharp(0,L) \text{ and } \sqrt{\delta}\dot\eta^0 \in H^2_\sharp(0,L)\},$$
$$X_f^0 := \{v^0 \in H^1_\sharp(\Omega_0) \text{ s.t. } \mathrm{div}(B_b^\top v^0) = 0 \text{ and } v^0(\cdot,0) = 0\}$$

and

$$X^0 := \{(\eta^0,\dot\eta^0,v^0) \in X_s^0 \times X_f^0 \text{ s.t. } v^0(\cdot,1) = \dot\eta^0 e_2\}.$$

These spaces are endowed with the product-norm:

$$\|(\eta^0,\dot\eta^0,v^0)\|_{X^0} := \|\eta^0\|_{H^2_\sharp(0,L)} + \|\sqrt{\alpha}\eta^0\|_{H^3_\sharp(0,L)} + \|\dot\eta^0\|_{H^1_\sharp(0,L)} + \|\sqrt{\delta}\dot\eta^0\|_{H^2_\sharp(0,L)} + \|v^0\|_{H^1_\sharp(\Omega_0)}.$$

Correspondingly, given $T > 0$ we define spaces to which our fluid structure unknowns belong to:

$$X_{s,T} := \left\{ \begin{array}{l} \eta \in H^2(0,T;L^2_\sharp(0,L)) \cap W^{1,\infty}(0,T;H^1_\sharp(0,L)) \cap L^\infty(0,T;H^2_\sharp(0,L)) \\ \quad \text{s.t. } \sqrt{\alpha}\,\eta \in L^\infty(0,T;H^3_\sharp(0,L)), \quad \sqrt{\gamma}\,\eta \in H^1(0,T;H^2_\sharp(0,L)), \\ \quad \text{and } \sqrt{\delta}\eta \in H^2(0,T;H^1_\sharp(0,L)) \cap W^{1,\infty}(0,T;H^2_\sharp(0,L)), \end{array} \right\},$$

$$X_{f,T} := [H^1(0,T;L^2_\sharp(\Omega_0)) \cap C([0,T];H^1_\sharp(\Omega_0)) \cap L^2(0,T;H^2_\sharp(\Omega_0))] \times L^2(0,T;H^1_\sharp(\Omega_0)).$$

Again, these spaces are endowed with the obvious product/intersection norms $\|\cdot\|_{X_{s,T}}$, $\|\cdot\|_{X_{f,T}}$. We note that, concerning the pressure q, the Poincaré-Wirtinger inequality on Ω_0 implies that we have the equivalence of norms:

$$\frac{1}{C_{PW}}\|q\|_{H^1_\sharp(\Omega_0)} \le \|\nabla r_0\|_{L^2_\sharp(\Omega_0)} + |d| \le C_{PW}\|q\|_{H^1_\sharp(\Omega_0)}$$

where $q = r_0 + d$ is the above decomposition (3.11)-(3.12).

Finally, the forcing terms f, g, h of our problem satisfy

$$f \in L^2(0,T;L^2_\sharp(\Omega_0)), \tag{3.13}$$

$$g \in L^2(0,T;H^1_\sharp(\Omega_0) \cap L^2_{\sharp,0}(\Omega_0)) \cap H^1_{0,0}(0,T;[H^1_\sharp(\Omega_0)]'), \tag{3.14}$$

$$h \in L^2(0,T;H^1_\sharp(\Omega_0)), \tag{3.15}$$

which we gather in the following space:

$$S_T = \left\{ (j,k,l) \text{ s.t. } \begin{array}{l} j \in L^2(0,T;L^2_\sharp(\Omega_0)) \\ k \in L^2(0,T;H^1_\sharp(\Omega_0) \cap L^2_{\sharp,0}(\Omega_0)) \cap H^1_{0,0}(0,T;[H^1_\sharp(\Omega_0)]') \\ l \in L^2(0,T;H^1_\sharp(\Omega_0)) \end{array} \right\}.$$

Concerning g we introduce the space

$$H^1_{0,0}(0,T;[H^1_\sharp(\Omega_0)]') = \{g \in H^1(0,T;[H^1_\sharp(\Omega_0)]') \text{ s.t. } g(\cdot,0) = 0\}.$$

We emphasize that we enforce the vanishing condition at initial time only. We endow again S_T with the product norm.

Remark 3.1.

1. *We underline, once again, that the regularity of* b *is chosen to be the worst regularity and corresponds to the case of the wave equation* $(\mathbf{C}_\beta)$ *for which we have* $\eta^0 \in H^2_\sharp(0,L)$ *and only a control of the structure displacement in* $L^\infty(0,T;H^2_\sharp(0,L))$*. In the cases* $(\mathbf{C}_{\alpha,\delta})$*,* $(\mathbf{C}_{\alpha,\gamma})$ *of a beam equation we could work with* b *in* $H^3_\sharp(0,L)$*.*
2. *In our fixed point method, we solve the linear problem above with* g *given by formula* (2.32)*. Hence, assuming that the solution* (η,v,q) *lies in* $X_{s,T}\times X_{f,T}$*, we have* $g = div G$ *with* $G \in L^2(0,T;H^1_\sharp(\Omega_0))\cap H^1(0,T;L^2_\sharp(\Omega_0))$ *vanishing on the top and bottom boundaries of* Ω_0*. In this particular case, there exists a universal constant* C *(independant of* T*) for which:*

$$\|(f,g,h)\|_{S_T} \leq \|f\|_{L^2(0,T;L^2_\sharp(\Omega_0))} + \|\nabla g\|_{L^2(0,T;L^2_\sharp(\Omega_0))} + C\|\partial_t G\|_{L^2(0,T;L^2_\sharp(\Omega_0))} + \|h\|_{L^2(0,T;H^1_\sharp(\Omega_0))}.$$

Further details on these computations are provided in Appendix A.

First we consider the case $g = 0$. Then by a lifting argument the case $g \neq 0$ is studied.

3.2. **Homogeneous divergence constraint.** The aim of this subsection is to prove the following proposition.

Proposition 3.2. *Let us consider* b *in* $H^2_\sharp(0,L)$*, s.t.* $\min_{x\in(0,L)}(1+b(x))>0$*, initial data* $(\eta^0,\dot\eta^0,v^0)$ *in* X^0 *satisfying* (3.9) *and* (3.10) *and* f *and* h *satisfying resp.* (3.13), (3.15)*. Given* $T\in(0,1)$*, there exists a unique solution* $(\eta,v,q)\in X_{s,T}\times X_{f,T}$ *of* (3.1)*–*(3.3)*, satisfying*

- *equations* (3.1), (3.2) *a.e. in* $(0,T)\times\Omega_0$*,*
- *equations* (3.3) *in* $L^2(0,T;H^{-1}_\sharp(0,L))$*,*
- *equations* (3.4), (3.5) *a.e. in* $(0,T)\times(0,L)$*,*
- *equations* (3.6), (3.7), (3.8) *a.e. in* $(0,L)$ *and* Ω_0*,*
- *equations* (3.11), (3.12) *a.e. in* $(0,T)\times\Omega_0$ *and* $(0,T)$ *respectively.*

Moreover, there exists a non-decreasing function $C:[0,\infty)\to[0,\infty)$ *such that, assuming* $\|b\|_{H^2_\sharp(0,L)} + \|(1+b)^{-1}\|_{L^\infty_\sharp(0,L)} \leq R_1$*, the solution* (η,v,q) *satisfies*

$$\|(v,q)\|_{X_{f,T}} + \|\eta\|_{X_{s,T}} \leq C(R_1)\left(\|(v^0,\eta^0,\dot\eta^0)\|_{X^0} + \|(f,0,h)\|_{S_T}\right), \tag{3.16}$$

The remainder of this section is devoted to the proof of this proposition which is split into three main steps. First we obtain the existence of a strong solution for a regularized problem, where we add a parabolic regularization to the structure equation. Then, we derive, for the solution of this regularized problem, additionnal regularity estimates, not depending on the regularization parameter. These estimates rely strongly on the elliptic result for the Stokes–like problem (see Section 2.4) and take advantage of the dissipation coming from the fluid in order to control the structure velocity, in particular in the cases of the wave equation $(\mathbf{C}_\beta)$ and of the beam with inertia of rotation $(\mathbf{C}_{\alpha,\delta})$ (where additional estimates are needed compared to the case $(\mathbf{C}_{\alpha,\gamma})$). Finally we pass to the limit as the regularization parameter tends to zero and prove uniqueness.

Regularized problem. Let $\varepsilon>0$. We add to the structure equation the viscous term $\varepsilon\partial_{xxxxt}\eta$ and we look for $(v_\varepsilon,q_\varepsilon,\eta_\varepsilon)$ solution of coupled problem where the structure equation (3.3) is replaced by

$$\begin{aligned}\rho_s\partial_{tt}\eta_\varepsilon - \delta\partial_{xxtt}\eta_\varepsilon + \alpha\,\partial_{xxxx}\eta_\varepsilon - \beta\partial_{xx}\eta_\varepsilon - \gamma\,\partial_{xxt}\eta_\varepsilon + \varepsilon\partial_{xxxxt}\eta_\varepsilon =& \\ -\,\mu((A_b\nabla)v_\varepsilon\, e_2)\cdot e_2 + q_\varepsilon(B_b\, e_2)\cdot e_2 - (he_2)\cdot e_2, &\quad \text{on } (0,L)\,,\end{aligned} \tag{3.17}$$

Note that we choose here a regularization that works with any of the three cases. Nevertheless, in the case $(\mathbf{C}_\beta)$ we could only consider that $\gamma>0$ and then let γ goes to zero.

Due to the regularization term we need also to regularize the structure initial velocity $\dot\eta^0$ and initial displacement η^0. We denote by $\dot\eta^0_\varepsilon$ the approximate initial velocity and η^0_ε the approximate initial displacement. It is such that

$$\dot\eta^0_\varepsilon \in H^3_\sharp(0,L) \text{ s.t. } \int_0^L \dot\eta^0_\varepsilon = 0, \qquad \eta^0_\varepsilon \in H^4_\sharp(0,L)$$

and

$$(\eta_\varepsilon^0, \dot\eta_\varepsilon^0) \longrightarrow (\eta^0, \dot\eta^0) \text{ in } X_s^0 \text{ when } \varepsilon \to 0$$
$$\varepsilon\left(\|\eta_\varepsilon^0\|^2_{H^4_\sharp(0,L)} + \|\dot\eta_\varepsilon^0\|^2_{H^3_\sharp(0,L)}\right) \text{ bounded independently of } \varepsilon > 0.$$

It is easy to construct such an approximation by projecting η^0 and $\dot\eta^0$ on the first Fourier modes for instance. Due to the compatibility conditions (2.24) that must be satisfied by the initial data, we build also an initial fluid velocity v_ε^0 such that

$$v_\varepsilon^0(x,1) = \dot\eta_\varepsilon^0(x)e_2\,,\ \forall\, x \in (0,L) \quad \text{and} \quad \mathrm{div}({B_b}^\top v_\varepsilon^0) = 0 \text{ in } \Omega^0.$$

To that purpose, we consider the linear lifting of $\dot\eta_\varepsilon^0$ defined as the solution of the Stokes–like problem (2.34)–(2.37) with $f = 0$, $g = 0$ and $\dot\eta = \dot\eta_\varepsilon^0$. We obtain a velocity denoted $U_b(\dot\eta_\varepsilon^0)$ and we define v_ε^0 as $v^0 - U_b(\dot\eta^0) + U_b(\dot\eta_\varepsilon^0)$. It belongs to $H^1_\sharp(\Omega_0)$, satisfies the required compatibility conditions and converges toward v^0 in $H^1_\sharp(0,L)$ when ε goes to zero.

A weak formulation for our regularized problem is obtained by taking any test-function $(w,\xi) \in L^2(0,T;H^1_\sharp(\Omega_0)) \times L^2(0,T;H^2_\sharp(\Omega_0))$, such that $w(\cdot,1,\cdot) = \xi e_2$ and $w(\cdot,0,\cdot) = 0$ on $(0,L)\times(0,T)$ with $\mathrm{div}\,(B_b^\top w) = 0$ on Ω_0. Then, multiplying the fluid equation (3.1) by w and structure equation (3.17) by ξ yields, after integration by parts:

$$\begin{aligned}
&\int_{\Omega_0} \rho_{f,b}\partial_t v_\varepsilon \cdot w + \rho_s \int_0^L \partial_{tt}\eta_\varepsilon \xi + \delta \int_0^L \partial_{xtt}\eta_\varepsilon \partial_x \xi \\
&+\mu \int_{\Omega_0} A_b \nabla v_\varepsilon : \nabla w + \gamma \int_0^L \partial_{xt}\eta_\varepsilon \partial_x\xi + \varepsilon \int_0^L \partial_{xxt}\eta_\varepsilon \partial_{xx}\xi \\
&\qquad +\alpha \int_0^L \partial_{xx}\eta_\varepsilon \partial_{xx}\xi + \beta \int_0^L \partial_x \eta_\varepsilon \partial_x \xi \quad = \quad \int_{\Omega_0} f\cdot w + \int_0^L h : \nabla w.
\end{aligned} \tag{3.18}$$

Note that, due to the kinematic coupling conditions satisfied by the test functions and to the specific form of the structure right-hand side, the boundary terms on the fluid-structure interface cancels. We may thus prove existence to this problem by performing a Galerkin method in the space

$$\{(\xi,w) \in H^2_\sharp(0,L) \cap H^1_\sharp(0,L) \text{ s.t. } \mathrm{div}(B_b^\top w) = 0, \quad w(\cdot,0) = 0, \quad w(\cdot,1) = \xi e_2\}.$$

To build a Galerkin basis of this space, and prove existence of solutions to the approximate problems, one follows exactly the same lines as in [5]. For the sake of conciseness we skip this step here. We focus on estimates that are obtained through this method. We note in particular that in the next subsection we obtain the estimates for a pair $(\eta_\varepsilon, v_\varepsilon)$ satisfying the weak-formulation (3.18) associated with our coupled problem. However, these estimates are can be justified on approximated problems for which any time-derivative of the solution is a valid multiplier.

Energy estimates. By first taking $(w,\xi) = (v_\varepsilon, \partial_t\eta_\varepsilon)$ as test-function in (3.18) we derive a standard energy estimate. We obtain, using the fact that the given domain displacement b does not depend on time,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\int_{\Omega_0} \rho_{f,b}|v_\varepsilon|^2 + \rho_s\int_0^L |\partial_t\eta_\varepsilon|^2 + \delta\int_0^L |\partial_{xt}\eta_\varepsilon|^2 + \alpha\int_0^L |\partial_{xx}\eta_\varepsilon|^2 + \beta\int_0^L |\partial_x\eta_\varepsilon|^2\right] \\
&+\mu\int_{\Omega_0} A_b\nabla v_\varepsilon : \nabla v_\varepsilon + \gamma\int_0^L |\partial_{xt}\eta_\varepsilon|^2 + \varepsilon\int_0^L |\partial_{xxt}\eta_\varepsilon|^2 \\
= &\int_{\Omega_0} f\cdot v^\varepsilon - \int_{\Omega_0} h : \nabla v_\varepsilon.
\end{aligned}$$

Thus the following energy equality holds true

$$\frac{d}{dt}\Big(\mathcal{E}_\delta(v_\varepsilon,\partial_t\eta_\varepsilon) + \mathcal{E}_{\alpha,\beta}(\eta_\varepsilon)\Big)(t) + \mathcal{D}_{\mu,\gamma}(v_\varepsilon,\partial_t\eta_\varepsilon)(t) + \varepsilon\int_0^L |\partial_{xxt}\eta_\varepsilon|^2 = \int_{\Omega_0} f\cdot v^\varepsilon - \int_{\Omega_0} h : \nabla v_\varepsilon. \tag{3.19}$$

with the kinetic energy of the coupled system defined by

$$\mathcal{E}_\delta(w,\xi) = \frac{1}{2}\left(\int_{\Omega_0} \rho_{f,b}|w|^2 + \rho_s\int_0^L |\xi|^2 + \delta\int_0^L |\partial_x\xi|^2\right),$$

the structure mechanical energy defined by

$$\mathcal{E}_{\alpha,\beta}(\zeta) = \frac{\alpha}{2}\int_0^L |\partial_{xx}\zeta|^2 + \frac{\beta}{2}\int_0^L |\partial_x\zeta|^2,$$

and the dissipated energy of the coupled system

$$\mathcal{D}_{\mu,\gamma}(w,\xi) = \mu\int_{\Omega_0} A_b\nabla w : \nabla w + \gamma\int_0^L |\partial_x\xi|^2.$$

The assumptions on b imply first that A_b is coercive and that there exists a positive constant λ depending on R_1 such that $\lambda(R_1)I \le A_b$ in the sense of symmetric matrices and second, that $\rho_{f,b}$ is bounded away from zero, namely $\rho_{f,b} \ge \frac{\rho_f}{R_1}$. Thus, thanks to Cauchy-Schwarz, Poincaré and Young inequalities, we have

$$\begin{aligned}(3.20)\quad & \frac{1}{2}\frac{d}{dt}\left[\int_{\Omega_0} \rho_{f,b}|v_\varepsilon|^2 + \rho_s\int_0^L |\partial_t\eta_\varepsilon|^2 + \delta\int_0^L |\partial_{xt}\eta_\varepsilon|^2 + \alpha\int_0^L |\partial_{xx}\eta_\varepsilon|^2 + \beta\int_0^L |\partial_x\eta_\varepsilon|^2\right] \\ & + \frac{\mu\lambda(R_1)}{2}\int_{\Omega_0} |\nabla v_\varepsilon|^2 + \gamma\int_0^L |\partial_{xt}\eta_\varepsilon|^2 + \varepsilon\int_0^L |\partial_{xxt}\eta_\varepsilon|^2 \le \frac{C}{\mu\lambda(R_1)}\left(\|f\|^2_{L^2_\sharp(\Omega_0)} + \|h\|^2_{L^2_\sharp(\Omega_0)}\right).\end{aligned}$$

Consequently, we obtain, thanks to the assumption on the regularized initial data,

(3.21)

$$\begin{aligned} & \frac{\rho_f}{2R_1}\|v_\varepsilon\|^2_{L^\infty(0,t;L^2_\sharp(\Omega_0))} + \frac{\rho_s}{2}\|\partial_t\eta_\varepsilon\|^2_{L^\infty(0,t;L^2_\sharp(0,L))} + \frac{\delta}{2}\|\partial_{xt}\eta_\varepsilon\|^2_{L^\infty(0,t;L^2_\sharp(0,L))} \\ & + \frac{\alpha}{2}\|\partial_{xx}\eta_\varepsilon\|^2_{L^\infty(0,t;L^2_\sharp(0,L))} + \frac{\beta}{2}\|\partial_x\eta_\varepsilon\|^2_{L^\infty(0,t;L^2_\sharp(0,L))} \\ & + \frac{\mu\lambda(R_1)}{2}\|\nabla v_\varepsilon\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))} + \gamma\,\|\partial_{xt}\eta_\varepsilon\|^2_{L^2(0,t;L^2_\sharp(0,L))} + \varepsilon\|\partial_{xxt}\eta_\varepsilon\|^2_{L^2(0,t;L^2_\sharp(0,L))} \\ \le\ & \frac{C}{\mu\lambda(R_1)}\left(\|f\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))} + \|h\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))}\right) \\ & + C\left(\frac{\rho_f}{2}(R_1+1)\|v^0_\varepsilon\|^2_{L^2_\sharp(\Omega_0)} + \frac{\rho_s}{2}\|\dot\eta^0_\varepsilon\|^2_{L^2_\sharp(0,L)} + \frac{\delta}{2}\|\partial_x\dot\eta^0_\varepsilon\|^2_{L^2_\sharp(0,L)} + \frac{\alpha}{2}\|\partial_{xx}\eta^0_\varepsilon\|^2_{L^2_\sharp(0,L)} + \frac{\beta}{2}\|\partial_x\eta^0_\varepsilon\|^2_{L^2_\sharp(0,L)}\right).\end{aligned}$$

Thus, we obtain uniform bounds (in ε) for the solution in the following spaces:

- v_ε in $L^\infty(0,T;L^2_\sharp(\Omega_0)) \cap L^2(0,T;H^1_\sharp(\Omega_0))$,
- η_ε in $L^\infty(0,T;H^1_\sharp(0,L)) \cap W^{1,\infty}(0,T;L^2_\sharp(0,L))$,
- $\sqrt{\alpha}\,\eta_\varepsilon$ in $L^\infty(0,T;H^2_\sharp(0,L))$,
- $\sqrt{\delta}\,\eta_\varepsilon$ in $W^{1,\infty}(0,T;H^1_\sharp(0,L))$,
- $\sqrt{\gamma}\,\eta_\varepsilon$ in $H^1(0,T;H^1_\sharp(0,L))$,

and also:

- $\sqrt{\varepsilon}\eta_\varepsilon$ in $H^1(0,T;H^2_\sharp(0,L))$.

Furthermore the constant appearing in estimate (3.20) does not depend on $T > 0$.

Additional estimates for $\varepsilon > 0$. Next we take $(w, \xi) = (\partial_t v_\varepsilon, \partial_{tt}\eta_\varepsilon)$ as test-function in (3.18). This yields:

$$\begin{aligned}(3.22)\quad &\int_{\Omega_0} \rho_{f,b}|\partial_t v_\varepsilon|^2 + \rho_s \int_0^L |\partial_{tt}\eta_\varepsilon|^2 + \delta \int_0^L |\partial_{ttx}\eta_\varepsilon|^2 + \frac{\gamma}{2}\frac{d}{dt}\int_0^L |\partial_{xt}\eta_\varepsilon|^2 + \frac{\varepsilon}{2}\frac{d}{dt}\int_0^L |\partial_{xxt}\eta_\varepsilon|^2 \\ &+ \mu \int_{\Omega_0} A_b \nabla v_\varepsilon : \nabla \partial_t v_\varepsilon + \alpha \int_0^L \partial_{xx}\eta_\varepsilon \partial_{xxtt}\eta_\varepsilon + \beta \int_0^L \partial_x \eta_\varepsilon \partial_{xtt}\eta_\varepsilon \\ &= \int_{\Omega_0} f \cdot \partial_t v_\varepsilon + \int_{\Omega_0} (\operatorname{div} h) \cdot \partial_t v_\varepsilon + \int_0^L ((he_2) \cdot e_2)\partial_{tt}\eta_\varepsilon.\end{aligned}$$

Note that here we have integrated by parts the term involving h in order to avoid the term $\nabla \partial_t v_\varepsilon$ which is not regular enough to be bounded. Since b does not depend on time and A_b is symmetric, we have

$$\mu \int_{\Omega_0} A_b \nabla v_\varepsilon : \nabla \partial_t v_\varepsilon = \frac{\mu}{2}\frac{d}{dt}\int_{\Omega_0} A_b \nabla v_\varepsilon : \nabla v_\varepsilon.$$

Moreover, integration by parts entails that:

$$(3.23)\quad \int_0^t\int_0^L \partial_{xx}\eta_\varepsilon \partial_{xxtt}\eta_\varepsilon = -\int_0^t\int_0^L |\partial_{xxt}\eta_\varepsilon|^2 + \int_0^L \partial_{xx}\eta_\varepsilon(t)\partial_{xxt}\eta_\varepsilon(t) + \int_0^L \partial_{xxx}\eta^0 \partial_x \dot\eta_\varepsilon^0,$$

$$(3.24)\quad \int_0^t\int_0^L \partial_x\eta_\varepsilon \partial_{xtt}\eta_\varepsilon = -\int_0^t\int_0^L |\partial_{xt}\eta_\varepsilon|^2 + \int_0^L \partial_x\eta_\varepsilon(t)\partial_{tx}\eta_\varepsilon(t) + \int_0^L \partial_{xx}\eta^0 \dot\eta_\varepsilon^0,$$

Eventually, equality (3.22) writes, after integration over any time interval $(0,t)$, $0 < t \le T$

$$\begin{aligned}(3.25)\quad &\int_0^t\int_{\Omega_0} \rho_{f,b}|\partial_t v_\varepsilon|^2 + \rho_s \int_0^t\int_0^L |\partial_{tt}\eta_\varepsilon|^2 + \delta \int_0^t\int_0^L |\partial_{ttx}\eta_\varepsilon|^2 \\ &+ \frac{\mu}{2}\int_{\Omega_0} A_b\nabla v_\varepsilon(t) : \nabla v_\varepsilon(t) + \frac{\gamma}{2}\int_0^L |\partial_{xt}\eta_\varepsilon(t)|^2 + \frac{\varepsilon}{2}\int_0^L |\partial_{xxt}\eta_\varepsilon(t)|^2 \\ &= \int_0^t\int_{\Omega_0} f\cdot\partial_t v_\varepsilon + \int_0^t\int_{\Omega_0} (\operatorname{div} h)\cdot \partial_t v_\varepsilon + \int_0^t\int_0^L ((he_2)\cdot e_2)\partial_{tt}\eta_\varepsilon \\ &+ \frac{\mu}{2}\int_{\Omega_0} A_b \nabla v_\varepsilon^0 : \nabla v_\varepsilon^0 + \frac{\gamma}{2}\int_0^L |\partial_x \dot\eta_\varepsilon^0|^2 + \frac{\varepsilon}{2}\int_0^L |\partial_{xx}\dot\eta_\varepsilon^0|^2 + \alpha\int_0^t\int_0^L |\partial_{xxt}\eta_\varepsilon|^2 + \beta \int_0^t\int_0^L |\partial_{xt}\eta_\varepsilon|^2 \\ &- \alpha\int_0^L \partial_{xx}\eta_\varepsilon(t)\partial_{xxt}\eta_\varepsilon(t) - \beta\int_0^L \partial_x\eta_\varepsilon(t)\partial_{xt}\eta_\varepsilon(t) - \alpha\int_0^L \partial_{xxx}\eta^0\partial_x\dot\eta_\varepsilon^0 - \beta\int_0^L \partial_{xx}\eta^0\dot\eta_\varepsilon^0.\end{aligned}$$

Remark 3.3. *At this stage, in the case* ($\mathbf{C}_\beta$) *of the wave equation* ($\beta > 0$, $\delta = \gamma = \alpha = 0$) *we need to have either* $\gamma > 0$ *or* $\varepsilon > 0$ *to control the terms* $\beta\int_0^t\int_0^L |\partial_{xt}\eta_\varepsilon|^2$ *and* $-\beta\int_0^L \partial_x\eta_\varepsilon(t)\partial_{tx}\eta_\varepsilon(t)$. *In the case* ($\mathbf{C}_{\alpha,\delta}$) *of the beam equation with inertia of rotation* ($\alpha > 0, \delta > 0$ $\gamma = \beta = 0$)*, or in the* ($\mathbf{C}_{\alpha,\gamma}$) *of the damped beam* ($\alpha > 0, \gamma > 0$ $\delta = \beta = 0$)*, we need* $\varepsilon > 0$ *to control the terms* $\alpha\int_0^t\int_0^L |\partial_{xxt}\eta_\varepsilon|^2$, $-\alpha\int_0^L \partial_{xx}\eta_\varepsilon(t)\partial_{xxt}\eta_\varepsilon(t)$ *in the right hand side. Nevertheless, these terms are controled independently of* ε *(or* γ*) thanks to additional estimates we derive later on.*

The assumptions on the data in the three different cases and the way η_ε^0 is build imply that all the terms involving initial data are bounded independently of ε. Consequenlty, thanks to Cauchy Schwarz, Young and trace inequalities and to the assumptions on b, we have

$$\begin{aligned}&\frac{\rho_f}{2R_1}\int_0^t\int_{\Omega_0} |\partial_t v_\varepsilon|^2 + \frac{\rho_s}{2}\int_0^t\int_0^L |\partial_{tt}\eta_\varepsilon|^2 + \delta\int_0^t\int_0^L |\partial_{ttx}\eta_\varepsilon|^2 \\ &+\frac{\mu\lambda(R_1)}{2}\int_{\Omega_0} \nabla v_\varepsilon(t) : \nabla v_\varepsilon(t) + \frac{\gamma}{2}\int_0^L |\partial_{xt}\eta_\varepsilon(t)|^2 + \frac{\varepsilon}{4}\int_0^L |\partial_{xxt}\eta_\varepsilon(t)|^2 \\ \le\ & C(R_1)\left(\|f\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))} + \|\operatorname{div} h\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))}\right) + C\|h\|^2_{L^2(0,t;H^1_\sharp(\Omega_0))} \\ &+\alpha\int_0^t\int_0^L |\partial_{xxt}\eta_\varepsilon|^2 + \beta\int_0^t\int_0^L |\partial_{xt}\eta_\varepsilon|^2 + \frac{C\alpha}{\varepsilon}\int_0^L |\partial_{xx}\eta_\varepsilon(t)|^2 + \frac{C\beta}{\varepsilon}\int_0^L |\partial_x\eta_\varepsilon(t)|^2 + C_0,\end{aligned}$$

where C_0 depends only on the initial data and on R_1, and C is a generic constant that may depend on the domain Ω_0. Thanks to the energy estimate (3.20), we end up with

$$\begin{aligned} & \frac{\rho_f}{2R_1}\int_0^t\int_{\Omega_0}|\partial_t v_\varepsilon|^2+\frac{\rho_s}{2}\int_0^t\int_0^L|\partial_{tt}\eta_\varepsilon|^2+\delta\int_0^t\int_0^L|\partial_{ttx}\eta_\varepsilon|^2 \\ & +\frac{\mu\lambda(R_1)}{2}\int_{\Omega_0}\nabla v_\varepsilon(t):\nabla v_\varepsilon(t)+\frac{\gamma}{2}\int_0^L|\partial_{xt}\eta_\varepsilon(t)|^2+\frac{\varepsilon}{4}\int_0^L|\partial_{xxt}\eta_\varepsilon(t)|^2 \\ \leq\ & C(R_1)\left(\|f\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))}+\|\operatorname{div}h\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))}\right)+C\|h\|^2_{L^2(0,t;H^1_\sharp(\Omega_0))} \\ & +\alpha\int_0^t\int_0^L|\partial_{xxt}\eta_\varepsilon|^2+\beta\int_0^t\int_0^L|\partial_{xt}\eta_\varepsilon|^2+C_\varepsilon+C_0, \end{aligned}$$

where C_ε depends on the initial data and $1/\varepsilon$. Consequently by applying Gronwall lemma, we obtain bounds on the solution in the following spaces:

- v_ε in $H^1(0,T;L^2_\sharp(\Omega_0))\cap L^\infty(0,T;H^1_\sharp(\Omega_0))$,
- η_ε in $H^2(0,T;L^2_\sharp(0,L))\cap W^{1,\infty}(0,T;H^2_\sharp(0,L))$,
- $\sqrt{\varepsilon}\,\eta_\varepsilon$ in $H^2(0,T;H^1_\sharp(0,L))$.

Final remark on the Galerkin method. As already stated, the previous calculations are justified by performing them on a Galerkin approximation of the regularized coupled system. They furnish uniform bounds satisfied by approximate solutions that are sufficient to pass to the limit in the approximated problems. We obtain the existence of a pair $(\eta_\varepsilon, v_\varepsilon)$ such that:

$$\eta_\varepsilon\in H^2(0,T;L^2_\sharp(0,L))\cap W^{1,\infty}(0,T;H^2_\sharp(0,L)),\qquad v_\varepsilon\in H^1(0,T;L^2_\sharp(\Omega_0))\cap L^\infty(0,T;H^1_\sharp(\Omega_0)).$$

and satisfying the weak-formulation (3.18) for any admissible test-function. We emphasize that, in the weak limit the uniform estimates deriving from (3.21) are still valid.

We note that, setting $\xi = 0$ in the weak-formulation, we obtain that v_ε is a weak solution to the incompressible Stokes system (3.1)-(3.2) with boundary conditions (3.4)-(3.5) (and periodic boundary conditions on the lateral boundaries of Ω_0). The time-regularity of v_ε ensures that we may reconstruct a mean-free pressure r_ε such that (3.1) is satisfied a.e. on $(0,T)\times\Omega_0$. Then, thanks to the elliptic result of Lemma 2.10, for the Stokes-like system, we deduce from the previous regularities that v_ε together with a pressure r_ε that satisfies:

$$v_\varepsilon\in L^2(0,T;H^2_\sharp(\Omega_0)),\qquad q_\varepsilon\in L^2(0,T;H^1_\sharp(\Omega_0)\cap L^2_{\sharp,0}(\Omega_0)).$$

Finally we may take an arbitrary $\xi\in L^2(0,T;H^2_\sharp(0,L)\cap L^2_{\sharp,0}(0,L))$ with associated lifting velocity $w\in L^2(0,T;H^1_\sharp(0,L))$ obtained by solving the corresponding Stokes-like problem for instance as test-function in (3.18). Thanks to the previous computations, we obtain that there exists a right-hand side $\Sigma\in L^2(0,T;H^{1/2}_\sharp(0,L))$ for which η_ε satisfies:

$$\begin{aligned} & \rho_s\int_0^L\partial_{tt}\eta_\varepsilon\xi+\delta\int_0^L\partial_{xtt}\eta_\varepsilon\partial_x\xi+\gamma\int_0^L\partial_{xt}\eta_\varepsilon\partial_x\xi \\ & \qquad +\varepsilon\int_0^L\partial_{xxt}\eta_\varepsilon\partial_{xx}\xi+\alpha\int_0^L\partial_{xx}\eta_\varepsilon\partial_{xx}\xi+\beta\int_0^L\partial_x\eta_\varepsilon\partial_x\xi=\int_0^L\Sigma\,\xi \end{aligned}$$

Up to change the fluid pressure for $q_\varepsilon = r_\varepsilon + d_\varepsilon$ where d_ε is given by (3.12) we may extend the weak formulation to any $\xi\in L^2(0,T;H^2_\sharp(0,L))$. Consequently, taking as test-function the projection of $\partial_{xxxxt}\eta_\varepsilon$ on the first Fourier modes, we obtain an identity which entails $\eta_\varepsilon\in H^2(0,T;L^2_\sharp(0,L))\cap H^1(0,T;H^4_\sharp(0,L))$ by letting the number of modes go to infinity. Then, the equation (3.17) is satisfied a.e. which yields that $\sqrt{\delta}\partial_{tt}\eta_\varepsilon\in L^2(0,T;H^2_\sharp(0,L))$.

Remark 3.4. *At this stage, to obtain the previous regularities on the structure displacement the regularization term is needed. All these estimates and regularities thus depend strongly on the parameter of regularization $\varepsilon > 0$. Note that quite similar calculations are performed later on, but, combined with other estimates, they enable us to obtain bounds independent of ε.*

Estimates independent of ε. Now we derive estimates independent of the regularization parameter ε. As already stated, the energy bounds given by (3.21) are independent of ε. In what follows, we consider once again $(\partial_t v_\varepsilon, \partial_{tt}\eta_\varepsilon)$ as a couple of multipliers of the fluid equations and of the structure equation but also $(-\partial_{xx} v_\varepsilon, -\partial_{xxt}\eta_\varepsilon)$. Then, a well chosen linear combinaison of both derived estimates lead to an estimate independent of ε. A key argument, at this step, is the elliptic estimate on the Stokes–like system stated in lemma 2.10.

Remark 3.5. *In the case* $(\mathbf{C}_{\alpha,\gamma})$ *of a damped beam, it is sufficient to take* $-\partial_{xxt}\eta_\varepsilon$ *as a test function for the structure equation, whereas, for the two other cases* $(\mathbf{C}_\beta)$ *of a wave equation and* $(\mathbf{C}_{\alpha,\delta})$ *of a beam with inertia of rotation, where no viscosity is added to the structure equation, we need to take also its fluid counterpart* $-\partial_{xx} v_\varepsilon$ *and to take advantage of the viscosity of the fluid. The main reason is that, in the damped case, we have* $\gamma > 0$*, and this additional viscosity enables to control the structure velocity in* $L^2(0,T; H^{3/2}_\sharp(0,L))$*. In the two other cases, the structure velocity is bounded in* $L^2(0,T; H^{3/2}_\sharp(0,L))$ *only because it is the trace of the fluid velocity on the boundary.*

Before entering into the details of the derivations of the additional regularity estimates, let us underline some of the difficulties to be treated. We have that $\partial_t v_\varepsilon$, and $-\partial_{xx} v_\varepsilon$ belong both to $L^2(0,T; L^2_\sharp(\Omega_0))$ and are thus multipliers of (3.1). In the same way $\partial_{tt}\eta_\varepsilon$ and $-\partial_{xxt}\eta_\varepsilon$ are multipliers of (3.17). Formally, both couples of multipliers $(\partial_t v_\varepsilon, \partial_{tt}\eta_\varepsilon)$ and $(-\partial_{xx} v_\varepsilon, -\partial_{xxt}\eta_\varepsilon)$ match at the fluid-structure interface. Nevertheless with the derived regularities, the trace of $\partial_t v_\varepsilon$, and $-\partial_{xx} v_\varepsilon$ over the fluid-structure interface are not well defined. Furthermore, a second difficulty for the second pair of multipliers is that it involves pressure terms since $\mathrm{div}\,(B_b^\top \partial_{xx} v_\varepsilon) \neq 0$.

Estimate of the fluid and structure accelerations. Let us first multiply (3.1) by $\partial_t v_\varepsilon$ and integrate over $\Omega_0 \times (0,t)$. This yields:

$$\begin{aligned}(3.26)\quad \int_0^t \int_{\Omega_0} \rho_{f,b} |\partial_t v_\varepsilon|^2 - \mu \int_0^t \int_{\Omega_0} \mathrm{div}\,((A_b\nabla) v_\varepsilon)\cdot \partial_t v_\varepsilon + \int_0^t \int_{\Omega_0} (B_b \nabla) q_\varepsilon \cdot \partial_t v_\varepsilon \\ = \int_0^t \int_{\Omega_0} f \cdot \partial_t v_\varepsilon + \int_0^t \int_{\Omega_0} (\mathrm{div}\, h)\cdot \partial_t v_\varepsilon.\end{aligned}$$

To integrate by parts in space the second and the third terms on the first line of this equality, we have the following lemma:

Lemma 3.6. *For* $w \in L^2(0,T; H^2_\sharp(\Omega_0)) \cap H^1(0,T; L^2_\sharp(\Omega_0))$*, such that*

$$\begin{aligned} &div\,(B_b^\top w) = 0, && on\ \Omega_0,\\ &w(x,1,t) = \partial_t \xi(x,t) e_2, && for\ some\ \xi \in H^2(0,T; L^2_\sharp(0,L)),\\ &w(0,1,t) = 0, && on\ (0,L),\end{aligned}$$

and for $q \in L^2(0,T; H^1_\sharp(\Omega_0))$ *we have*

$$\begin{aligned}-\mu \int_0^t \int_{\Omega_0} div\,((A_b\nabla)w)\cdot \partial_t w + \int_0^t \int_{\Omega_0} (B_b\nabla) q \cdot \partial_t w = \frac{\mu}{2}\int_{\Omega_0} A_b \nabla w : \nabla w(t) - \frac{\mu}{2}\int_{\Omega_0} A_b\nabla w : \nabla w(0) \\ - \int_0^t \int_0^L (\mu((A_b\nabla) w\, e_2)\cdot e_2 - q(B_b\, e_2)\cdot e_2)\, \partial_{tt}\xi.\end{aligned}$$

Proof. Let $w_n \in C^\infty([0,T]; H^2_\sharp(\Omega_0))$ be obtained from w by acting a convolution in time with an approximation of unity. We have:

- $w_n \to w$ in $L^2(0,T; H^2_\sharp(\Omega_0)) \cap H^1(0,T; L^2_\sharp(\Omega_0))$,
- $\mathrm{div}(B_b^\top w_n) = 0$ on $(0,T)\times\Omega_0$, and $w_n = 0$ on $y = 0$,
- $w_n = \xi_n'$ on $y = 1$ with $\xi_n' \to \partial_t \xi$ in $H^1(0,T; L^2_\sharp(0,L))$.

For such a regular vector-field w_n, the identity under consideration is a simple integration by parts. We note then that, since $(A_b, B_b) \in L^\infty((0,T); L^\infty_\sharp(\Omega_0))$ with

$$\sup_{y\in(0,1)} \|\nabla A_b\|_{L^2_\sharp(0,L)} < \infty.$$

and $L^2(0,T; H^2_\sharp(\Omega_0)) \cap H^1(0,T; L^2_\sharp(\Omega_0))$ embeds in $C([0,T]; H^1_\sharp(\Omega_0))$, all the integrals involved in this identity are continuous with respect to the topology for which w_n and ξ'_n converge. This ends the proof. □

By applying the previous lemma with $w = v_\varepsilon$, $q = q_\varepsilon$, (3.26) becomes

$$\int_0^t \int_{\Omega_0} \rho_{f,b} |\partial_t v_\varepsilon|^2 + \frac{\mu}{2} \int_{\Omega_0} A_b \nabla v_\varepsilon(t) : \nabla v_\varepsilon(t) - \int_0^t \int_0^L ((\mu A_b \nabla v_\varepsilon - B_b p_\varepsilon) e_2) \cdot e_2 \partial_{tt}\eta_\varepsilon \\ = \int_0^t \int_{\Omega_0} f \cdot \partial_t v_\varepsilon + \int_0^t \int_{\Omega_0} (\operatorname{div} h) \cdot \partial_t v_\varepsilon + \frac{\mu}{2} \int_{\Omega_0} A_b \nabla v^0_\varepsilon : \nabla v^0_\varepsilon. \tag{3.27}$$

Next, we multiply the structure equation (3.17) by $\partial_{tt}\eta_\varepsilon$ which belongs to $L^2(0,T; L^2_\sharp(0,L))$ (in the case $\delta > 0$, it even belongs to $L^2(0,T; H^2_\sharp(0,L))$) and integrate over $(0,L)\times(0,T)$. This yields:

$$\rho_s \int_0^t \int_0^L |\partial_{tt}\eta_\varepsilon|^2 + \delta \int_0^t \int_0^L |\partial_{ttx}\eta_\varepsilon|^2 - \gamma \int_0^t \int_0^L \partial_{xxt}\eta_\varepsilon \partial_{tt}\eta_\varepsilon + \varepsilon \int_0^t \int_0^L \partial_{xxxxt}\eta_\varepsilon \partial_{tt}\eta_\varepsilon \\ + \alpha \int_0^t \int_0^L \partial_{xxxx}\eta_\varepsilon \partial_{tt}\eta_\varepsilon - \beta \int_0^t \int_0^L \partial_{xx}\eta_\varepsilon \partial_{tt}\eta_\varepsilon \\ = \int_0^t \int_0^L \left(-\mu((A_b\nabla) v_\varepsilon\, e_2) \cdot e_2 + q_\varepsilon (B_b\, e_2) \cdot e_2\right) \partial_{tt}\eta_\varepsilon - \int_0^t \int_0^L ((h e_2) \cdot e_2) \partial_{tt}\eta_\varepsilon. \tag{3.28}$$

We would like to integrate by parts in space. But $\partial_{tt}\eta_\varepsilon$ is not regular enough. Nevertheless we can easily prove by a regularization argument that, for $t \in (0,T)$:

$$\begin{aligned}
-\gamma \int_0^t \int_0^L \partial_{xxt}\eta_\varepsilon \partial_{tt}\eta_\varepsilon &= \frac{\gamma}{2} \int_0^L |\partial_{xt}\eta_\varepsilon(t)|^2 - \frac{\gamma}{2} \int_0^L |\partial_{xt}\eta^0_\varepsilon|^2, \\
\varepsilon \int_0^t \int_0^L \partial_{xxxxt}\eta_\varepsilon \partial_{tt}\eta_\varepsilon &= \frac{\varepsilon}{2} \int_0^L |\partial_{xxt}\eta_\varepsilon(t)|^2 - \frac{\varepsilon}{2} \int_0^L |\partial_{xxt}\eta^0_\varepsilon|^2, \\
\alpha \int_0^t \int_0^L \partial_{xxxx}\eta_\varepsilon \partial_{tt}\eta_\varepsilon &= -\alpha \int_0^t \int_0^L |\partial_{xxt}\eta_\varepsilon|^2 - \alpha \int_0^L \partial_{xxx}\eta_\varepsilon(t) \partial_{xt}\eta_\varepsilon(t) + \alpha \int_0^L \partial_{xxx}\eta^0 \partial_x \dot\eta^0_\varepsilon, \\
-\beta \int_0^t \int_0^L \partial_{xx}\eta_\varepsilon \partial_{tt}\eta_\varepsilon &= -\beta \int_0^t \int_0^L |\partial_{xt}\eta_\varepsilon|^2 - \beta \int_0^L \partial_{xx}\eta_\varepsilon(t) \partial_t\eta_\varepsilon(t) + \beta \int_0^L \partial_{xx}\eta^0 \dot\eta^0_\varepsilon,
\end{aligned}$$

Note that the two last expressions differ slightly from (3.23)-(3.24) that we used to derive the L^2 estimates on the fluid and structure accelerations for a given ε. This enables us to derive estimates independent of ε. Thus, from the four previous equalities, (3.28) is transformed in

$$\rho_s \int_0^t \int_0^L |\partial_{tt}\eta_\varepsilon|^2 + \delta \int_0^t \int_0^L |\partial_{ttx}\eta_\varepsilon|^2 + \frac{\gamma}{2} \int_0^L |\partial_{xt}\eta_\varepsilon(t)|^2 + \frac{\varepsilon}{2} \int_0^L |\partial_{xxt}\eta_\varepsilon(t)|^2 \\ = \alpha \int_0^t \int_0^L |\partial_{xxt}\eta_\varepsilon|^2 + \beta \int_0^t \int_0^L |\partial_{xt}\eta_\varepsilon|^2 + \alpha \int_0^L \partial_{xxx}\eta_\varepsilon(t) \partial_{xt}\eta_\varepsilon(t) + \beta \int_0^L \partial_{xx}\eta_\varepsilon(t) \partial_t\eta_\varepsilon(t) \\ - \alpha \int_0^L \partial_{xxx}\eta^0_\varepsilon \partial_x \dot\eta^0_\varepsilon - \beta \int_0^L \partial_{xx}\eta^0_\varepsilon \dot\eta^0_\varepsilon + \int_0^t \int_0^L \left(-\mu((A_b\nabla) v_\varepsilon\, e_2) \cdot e_2 + q_\varepsilon (B_b\, e_2) \cdot e_2\right) \partial_{tt}\eta_\varepsilon \\ - \int_0^t \int_0^L ((h e_2) \cdot e_2) \partial_{tt}\eta_\varepsilon + \frac{\gamma}{2} \int_0^L |\partial_x \dot\eta^0_\varepsilon|^2 + \frac{\varepsilon}{2} \int_0^L |\partial_{xx}\dot\eta^0_\varepsilon|^2. \tag{3.29}$$

Note that all the terms of (3.29) make sense for the regularities we have derived for η_ε.

By adding (3.27) and (3.29), we obtain:

$$\begin{aligned}
&\int_0^t\int_{\Omega_0}\rho_{f,b}|\partial_t v_\varepsilon|^2+\rho_s\int_0^t\int_0^L|\partial_{tt}\eta_\varepsilon|^2+\delta\int_0^t\int_0^L|\partial_{ttx}\eta_\varepsilon|^2\\
&+\frac{\mu}{2}\int_{\Omega_0}A_b\nabla v_\varepsilon(t):\nabla v_\varepsilon(t)+\frac{\gamma}{2}\int_0^L|\partial_{xt}\eta_\varepsilon(t)|^2+\frac{\varepsilon}{2}\int_0^L|\partial_{xxt}\eta_\varepsilon(t)|^2\\
=\;&\int_0^t\int_{\Omega_0}f\cdot\partial_t v_\varepsilon+\int_0^t\int_{\Omega_0}(\operatorname{div}h)\cdot\partial_t v_\varepsilon+\frac{\mu}{2}\int_{\Omega_0}A_b\nabla v_\varepsilon^0:\nabla v_\varepsilon^0\\
&+\alpha\int_0^t\int_0^L|\partial_{xxt}\eta_\varepsilon|^2+\beta\int_0^t\int_0^L|\partial_{xt}\eta_\varepsilon|^2+\alpha\int_0^L\partial_{xxx}\eta_\varepsilon(t)\partial_{xt}\eta_\varepsilon(t)+\beta\int_0^L\partial_{xx}\eta_\varepsilon(t)\partial_t\eta_\varepsilon(t)\\
&-\alpha\int_0^L\partial_{xxx}\eta_\varepsilon^0\partial_x\dot\eta_\varepsilon^0-\beta\int_0^L\partial_{xx}\eta_\varepsilon^0\dot\eta_\varepsilon^0-\int_0^t\int_0^L((he_2)\cdot e_2)\partial_{tt}\eta_\varepsilon+\frac{\gamma}{2}\int_0^L|\partial_x\dot\eta_\varepsilon^0|^2+\frac{\varepsilon}{2}\int_0^L|\partial_{xx}\dot\eta_\varepsilon^0|^2.
\end{aligned}$$

We recover (3.25) (except for two terms which have been integrated by parts) that had been derived on a Galerkin approximation of the solution $(v_\varepsilon,\eta_\varepsilon)$. Next using the assumptions on b, Cauchy-Schwartz, Young and trace inequalities, and the assumptions on the initial data, we have

$$\begin{aligned}
(3.30)\quad &\frac{\rho_f}{2R_1}\int_0^t\int_{\Omega_0}|\partial_t v_\varepsilon|^2+\frac{\rho_s}{2}\int_0^t\int_0^L|\partial_{tt}\eta_\varepsilon|^2+\delta\int_0^t\int_0^L|\partial_{ttx}\eta_\varepsilon|^2\\
&+\frac{\mu\lambda(R_1)}{2}\int_{\Omega_0}|\nabla v_\varepsilon(t)|^2+\frac{\gamma}{2}\int_0^L|\partial_{xt}\eta_\varepsilon(t)|^2+\frac{\varepsilon}{2}\int_0^L|\partial_{xxt}\eta_\varepsilon(t)|^2\\
&\le C(R_1)\left(\|f\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))}+\|h\|^2_{L^2(0,t;H^1_\sharp(\Omega_0))}\right)+C_0(R_1)\\
&+\alpha\int_0^t\int_0^L|\partial_{xxt}\eta_\varepsilon|^2+\beta\int_0^t\int_0^L|\partial_{xt}\eta_\varepsilon|^2+\alpha\int_0^L\partial_{xxx}\eta_\varepsilon(t)\partial_{xt}\eta_\varepsilon(t)+\beta\int_0^L\partial_{xx}\eta_\varepsilon(t)\partial_t\eta_\varepsilon(t)
\end{aligned}$$

where $C_0(R_1)$ depends only on the initial data and R_1 and do not depend on ε. Note that here we have used the assumptions made on the regularized initial structure velocity that ensures that $\varepsilon\|\partial_{xx}\dot\eta_\varepsilon^0\|^2_{L^2_\sharp(0,L)}$ is bounded independently of ε. Now we want to control the four terms (whenever they make sense) on the last line of this inequality without using the additional viscosity coming from the regularization term. For that purpose, we consider below $(-\partial_{xx}v_\varepsilon,-\partial_{xxt}\eta_\varepsilon)$ as "test functions". They are multipliers of the fluid and structure equations respectively, since

$$(-\partial_{xx}v_\varepsilon,-\partial_{xxt}\eta_\varepsilon)\in L^2(0,T;L^2_\sharp(\Omega_0))\times L^2(0,T;L^2_\sharp(0,L)).$$

Furthermore, at least formally, they match at the interface. Yet $\operatorname{div}(B_b^\top\partial_{xx}v_\varepsilon)\neq 0$, leading to the apparition of pressure terms we need to control.

A first regularity estimate on the structure alone. Let us first multiply the structure equation (3.3) by $-\partial_{xxt}\eta_\varepsilon$ and integrate over $(0,L)$. We obtain

$$\begin{aligned}
&-\rho_s\int_0^L\partial_{tt}\eta_\varepsilon\partial_{xxt}\eta_\varepsilon+\delta\int_0^L\partial_{xxtt}\eta_\varepsilon\partial_{xxt}\eta_\varepsilon-\alpha\int_0^L\partial_{xxxx}\eta_\varepsilon\partial_{xxt}\eta_\varepsilon+\beta\int_0^L\partial_{xx}\eta_\varepsilon\partial_{xxt}\eta_\varepsilon\\
&+\gamma\int_0^L\partial_{xxt}\eta_\varepsilon\partial_{xxt}\eta_\varepsilon-\varepsilon\int_0^L\partial_{xxxxt}\eta_\varepsilon\partial_{xxt}\eta_\varepsilon\\
=\;&\int_0^L(\mu((A_b\nabla)v_\varepsilon\,e_2)\cdot e_2-q_\varepsilon(B_b\,e_2)\cdot e_2)\partial_{xxt}\eta_\varepsilon+\int_0^L(he_2)\cdot e_2\partial_{xxt}\eta_\varepsilon,
\end{aligned}$$

which leads after integration by parts in space to

$$\begin{aligned}
(3.31)\quad &\frac{\rho_s}{2}\frac{d}{dt}\int_0^L|\partial_{xt}\eta_\varepsilon|^2+\frac{\delta}{2}\frac{d}{dt}\int_0^L|\partial_{xxt}\eta_\varepsilon|^2+\frac{\alpha}{2}\frac{d}{dt}\int_0^L|\partial_{xxx}\eta_\varepsilon|^2+\frac{\beta}{2}\frac{d}{dt}\int_0^L|\partial_{xx}\eta_\varepsilon|^2\\
&+\gamma\int_0^L|\partial_{xxt}\eta_\varepsilon|^2+\varepsilon\int_0^L|\partial_{xxxt}\eta_\varepsilon|^2=\int_0^L(\mu((A_b\nabla)v_\varepsilon\,e_2)\cdot e_2-q_\varepsilon(B_b\,e_2)\cdot e_2)\partial_{xxt}\eta_\varepsilon+\int_0^L(he_2)\cdot e_2\partial_{xxt}\eta_\varepsilon,
\end{aligned}$$

Remark 3.7. *In the case* $(\mathbf{C}_{\alpha,\gamma})$*, we may play a little further with this inequality. Indeed, on the right-hand side, we have:*

$$
\begin{aligned}
(3.32)\quad & \left|\int_0^L (\mu((A_b\nabla)v_\varepsilon\, e_2)\cdot e_2 - q_\varepsilon(B_b\, e_2)\cdot e_2)\partial_{xxt}\eta_\varepsilon + \int_0^L (he_2)\cdot e_2\partial_{xxt}\eta_\varepsilon\right| \\
&\le \|\mu((A_b\nabla)v_\varepsilon\, e_2)\cdot e_2 - q_\varepsilon(B_b\, e_2)\cdot e_2\|_{H^{1/2}_\sharp(0,L)}\|\partial_t\eta_\varepsilon\|_{H^{3/2}_\sharp(0,L)} + \|h\|_{H^{1/2}_\sharp(0,L)}\|\partial_t\eta_\varepsilon\|_{H^{3/2}_\sharp(0,L)} \\
&\le \Big(\|\mu((A_b\nabla)v_\varepsilon\, e_2)\cdot e_2 - q_\varepsilon(B_b\, e_2)\cdot e_2\|_{H^{1/2}_\sharp(0,L)} + C\|h\|_{H^1_\sharp(\Omega_0)}\Big)\|\partial_t\eta_\varepsilon\|_{H^{3/2}_\sharp(0,L)}
\end{aligned}
$$

The term $\|\mu((A_b\nabla)v_\varepsilon\, e_2)\cdot e_2 - q_\varepsilon(B_b\, e_2)\cdot e_2\|_{H^{1/2}_\sharp(0,L)}$ *is bounded by applying Lemma 2.10. We have:*

$$
(3.33)\quad \|\mu((A_b\nabla)v_\varepsilon\, e_2)\cdot e_2 - q_\varepsilon(B_b\, e_2)\cdot e_2\|_{H^{1/2}_\sharp(0,L)} \le K^{\mathrm{s}}(R_1)\Big(\|\partial_t v_\varepsilon\|_{L^2(\Omega_0)} + \|\partial_t\eta_\varepsilon\|_{H^{3/2}_\sharp(0,L)}\Big)
$$

By interpolating the $H^{3/2}$ *norm of* $\partial_t\eta$ *between its* H^1 *and* H^2 *norms,* (3.32) *together with* (3.33) *lead to (remember that we consider* $\gamma > 0$*):*

$$
\begin{aligned}
(3.34)\quad & \left|\int_0^L (\mu((A_b\nabla)v_\varepsilon\, e_2)\cdot e_2 - q_\varepsilon(B_b\, e_2)\cdot e_2)\partial_{xxt}\eta_\varepsilon - \int_0^L (he_2)\cdot e_2\partial_{xxt}\eta_\varepsilon|\right| \\
&\le \kappa\|\partial_t v_\varepsilon\|^2_{L^2(\Omega_0)} + \frac{\gamma}{2}\|\partial_t\eta_\varepsilon\|^2_{H^2_\sharp(0,L)} + C\left(\frac{1}{\kappa},\frac{1}{\gamma},R_1\right)\|\partial_t\eta_\varepsilon\|^2_{H^1_\sharp(0,L)} + C\|h\|^2_{H^1_\sharp(\Omega_0)},
\end{aligned}
$$

with $\kappa > 0$ *arbitrary small.*

Finally, (3.34) *and* (3.31) *imply that:*

$$
\begin{aligned}
&\frac{\rho_s}{2}\frac{d}{dt}\int_0^L |\partial_{xt}\eta_\varepsilon|^2 + \frac{\delta}{2}\frac{d}{dt}\int_0^L |\partial_{xxt}\eta_\varepsilon|^2 + \frac{\alpha}{2}\frac{d}{dt}\int_0^L |\partial_{xxx}\eta_\varepsilon|^2 + \frac{\beta}{2}\frac{d}{dt}\int_0^L |\partial_{xx}\eta_\varepsilon|^2 \\
&+\frac{\gamma}{2}\int_0^L |\partial_{xxt}\eta_\varepsilon|^2 + \varepsilon\int_0^L |\partial_{xxxt}\eta_\varepsilon|^2 \le \kappa\|\partial_t v_\varepsilon\|^2_{L^2(\Omega_0)} + C(\frac{1}{\kappa},\frac{1}{\gamma},R_1)\|\partial_t\eta_\varepsilon\|^2_{H^1_\sharp(0,L)} + C\|h\|^2_{H^1_\sharp(\Omega_0)},
\end{aligned}
$$

which is further integrated in time into:

$$
\begin{aligned}
(3.35)\quad & \frac{\rho_s}{2}\int_0^L |\partial_{xt}\eta_\varepsilon|^2(t) + \frac{\delta}{2}\int_0^L |\partial_{xxt}\eta_\varepsilon|^2(t) + \frac{\alpha}{2}\int_0^L |\partial_{xxx}\eta_\varepsilon|^2(t) + \frac{\beta}{2}\int_0^L |\partial_{xx}\eta_\varepsilon|^2(t) \\
&\qquad + \frac{\gamma}{2}\int_0^t\int_0^L |\partial_{xxt}\eta_\varepsilon|^2 + \varepsilon\int_0^t\int_0^L |\partial_{xxxt}\eta_\varepsilon|^2 \\
&\le \kappa\|\partial_t v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_0))} + C\left(\frac{1}{\kappa},\frac{1}{\gamma},R_1\right)\|\partial_t\eta_\varepsilon\|^2_{L^2(0,t;H^1_\sharp(0,L))} + C\|h\|^2_{L^2(0,t;H^1_\sharp(\Omega_0))} \\
&\qquad + C\left(\frac{\rho_s}{2}\|\dot\eta^0_\varepsilon\|^2_{H^1_\sharp(0,L)} + \frac{\delta}{2}\|\dot\eta^0_\varepsilon\|^2_{H^2_\sharp(0,L)} + \frac{\alpha}{2}\|\eta^0_\varepsilon\|^2_{H^3_\sharp(0,L)} + \frac{\beta}{2}\|\eta^0_\varepsilon\|^2_{H^2_\sharp(0,L)}\right),
\end{aligned}
$$

The previous estimate (3.35)*, combined with* (3.30) *with a well chosen* κ*, provides regularity estimates for the solution of fluid–structure system uniformly in* ε *in the case* $(\mathbf{C}_{\alpha,\gamma})$*. Here we used strongly the fact that an extra viscosity has been added to the structure equation to estimate the forcing term applied by the fluid on the structure. In the case where no extra viscosity is added we need to take advantage of the viscosity coming from the fluid.*

A regularity estimate for the coupled fluid–structure system. To obtain a regularity result for the fluid independently of ε valid in the cases $(\mathbf{C}_\beta)$ of the wave equation and $(\mathbf{C}_{\alpha,\delta})$ of a beam equation with inertia of rotation, we multiply the fluid equations (3.1) by $-\partial_{xx}v_\varepsilon$ which is the fluid counterpart of $-\partial_{xxt}\eta_\varepsilon$. As already stated, at least formally these functions match at the interface. We have,

$$-\int_0^t\int_{\Omega_0}\rho_{f,b}\partial_t v_\varepsilon\cdot\partial_{xx}v_\varepsilon+\int_0^t\int_{\Omega_0}(\mu\mathrm{div}\,((A_b\nabla)v_\varepsilon)-(B_b\nabla)q_\varepsilon)\cdot\partial_{xx}v_\varepsilon=$$
$$-\int_0^t\int_{\Omega_0} f\cdot\partial_{xx}v_\varepsilon-\int_0^t\int_{\Omega_0}\mathrm{div}\,h\cdot\partial_{xx}v_\varepsilon.$$

At this stage we remark that $-\partial_{xx}v_\varepsilon$ is not regular enough to perform the desired integration by parts and moreover $-\partial_{xx}v_\varepsilon$ does not satisfied $\mathrm{div}\,(B_b^\top\partial_{xx}v_\varepsilon)=0$. Nevertheless the following lemma holds true

Lemma 3.8. *For $w\in L^2(0,T;H^2_\sharp(\Omega_0))\cap H^1(0,T;L^2_\sharp(\Omega_0))$, such that*

$$\begin{aligned} &w(x,1,t)=\partial_t\xi(x,t)e_2, && \text{for some } \xi\in H^1(0,T;H^2_\sharp(0,L)),\\ &w(x,0,t)=0, && \text{on } (0,L),\end{aligned}$$

and for $q\in L^2(0,T;H^1_\sharp(\Omega_0))$ we have

$$\begin{aligned} &-\int_0^t\int_{\Omega_0}\rho_{f,b}\partial_t w\cdot\partial_{xx}w+\int_0^t\int_{\Omega_0}(\mu\,div\,((A_b\nabla)w)-(B_b\nabla)q)\cdot\partial_{xx}w\\ =\ &\frac12\int_{\Omega_0}\rho_{f,b}|\partial_x w|^2(t)-\frac12\int_{\Omega_0}\rho_{f,b}|\partial_x w|^2(0)+\int_0^t\int_{\Omega_0}\rho_f\partial_x b\,\partial_t w\cdot\partial_x w\\ &+\int_0^t\Big\langle\mu((A_b\nabla)w\,e_2)\cdot e_2-q(B_b\,e_2)\cdot e_2,\partial_{xxt}\xi\Big\rangle_{H^{\frac12}_\sharp(0,L),H^{-\frac12}_\sharp(0,L)}+\mu\int_0^t\int_{\Omega_0}A_b\nabla\partial_x w:\nabla\partial_x w\\ &+\mu\int_0^t\int_{\Omega_0}(\partial_x A_b\nabla)w:\nabla\partial_x w-\int_0^t\int_{\Omega_0}\partial_x q\,B_b:\nabla\partial_x w-\int_0^t\int_{\Omega_0}q\,\partial_x B_b:\nabla\partial_x w.\end{aligned}$$

Moreover if $div\,(B_b^\top w)=0$ then

$$\int_0^t\int_{\Omega_0}\partial_x q\,B_b:\nabla\partial_x w=-\int_0^t\int_{\Omega_0}\partial_x q\,\partial_x B_b:\nabla w.$$

Proof. By convolution (in time and space), we can approximate w by a family $(w_n)_{n\in\mathbb{N}}$ of smooth vector-fields such that:

$$w_n\to w\text{ in }L^2(0,T;H^2_\sharp(0,L))\cap H^1(0,T;L^2_\sharp(\Omega_0))$$

We have then that, when $n\to\infty$, the following convergences hold:

$w_n\to 0$ in $L^2(0,T;H^{3/2}_\sharp(0,L))$ so that $\partial_{xx}w_n\to 0$ in $L^2(0,T;H^{-1/2}_\sharp(0,L))$ on $y=0$,

$w_n\to\partial_t\xi e_2$ in $L^2(0,T;H^{3/2}_\sharp(0,L))$ so that $\partial_{xx}w_n\to\partial_{xxt}\xi e_2$ in $L^2(0,T;H^{-1/2}_\sharp(0,L))$ on $y=1$,

$(A_b\nabla)w_n e_2\to(A_b\nabla)we_2$ in $L^2(0,T;H^{1/2}_\sharp(0,L))$ on $y=1$ and $y=0$.

Again, the identity to be proven is a simple integration by parts for w_n. We may extend the identity to w by remarking that all integrals involved in these identities are continuous with respect to the topology for which w_n converges to w.

□

We apply the previous lemma for $w=v_\varepsilon$ and we obtain

$$\begin{aligned}(3.36)\quad &\frac12\int_{\Omega_0}\rho_{f,b}|\partial_x v_\varepsilon|^2(t)+\mu\int_0^t\int_{\Omega_0}A_b\nabla\partial_x v_\varepsilon:\nabla\partial_x v_\varepsilon\\ &=-\int_0^t\int_0^L(\mu((A_b\nabla)v_\varepsilon\,e_2)\cdot e_2-q(B_b\,e_2)\cdot e_2)\,\partial_{xxt}\eta_\varepsilon-\int_0^t\int_{\Omega_0}f\cdot\partial_{xx}v_\varepsilon-\int_0^t\int_{\Omega_0}\mathrm{div}\,h\cdot\partial_{xx}v_\varepsilon\\ &\quad+\frac12\int_{\Omega_0}\rho_{f,b}|\partial_x u^0_\varepsilon|^2-\int_0^t\int_{\Omega_0}\rho_f\partial_x b\,\partial_t v_\varepsilon\cdot\partial_x v_\varepsilon-\mu\int_0^t\int_{\Omega_0}\partial_x A_b\nabla v_\varepsilon:\nabla\partial_x v_\varepsilon\\ &\quad+\int_0^t\int_{\Omega_0}\partial_x q_\varepsilon\,B_b:\nabla\partial_x v_\varepsilon+\int_0^t\int_{\Omega_0}q_\varepsilon\,\partial_x B_b:\nabla\partial_x v_\varepsilon.\end{aligned}$$

The forcing term involving h can be integrated by parts

$$-\int_0^t \int_{\Omega_0} \operatorname{div} h \cdot \partial_{xx} v_\varepsilon = -\int_0^t \int_{\Omega_0} \partial_x h : \nabla \partial_x v_\varepsilon - \int_0^t \int_0^L h e_2 \cdot e_2 \partial_{xxt} \eta_\varepsilon .$$

From (3.36) we deduce

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega_0} \rho_{f,b} |\partial_x v_\varepsilon|^2(t) + \mu \int_0^t \int_{\Omega_0} A_b \nabla \partial_x v_\varepsilon : \nabla \partial_x v_\varepsilon \\
= & -\int_0^t \int_0^L \left(\mu ((A_b \nabla) v_\varepsilon \, e_2) \cdot e_2 - q_\varepsilon (B_b \, e_2) \cdot e_2 \right) \partial_{xxt} \eta_\varepsilon - \int_0^t \int_0^L h e_2 \cdot e_2 \partial_{xxt} \eta_\varepsilon \\
& - \int_0^t \int_{\Omega_0} f \cdot \partial_{xx} v_\varepsilon - \int_0^t \int_{\Omega_0} \partial_x h : \nabla \partial_x v_\varepsilon + \frac{1}{2} \int_{\Omega_0} \rho_{f,b} |\partial_x u_\varepsilon^0|^2 \\
& - \int_0^t \int_{\Omega_0} \rho_f \partial_x b \, \partial_t v_\varepsilon \cdot \partial_x v_\varepsilon - \mu \int_0^t \int_{\Omega_0} \partial_x A_b \nabla v_\varepsilon : \nabla \partial_x v_\varepsilon \\
& + \int_0^t \int_{\Omega_0} \partial_x q_\varepsilon \, B_b : \nabla \partial_x v_\varepsilon + \int_0^t \int_{\Omega_0} q_\varepsilon \, \partial_x B_b : \nabla \partial_x v_\varepsilon .
\end{aligned} \tag{3.37}$$

The first (respectively the second) term in the right-hand side correspond to the forcing term applied by the fluid on the structure (respectively to the external forcing term on the structure), namely the opposite of the first term (respectively the second term) in the right hand side of (3.31). By adding (3.37) and (3.31) (that has been previously integrated with respect to time) and by recalling that $A_b \geq \lambda(R_1) \mathbb{I}_2$, we thus obtain, after some rearrangement of the terms

$$\begin{aligned}
& \mathcal{H}_r(t) + \mu \lambda(R_1) \int_0^t \int_{\Omega_0} |\nabla \partial_x v_\varepsilon|^2 + \gamma \int_0^t \int_0^L |\partial_{xxt} \eta_\varepsilon|^2 + \varepsilon \int_0^t \int_0^L |\partial_{xxxt} \eta_\varepsilon|^2 \\
\leq \; & \mathcal{H}_r^0 - \int_0^t \int_{\Omega_0} f \cdot \partial_{xx} v_\varepsilon - \int_0^t \int_{\Omega_0} \partial_x h \cdot \nabla \partial_x v_\varepsilon - \int_0^t \int_{\Omega_0} \rho_f \partial_x b \, \partial_t v_\varepsilon \cdot \partial_x v_\varepsilon \\
& - \int_0^t \int_{\Omega_0} \mu (\partial_x A_b \nabla) v_\varepsilon : \nabla \partial_x v_\varepsilon + \int_0^t \int_{\Omega_0} \partial_x q_\varepsilon \, B_b : \nabla \partial_x v_\varepsilon + \int_0^t \int_{\Omega_0} q_\varepsilon \, \partial_x B_b : \nabla \partial_x v_\varepsilon .
\end{aligned} \tag{3.38}$$

where :

$$\begin{aligned}
\mathcal{H}_r(t) \; = \; & \frac{1}{2} \int_{\Omega_0} \rho_{f,b} |\partial_x v_\varepsilon|^2(t) + \frac{\rho_s}{2} \int_0^L |\partial_{xt} \eta_\varepsilon|^2(t) + \frac{\delta}{2} \int_0^L |\partial_{xxt} \eta_\varepsilon|^2(t) \\
& + \frac{\alpha}{2} \int_0^L |\partial_{xxx} \eta_\varepsilon|^2(t) + \frac{\beta}{2} \int_0^L |\partial_{xx} \eta_\varepsilon|^2(t),
\end{aligned}$$

and $\mathcal{H}_r^0 = \mathcal{H}_r(0)$. To proceed with (3.38), we compute bounds for the right-hand side. The most intricated terms are the four last ones on the right-hand side, which are the remainder terms coming from the derivation of the geometry with respect to x (and thus involving b). We denote these terms $T_i, i = 1, 2, 3, 4$.

So, let us consider the right-hand side terms in their order of appearence in equation (3.38). First, for the initial conditions, we have

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega_0} \rho_{f,b} |\partial_x u_\varepsilon^0|^2 + \frac{\rho_s}{2} \int_0^L |\partial_x \dot{\eta}_\varepsilon^0|^2 + \frac{\delta}{2} \int_0^L |\partial_{xx} \dot{\eta}_\varepsilon^0|^2 + \frac{\alpha}{2} \int_0^L |\partial_{xxx} \eta_\varepsilon^0|^2 + \frac{\beta}{2} \int_0^L |\partial_{xx} \eta_\varepsilon^0|^2 \\
& \leq C \frac{\rho_f R_1}{2} \|v_\varepsilon^0\|^2_{H^1_\sharp(\Omega_0)} + \frac{\rho_s}{2} \|\dot{\eta}_\varepsilon^0\|^2_{H^1_\sharp(0,L)} + \frac{\delta}{2} \|\dot{\eta}_\varepsilon^0\|^2_{H^2_\sharp(0,L)} + \frac{\alpha}{2} \|\eta_\varepsilon^0\|^2_{H^3_\sharp(0,L)} + \frac{\beta}{2} \|\eta_\varepsilon^0\|^2_{H^2_\sharp(0,L)} .
\end{aligned} \tag{3.39}$$

Due to our assumptions on initial data, this last right-hand side is bounded whatever the value of $(\alpha, \beta, \gamma, \delta)$ in the three different cases under consideration. Then, the external forcing terms are bounded with classical inequalities:

$$\left|\int_0^t\int_{\Omega_0} f\cdot\partial_{xx}v_\varepsilon + \int_0^t\int_{\Omega_0}\partial_x h\cdot\nabla\partial_x v_\varepsilon\right| \leq \int_0^t \left(\|f\|_{L^2_\sharp(\Omega_0)}+\|h\|_{H^1_\sharp(\Omega_0)}\right)\|\nabla\partial_x v_\varepsilon\|_{L^2_\sharp(\Omega_0)} \tag{3.40}$$
$$\leq C\left(R_1,\frac{1}{\mu}\right)\left(\|f\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))}+\|h\|^2_{L^2(0,t;H^1_\sharp(\Omega_0))}\right)+\frac{\mu\lambda(R_1)}{16}\|\nabla\partial_x v_\varepsilon\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))}$$

We turn now to the geometrical terms T_i. The first one is given by

$$T_1 = -\int_0^t\int_{\Omega_0}\rho_f b_x\,\partial_t v_\varepsilon\cdot\partial_x v_\varepsilon.$$

We recall that $b\in H^2_\sharp(0,L)\subset C^1([0,L])$. Then, classically, we have that, for $\kappa>0$ to be fixed later,

$$|T_1| \leq \kappa\|\partial_t v_\varepsilon\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))} + C\left(\rho_f,R_1,\frac{1}{\kappa}\right)\|\nabla v_\varepsilon\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))}. \tag{3.41}$$

Note that $\|v_\varepsilon\|^2_{L^2(0,t;H^1_\sharp(\Omega_0))}$ is bounded independently of ε thanks to the energy estimate (3.20).

The second term is defined by

$$T_2 = -\int_0^t\int_{\Omega_0}\partial_x A_b\nabla v_\varepsilon : \nabla\partial_x v_\varepsilon,$$

As we stated previously $(\partial_x A_b,\partial_x B_b)\in L^2_\sharp((0,L),H^s(0,1))$, for arbitrary $s\geq 0$, (with norms bounded by a function of R_1) and $H^1_\sharp((0,L)\times(0,1))\subset L^\infty_\sharp((0,L);L^2(0,1))$. Hence, in the spirit of [13, Lemma 6] and of the proof of Lemma 2.10 (that can be found in [15]), we bound T_2 as follows

(3.42)

$$\begin{aligned}|T_2| &\leq C\int_0^t\|\partial_x A_b\|_{L^\infty(0,1;L^2_\sharp(0,L))}\|\nabla v_\varepsilon\|_{L^2(0,1;L^\infty_\sharp(0,L))}\|\nabla\partial_x v_\varepsilon\|_{L^2_\sharp(\Omega_0)}\\ &\leq \int_0^t\|\partial_x A_b\|_{L^\infty(0,1;L^2_\sharp(0,L))}\left[\|\nabla v_\varepsilon\|^{1/2}_{L^2_\sharp(\Omega_0)}\|\nabla\partial_x v_\varepsilon\|^{3/2}_{L^2_\sharp(\Omega_0)}+\|\nabla v_\varepsilon\|_{L^2_\sharp(\Omega_0)}\|\nabla\partial_x v_\varepsilon\|_{L^2_\sharp(\Omega_0)}\right]\\ &\leq C\left(R_1,\frac{1}{\mu}\right)\|\nabla v_\varepsilon\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))}+\frac{\mu\lambda(R_1)}{8}\|\nabla\partial_x v_\varepsilon\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))}.\end{aligned}$$

The first term in the right hand side is bounded independently of ε thanks to the energy estimate (3.20) and the second term can be absorbed by the second term of (3.38).

The two last terms involve the pressure for which we have no bound so far. Consequently we need, at this step, to use Lemma 2.10 on ellipticity of the Stokes-like problem. The third term is defined by

$$T_3 = \int_0^t\int_{\Omega_0}\partial_x q_\varepsilon\,B_b : \nabla\partial_x v_\varepsilon.$$

As stated in Lemma 3.8, T_3 reads:

$$T_3 = -\int_0^t\int_{\Omega_0}\partial_x q_\varepsilon\,\partial_x B_b : \nabla v_\varepsilon.$$

Once again, following the same lines as in [13, Lemma 6] and in the proof of Lemma 2.10 (see [15]), we obtain

$$\begin{aligned}|T_3| &\leq \int_0^t\|\partial_x q_\varepsilon\|_{L^2_\sharp(\Omega_0)}\|\partial_x B_b:\nabla v_\varepsilon\|_{L^2_\sharp(\Omega_0)},\\ &\leq C(R_1)\int_0^t\|\partial_x q_\varepsilon\|_{L^2_\sharp(\Omega_0)}\left[\|\nabla v_\varepsilon\|^{1/2}_{L^2_\sharp(\Omega_0)}\|\nabla\partial_x v_\varepsilon\|^{1/2}_{L^2_\sharp(\Omega_0)}+\|\nabla v_\varepsilon\|_{L^2_\sharp(\Omega_0)}\right].\end{aligned}$$

Using the elliptic result of Lemma 2.10, and the equality of the fluid and structure velocities at the interface, we have (since $\partial_t \eta_\varepsilon \in H^{\frac{3}{2}}_\sharp(0,L) \cap L^2_{\sharp,0}(0,L)$):

$$\begin{aligned}\|\partial_x q_\varepsilon\|_{L^2_\sharp(\Omega_0)} &\leq K^s(R_1)\left(\|\partial_t v_\varepsilon\|_{L^2_\sharp(\Omega_0)} + \|\partial_{tx}\eta_\varepsilon\|_{H^{1/2}_\sharp(0,L)}\right)\\ &\leq K^s(R_1)\left(\|\partial_t v_\varepsilon\|_{L^2_\sharp(\Omega_0)} + \|\partial_x v_\varepsilon\|_{H^1_\sharp(\Omega_0)}\right).\end{aligned}$$

Therefore, we have the bound:

$$|T_3| \leq \kappa \|\partial_t v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_0))} + \frac{\mu\lambda(R_1)}{8}\|\nabla\partial_x v_\varepsilon\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))} + C\left(R_1, \frac{1}{\mu}, \frac{1}{\kappa}\right)\|\nabla v_\varepsilon(t)\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))}. \tag{3.43}$$

The last term T_4 is

$$T_4 = \int_0^t \int_{\Omega_0} q_\varepsilon\, \partial_x B_b : \nabla\partial_x v_\varepsilon.$$

At this point, we note that $\partial_x B_b \in L^\infty(0,1;L^2_\sharp(0,L))$, $\nabla\partial_x v_\varepsilon \in L^2(0,T;L^2_\sharp(\Omega_0))$ and $q_\varepsilon \in L^2(0,T;H^1_\sharp(\Omega_0)) \subset L^2(0,T;L^2(0,1;L^\infty_\sharp(0,L)))$. Hence, again with the same trick of tensorizing the space-integral, we have that T_4 is well-defined and a continuous mapping of its argument in the mentioned spaces. Up to an approximation argument, we may thus assume that $b \in C^\infty_\sharp(0,L)$ so that $B_b \in C^\infty_\sharp(\Omega_0)$. Then, since $\operatorname{div}(B_b) = 0$ due to Piola identity we have $\operatorname{div}(\partial_x B_b) = 0$. Consequently, we have:

$$T_4 = -\int_0^t \int_{\Omega_0} \nabla q_\varepsilon\, \partial_x B_b^\top \partial_x v_\varepsilon + \int_0^t \int_0^L [q_\varepsilon \partial_x B_b^\top \partial_x v_\varepsilon \cdot e_2(x,1)]_{|y=1} \mathrm{d}x.$$

The first term can be estimated exactly as T_3:

$$\left|\int_0^t \int_{\Omega_0} \nabla q_\varepsilon\, \partial_x B_b^\top \partial_x v_\varepsilon\right| \leq \kappa\|\partial_t v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_0))} + \frac{\mu\lambda(R_1)}{8}\|\nabla\partial_x v_\varepsilon\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))} + C\left(R_1, \frac{1}{\mu}, \frac{1}{\kappa}\right)\|\nabla v_\varepsilon\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))} \tag{3.44}$$

As for the boundary term, because the change of variables χ_b maps Ω_0 to Ω_b, we have in particular, thanks to the Nanson formula in $(0,L)$ that

$$B_b e_2 = -\partial_x b e_1 + e_2,$$

therefore, we obtain $\partial_x(B_b e_2) = -\partial_{xx} b e_1$. On the other hand, the boundary condition at the interface fluid/structure, that is for $x \in (0,L)$ is $v_\varepsilon(t,x,1) = \partial_t\eta_\varepsilon(t,x)e_2$, which makes sense in $L^2(0,T;H^{3/2}_\sharp(\Omega_0))$ therefore, we obtain $\partial_x v_\varepsilon(t,x) = \partial_{tx}\eta_\varepsilon(t,x)e_2$ in $L^2(0,T;H^{1/2}_\sharp(\Omega_0))$. Thus, we have

$$\partial_x B_b e_2 \cdot \partial_x v_\varepsilon = 0 \quad \text{ in } L^2(0,T;H^{1/2}(0,L)).$$

and finally, the boundary integral is thus zero. Then, from (3.44), we deduce

$$|T_4| \leq \kappa\|\partial_t v_\varepsilon\|^2_{L^2(0,t;L^2(\Omega_0))} + \frac{\mu\lambda(R_1)}{8}\|\nabla\partial_x v_\varepsilon\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))} + C\left(R_1, \frac{1}{\mu}, \frac{1}{\kappa}\right)\|\nabla v_\varepsilon\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))}. \tag{3.45}$$

Consequently, from (3.38), and taking into account (3.40), (3.39), (3.41), (3.42), (3.43), (3.45), we end up with

(3.46)
$$\begin{aligned}\frac{1}{2}\int_{\Omega_0}\rho_{f,b}|\partial_x v_\varepsilon|^2(t)+\frac{\rho_s}{2}\int_0^L|\partial_{xt}\eta_\varepsilon|^2(t)+\frac{\delta}{2}\int_0^L|\partial_{xxt}\eta_\varepsilon|^2(t)+\frac{\alpha}{2}\int_0^L|\partial_{xxx}\eta_\varepsilon|^2(t)+\frac{\beta}{2}\int_0^L|\partial_{xx}\eta_\varepsilon|^2(t)\\ +\frac{\mu\lambda(R_1)}{2}\int_0^t\int_{\Omega_0}|\nabla\partial_x v_\varepsilon|^2+\gamma\int_0^t\int_0^L|\partial_{xxt}\eta_\varepsilon|^2+\varepsilon\int_0^t\int_0^L|\partial_{xxxt}\eta_\varepsilon|^2\\ \leq 4\kappa\|\partial_t v_\varepsilon\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))}+C(\rho_f,R_1,\frac{1}{\mu},\frac{1}{\kappa})\Big(\|f\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))}+\|h\|^2_{L^2(0,t;H^1_\sharp(\Omega_0))}\\ +\rho_f\|u^0\|^2_{H^1_\sharp(\Omega_0)}+\frac{\rho_s}{2}\|\dot\eta^0\|^2_{H^1_\sharp(0,L)}+\frac{\delta}{2}\|\dot\eta^0\|^2_{H^2_\sharp(0,L)}+\frac{\alpha}{2}\|\eta^0\|^2_{H^3_\sharp(0,L)}+\frac{\beta}{2}\|\eta^0\|^2_{H^2_\sharp(0,L)}\Big),\end{aligned}$$

Final regularity estimates. We are now in a position to close a regularity estimate independent of the parameter of regularization ε. We recall that, until now, we obtained two inequalities. The first one, (3.30) is obtained by taking $(\partial_t v_\varepsilon, \partial_{tt}\eta_\varepsilon)$ as test functions. It reads:

$$\begin{aligned}\frac{\rho_f}{2R_1}\int_0^t\int_{\Omega_0}|\partial_t v_\varepsilon|^2+\frac{\rho_s}{2}\int_0^t\int_0^L|\partial_{tt}\eta_\varepsilon|^2+\delta\int_0^t\int_0^L|\partial_{ttx}\eta_\varepsilon|^2\\ +\frac{\mu\lambda(R_1)}{2}\int_{\Omega_0}|\nabla v_\varepsilon(t)|^2+\frac{\gamma}{2}\int_0^L|\partial_{xt}\eta_\varepsilon(t)|^2+\frac{\varepsilon}{2}\int_0^L|\partial_{xxt}\eta_\varepsilon(t)|^2\\ \leq C(R_1)\Big(\|f\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))}+\|h\|^2_{L^2(0,t;H^1_\sharp(\Omega_0))}\Big)+C_0(R_1)\\ +\alpha\int_0^t\int_0^L|\partial_{xxt}\eta_\varepsilon|^2+\beta\int_0^t\int_0^L|\partial_{xt}\eta_\varepsilon|^2+\alpha\int_0^L\partial_{xxx}\eta_\varepsilon(t)\partial_{xt}\eta_\varepsilon(t)+\beta\int_0^L\partial_{xx}\eta_\varepsilon(t)\partial_t\eta_\varepsilon(t)\end{aligned}$$

Second, we have the above coupled regularity estimate (3.46):

$$\begin{aligned}\frac{1}{2}\int_{\Omega_0}\rho_{f,b}|\partial_x v_\varepsilon|^2(t)+\frac{\rho_s}{2}\int_0^L|\partial_{xt}\eta_\varepsilon|^2(t)+\frac{\delta}{2}\int_0^L|\partial_{xxt}\eta_\varepsilon|^2(t)+\frac{\alpha}{2}\int_0^L|\partial_{xxx}\eta_\varepsilon|^2(t)+\frac{\beta}{2}\int_0^L|\partial_{xx}\eta_\varepsilon|^2(t)\\ +\frac{\mu\lambda(R_1)}{2}\int_0^t\int_{\Omega_0}|\nabla\partial_x v_\varepsilon|^2+\gamma\int_0^t\int_0^L|\partial_{xxt}\eta_\varepsilon|^2+\varepsilon\int_0^t\int_0^L|\partial_{xxxt}\eta_\varepsilon|^2\\ \leq 4\kappa\|\partial_t v_\varepsilon\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))}+C(\rho_f,R_1,\frac{1}{\mu},\frac{1}{\kappa})\Big(\|f\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))}+\|h\|^2_{L^2(0,t;H^1_\sharp(\Omega_0))}\\ +\rho_f\|u^0\|^2_{H^1_\sharp(\Omega_0)}+\frac{\rho_s}{2}\|\dot\eta^0\|^2_{H^1_\sharp(0,L)}+\frac{\delta}{2}\|\dot\eta^0\|^2_{H^2_\sharp(0,L)}+\frac{\alpha}{2}\|\eta^0\|^2_{H^3_\sharp(0,L)}+\frac{\beta}{2}\|\eta^0\|^2_{H^2_\sharp(0,L)}\Big),\end{aligned}$$

We propose to write the combination $(E_\Lambda) = \Lambda(3.30)+(3.46)$ with a parameter Λ that we fix below. On the right-hand side of this inequality, we obtain then at most 5 terms to bound:

$$I_1 := \Lambda\alpha\int_0^t\int_0^L|\partial_{xxt}\eta_\varepsilon|^2 \qquad I_2 := \Lambda\beta\int_0^t\int_0^L|\partial_{xt}\eta_\varepsilon|^2$$
$$I_3 := \Lambda\alpha\int_0^L\partial_{xxx}\eta_\varepsilon(t)\partial_{xt}\eta_\varepsilon(t) \qquad I_4 := \Lambda\beta\int_0^L\partial_{xx}\eta_\varepsilon(t)\partial_t\eta_\varepsilon(t)$$
$$I_5 := 4\kappa\int_0^t\int_{\Omega_0}|\partial_t v_\varepsilon|^2$$

First, we restrict Λ to satisfy $\Lambda\alpha \leq (\gamma+\delta)/2$ (we remind that, with our assumptions, if $\gamma=\delta=0$ then $\alpha=0$ so that this inequality is not a restriction on Λ in this case). We may then bound:

$$I_1 \leq \frac{\gamma+\delta}{2}\int_0^t\int_0^L|\partial_{xxt}\eta_\varepsilon|^2,$$

Second, we restrict Λ to satisfy $\Lambda \leq 1/2$ and $\Lambda\alpha \leq \rho_s/4$. We have then:

$$I_3 \leq \frac{\alpha}{4}\int_0^L |\partial_{xxx}\eta_\varepsilon|^2(t) + \frac{\rho_s}{8}\int_0^L |\partial_{xt}\eta_\varepsilon(t)|^2$$

Finally, we introduce C_{PW} the optimal constant for the Poincaré-Wirtinger in $H^1_\sharp(0,L)\cap L^2_{\sharp,0}(0,L)$. We restrict then Λ to satisfy $\Lambda|C_{PW}|^2\beta \leq \rho_s/4$. We obtain then (since $\partial_t\eta \in H^1_\sharp(0,L)\cap L^2_{\sharp,0}(0,L)$ for a.e. $t\in(0,T)$ and $\Lambda \leq 1/2$):

$$I_4 \leq \frac{\beta}{4}\int_0^L |\partial_{xx}\eta_\varepsilon(t)|^2 + \frac{\rho_s}{8}\int_0^L |\partial_{xt}\eta_\varepsilon(t)|^2$$

Finally, we choose (with the convention that if 0 appears as a denominator of a fraction of the list on right-hand side, then the quantity must be deleted from the list):

$$\Lambda = \min\left(\frac{\gamma+\delta}{2\alpha}, \frac{1}{2}, \frac{\rho_s}{4\alpha}, \frac{\rho_s}{4|C_{PW}|^2\beta}\right)$$

With this choice, we note that $\Lambda > 0$ so that we may choose

$$\kappa = \frac{\rho_f}{16\Lambda R_1}.$$

Taking into account the bounds for I_1, I_3, I_4 mentioned above, we have that the combination (E_Λ) implies:

$$\begin{aligned}
&\frac{\Lambda\rho_f}{4R_1}\int_0^t\int_{\Omega_0}|\partial_t v_\varepsilon|^2 + \frac{\Lambda\rho_s}{2}\int_0^t\int_0^L|\partial_{tt}\eta_\varepsilon|^2 + \delta\Lambda\int_0^t\int_0^L|\partial_{ttx}\eta_\varepsilon|^2\\
&+\frac{1}{2}\int_{\Omega_0}\rho_{f,b}|\partial_x v_\varepsilon|^2(t) + \frac{\rho_s}{4}\int_0^L|\partial_{xt}\eta_\varepsilon|^2(t) + \frac{\delta}{2}\int_0^L|\partial_{xxt}\eta_\varepsilon|^2(t) + \frac{\alpha}{4}\int_0^L|\partial_{xxx}\eta_\varepsilon|^2(t) + \frac{\beta}{4}\int_0^L|\partial_{xx}\eta_\varepsilon|^2(t)\\
&\quad + \frac{\mu\Lambda\lambda(R_1)}{2}\int_{\Omega_0}|\nabla v_\varepsilon(t)|^2 + \frac{\Lambda\gamma}{2}\int_0^L|\partial_{xt}\eta_\varepsilon(t)|^2 + \frac{\Lambda\varepsilon}{2}\int_0^L|\partial_{xxt}\eta_\varepsilon(t)|^2\\
&\quad + \frac{\mu\lambda(R_1)}{2}\int_0^t\int_{\Omega_0}|\nabla\partial_x v_\varepsilon|^2 + \frac{\gamma}{2}\int_0^t\int_0^L|\partial_{xxt}\eta_\varepsilon|^2 + \varepsilon\int_0^t\int_0^L|\partial_{xxxt}\eta_\varepsilon|^2\\
&\quad \leq C\left(\rho_f,\rho_s,R_1,\frac{1}{\mu},\frac{1}{\alpha},\frac{1}{\beta},\gamma,\delta\right)\Big(\|f\|^2_{L^2(0,t;L^2_\sharp(\Omega_0))} + \|h\|^2_{L^2(0,t;H^1_\sharp(\Omega_0))}\\
&+\rho_f\|u^0\|^2_{H^1_\sharp(\Omega_0)} + \frac{\rho_s}{2}\|\dot\eta^0\|^2_{H^1_\sharp(0,L)} + \frac{\delta}{2}\|\dot\eta^0\|^2_{H^2_\sharp(0,L)} + \frac{\alpha}{2}\|\eta^0\|^2_{H^3_\sharp(0,L)} + \frac{\beta}{2}\|\eta^0\|^2_{H^2_\sharp(0,L)}\Big),\\
&\qquad + \frac{\delta}{2}\int_0^t\int_0^T|\partial_{xxt}\eta_\varepsilon|^2 + \Lambda\beta\int_0^t\int_0^T|\partial_{xt}\eta_\varepsilon|^2.
\end{aligned}$$

We skip the dependencies of the constant C and we denote by C_0 the quantity involving initial data on the right-hand side to lighten notations. Finally, we remark that, by keeping all the necessary terms, the quantity:

$$Y(t) = \frac{\delta}{2}\int_0^L|\partial_{xxt}\eta_\varepsilon|^2 + \frac{\rho_s}{4}\int_0^t\int_0^L|\partial_{xt}\eta_\varepsilon|^2,$$

satisfies the Gronwall-type inequality:

$$Y(t) \leq C\left(\|f\|^2_{L^2(0,T;L^2_\sharp(\Omega_0))} + \|h\|^2_{L^2(0,T;H^1_\sharp(\Omega_0))} + C_0\right) + \left(1+\frac{4\Lambda\beta}{\rho_s}\right)\int_0^t Y(s)\mathrm{d}s$$

We obtain thus that:

$$Y(t) \leq C\left(C_0 + \|f\|^2_{L^2(0,T;L^2_\sharp(\Omega_0))} + \|h\|^2_{L^2(0,T;H^1_\sharp(\Omega_0))}\right)\exp\left[\left(1+\frac{4\Lambda\beta}{\rho_s}\right)t\right] \quad \forall t>0.$$

We may plug this inequality in (E_Λ) to get, under the restriction that $T<1$:

$$\frac{\Lambda\rho_f}{4R_1}\int_0^t\int_{\Omega_0}|\partial_t v_\varepsilon|^2+\frac{\Lambda\rho_s}{2}\int_0^t\int_0^L|\partial_{tt}\eta_\varepsilon|^2+\delta\Lambda\int_0^t\int_0^L|\partial_{ttx}\eta_\varepsilon|^2$$
$$+\frac{1}{2}\int_{\Omega_0}\rho_{f,b}|\partial_x v_\varepsilon|^2(t)+\frac{\rho_s}{4}\int_0^L|\partial_{xt}\eta_\varepsilon|^2(t)+\frac{\delta}{2}\int_0^L|\partial_{xxt}\eta_\varepsilon|^2(t)+\frac{\alpha}{4}\int_0^L|\partial_{xxx}\eta_\varepsilon|^2(t)+\frac{\beta}{4}\int_0^L|\partial_{xx}\eta_\varepsilon|^2(t)$$
$$+\frac{\mu\Lambda\lambda(R_1)}{2}\int_{\Omega_0}|\nabla v_\varepsilon(t)|^2+\frac{\Lambda\gamma}{2}\int_0^L|\partial_{xt}\eta_\varepsilon(t)|^2+\frac{\Lambda\varepsilon}{2}\int_0^L|\partial_{xxt}\eta_\varepsilon(t)|^2$$
$$+\frac{\mu\lambda(R_1)}{2}\int_0^t\int_{\Omega_0}|\nabla\partial_x v_\varepsilon|^2+\frac{\gamma}{2}\int_0^t\int_0^L|\partial_{xxt}\eta_\varepsilon|^2+\varepsilon\int_0^t\int_0^L|\partial_{xxxt}\eta_\varepsilon|^2$$
$$\leq C\left(C_0+\|f\|^2_{L^2(0,T;L^2_\sharp(\Omega_0))}+\|h\|^2_{L^2(0,T;H^1_\sharp(\Omega_0))}\right),$$

where C depends again only on $\rho_f,\rho_s,R_1,\mu,\alpha,\beta,\gamma,\delta$.

Limit as ε goes to zero and uniqueness. By standard compactness arguments, we may extract a weakly converging subsequence in the family of ε-solutions $(\eta_\varepsilon,v_\varepsilon,q_\varepsilon)_{\varepsilon>0}$. Applying classical arguments, we obtain that the weak limit (η,v,q) is a solution to our linear problem with the expected regularity. Estimate (3.16) yields as the weak limit of the last inequality of the previous section. All these computations are completly classical and left to the reader.

Finally, in the smooth-solution setting that we consider herein, we note that we can derive the energy equality (3.19) by a multiplier argument. Consequently, if we were having two solutions to (3.1)–(3.3) with the same data, the difference would satisfy (3.19) with vanishing data. This implies that both solutions are equal: we have uniqueness of the solution to (3.1)–(3.3).

This ends the proof of Proposition 3.2

3.3. **Non homogeneous divergence constraint.** In this section, we extend the analysis of the linear problem to the case in which the right-hand side g of the divergence constraint is not equal to zero. We have the analog of Proposition 3.2:

Proposition 3.9. *Let us consider b in $H^2_\sharp(0,L)$, s.t. $\min(1+b)>0$, intial data $(\eta^0,\dot\eta^0,v^0)\in X^0$ satisfying* (3.9) *and* (3.10) *and f, g and h satisfying resp.* (3.13), (3.14), (3.15). *Given $T\in(0,1)$, there exists a unique solution $(\eta,v,q)\in X_{s,T}\times X_{f,T}$ of* (3.1)–(3.3), *satisfying*

- *equations* (3.1), (3.2) *a.e. in $(0,T)\times\Omega_0$,*
- *equations* (3.3) *in $L^2(0,T;H^{-1}_\sharp(0,L))$,*
- *equations* (3.4), (3.5) *a.e. in $(0,T)\times(0,L)$,*
- *equations* (3.6), (3.7), (3.8) *a.e. in $(0,L)$ and Ω_0.*
- *equations* (3.11), (3.12) *a.e. in $(0,T)\times\Omega_0$ and $(0,T)$ respectively.*

Moreover, there exists a non-decreasing function $C:[0,\infty)\to[0,\infty)$ such that, assuming $\|b\|_{H^2_\sharp(0,L)}+\|(1+b)^{-1}\|_{L^\infty_\sharp(0,L)}\leq R_1$ the solution (η,v,q,d) satisfies

$$\|(v,q)\|_{X_{f,T}}+\|\eta\|_{X_{s,T}}\leq C(R_1)\left(\|(v^0,\eta^0,\dot\eta^0)\|_{X^0}+\|(f,g,h)\|_{S_T}\right),$$

Proof. We first note that uniqueness is proven as in the case of vanishing data in the divergence equation, so that existence only requires special attention. We note also that, since $g(\cdot,0)=0$, we may restrict at first to data g with compact support in $(0,T)$ (*i.e.* in time) and then apply a compactness argument.

In the case g has compact support in $(0,T)$, we transform the source term in the divergence equation into a source term in the Navier Stokes equation by introducing a suitable lifting of the divergence. Namely, we construct v_g (see below) such that $\mathrm{div}(B_b^\top v_g)=g$ and we look for a solution (η,v,q) to (3.1)–(3.3) of the form $v=v_g+v'$.

Construction of v_g. For a.e. $t \in (0,T)$, as $g(\cdot,t)$ is mean free, we may apply Lemma 2.10 to obtain that there exists a unique $(w_t,\pi_t) \in H^2_\sharp(\Omega_0) \times (H^1_\sharp(\Omega_0) \cap L^2_{\sharp,0}(\Omega_0))$ solution to:

$$\begin{aligned} -\mathrm{div}[(A_b\nabla)w_t] + (B_b\nabla)\pi_t &= 0\,, \quad \text{in } \Omega_0\,,\\ \mathrm{div}(B_b^\top w_t) &= g(\cdot,t)\,, \quad \text{in } \Omega_0\,,\end{aligned}$$

with the boundary conditions:

$$w_t(x,1) = 0\,, \quad w_t(x,0) = 0\,, \quad \forall\, x \in (0,L)\,.$$

We set then:

$$v_g(x,y,t) = w_t(x,y)\,, \qquad q_g(x,y,t) = \pi_t(x,y)\,, \qquad \text{for a.e. } (x,y,t) \in \Omega_0 \times (0,T).$$

Applying Lemma 2.10 again and our assumption on g, we obtain that

$$(v_g,q_g) \in L^2(0,T;H^2_\sharp(\Omega_0)) \times L^2(0,T;H^1_\sharp(\Omega_0))$$

with

$$\|v_g\|_{L^2(0,T;H^2_\sharp(\Omega_0))} + \|q_g\|_{L^2(0,T;H^1_\sharp(\Omega_0))} \leq K^s(R_1)\|g\|_{L^2(0,T;H^1_\sharp(\Omega_0))}.$$

To obtain further time-regularity on v_g, we note that it satisfies:

$$\int_0^T\int_{\Omega_0} v_g\cdot[-\mathrm{div}(A_b\nabla)z + (B_b\nabla)r] = -\int_0^T\int_{\Omega_0} gr. \tag{3.47}$$

for any pair (z,r) such that:

$$\begin{aligned}&z \in L^2(0,T;H^2_\sharp(\Omega_0)) \quad \text{and} \quad r \in L^2(0,T;H^1_\sharp(\Omega)),\\ &\mathrm{div}(B_b^\top z) = 0 \quad \text{on } \Omega_0\times(0,T), \qquad z(x,1,t) = z(x,0,t) = 0 \quad \text{on } (0,L)\times(0,T).\end{aligned}$$

On the other hand, given $\zeta \in C^\infty_c(\Omega_0\times(0,T))$ we apply Lemma 2.10 again in order to construct a pair $(z_\zeta,r_\zeta) \in C^1_c([0,T];H^2_\sharp(\Omega_0)) \times C^1_c([0,T];H^1_\sharp(\Omega_0)\cap L^2_{\sharp,0}(\Omega_0))$ satisfying:

$$\begin{aligned} -\mathrm{div}\,((A_b\nabla)z_\zeta) + (B_b\nabla)r_\zeta = \zeta, &\qquad \text{in } \Omega_0\times(0,T),\\ \mathrm{div}\,(B_b^\top z_\zeta) = 0, &\qquad \text{in } \Omega_0\times(0,T),\\ z_\zeta(x,1,t) = z_\zeta(x,0,t) = 0, &\qquad \text{on } (0,L)\times(0,T)\end{aligned}$$

and

$$\|z_\zeta\|_{L^2(0,T;H^2_\sharp(\Omega_0))} + \|r_\zeta\|_{L^2(0,T;H^1_\sharp(\Omega_0))} \leq K^s(R_1)\|\zeta\|_{L^2(0,T;L^2_\sharp(\Omega_0))}.$$

Introducing the definition of $(\partial_t z_\zeta, \partial_t r_\zeta)$ in (3.47), we obtain:

$$\begin{aligned}\int_0^T\int_{\Omega_0} v_g\partial_t\zeta &= \int_0^T\int_{\Omega_0} v_g\cdot[-\mathrm{div}\,((A_b\nabla)\partial_t z_\zeta) + (B_b\nabla)\partial_t r_\zeta]\\ &= -\int_0^T\int_{\Omega_0} g\partial_t r_\zeta,\end{aligned}$$

where, thanks to our assumptions on g:

$$\begin{aligned}\left|\int_0^T\int_{\Omega_0} g\partial_t r_\zeta\right| &\leq \|\partial_t g\|_{L^2(0,T;H^1_\sharp(\Omega_0)')}\|r_\zeta\|_{L^2(0,T;H^1_\sharp(\Omega_0))}\\ &\leq K^s(R_1)\|\partial_t g\|_{L^2(0,T;H^1_\sharp(\Omega_0)')}\|\zeta\|_{L^2(0,T;L^2_\sharp(\Omega_0))}.\end{aligned}$$

Eventually, we obtain that $v_g \in H^1(0,T;L^2_\sharp(\Omega_0)) \cap L^2(0,T;H^2_\sharp(\Omega_0))$ with:

$$\begin{aligned}&\|v_g\|_{H^1(0,T;L^2_\sharp(\Omega_0))} + \|v_g\|_{L^2(0,T;H^2_\sharp(\Omega_0))}\\ &\qquad \leq K^s(R_1)\left[\|g\|_{L^2(0,T;H^1_\sharp(\Omega_0))} + \|\partial_t g\|_{L^2(0,T;H^1_\sharp(\Omega_0)')}\right].\end{aligned} \tag{3.48}$$

Conclusion. On the one-hand, we have:

$$\|v\|_{H^1(0,T;L^2_\sharp(\Omega_0))} + \|v\|_{L^2(0,T;H^2_\sharp(\Omega_0))} \leq \|v_g\|_{H^1(0,T;L^2_\sharp(\Omega_0))} + \|v_g\|_{L^2(0,T;H^2_\sharp(\Omega_0))} + \|v'\|_{H^1(0,T;L^2_\sharp(\Omega_0))} + \|v'\|_{L^2(0,T;H^2_\sharp(\Omega_0))}. \tag{3.49}$$

On the other hand, we have that (η, v', q) is a solution to (3.1)–(3.3) with initial data $(\eta^0, \dot\eta^0, v^0)$ and source term

$$f' = f - \partial_t v_g, \quad g' = 0, \quad h' = h - (A_b\nabla)v_g.$$

Hence, applying Proposition 3.2, we obtain that:

$$\|\eta\|_{X_{s,T}} + \|(v',q)\|_{X_{f,T}} \leq C(R_1)\left(\|(v^0,\eta^0,\dot\eta^0)\|_{X^0} + \|(f,0,h)\|_{S_T} + \|\partial_t v_g\|_{L^2(0,T;L^2_\sharp(\Omega_0))} + \|A_b\nabla v_g\|_{L^2(0,T;H^1_\sharp(\Omega_0))}\right).$$

Applying that $\nabla v_g \in L^2(0,T;H^1_\sharp(\Omega_0))$ with (3.48) and that A_b belongs to a multiplier space of H^1 we obtain that:

$$\|\eta\|_{X_{s,T}} + \|(v',q)\|_{X_{f,T}} \leq C(R_1)\left(\|(v^0,\eta^0,\dot\eta^0)\|_{X^0} + \|(f,g,h)\|_{S_T}\right).$$

Finally, combining (3.48), (3.49) and this last inequality (we recall that $\|v\|_{C([0,T];H^1_\sharp(\Omega_0))}$ is computed by interpolating $\|\partial_t v\|_{L^2(0,T;L^2_\sharp(\Omega_0)}$ and $\|v\|_{L^2(0,T;H^2_\sharp(\Omega_0))}$), we obtain:

$$\|\eta\|_{X_{s,T}} + \|(v,q)\|_{X_{f,T}} \leq C(R_1)\left[\|(v^0,\eta^0,\dot\eta^0)\|_{X^0} + \|(f,g,h)\|_{S_T}\right].$$

This ends the proof. □

4. Fixed Point. Proof of Theorem 2.9

In the whole section $(\alpha,\beta,\gamma,\delta) \in [0,\infty)^4$ and the initial data $(\eta^0,\dot\eta^0,v^0)$ are fixed. The parameters $\alpha,\beta,\gamma,\delta$ satisfy one of the assumptions $(\mathbf{C}_{\alpha,\gamma})$, $(\mathbf{C}_{\alpha,\delta})$ or $(\mathbf{C}_{\beta})$ and the initial data satisfy the assumptions of Theorem 2.9. With our notations, we have then that $(\eta^0,\dot\eta^0,v^0) \in X^0$ and we denote:

$$R_1 := \|(\eta^0,\dot\eta^0,v^0)\|_{X^0} + \|(1+\eta^0)^{-1}\|_{L^\infty_\sharp(0,L)}.$$

We fix also $\mathcal{K} > 0$ as in the data of our theorem.

To handle the fixed-point strategy we introduce two mappings. First, we note that Proposition 3.9 may be interpreted as follows:

Proposition 4.1. *Given $T \in (0,1)$ there exists a mapping*

$$\mathcal{L}_T : (f,g,h) \in S_T \longmapsto (\eta,v,q) \in X_{s,T} \times X_{f,T}$$

such that, for any $(f,g,h) \in S_T$ the triplet (η,v,q) is the unique strong solution to (3.1)–(3.10) *with $b = \eta^0$, initial data $(\eta^0,\dot\eta^0,v^0)$ and right-hand side (f,g,h). Furthermore, there exists a non-decreasing mapping $C_\mathcal{L} : (0,\infty) \to (0,\infty)$ such that*

(1) given $(f,g,h) \in S_T$,

$$\|\mathcal{L}_T(f,g,h)\|_{X_{s,T}\times X_{f,T}} \leq C_\mathcal{L}(R_1)\left(\|(\eta^0,\dot\eta^0,v^0)\|_{X^0} + \|(f,g,h)\|_{S_T}\right). \tag{4.1}$$

(2) given (f_1,g_1,h_1) and (f_2,g_2,h_2) in S_T, then

$$\|\mathcal{L}_T(f_1,g_1,h_1) - \mathcal{L}_T(f_2,g_2,h_2)\|_{X_{s,T}\times X_{f,T}} \leq C_\mathcal{L}(R_1)\|(f_1,g_1,h_1)-(f_2,g_2,h_2)\|_{S_T}. \tag{4.2}$$

Proof. Applying Proposition 3.9, we obtain the existence of the mapping and estimate (4.1). Moreover, for (f_1,g_1,h_1), (f_2,g_2,h_2) in S_T, we denote (η_1,v_1,q_1) (respectively (η_2,v_2,q_2)) the solution to (3.1)–(3.10) with initial data $(\eta^0,\dot\eta^0,v^0)$ and right-hand side (f_1,g_1,h_1) (resp. (f_2,g_2,h_2)). By linearity, $(v_1-v_2,q_1-q_2,\eta_1-\eta_2)$ is the unique solution to (3.1)–(3.10) with zero initial data and $(f_1-f_2,g_1-g_2,h_1-h_2)$ as right-hand side. Applying Proposition 3.9 with these data yields estimate (4.2). □

Second, we introduce the computations of nonlinearities (2.30)–(2.31)–(2.32)–(2.33) arising in the linearization process depicted in Section 2.3. Namely, we fix:

$$E^0_{s,T} = \left\{\overline{\zeta} \in L^\infty(0,T;H^2_\sharp(0,L)) \cap W^{1,\infty}(0,T;H^1_\sharp(0,L)) \text{ s.t. } \overline{\zeta}(0)=0\right\}.$$

and we denote:

$$\mathcal{S} : (\overline{\zeta},z,r) \in E^0_{s,T} \times X_{f,T} \longmapsto (f[z,\overline{\zeta}+\eta^0], g[z,\overline{\zeta}+\eta^0], h[z,r,\overline{\zeta}+\eta^0]) \in S_T$$

where, introducing $\eta = \overline{\zeta}+\eta^0$, we have:

$$\begin{aligned} f[v,\eta] &= \rho_f(\det\nabla\chi_{\eta^0} - \det\nabla\chi_\eta)\partial_t v - \rho_f((v-\partial_t\chi_\eta)\cdot(B_\eta\nabla))v,\\ g[v,\eta] &= \operatorname{div}((B^\top_{\eta^0} - B^\top_\eta)v),\\ h[v,q,\eta] &= -\mu((A_{\eta^0}-A_\eta)\nabla)v + q(B_{\eta^0}-B_\eta), \end{aligned}$$

with:

$$B_\eta = \operatorname{cof}\nabla\chi_\eta, \quad A_\eta = \frac{1}{\det\nabla\chi_\eta}B_\eta {B_\eta}^\top.$$

We refer the reader to Section 2.2 for the relations between η and χ_η. The properties of this mapping are analyzed below.

Finally, we introduce the projection mappping $\mathcal{L}^0_T : (f,g,h) \in S_T \mapsto (\overline{\zeta},v,q) \in E^0_{s,T} \times X_{f,T}$ where $\overline{\zeta} = \eta - \eta^0$ (as above) and $(\eta,v,q) = \mathcal{L}_T(f,g,h)$ is defined in Proposition 4.1. Therefore, $\mathcal{L}^0_T$ satisfies obviously, with the same notations as in Proposition 4.1, the following properties inherated from $\mathcal{L}_T$:

(1) given $(f,g,h) \in S_T$,

$$\|\mathcal{L}^0_T(f,g,h)\|_{E^0_{s,T}\times X_{f,T}} \le \|\eta^0\|_{H^2_\sharp(0,L)} + C_{\mathcal{L}}(R_1)\left(\|(\eta^0,\dot\eta^0,v^0)\|_{X^0} + \|(f,g,h)\|_{S_T}\right).$$

(2) given (f_1,g_1,h_1) and (f_2,g_2,h_2) in S_T, then

$$\left\|\mathcal{L}^0_T(f_1,g_1,h_1) - \mathcal{L}^0_T(f_2,g_2,h_2)\right\|_{E^0_{s,T}\times X_{f,T}} \le C_{\mathcal{L}}(R_1)\|(f_1,g_1,h_1)-(f_2,g_2,h_2)\|_{S_T}.$$

We note that (η,v,q) is a solution to $(\mathrm{FS})_{ref}$ on $(0,T)$ if and only $(\overline{\zeta},v,q) = (\eta-\eta^0,v,q)$ is a fixed point of the mapping $\mathcal{L}^0_T \circ \mathcal{S}$.

We have obtained above that $\mathcal{L}^0_T$ is a Lipschitz mapping with a constant $C_{\mathcal{L}}$. To proceed, we prove in the following proposition that $\mathcal{S}$ is a well-defined Lipschitz mapping, for T sufficiently small, and that the Lipschitz constant of $\mathcal{S}$ converges to 0 when $T\to 0$.

Proposition 4.2. *Given $M \in (0,\infty)$ there exists $T(R_1,M) > 0$ such that, for $T < T(R_1,M)$ the mapping $\mathcal{S}$ is well-defined on $B_{E^0_{s,T}\times X_{f,T}}(M)$ and there exists a constant $C_{\mathcal{S}}(R_1,M) > 0$ such that, for every $(\overline{\zeta},z,r) \in B_{E^0_{s,T}\times X_{f,T}}(M)$, there holds:*

$$\|\mathcal{S}(\overline{\zeta},z,r)\|_{S_T} \le C_{\mathcal{S}}(R_1,M)T^{\frac14}M^2.$$

Furthermore, there exists a constant $P_{\mathcal{S}}(R_1,M) > 0$ such that, for every $(\overline{\zeta}_1,z_1,r_1)$, $(\overline{\zeta}_2,z_2,r_2)$ in $B_{E^0_{s,T}\times X_{f,T}}(M)$, there holds:

$$\|\mathcal{S}(\overline{\zeta}_1,z_1,r_1) - \mathcal{S}(\overline{\zeta}_2,z_2,r_2)\|_{S_T} \le T^{\frac14}P_{\mathcal{S}}(R_1,M)\left\|(\overline{\zeta}_1-\overline{\zeta}_2, z_1-z_2, r_1-r_2)\right\|_{E^0_{s,T}\times X_{f,T}}.$$

Remark 4.3. *Note that the exponent $\frac14$ in the previous inequality corresponds with the explicit choice of $\varepsilon_0 = \frac14$ in the regularity results for the change of variables (see Remark 2.8).*

The proof of this proposition is postponed to Appendix A. First, we explain how we conclude the proof of Theorem 2.9. Since $C_{\mathcal{L}}$ does not depend on T, we obtain that there exists $M_0 := M_0(R_1,\eta^0,\dot\eta^0,v^0)$ (for instance, take $M_0 = 2\left(\|\eta^0\|_{H^2_\sharp(0,L)} + C_{\mathcal{L}}(R_1)\|(\eta^0,\dot\eta^0,v^0)\|_{X^0}\right) > 0$) such that, for $T\in(0,1)$ we have $\|\mathcal{L}^0_T\circ\mathcal{S}(0)\|_{E^0_{s,T}\times X_{f,T}} \le \frac{M_0}{2}$. Applying classical arguments, Proposition 4.2 implies then that there exists $T(R_1)$ such that, for $T \le T(R_1)$, the mapping $\mathcal{L}^0_T\circ\mathcal{S}$ is a contraction on $B_{E^0_{s,T}\times X_{f,T}}(M_0)$. Consequently, for $T\le T(R_1)$, the mapping $\mathcal{L}^0_T\circ\mathcal{S}$ admits a unique fixed point on $B_{E^0_{s,T}\times X_{f,T}}(M_0)$, denoted $(\overline{\zeta},v,q)$. We fix now $T_0 = T(R_1)/2$ and we have existence of a strong solution $(\eta, v, q) = (\overline{\zeta}+\eta^0, v, q)$ to $(\mathrm{FS})_{ref}$ on $(0, T_0)$. As expected, T_0 depends only on:

$$\|v^0\|_{H^1_\sharp(\mathcal{F}^0)} + \|\eta^0\|_{H^2_\sharp(0,L)} + \|\sqrt{\alpha}\,\eta^0\|_{H^3_\sharp(0,L)} + \|\dot\eta^0\|_{H^1_\sharp(0,L)} + \|\sqrt{\delta}\,\dot\eta^0\|_{H^2_\sharp(0,L)} + \|(1+\eta^0)^{-1}\|_{L^\infty_\sharp(0,L)},$$

By restriction, $(\overline{\zeta}, v, q)$ is the unique fixed-point of $\mathcal{L}^0_T \circ \mathcal{S}$ on $B_{E^0_{s,T}\times X_{f,T}}(M_0)$ for $T \le T_0$. Conversely, assume that $(\tilde\eta, \tilde v, \tilde q)$ is a strong solution to $(\mathrm{FS})_{ref}$ on $(0, T_0)$. Then, for some $T_+ \le T < 1$ we have that $(\tilde\eta - \eta^0, \tilde v, \tilde q) \in B_{E^0_{s,T_+}\times X_{f,T_+}}(M_0)$ and is a fixed point of $\mathcal{L}^0_{T_+} \circ \mathcal{S}$. By uniqueness of the fixed point, we have that $(\tilde\eta, \tilde v, \tilde q) = (\eta, v, q)$ on $(0, T_+)$ and we are in position to initiate a continuation argument in order to prove that $(\tilde\eta, \tilde v, \tilde q) = (\eta, v, q)$ on $(0, T_0)$.

Finally, by interpolating the regularity $\eta \in L^\infty(0, T_0; H^2_\sharp(0, L))$ and $\eta \in W^{1,\infty}(0, T_0; H^1_\sharp(0, L))$ with $\eta(\cdot, 0) = \eta^0$, we obtain that:

$$\|\eta - \eta^0\|_{L^\infty(0,T_0;H^{7/4}(0,L))} \le T_0^{1/4} M_0.$$

Up to take T_0 smaller (but depending again only on R_1 and $\mathcal{K}$) we may reach the condition:

$$\|\eta - \eta^0\|_{L^\infty(0,T_0;H^{7/4}(0,L))} \le \mathcal{K}.$$

Appendix A. Proof of Proposition 4.2

In this section, we prove Proposition 4.2. We recall that the purposes of this proposition are twofold. First, we consider data (η, v, q) such that

- $\eta = \eta^0 + \overline{\zeta}$ where $1 + \eta^0 \in H^2_\sharp(0, L)$ is bounded from below by a positive constant and $\overline{\zeta} \in L^\infty(0, T; H^2_\sharp(0, L)) \cap W^{1,\infty}(0, T; H^1_\sharp(0, L))$ with:
$$\|\eta^0\|_{H^2_\sharp(0,L)} + \|(1+\eta^0)^{-1}\|_{L^\infty_\sharp(0,L)} \le R_1,$$
$$\|\overline{\zeta}\|_{L^\infty(0,T;H^2_\sharp(0,L))} + \|\overline{\zeta}\|_{W^{1,\infty}(0,T;H^1_\sharp(0,L))} \le M.$$
- $v \in H^1(0, T; L^2_\sharp(\Omega_0)) \cap C([0, T]; H^1_\sharp(0, L)) \cap L^2(0, T; H^2_\sharp(\Omega_0))$ with:
$$\|\partial_t v\|_{L^2(0,T;L^2_\sharp(\Omega_0))} + \|v\|_{L^\infty(0,T;H^1_\sharp(\Omega_0))} + \|v\|_{L^2(0,T;H^2_\sharp(\Omega_0))} \le M.$$
- $q \in L^2(0, T; H^1_\sharp(\Omega_0))$ with:
$$\|q\|_{L^2(0,T;H^1_\sharp(\Omega_0))} \le M.$$

We prove that the following formulas:

$$\begin{aligned} f[v, \eta] &= \rho_f(\det\nabla\chi_{\eta^0} - \det\nabla\chi_\eta)\partial_t v - \rho_f((v - \partial_t\chi_\eta)\cdot(B_\eta\nabla))v,\\ g[v, \eta] &= \mathrm{div}\,((B^\top_{\eta^0} - B^\top_\eta)v),\\ h[v, q, \eta] &= -\mu((A_{\eta^0} - A_\eta)\nabla)v + q(B_{\eta^0} - B_\eta),\end{aligned}$$

make sense with:

$$f \in L^2(0, T; L^2_\sharp(\Omega_0)), \quad h \in L^2(0, T; H^1_\sharp(\Omega_0)), \quad g \in L^2(0, T; H^1_\sharp(\Omega_0)) \cap H^1_{0,0}(0, T; (H^1_\sharp(\Omega_0))')$$

satisfying furhtermore the estimates of Proposition 4.2.

Second, we consider two sets of datas (η_1, v_1, q_1) and (η_2, v_2, q_2) satisfying the items above and we want to control the difference between the respective images (f_1, g_1, h_1) and (f_2, g_2, h_2) with obvious notations. We recall that we are interested in proving such properties in the spaces mentioned above which are summarized by $(\overline{\zeta}, v, q) \in E^0_{s,T} \times X_{f,T}$ for the data and $(f, g, h) \in S_T$ for the image. We recall also that, since $\overline{\zeta} = 0$ initially, we have the interpolation inequality

$$\begin{aligned}\|\overline{\zeta}\|_{C([0,T];H^{7/4}_\sharp(\Omega_0))} &\le T^{\frac14}\|\overline{\zeta}\|^{3/4}_{L^\infty(0,T;H^2_\sharp(\Omega_0))}\|\overline{\zeta}\|^{1/4}_{W^{1,\infty}(0,T;H^1_\sharp(\Omega_0))}\\ &\le T^{\frac14}\|\overline{\zeta}\|_{E^0_{s,T}}.\end{aligned}$$

We use this inequality extensively below without mention.

Before analyzing f, g, h we first consider the properties of the change of variable χ_η. We recall that

$$\begin{aligned}\chi_\eta(x,y,t) &= \chi^1_{\eta^0}(x,y) + \mathcal{R}_{\overline{\zeta}}(x,y,t)\\ &= \begin{pmatrix} x \\ (1+\eta^0(x))y\end{pmatrix} + \begin{pmatrix} 0 \\ \mathcal{R}^2\overline{\zeta}(x,y,t)\end{pmatrix},\end{aligned}$$

where $\mathcal{R}^2$ is a (continuous linear) lifting $H^{7/4}_\sharp(0,L) \to H^{9/4}_\sharp(\Omega_0)$ of the boundary condition:

$$\mathcal{R}^2\overline{\zeta}(x,1,t) = \overline{\zeta}(x,t), \qquad \mathcal{R}^2\overline{\zeta}(x,0,t) = 0, \qquad \text{on } (0,L).$$

Consequently, we have that

$$\nabla\chi_\eta(x,y,t) = \begin{pmatrix} 1 & 0 \\ y\partial_x\eta^0 + \partial_x\mathcal{R}^2\overline{\zeta} & (1+\eta^0) + \partial_y\mathcal{R}^2\overline{\zeta}\end{pmatrix}.$$

In these formula we remark that

$$\partial_x\eta^0 \in H^1_\sharp(0,L) \subset C_\sharp(0,L), \qquad \nabla\mathcal{R}^2\overline{\zeta} \in H^{5/4}_\sharp(\Omega_0) \subset C_\sharp(\Omega_0). \tag{A.1}$$

We infer from the previous remark and formula that:

$$\begin{aligned}&\nabla\chi_\eta \in C([0,T]; H^1_\sharp(\Omega_0)) \cap C([0,T]; C_\sharp(\Omega_0)),\\ &\|\nabla\chi_{\eta^0}\|_{H^1_\sharp(\Omega_0)} + \|\nabla\chi_{\eta^0}\|_{C_\sharp(\Omega_0)} \le C(R_1),\\ &\|\nabla\chi_\eta - \nabla\chi_{\eta^0}\|_{C([0,T];H^1_\sharp(\Omega_0))} + \|\nabla\chi_\eta - \nabla\chi_{\eta^0}\|_{C([0,T];C_\sharp(\Omega_0))} \le CT^{\frac14}\|\overline{\zeta}\|_{E^0_{s,T}},\\ &\|\nabla\chi_{\eta_1} - \nabla\chi_{\eta_2}\|_{C([0,T];H^1_\sharp(\Omega_0))} + \|\nabla\chi_{\eta_1} - \nabla\chi_{\eta_2}\|_{C([0,T];C_\sharp(\Omega_0))} \le CT^{\frac14}\|\overline{\zeta}_1 - \overline{\zeta}_2\|_{E^0_{s,T}}.\end{aligned}$$

Since (up to signs) the matrix $B_\eta := \operatorname{cof}\nabla\chi_\eta$ is obtained by reordering the components of $\nabla\chi_\eta$ we also have that:

$$B_\eta \in C([0,T]; H^1_\sharp(\Omega_0)) \cap C([0,T]; C_\sharp(\Omega_0)), \tag{A.2}$$

$$\|B_{\eta^0}\|_{H^1_\sharp(\Omega_0)} + \|B_{\eta^0}\|_{C_\sharp(\Omega_0)} \le C(R_1), \tag{A.3}$$

$$\|B_\eta - B_{\eta^0}\|_{C([0,T];H^1_\sharp(\Omega_0))} + \|B_\eta - B_{\eta^0}\|_{C([0,T];C_\sharp(\Omega_0))} \le CT^{\frac14}\|\overline{\zeta}\|_{E^0_{s,T}}, \tag{A.4}$$

$$\|B_{\eta_1} - B_{\eta_2}\|_{C([0,T];H^1_\sharp(\Omega_0))} + \|B_{\eta_1} - B_{\eta_2}\|_{C([0,T];C_\sharp(\Omega_0))} \le CT^{\frac14}\|\overline{\zeta}_1 - \overline{\zeta}_2\|_{E^0_{s,T}}. \tag{A.5}$$

Concerning time-derivatives, as η^0 does not depend on time, we note that $\partial_t\chi_\eta = (0, \partial_t\mathcal{R}^2\overline{\zeta})$ which is the lifting of $\partial_t\overline{\zeta} \in L^\infty(0,T; H^1_\sharp(\Omega_0))$. Thanks to the smoothing properties of the lifting operator, we have then that $\partial_t\chi_\eta$ is $L^\infty(0,T; H^{3/2}_\sharp(\Omega_0))$ which enables to state:

$$\partial_t\chi_\eta \in L^\infty(0,T; H^1_\sharp(\Omega_0)) \cap L^\infty(0,T; L^\infty_\sharp(\Omega_0)), \tag{A.6}$$

$$\|\partial_t\chi_\eta\|_{L^\infty(0,T;H^1_\sharp(\Omega_0))} + \|\partial_t\chi_\eta\|_{L^\infty(0,T;L^\infty_\sharp(\Omega_0))} \le C\|\overline{\zeta}\|_{E^0_{s,T}}, \tag{A.7}$$

$$\|\partial_t\chi_{\eta_1} - \partial_t\chi_{\eta_2}\|_{L^\infty(0,T;H^1_\sharp(\Omega_0))} + \|\partial_t\chi_{\eta_1} - \partial_t\chi_{\eta_2}\|_{L^\infty(0,T;L^\infty_\sharp(\Omega_0))} \le C\|\overline{\zeta}_1 - \overline{\zeta}_2\|_{E^0_{s,T}}, \tag{A.8}$$

and, remarking also that $\partial_t B_\eta = B_{\partial_t\eta}$, we apply the embedding $H^{1/2}_\sharp(\Omega_0) \subset L^4_\sharp(\Omega_0)$ yielding:

$$\partial_t B_\eta \in L^\infty(0,T; L^4_\sharp(\Omega_0)), \tag{A.9}$$

$$\|\partial_t B_\eta\|_{L^\infty(0,T;L^4_\sharp(\Omega_0))} \le C\|\overline{\zeta}\|_{E^0_{s,T}}, \tag{A.10}$$

$$\|\partial_t B_{\eta_1} - \partial_t B_{\eta_2}\|_{L^\infty(0,T;L^4_\sharp(\Omega_0))} \le C\|\overline{\zeta}_1 - \overline{\zeta}_2\|_{E^0_{s,T}}. \tag{A.11}$$

We proceed with the computations of nonlinear quantities. To this end, we note that (A.1), yields that all the components of $\nabla\chi_\eta$ belong to $C_\sharp(\Omega_0) \cap H^1_\sharp(\Omega_0)$ which is an algebra. This remark is crucial in the following computations and is used without mention. We infer directly from this remark and the explicit formula for $\nabla\chi_\eta$ that:

$$\det\nabla\chi_\eta \in C([0,T]; H^1_\sharp(\Omega_0)) \cap C([0,T]; C_\sharp(\Omega_0)), \tag{A.12}$$

$$\|\det\nabla\chi_{\eta^0}\|_{H^1_\sharp(\Omega_0)} + \|\det\nabla\chi_{\eta^0}\|_{C_\sharp(\Omega_0)} \le C(R_1), \tag{A.13}$$

(A.14) $$\|\det\nabla\chi_\eta-\det\nabla\chi_{\eta^0}\|_{C([0,T];H^1_\sharp(\Omega_0))}+\|\det\nabla\chi_\eta-\det\nabla\chi_{\eta^0}\|_{C([0,T];C_\sharp(\Omega_0))}\leq C(R_1)T^{\frac14}\|\overline{\zeta}\|_{E^0_{s,T}},$$

(A.15) $$\|\det\nabla\chi_{\eta_1}-\det\nabla\chi_{\eta_2}\|_{C([0,T];H^1_\sharp(\Omega_0))}+\|\det\nabla\chi_{\eta_1}-\det\nabla\chi_{\eta_2}\|_{C([0,T];C_\sharp(\Omega_0))}\leq C(R_1)T^{\frac14}\|\overline{\zeta}_1-\overline{\zeta}_2\|_{E^0_{s,T}}.$$

By construction, we note that (A.13) reads more precisely: $\det\nabla\chi_{\eta^0}=(1+\eta^0)\in C(\overline{\Omega}_0)$. So, we have:

$$\frac{1}{C(R_1)}\leq\det\nabla\chi_{\eta^0}\leq C(R_1)\text{ on }\overline{\Omega}_0.$$

Adding that the inequality (A.14) implies

$$\|\det\nabla\chi_\eta-\det\nabla\chi_{\eta^0}\|_{L^\infty(0,T;L^\infty_\sharp(\Omega_0))}\leq C(R_1)T^{\frac14}M$$

we obtain that we can restrict to $T\leq T(R_1,M)$ in order that:

$$\frac{1}{2C(R_1)}\leq\det\nabla\chi_\eta\leq 2C(R_1)\quad\text{on }\Omega_0\times(0,T).$$

From now on this value of $T(R_1,M)$ is fixed and we assume $T\leq T(R_1,M)$. A first consequence of this last inequality is that χ_η remains a C^1-diffeomorphism from Ω_0 onto its image for $t\leq T$.

Finally, we have that the formula:

$$A_\eta=\frac{1}{\det\nabla\chi_\eta}B_\eta^\top B_\eta$$

satisfies:

(A.16) $$A_\eta\in C([0,T];H^1_\sharp(\Omega_0))\cap C([0,T];C_\sharp(\Omega_0)),$$
$$\|A_{\eta^0}\|_{H^1_\sharp(\Omega_0)}+\|A_{\eta^0}\|_{C_\sharp(\Omega_0)}\leq C(R_1),$$

(A.17) $$\|A_\eta-A_{\eta^0}\|_{C([0,T];H^1_\sharp(\Omega_0))}+\|A_\eta-A_{\eta^0}\|_{C([0,T];C_\sharp(\Omega_0))}\leq C(R_1,M)T^{\frac14}\|\overline{\zeta}\|_{E^0_{s,T}},$$

(A.18) $$\|A_{\eta_1}-A_{\eta_2}\|_{C([0,T];H^1_\sharp(\Omega_0))}+\|A_{\eta_1}-A_{\eta_2}\|_{C([0,T];C_\sharp(\Omega_0))}\leq C(R_1,M)T^{\frac14}\|\overline{\zeta}_1-\overline{\zeta}_2\|_{E^0_{s,T}}.$$

Source term f. We remind that we want to prove that $f\in L^2(0,T;L^2_\sharp(\Omega_0))$ and compute lipschitz estimate with respect to the data $(\overline{\zeta},v,q)$. To this end, we split f into $f^a-f^b+f^c$ with:

$$f^a=\rho_f(\det\nabla\chi_{\eta^0}-\det\nabla\chi_\eta)\partial_t v,\quad f^b=\rho_f v\cdot(B_\eta\nabla)v,\quad f^c=\rho_f\partial_t\chi_\eta\cdot(B_\eta\nabla)v.$$

For the first term, we note that:

$$\|f^a\|_{L^2(0,T;L^2_\sharp(\Omega_0))}\leq C(R_1,\rho_f)\|\det\nabla\chi_{\eta^0}-\det\nabla\chi_\eta\|_{L^\infty(0,T;L^\infty_\sharp(0,L))}\|\partial_t v\|_{L^2(0,T;L^2_\sharp(\Omega))}.$$

Recalling (A.12)-(A.13) yields then that f^a is well-defined. Since f^a is bilinear in "$\det\nabla\chi_{\eta^0}-\det\nabla\chi_\eta$" and "$\partial_t v$") we may apply then (A.14)-(A.15) to obtain that:

(A.19) $$\|f^a_1-f^a_2\|_{L^2(0,T;L^2_\sharp(\Omega_0))}\leq C(R_1,\rho_f)MT^{\frac14}\left(\|\overline{\zeta}_1-\overline{\zeta}_2\|_{E^0_{s,T}}+\|\partial_t v_1-\partial_t v_2\|_{L^2(0,T;L^2_\sharp(\Omega))}\right).$$

For the second term, we make repeated use of the interpolation inequality $H^1_\sharp(\Omega_0)\subset L^4_\sharp(\Omega_0)$:

$$\|w\|_{L^4_\sharp(\Omega_0)}\leq C\|w\|^{\frac12}_{L^2_\sharp(\Omega_0)}\|w\|^{\frac12}_{H^1_\sharp(\Omega_0)},\quad\forall\,w\in H^1_\sharp(\Omega_0).$$

We obtain:

$$\begin{aligned}\|f^b\|_{L^2(0,T;L^2_\sharp(\Omega_0))}&\leq\rho_f\|B_\eta\|_{L^\infty(0,T;L^\infty_\sharp(\Omega_0))}\|v\nabla v\|_{L^2(0,T;L^2_\sharp(\Omega))}\\&\leq C\rho_f\|B_\eta\|_{L^\infty(0,T;L^\infty_\sharp(\Omega_0))}\|v\|^{3/2}_{L^\infty(0,T;H^1_\sharp(\Omega_0))}\|v\|^{1/2}_{L^1(0,T;H^2_\sharp(\Omega_0))}\\&\leq C\rho_f T^{\frac14}\|B_\eta\|_{L^\infty(0,T;L^\infty_\sharp(\Omega_0))}\|v\|^{3/2}_{L^\infty(0,T;H^1_\sharp(\Omega_0))}\|v\|^{1/2}_{L^2(0,T;H^2_\sharp(\Omega_0))}.\end{aligned}$$

Identity (A.2) implies then that f^b is well defined. Noting the multilinearity of f^b, we conclude again, applying (A.3)-(A.4)-(A.5), that:

$$\text{(A.20)}\quad \|f_1^b - f_2^b\|_{L^2(0,T;L^2_\sharp(\Omega_0))} \leq C(M,R_1,\rho_f)T^{\frac14}\left(T^{\frac14}\|\overline{\zeta}_1-\overline{\zeta}_2\|_{E^0_{s,T}} + \|v_1-v_2\|_{L^\infty(0,T;H^1_\sharp(\Omega_0))} + \|v_1-v_2\|_{L^2(0,T;H^2_\sharp(\Omega_0))}\right).$$

Finally, we have:

$$\begin{aligned}\|f^c\|_{L^2(0,T;L^2_\sharp(\Omega_0))} &\leq \rho_f\|\partial_t\chi_\eta\|_{L^\infty(0,T;L^\infty_\sharp(\Omega_0))}\|B_\eta\|_{L^\infty(0,T;L^\infty_\sharp(\Omega_0))}\|\nabla v\|_{L^2(0,T;L^2_\sharp(\Omega))}\\ &\leq C\rho_f T^{\frac12}\|\partial_t\chi_\eta\|_{L^\infty(0,T;L^\infty_\sharp(\Omega_0))}\|B_\eta\|_{L^\infty(0,T;L^\infty_\sharp(\Omega_0))}\|v\|_{L^\infty(0,T;H^1_\sharp(\Omega_0))}.\end{aligned}$$

Hence (A.2) with (A.6) imply that f^c is well-defined and we apply (A.3)-(A.4)-(A.5) with (A.7)-(A.8) to prove:

$$\text{(A.21)}\quad \|f_1^c - f_2^c\|_{L^2(0,T;L^2_\sharp(\Omega_0))} \leq C(M,R_1,\rho_f)T^{\frac12}\left(\|\overline{\zeta}_1-\overline{\zeta}_2\|_{E^0_{s,T}} + \|v_1-v_2\|_{L^\infty(0,T;H^1_\sharp(\Omega_0))}\right).$$

Finally, we have indeed that f is well-defined and combining (A.19)-(A.20)-(A.21), we obtain the lipschitz estimate (we remind that $T < T(R_1, M)$):

$$\|f_1 - f_2\|_{L^2(0,T;L^2_\sharp(\Omega_0))} \leq C(M,R_1,\rho_f)T^{\frac14}\left(\|\overline{\zeta}_1-\overline{\zeta}_2\|_{E^0_{s,T}} + \|(v_1,q_1)-(v_2,q_2)\|_{X_{f,T}}\right).$$

Source term h. We remind that we want to prove that $h \in L^2(0,T;H^1_\sharp(\Omega_0))$ and to compute lipschitz estimate with respect to the data $(\overline{\zeta}, v, q)$. To this end we note split again $h = h^a + h^b$ with

$$h^a = (A_\eta - A_{\eta^0})\nabla v, \qquad h^b = (B_{\eta^0} - B_\eta)q.$$

For the first term, we apply (A.16) and bound:

$$\begin{aligned}\|h^a\|_{L^2(0,T;H^1_\sharp(\Omega_0))} \leq{}& \|A_\eta - A_{\eta^0}\|_{L^\infty(0,T;L^\infty_\sharp(0,L))}\|v\|_{L^2(0,T;H^2_\sharp(\Omega_0))}\\ &+ T^{\frac12}\|A_\eta - A_{\eta^0}\|_{L^\infty(0,T;H^1_\sharp(0,L))}\|v\|_{L^\infty(0,T;H^1_\sharp(\Omega_0))}.\end{aligned}$$

This yields that h^a is well-defined. We may then use that h^a is a multilinear combination of $A_\eta - A_{\eta^0}$ and ∇v in order to apply (A.17)-(A.18). This yields:

$$\begin{aligned}\|h_1^a - h_2^a\|_{L^2(0,T;H^1_\sharp(\Omega_0))} \leq{}& C(M,R_1)T^{\frac14}(\|\overline{\zeta}_1-\overline{\zeta}_2\|_{E^0_{s,T}}\\ &+ \|v_1-v_2\|_{L^2(0,T;H^2_\sharp(\Omega_0))} + \|v_1-v_2\|_{L^\infty(0,T;H^1_\sharp(\Omega_0))}).\end{aligned}$$

Similarly, we refer to (A.2)-(A.3) to bound:

$$\|h^b\|_{L^2(0,T;H^1_\sharp(\Omega_0))} \leq \left(\|B_\eta - B_{\eta^0}\|_{L^\infty(0,T;L^\infty_\sharp(\Omega_0))} + \|B_\eta - B_{\eta^0}\|_{L^\infty(0,T;H^1_\sharp(\Omega_0))}\right)\|q\|_{L^2(0,T;H^1_\sharp(\Omega_0))}.$$

and, applying (A.4)-(A.5):

$$\|h_1^b - h_2^b\|_{L^2(0,T;H^1_\sharp(\Omega_0))} \leq C(M)T^{\frac14}\left(\|\overline{\zeta}_1-\overline{\zeta}_2\|_{E^0_{s,T}} + \|q_1-q_2\|_{L^2(0,T;H^1_\sharp(\Omega_0))}\right).$$

Consequently, we have that h is well-defined and the lipschitz estimate:

$$\|h_1 - h_2\|_{L^2(0,T;H^1_\sharp(\Omega_0))} \leq C(M,R_1)T^{\frac14}\left(\|\overline{\zeta}_1-\overline{\zeta}_2\|_{E^0_{s,T}} + \|(v_1,q_1)-(v_2,q_2)\|_{X_{f,T}}\right).$$

Source term g. Finally, we want to prove that $g \in L^2(0,T;H^1_\sharp(\Omega_0)) \cap H^1_{0,0}(0,T;(H^1_\sharp(\Omega_0))')$ and obtain Lipschitz estimates. We recall that, applying Piolà formula, we have:

$$g = \operatorname{div}((B_\eta^\top - B_{\eta^0}^\top)v) = (B_\eta^\top - B_{\eta^0}^\top) : \nabla v.$$

Hence, we apply (A.2) to bound:

$$\|g\|_{L^2(0,T;H^1_\sharp(\Omega_0))} \leq \left(\|B_\eta - B_{\eta^0}\|_{L^\infty(0,T;L^\infty_\sharp(\Omega_0))} + \|B_\eta - B_{\eta^0}\|_{L^\infty(0,T;H^1_\sharp(\Omega_0))}\right)\|v\|_{L^2(0,T;H^2_\sharp(\Omega_0))}$$

and also (A.4)-(A.5) to compute the lipschitz estimate:

$$\|g_1 - g_2\|_{L^2(0,T;H^1_\sharp(\Omega_0))} \leq C(M,R_1)T^{\frac14}\left(\|\overline{\zeta}_1-\overline{\zeta}_2\|_{E^0_{s,T}} + \|v_1-v_2\|_{L^2(0,T;H^2_\sharp(\Omega_0))}\right).$$

Then, we note that $B_\eta \in C([0,T];H^1_\sharp(\Omega_0))$ and $v \in C([0,T];H^1_\sharp(\Omega_0))$ so that $g \in C([0,T];L^2_\sharp(\Omega_0)) \subset C([0,T];(H^1_\sharp(\Omega_0))')$ with (since $B_{\eta(\cdot,0)} = B_{\eta^0}$): $g(\cdot,0)=0$.

Furthermore, we recall that v vanishes on $y=0$ and is directed along e_2 on $y=1$. We recall also that Nanson formula yields:

$$((B_\eta^\top - B_{\eta^0}^\top)e_2)\cdot e_2 = 0, \qquad \text{on } y=1.$$

Consequently, for arbitrary $w \in H^1_\sharp(\Omega_0)$ and $\overline{\zeta} \in C^\infty_c(0,T)$ we have:

$$\begin{aligned}\int_0^T\int_{\Omega_0} gw\partial_t\overline{\zeta} &= \int_0^T\int_{\Omega_0} \mathrm{div}((B_\eta^\top - B_{\eta^0}^\top)v)w\partial_t\overline{\zeta}\\ &= -\int_0^T\int_{\Omega_0}((B_\eta^\top - B_{\eta^0}^\top)v)\cdot\nabla w\partial_t\overline{\zeta}\\ &= \int_0^T\int_{\Omega_0}(\partial_t B_\eta v + (((B_\eta^\top - B_{\eta^0}^\top)\partial_t v)\cdot\nabla w\overline{\zeta}.\end{aligned}$$

Consequently, we have that

$$\langle\partial_t g, w\rangle_{(H^1_\sharp(\Omega_0))',H^1_\sharp(\Omega_0)} = \int_{\Omega_0}(\partial_t B_\eta v + (((B_\eta^\top - B_{\eta^0}^\top)\partial_t v)\cdot\nabla w, \quad \forall\, w \in H^1_\sharp(\Omega_0), \quad \text{for a.e. } t\in(0,T),$$

and, there holds:

$$\begin{aligned}&\|\partial_t g\|_{L^2(0,T;(H^1_\sharp(\Omega_0))')}\\ &\le T^{\frac12}\|\partial_t B_\eta\|_{L^\infty(0,T;L^4_\sharp(\Omega_0))}\|v\|_{L^\infty(0,T;H^1_\sharp(\Omega_0))} + \|B_\eta - B_{\eta^0}\|_{L^\infty(0,T;L^\infty_\sharp(\Omega_0))}\|\partial_t v\|_{L^2(0,T;L^2_\sharp(\Omega_0))}.\end{aligned}$$

Thanks to (A.9) and (A.2) we have that $g \in H^1_{0,0}(0,T;(H^1(\Omega_0))')$. We may then conclude by recalling (A.10)-(A.11) and (A.3)-(A.4) that (for $T < T(R_1,M)$):

$$\begin{aligned}&\|g_1 - g_2\|_{H^1(0,T;(H^1_\sharp(\Omega_0))')}\\ &\le C(M,R_1)T^{\frac14}\left(\|\overline{\zeta}_1 - \overline{\zeta}_2\|_{L^\infty(0,T;L^4_\sharp(\Omega_0))} + \|v_1 - v_2\|_{L^\infty(0,T;H^1_\sharp(\Omega_0))} + \|\partial_t v_1 - \partial_t v_2\|_{L^2(0,T;L^2_\sharp(\Omega_0))}\right).\end{aligned}$$

Finally, we obtain that we have indeed $g \in L^2(0,T;H^1_\sharp(\Omega_0)) \cap H^1_{0i}(0,T;(H^1_\sharp(\Omega_0))')$ with the lipschitz estimate:

$$\begin{aligned}&\|g_1 - g_2\|_{L^2(0,T;H^1_\sharp(\Omega_0))} + \|g_1 - g_2\|_{H^1(0,T;(H^1_\sharp(\Omega_0))')}\\ &\qquad\qquad \le C(M,R_1)T^{\frac14}\left(\|\overline{\zeta}_1 - \overline{\zeta}_2\|_{E^0_{s,T}} + \|(v_1,q_1)-(v_2,q_2)\|_{X_{f,T}}\right).\end{aligned}$$

This ends the proof of Proposition 4.2.

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