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Research Article

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On improvements of Opial-type inequalities

Abstract: In the present paper, we establish some new Opial-type integral inequalities in two variables. The results in special cases yield some of the interrelated results on Godunova–Levin’s and Mitrinović–Pečarić’s inequalities. These results provide new estimates on inequalities of this type.

Keywords: Opial’s inequality, Hölder’s inequality, Jensen’s inequality

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1 Introduction

In 1960, Opial [15] established the following inequality:

Theorem 1.1. *Suppose $f \in C^1[0, h]$ satisfies $f(0) = f(h) = 0$ and $f(x) > 0$ for all $x \in (0, h)$. Then*

$$\int_0^h |f(x)f'(x)|dx \leq \frac{h}{4} \int_0^h (f'(x))^2 dx. \quad (1.1)$$

The Opial-type inequality was first established by Willett [16]:

Theorem 1.2. *Let $x(t)$ be absolutely continuous on $[0, a]$, and $x(0) = 0$. Then*

$$\int_0^a |x(t)x'(t)|dt \leq \frac{a}{2} \int_0^a |x'(t)|^2 dt. \quad (1.2)$$

A non-trivial generalization of Theorem 1.2 was established by Hua [12]:

Theorem 1.3. *Let $x(t)$ be absolutely continuous in $[0, a]$, and $x(0) = 0$. Further, let l be a positive integer. Then*

$$\int_0^a |x(t)x'(t)|dt \leq \frac{a^l}{l+1} \int_0^a |x'(t)|^{l+1} dt. \quad (1.3)$$

A sharper inequality was established by Godunova [9]:

Theorem 1.4. *Let $f(t)$ be a convex and increasing function on $[0, \infty)$ with $f(0) = 0$. Further, let $x(t)$ be absolutely continuous on $[0, \tau]$, and $x(\alpha) = 0$. Then, the following inequality holds:*

$$\int_\alpha^\tau f'(|x(t)|)|x'(t)|dt \leq f\left(\int_\alpha^\tau |x'(t)|dt\right). \quad (1.4)$$

Opial’s inequality and its generalizations, extensions and discretizations play an important role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations, see for example [1, 4–8, 10, 13, 17]. For Opial-type integral inequalities involving high-order partial derivatives, see [3, 18]. For an extensive survey on these inequalities, see [2]. Mitrinović and Pečarić [14] proved some new extensions of Opial-type inequalities. The aim of the present paper is to establish some Opial-type inequalities, which are some extensions of Godunova–Levin’s and Mitrinović–Pečarić inequalities.

2 Statement of the results

We shall extend some of the previous results for the functions which have an integral representation. For this, we say that a function $x(s, t)$ belongs to the class $U(y, K)$ if it can be represented in the form

$$x(s, t) = \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} K(s, t, \sigma, \tau) y(\sigma, \tau) d\sigma d\tau, \quad (s, t) \in [\alpha_1, \beta_1] \times [\alpha_2, \beta_2], \tag{2.1}$$

where $y(s, t)$ is a continuous function on $[\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$, and $K(s, t, \sigma, \tau)$ is an arbitrary non-negative kernel function defined on $[\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$ such that $x(s, t) > 0$ if $y(s, t) > 0$, $(s, t) \in [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$.

In particular, for $\lambda > 0$, we let

$$K(s, t, \sigma, \tau) = K_\lambda(t, s, \sigma, \tau) = \begin{cases} \frac{[(s-\alpha)+(t-\beta)]^{\lambda-1}}{\Gamma(\lambda)}, & s + t \geq \sigma + \tau, \\ 0, & s + t < \sigma + \tau. \end{cases}$$

Theorem 2.1. For $i = 1, 2, 3$, let $x_i(s, t) \in U(y_i, K)$, where $y_2(s, t) > 0$ for all $(s, t) \in [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$, let $p(s, t) > 0$ for all $(s, t) \in [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$, and let $f(x, y)$ be convex and increasing on $[0, \infty) \times [0, \infty)$. Then the following inequality holds:

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} p(s, t) f\left(\left|\frac{x_1(s, t)}{x_2(s, t)}\right|, \left|\frac{x_3(s, t)}{x_2(s, t)}\right|\right) ds dt \leq \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi(s, t) f\left(\left|\frac{y_1(s, t)}{y_2(s, t)}\right|, \left|\frac{y_3(s, t)}{y_2(s, t)}\right|\right) ds dt, \tag{2.2}$$

where

$$\phi(t, s) = y_2(s, t) \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \frac{p(\sigma, \tau) K(s, t, \sigma, \tau)}{x_2(\sigma, \tau)} d\sigma d\tau.$$

Remark 2.2. Let $x_i(s, t)$, $y_i(s, t)$ and $p(s, t)$ be reduced to $x_i(t)$, $y_i(t)$, $p(t)$, respectively, where $t \in (\alpha, \tau)$ and $i = 1, 2, 3$, with suitable modifications in Theorem 2.1. Then (2.2) becomes the following inequality established by Mitrinović and Pečarić [14]:

$$\int_{\alpha}^{\tau} p(t) f\left(\left|\frac{x_1(t)}{x_2(t)}\right|, \left|\frac{x_3(t)}{x_2(t)}\right|\right) dt \leq \int_{\alpha}^{\tau} \phi(t) f\left(\left|\frac{y_1(t)}{y_2(t)}\right|, \left|\frac{y_3(t)}{y_2(t)}\right|\right) dt, \tag{2.3}$$

where

$$\phi(t) = y_2(t) \int_{\alpha}^{\tau} \frac{p(s) K(t, s)}{x_2(s)} ds,$$

$y(t)$ is a continuous function on $[\alpha, \tau]$ and $K(t, s)$ is an arbitrary non-negative kernel defined on $[\alpha, \tau] \times [\alpha, \tau]$ such that $x(t) > 0$ if $y(t) > 0$, $t \in [\alpha, \tau]$.

Remark 2.3. Taking $K(s, t, \sigma, \tau) = K_\lambda(s, t, \sigma, \tau)$ in Theorem 2.1, (2.2) reduces to

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} p(s, t) f\left(\left|\frac{x_1(s, t)}{x_2(s, t)}\right|, \left|\frac{x_3(s, t)}{x_2(s, t)}\right|\right) ds dt \leq \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi(s, t) f\left(\left|\frac{y_1(s, t)}{y_2(s, t)}\right|, \left|\frac{y_3(s, t)}{y_2(s, t)}\right|\right) ds dt, \tag{2.4}$$

where

$$\phi(s, t) = y_2(s, t) \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \frac{[(s-\sigma)+(t-\tau)]^{\lambda-1}}{\Gamma(\lambda)} \frac{p(\sigma, \tau)}{x_2(\sigma, \tau)} d\sigma d\tau, \quad s + t \geq \sigma + \tau,$$

and $\phi(s, t) = 0$ if $s + t < \sigma + \tau$.

Let us change $K_\lambda(s, t, \sigma, \tau)$ to $K_\lambda(t, s)$. Namely, for $\lambda > 0$,

$$K_\lambda(t, s) = \begin{cases} \frac{(t-s)^{\lambda-1}}{\Gamma(\lambda)}, & s \leq t, \\ 0, & s > t. \end{cases}$$

Further, let $x_i(s, t)$, $y_i(s, t)$, $f(s, t)$ and $p(s, t)$ be reduced to $s_i(t)$, $y_i(t)$, $f(t)$ and $p(t)$, respectively, where $t \in (\alpha, \tau)$ and $i = 1, 2$. Taking $K(t, s) = K_\lambda(t, s)$ in (2.4), we reduce (2.4) to the result of Godunova and Levin [11].

Now, let $x(s, t) \in U(y, K)$, where $K(s, t, \sigma, \tau) = 0$ for $\sigma + \tau > s + t$. We shall say that such functions belong to the class $U_1(y, K)$. It is clear that in this case, (2.1) reduces to

$$x(s, t) = \int_{\alpha_1}^s \int_{\alpha_2}^t K(s, t, \sigma, \tau) y(\sigma, \tau) d\sigma d\tau. \tag{2.5}$$

Theorem 2.4. *Let the function $f(x)$ be differentiable on $[0, \infty)$ such that, for $v > 1$, the function $f(x^{1/v})$ is convex and $f(0) = 0$. Let $\frac{1}{\mu} + \frac{1}{v} = 1$, and let $x(s, t) \in U_1(y, K)$, where*

$$\left(\int_{\alpha_1}^s \int_{\alpha_2}^t (K(s, t, \sigma, \tau))^\mu d\sigma d\tau \right)^{1/\mu} \leq M.$$

Then

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} |x(s, t)|^{2(1-v)} \frac{\partial^2 f}{\partial s \partial t} (|x(s, t)|) |y(s, t)|^v ds dt \leq \frac{v^2}{M^{2v}} f \left(M \left(\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} |y(s, t)|^v ds dt \right)^{1/v} \right). \tag{2.6}$$

Remark 2.5. Let $x(s, t)$ and $y(s, t)$ be reduced to $x(t)$, $y(t)$, respectively, where $t \in (\alpha, \tau)$, with suitable modifications in Theorem 2.4. Then (2.6) becomes the following inequality:

$$\int_{\alpha}^{\tau} |x(t)|^{1-v} f'(|x(t)|) |y(t)|^v ds dt \leq \frac{v}{M^v} f \left(M \left(\int_{\alpha}^{\tau} |y(t)|^v dt \right)^{1/v} \right).$$

This is just a new result established by Mitrinović and Pečarić [14].

3 Proofs of the results

Proof of Theorem 2.1. From the hypotheses of Theorem 2.1, it turns out that

$$\begin{aligned} & \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} p(s, t) f \left(\left| \frac{x_1(s, t)}{x_2(s, t)} \right|, \left| \frac{x_3(s, t)}{x_2(s, t)} \right| \right) ds dt \\ &= \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} p(s, t) f \left(\left| \frac{\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} K(s, t, \sigma, \tau) y_2(\sigma, \tau) \frac{y_1(\sigma, \tau)}{y_2(\sigma, \tau)} d\sigma d\tau}{x_2(s, t)} \right|, \left| \frac{\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} K(s, t, \sigma, \tau) y_2(\sigma, \tau) \frac{y_3(\sigma, \tau)}{y_2(\sigma, \tau)} d\sigma d\tau}{x_2(s, t)} \right| \right) ds dt \\ &\leq \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} p(s, t) f \left(\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \frac{K(s, t, \sigma, \tau) y_2(\sigma, \tau)}{x_2(s, t)} \left| \frac{y_1(\sigma, \tau)}{y_2(\sigma, \tau)} \right| d\sigma d\tau, \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \frac{K(s, t, \sigma, \tau) y_2(\sigma, \tau)}{x_2(s, t)} \left| \frac{y_3(\sigma, \tau)}{y_2(\sigma, \tau)} \right| d\sigma d\tau \right) ds dt. \end{aligned}$$

By using Jensen’s integral inequality, we have

$$\begin{aligned} & \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} p(s, t) f \left(\left| \frac{x_1(s, t)}{x_2(s, t)} \right|, \left| \frac{x_3(s, t)}{x_2(s, t)} \right| \right) ds dt \leq \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} p(s, t) \left(\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \frac{K(s, t, \sigma, \tau) y_2(\sigma, \tau)}{x_2(s, t)} f \left(\left| \frac{y_1(\sigma, \tau)}{y_2(\sigma, \tau)} \right|, \left| \frac{y_3(\sigma, \tau)}{y_2(\sigma, \tau)} \right| \right) d\sigma d\tau \right) ds dt \\ &= \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} f \left(\left| \frac{y_1(\sigma, \tau)}{y_2(\sigma, \tau)} \right|, \left| \frac{y_3(\sigma, \tau)}{y_2(\sigma, \tau)} \right| \right) y_2(\sigma, \tau) \left(\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \frac{p(s, t) K(s, t, \sigma, \tau)}{x_2(s, t)} ds dt \right) d\sigma d\tau \\ &= \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi(\sigma, \tau) f \left(\left| \frac{y_1(\sigma, \tau)}{y_2(\sigma, \tau)} \right|, \left| \frac{y_3(\sigma, \tau)}{y_2(\sigma, \tau)} \right| \right) d\sigma d\tau, \end{aligned}$$

where

$$\phi(\sigma, \tau) = y_2(\sigma, \tau) \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \frac{p(s, t)K(s, t, \sigma, \tau)}{x_2(s, t)} ds dt.$$

Hence

$$\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} p(s, t) f\left(\left|\frac{x_1(s, t)}{x_2(s, t)}\right|, \left|\frac{x_3(s, t)}{x_2(s, t)}\right|\right) ds dt \leq \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \phi(s, t) f\left(\left|\frac{y_1(s, t)}{y_2(s, t)}\right|, \left|\frac{y_3(s, t)}{y_2(s, t)}\right|\right) ds dt,$$

where

$$\phi(s, t) = y_2(s, t) \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \frac{p(\sigma, \tau)K(s, t, \sigma, \tau)}{x_2(\sigma, \tau)} d\sigma d\tau.$$

This completes the proof. □

Proof of Theorem 2.4. From the hypotheses of Theorem 2.4 and in view of Hölder’s inequality, we obtain

$$\begin{aligned} |x(s, t)| &\leq \int_{\alpha_1}^s \int_{\alpha_2}^t K(s, t, \sigma, \tau) |y(\sigma, \tau)| d\sigma d\tau \\ &\leq \left(\int_{\alpha_1}^s \int_{\alpha_2}^t (K(s, t, \sigma, \tau))^\mu d\sigma d\tau \right)^{1/\mu} \left(\int_{\alpha_1}^s \int_{\alpha_2}^t |y(\sigma, \tau)|^v d\sigma d\tau \right)^{1/v} \\ &\leq M \left(\int_{\alpha_1}^s \int_{\alpha_2}^t |y(\sigma, \tau)|^v d\sigma d\tau \right)^{1/v}. \end{aligned}$$

Now, let

$$z(s, t) = \int_{\alpha_1}^s \int_{\alpha_2}^t |y(\sigma, \tau)|^v d\sigma d\tau.$$

Hence

$$\frac{\partial^2 z(s, t)}{\partial s \partial t} = |y(s, t)|^v.$$

Moreover, it is easy to see that

$$|x(s, t)| \leq M(z(s, t))^{1/v}.$$

Therefore

$$\begin{aligned} &\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} |x(s, t)|^{2(1-v)} \frac{\partial^2 f}{\partial s \partial t} (|x(s, t)|) |y(s, t)|^v ds dt \\ &\leq \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} M^{2(1-v)} (z(s, t))^{2(1/v-1)} \frac{\partial^2 f}{\partial s \partial t} (Mz(s, t)^{1/v}) \frac{\partial z(s, t)}{\partial s} \frac{\partial z(s, t)}{\partial t} ds dt \\ &= \frac{v^2}{M^{2v}} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \frac{\partial^2 f}{\partial s \partial t} (Mz(s, t)^{1/v}) \cdot \frac{M}{v} (z(s, t))^{1/v-1} \frac{\partial z(s, t)}{\partial s} \cdot \frac{M}{v} (z(s, t))^{1/v-1} \frac{\partial z(s, t)}{\partial t} ds dt \\ &= \frac{v^2}{M^{2v}} \int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} \frac{\partial^2 f}{\partial s \partial t} (Mz(s, t)^{1/v}) d(M(z(s, t))^{1/v})_s d(M(z(s, t))^{1/v})_t \\ &= \frac{v^2}{M^{2v}} f(M(z(\beta_1, \beta_2))^{1/v}) \\ &= \frac{v^2}{M^{2v}} f\left(M \left(\int_{\alpha_1}^{\beta_1} \int_{\alpha_2}^{\beta_2} |y(\sigma, \tau)|^v d\sigma d\tau \right)^{1/v}\right). \end{aligned}$$

This completes the proof. □

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References

- [1] R. P. Agarwal and V. Lakshmikantham, *Uniqueness and Nonuniqueness Criteria for Ordinary Differential Equations*, Ser. Real Anal. 6, World Scientific, River Edge, 1993.
- [2] R. P. Agarwal and P. Y. H. Pang, *Opial Inequalities with Applications in Differential and Difference Equations*, Math. Appl. 320, Kluwer Academic, Dordrecht, 1995.
- [3] R. P. Agarwal and P. Y. H. Pang, Sharp Opial-type inequalities in two variables, *Appl. Anal.* **56** (1995), no. 3–4, 227–242.
- [4] R. P. Agarwal and E. Thandapani, On some new integro-differential inequalities, *An. Ştiinţ. Univ. “Al. I. Cuza” Iaşi. Mat. (N.S.)* **28** (1982), no. 1, 123–126.
- [5] D. Bainov and P. Simeonov, *Integral Inequalities and Applications*, Math. Appl. (East Eur. Ser.) 57, Kluwer Academic, Dordrecht, 1992.
- [6] W.-S. Cheung, On Opial-type inequalities in two variables, *Aequationes Math.* **38** (1989), no. 2–3, 236–244.
- [7] W.-S. Cheung, Some generalized Opial-type inequalities, *J. Math. Anal. Appl.* **162** (1991), no. 2, 317–321.
- [8] K. M. Das, An inequality similar to Opial’s inequality, *Proc. Amer. Math. Soc.* **22** (1969), 258–261.
- [9] E. K. Godunova, Integral inequalities with derivatives and arbitrary convex functions (in Russian), *Moskov. Gos. Ped. Inst. Učen. Zap.* **460** (1972), 58–65.
- [10] E. K. Godunova and V. I. Levin, An inequality of Maroni (in Russian), *Mat. Zametki* **2** (1967), 221–224; translation in *Math. Notes* **2** (1967), 618–619.
- [11] E. K. Godunova and V. I. Levin, Certain integral inequalities that contain derivatives (in Russian), *Izv. Vysš. Učebn. Zaved. Matematika* **1969** (1969), no. 12(91), 20–24.
- [12] L. G. Hua, On an inequality of Opial, *Sci. Sinica* **14** (1965), 789–790.
- [13] J. D. Li, Opial-type integral inequalities involving several higher order derivatives, *J. Math. Anal. Appl.* **167** (1992), no. 1, 98–110.
- [14] D. S. Mitrinović and J. E. Pečarić, Generalizations of two inequalities of Godunova and Levin, *Bull. Pol. Acad. Sci. Math.* **36** (1988), no. 9–10, 645–648.
- [15] Z. Opial, Sur une inégalité, *Ann. Polon. Math.* **8** (1960), 29–32.
- [16] D. Willett, The existence-uniqueness theorem for an n th order linear ordinary differential equation, *Amer. Math. Monthly* **75** (1968), 174–178.
- [17] G. S. Yang, A note on an inequality similar to Opial inequality, *Tamkang J. Math.* **18** (1987), no. 4, 101–104.
- [18] C.-J. Zhao and W.-S. Cheung, Sharp integral inequalities involving high-order partial derivatives, *J. Inequal. Appl.* **2008** (2008), Article ID 571417.

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