On the Structure of BV Entropy Solutions for Hyperbolic Systems of Balance Laws with General Flux Function

(Article begins on next page)

On the Structure of BV Entropy Solutions
for Hyperbolic Systems of Balance Laws
with General Flux Function

FABIO ANCONA, LAURA CARAVENNA, ANDREA MARSON *
March 7, 2018

Abstract

The paper describes the qualitative structure of BV entropy solutions of a general strictly hyperbolic system of balance laws with characteristic fields either piecewise genuinely nonlinear or linearly degenerate. In particular, we provide an accurate description of the local and global wave-front structure of a BV solution generated by a fractional step scheme combined with a wave-front tracking algorithm. This extends the corresponding results in [7] for strictly hyperbolic systems of conservation laws.

Contents

---

*Dipartimento di Matematica 'Tullio Levi-Civita' Via Trieste, 63, 35121 - Padova, Italy;
e-mail: ancona@math.unipd.it, laura.caravenna@unipd.it, marson@math.unipd.it

1   Introduction

Consider the Cauchy problem for a general hyperbolic system of N quasilinear first order PDEs in one space dimension

$$u_t + A(u)u_x = g(t, x, u) \,, \tag{1.1}$$

$$u(0, x) = \overline{u}(x) \,. \tag{1.2}$$

Here the vector $u = u(t, x) = \big(u_1(t, x), \ldots, u_N(t, x)\big)$, and $A = A(u)$ is a smooth matrix-valued function defined on a domain $\Omega \subseteq \mathbb{R}^N$. Solution to (1.1)-(1.2) are considered as limits in L^1_{loc} of vanishing viscosity approximations

$$u^\varepsilon_t + A(u^\varepsilon)u^\varepsilon_x = g(t, x, u^\varepsilon) + \varepsilon u^\varepsilon_{xx} \tag{1.3}$$

as $\varepsilon \to 0+$. In case A is the jacobian matrix of a *flux function* $F : \Omega \to \mathbb{R}^N$, then (1.1) can be written as a system of balance laws, namely

$$u_t + F(u)_x = g(t, x, u) \,. \tag{1.4}$$

We assume that system (1.1) is strictly hyperbolic, i.e. that the matrix $A(u)$ has N real distinct eigenvalues

$$\lambda_1(u) < \cdots < \lambda_N(u) \qquad \forall\, u \,, \tag{1.5}$$

and we will denote by

$$r_1(u), \ldots, r_N(u) \,, \qquad l_1(u), \ldots, l_N(u) \tag{1.6}$$

corresponding bases of, respectively, right and left eigenvectors, normalized so that

$$|r_k(u)| \equiv 1 \,, \qquad \langle r_k(u), l_h(u) \rangle = \delta_{kh} \,, \tag{1.7}$$

where $\langle \cdot, \cdot \rangle$ stand for the usual scalar product in $\mathbb{R}^N$, and δ_{hk} is the usual Kronecker symbol. The limits of vanishing viscosity approximations to (1.4)-(1.2) turns out to be distributional solutions which are entropy admissible (see [10]). We consider the function $g : \mathbb{R} \times \Omega \to \mathbb{R}^N$ to be continuosly differentiable in the u variable and measurable in the x one. Moreover, we assume that

(G) the function g in (1.1) is continuous in t and Lipschitz continuous w.r.t. x and u, uniformly in t; moreover, there exists function $\alpha \in L^1(\mathbb{R})$ such that $|g_x(t, x, u)| \leq \alpha(x)$ for any t, u.

Regarding the assumptions on g, since the seminal papers [16, 26] and the first paper [13] on the well-posedness of the Cauchy problem, many papers appeared in the past years dealing with several existence results for first order hyperbolic inhomogeneous systems, both local and global in time, provided the initial datum $\overline{u}$ has suitably small total variation. See Subsection 1.1 below. Moreover, as long as one is interested in a local (both in time and space) existence result, the assumption $|g_x(t, x, u)| \leq \alpha(x)$ with $\alpha \in L^1(\mathbb{R})$ is not relevant.

Aim of this paper is to provide some preliminaries that will be used to prove that, for a.e. time t in the interval of existence of a solution $u = u(t, x)$ to (1.1)-(1.2), $u(t, \cdot)$ enjoys a SBV regularity, i.e. it is BV and $\partial_x u(t, \cdot)$ does not have a Cantor part. In order to pursue this result, in the present paper a couple of achievements are presented.

1. We briefly introduce a method to construct a piecewise constant approximate solution for a non conservative system (1.1), under the only regularity and strict hyperbolicity assumptions on the matrix A. The algorithm we introduce, that will lead to an existence result for (1.1)-(1.2) at least locally in time, follows the guidelines contained in [1] and [2] for inhomogeneous systems, and in [3] for systems without assumptions of genuine nonlinearity or linear degeneracy on the characteristic fields.

    Theorem 1. *Consider a strictly hyperbolic system of balance laws (1.1)-(1.2)-(1.5) and suppose Assumption (G) at Page 2 holds. Then there exists $\overline{\delta}, T > 0$ and a sub-domain $\mathfrak{D}$ of $\{\overline{v} \in L^1(\mathbb{R}; \mathbb{R}^N) \cap \mathrm{BV}(\mathbb{R}; \mathbb{R}^N) : \mathrm{Tot.Var.}(\overline{v}) < \overline{\delta}\}$ of initial data such that the following hold. Suppose u is the vanishing viscosity solution [10] of the Cauchy problem (1.1)-(1.2) with a given initial datum $\overline{u} \in \mathfrak{D}$. We construct a piecewise-constant, fractional step approximation u_ν of the vanishing viscosity solution u such that $u_\nu(t, \cdot)$ converges to $u(t, \cdot)$ in $L^1(\mathbb{R})$ for $0 \leq t < T$.*

    The precise statement is given in Theorems 11-14 below. We stress that

    - Our local in time result yields a piecewise constant approximation for two interesting physical models, see § 1.2. This presently was not available.
    - It is interesting to combine our local-in-time Theorem 1, or a vanishing-viscosity local-in-time version of the existence theorem in [10], with the works by Dafermos [15, 18, 17, 19] which allow to pass from local-in-time to global-in-time existence.

2. We define some measures related to the approximate solutions that eventually will converge weakly to $\partial_x u(t, \cdot)$. They will be fundamental for describing the qualitative structure of BV entropy solutions of a general strictly hyperbolic system of balance laws with characteristic fields either piecewise genuinely nonlinear or linearly degenerate. Details are available in § 1.3 and Theorem 5 below, extending the works [9, 8, 7] relative to the homogeneous system.

1.1 Few results available on balance laws

In the following, we sumarize a few results, and we refer to the original papers and the references therein for a complete treatment.

- In [2] the authors consider a system in conservation form (1.4) with each chracteristic field r_k genuinely nonlinear or linearly degenerate in the sense of Lax [23], and the source term g is assumed to depend only on u, and not on (t, x). They prove a couple of results.

  - A local existence theorem with the only assumptions that $g \in \mathcal{C}^2$ and $g(0) = 0$.
  - A global existence theorem assuming, besides the above assumptions on g, that the system is *diagonally dominant*, i.e., denoting by $R(u)$ the matrix whose columns are the vectors r_k, and letting

  $$G \doteq R^{-1}(0) \cdot D_u g(0) \cdot R(0) = (G_{ij})_{i,j=1,\dots,N}, \tag{1.8}$$

  the entries $G_{i,j}$ of G satisfy

  $$G_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{n} |G_{ij}| \leq -c \qquad \forall i = 1, \dots, n, \tag{1.9}$$

  for some positive constant c. Moreover, the authors provide a uniqueness result.

- [1] deals again with a system in conservation form with genuinely nonlinear and linearly degenerate characteristic fields. Moreover, the authors assume the system to be *non resonant*. i.e. that all the eigenvalues of $A(u) = DF(u)$ be bounded away from zero, i.e.

  $$|\lambda_i(u)| \geq c > 0 \qquad \forall i = 1, \dots, N, \tag{1.10}$$

  for some $c > 0$. Regarding the source term, it does not depend on t, so that $g = g(x, u)$, and, besides the regularity assumptions ($\mathcal{C}^2$ in u and measurable in x), the authors require the existence of a bounded L^1 function $\omega = \omega(x)$ such that

  $$|g(x, u)| + |D_u g(x, u)| \leq \omega(x) \qquad \forall x \in \mathbb{R}, \ \forall u \in \mathbb{R}^N. \tag{1.11}$$

  Under this assumptions, a global solution to a Cauchy problem for (1.4) is constructed by means of suitable front tracking approximations. Such a solution is proved to be unique.

- In [10] a global solution to (1.1)-(1.2) is constructed by means of vanishing viscosity approximations (1.3), following the approach contained in [6] for homogeneous systems. No assumptions are considered on the matrix $A = A(u)$, but the regularity and the strict hyperbolicity. Instead, the source term g is share the same assumptions in [2], and hence it enjoys a $\mathcal{C}^2$ regularity and it is diagonally dominant in the sense we explained above. In [11] the same author prove that the solution to a Cauchy problem as the one obtained in [10] is actually unique.

- In [15] and [18] two kinds of dissipative assumptions are considered, to weaken the one contained in [2].

  - In [15], instead of condition (1.9), it is simply required that $G_{ii} > 0$ for any $i = 1, \dots, N$. In order to obtain a weak entropy admissible solution to (1.4)-(1.2) globally defined in time, the authors assumes that a local solution exists fulfilling the estimate

  $$\int_{\mathbb{R}} u(t, x) \, dx \leq b \int_{\mathbb{R}} \overline{u}(x) \, dx, \tag{}$$

  for some $b > 0$ and for any $t \in [0, T[$ where the solution is defined.

  - In [18] system (1.4) is assumed to be endowed with a convex antrpy-entropy flux pair (η, q), and the matrix $D^2\eta(0)D_u g(0)$ is required to be positive definite. Then a global solution to (1.4)-(1.2) does exists, provided that the intial datum satisfies

  $$\int_{\mathbb{R}} (1 + |x|)^{2s} |\overline{u}(x)|^2 \, dx < \delta \tag{1.12}$$

  for some $s > 1$ and a sufficiently small $\delta > 0$. Here, a local solution does exist due to [16, Theorem 1], where a random choice method [22] is used.

We stress that in both [15] and [18] no assumption of genuine nonlinearity or linear degeneracy are made, while the source term g is assumed not to depend on x.

- In [17] and in the survey [19] the author deals with BV solution of inhomegenous systems endowed with convex entropy-entropy pair (η, q) satisfying a dissipative condition, and satisfying the Shizuta-Kawashima condition [31]

$$D_u g(0)\, \mathbf{r}_i(0) \neq \mathbf{0} \quad \forall i = 1, \ldots, n\,, \tag{1.13}$$

where g is independent on t and x, and $g(0) = 0$. Namely, the theorem is the following

Theorem 2 ([17, 19]). *Consider system* (1.4) *with g independent on (t, x), and assume it is in the form*

$$\begin{cases} V_t + K(V, W)_x = 0 \\ W_t + H(V, W)_x = C(V, W)W\,, \end{cases}$$

where $u = (V, W)$. Assume that there exists $V_0 \in \mathbb{R}$ such that the V-component of the initial datum $\overline{u} = (\overline{V}, \overline{W})$ satisfies

$$\int_{\mathbb{R}} \left(\overline{V}(x) - V_0 \right) dx = 0\,. \tag{1.14}$$

Moreover, assume that there exist a convex entropy-entropy pair (η, q) and positive constant a such that the dissipative condition

$$D\eta(u)\big[g(u) - g(V_0, 0)\big] \leq -a\big|g(u) - g(V_0, 0)\big|^2 \tag{1.15}$$

holds. Then, there exist $\delta_0, \sigma_0, \gamma, b, c_0, c_1 > 0$ such that, if

$$\mathrm{Tot.Var.}\overline{u} = \delta < \delta_0 \quad and \quad \int_{\mathbb{R}} (1 + x^2)\big(\big|\overline{V}(x) - V_0\big|^2 + \big|\overline{W}(x)\big|^2\big)\, dx = \sigma^2 < \sigma_0^2\,, \tag{1.16}$$

then the Cauchy problem (1.4)-(1.2) *has an admissible BV solution $u = u(t, x)$ on $[0, +\infty[\times\mathbb{R}$, and*

$$\int_{\mathbb{R}} \big(\big|\overline{V}(x) - V_0\big| + \big|\overline{W}(x)\big|\big)\, dx \leq b\sigma\,, \quad \mathrm{Tot.Var.}u(t, \cdot) \leq c_0\sigma + c_1\delta e^{-\gamma t} \quad \forall t \geq 0\,, \tag{1.17}$$

$$\int_{\mathbb{R}} \big(\big|\overline{V}(x) - V_0\big| + \big|\overline{W}(x)\big|\big)\, dx \to 0\,, \quad \mathrm{Tot.Var.}u(t, \cdot) \to 0 \quad as \quad t \to +\infty\,. \tag{1.18}$$

Again, in [17, 19] local in time solutions are provided thanks to [16, Theorem 1].

1.2   Motivating models and global in time solutions

Systems that do not satisfy the classical assumptions of genuine nonlinearity or linear degeneracy in the sense of Lax [23] may arise in several context. A first example is a system of balance laws arising in modelling elasticity,

$$\begin{cases} v_t - u_x = 0 \\ u_t - \sigma(v)_x = -\alpha u\,, \end{cases} \tag{1.19}$$

where the stress $\sigma = \sigma(v)$ satisfies $\sigma'(v) > 0$. Such a system has been diffusively studied (e.g., see [14, 21]). Its behavior resembles the p-system with dumping [14, 20], but its characteristic fields are not genuinely nonlinear, nor linearly degenerate. Indeed, the derivatives of the eigenvalues $\lambda_{1,2}(u, v) = \pm\sqrt{\sigma'(v)}$ of the jacobian matrix of the flux function $f(u, v) = (-u, -\sigma(v))$ along the corresponding right eigenvectors vanish whenever $\sigma''(v) = 0$. It can be easily seen that the Shizuta-Kawashima condition (1.13) is fulfilled, and that

$$\eta(v, u) = \int \sigma(v)\, dv + \frac{1}{2}u^2$$

is an entropy satisfying the dissipative condition (1.15), where $g(v, u) = (0, -\alpha u)$. Hence, once a local in time solution to a Cauchy problem for (1.19) has been provided, it can be prolonged to a global in time one by using Theorem 2.

Another 2×2 system of balance laws not fulfilling the classical Lax assumptions on characteristic fields is the generalized Cattaneo's model of heat conduction in high purity crystals [28, 29, 30],

$$\begin{cases} \rho e_t + q_x = 0 \\ (\alpha q)_t + \nu_x = -\dfrac{\nu'}{k}q\,. \end{cases} \tag{1.20}$$

In order to check that the assumptions of Theorem 2 are fulfilled, we rewrite (1.20) in terms of the conserved quantities e and $Q = \alpha q$:

$$\begin{cases} e_t + \left(\dfrac{Q}{\rho\alpha}\right)_x = 0 \\ Q_t + \nu_x = -\dfrac{\nu'}{\alpha k}Q. \end{cases} \tag{1.21}$$

Here, letting ϑ be the absolute temperature, we denote with $q = q(t,x)$ the heat flux, $e = e(\vartheta)$ the internal energy, ρ the density, which is assumed to be constant, $k = k(\vartheta)$ the heat conductivity, while $\alpha = \alpha(\vartheta)$ and $\nu = \nu(\vartheta)$ are constitutive functions. Regarding e, α, ν, up to a rescaling of the absolute temperature ϑ, we assume that [28]

$$e(\vartheta) = \vartheta^4, \qquad \alpha(\vartheta) = \frac{1}{\rho\vartheta U(\vartheta)\sqrt{e'(\vartheta)}}, \qquad \nu'(\vartheta) = \frac{U(\vartheta)}{\vartheta}\sqrt{e'(\vartheta)}, \tag{1.22}$$

where $U(\vartheta) > 0$ is the so called "second sound velocity", i.e. the velocity of small perturbations propagating into an equilibrium state. In the case studies contained in [28, 30] U takes the form

$$U(\vartheta) = \frac{1}{\sqrt{A + B\vartheta^n}}, \tag{1.23}$$

where the positive constants A and B and the exponent n depend on the material under observation. Moreover, following [12, (21)], for the heat conductivity $k = k(\vartheta)$ we can deduce the following expression

$$k(\vartheta) = \sqrt{\frac{e'(\vartheta)\nu'(\vartheta)}{\alpha(\vartheta)}}U(\vartheta). \tag{1.24}$$

A direct computation shows that (1.20) is piecewise genuinely nonlinear in the sense of Definition 4 below (see [29, 30]). Unfortunately (1.20) is only weakly diagonally dominant around an equilibrium state $(\bar{\vartheta}, 0)$, i.e. the entries of the corresponding matrix G at (1.8) do not satisfy (1.9), but

$$G_{ii} < 0, \qquad G_{ii} + \sum_{j \neq i}|G_{i,j}| = 0. $$

Theorem 11 below states the convergence of the algorithm described in §2 to a local in time solution to a Cauchy problem (1.1)-(1.2), which a Cauchy problem for (1.20) is a particular case of. In order to extend a local solution to a global one, it can be used the method described in [17, 19]. Now we need to check that the Shizuta-Kawashima condition (1.13) and the entropy dissipation condition (1.15) are fulfilled.

- *Shizuta-Kawashima condition.* A basis of right eigenvectors of the jacobian matrix of the flux function

$$F(e, Q) = (Q/\rho\alpha, \nu)$$

  is given by

$$\mathbf{r}_1(e, Q) = \left(-\frac{2\sqrt{\vartheta^3\alpha'^2 Q^2 + \rho\alpha^3\nu'} + \alpha'Q}{8\rho\vartheta^3\alpha^2\nu'}, 1\right),$$

$$\mathbf{r}_2(e, Q) = \left(\frac{2\sqrt{\vartheta^3\alpha'^2 Q^2 + \rho\alpha^3\nu'} - \alpha'Q}{8\rho\vartheta^3\alpha^2\nu'}, 1\right).$$

  Let $(e, Q) = (\bar{e}, 0)$ be an equilibrium of (1.21), corresponding to a constant temperature $\bar{\vartheta} = \sqrt[4]{\bar{e}}$. The Shizuta-Kawashima condition (1.13)

$$D_{e,Q}(0, -\nu'Q/\alpha k)\Big|_{(e,Q)=(\bar{e},0)} \mathbf{r}_i(\bar{e}, 0) \neq \mathbf{0} \quad i = 1, 2,$$

  reduces to

$$\frac{\nu'(\bar{\vartheta})}{\alpha(\bar{\vartheta})k(\bar{\vartheta})} \neq 0.$$

- *Entropy dissipation condition.* An entropy for system (1.21) is given by (see [28, (2.11)-(2.12)])

$$\eta(e, Q) = -\frac{4}{3}\rho e^{3/4} + \frac{1}{2\gamma}Q^2.$$

  Hence, the entropy dissipation condition [19, (2.7)] is written

$$D_{e,Q}\eta(e, Q) \cdot (0, -\nu'Q/\alpha k) \leq -a\frac{\nu'^2}{\alpha^2 k^2}Q^2$$
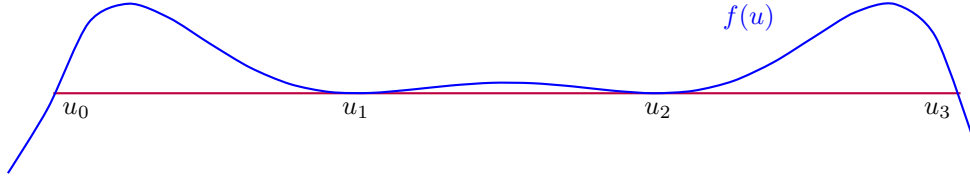
Figure 1: The wave $[u_0, u_3]$ is a composition of the simple waves $[u_0, u_1]$, $[u_1, u_2]$, $[u_2, u_3]$

for some $a > 0$, and (e, Q) in a neighbourhood of the equilibrium point $(\bar{e}, 0)$, so that

$$\frac{1}{\gamma} \geq a \frac{\nu'}{\alpha k} \tag{1.25}$$

must holds in a neighbourhood of $(\bar{e}, 0)$. By using the constitutive relationships (1.22)-(1.24), we get that (1.25) holds true once $a \leq 1/\gamma\sqrt{\rho}$.

It follows that we can apply Theorem 2, and, if an intial datum for (1.21) fulfills assumptions (1.14) and (1.16), then a global in time solution to (1.21) exists, and it satisfies (1.17). In particular, using the construction of §§ 2 and 3, we can provide a piecewise constant approximate solution to (1.21) globally defined in time.

1.3 Statement on the structure of solutions to balance laws

We describe in this paper the global local structure of solutions of balance laws whose characteristic fields are either piecewise-genuinely nonlinear or linearly degenerate, for Lipschitz continuous sources $g = g(u)$. The theorem extends the works [9, 8, 7] relative to the homogeneous system. One of the expected application is to extend to this setting the SBV and SBV-like regularity of solutions in a forthcoming paper.

Definition 3. *The ith-characteristic field is* linearly degenerate *if* $\quad \nabla \lambda_i(u) \cdot r_i(u) \equiv 0$.

Definition 4. *The ith-characteristic field is* piecewise genuinely nonlinear *if the set*

$$Z_i := \{u \quad : \quad \nabla \lambda_i(u) \cdot r_i(u) = 0\}$$

is the union of $(N-1)$-dimensional disjoint manifolds Z_i^j, for $j = 1, \ldots, J_i$, which are transversal to the field $r_i(u)$ and such that each i-rarefaction curve $R_i[u_0]$ crosses all the Z_i^j.

Let $S_i[u^-](s)$ denote the *i*-**th Hugoniot curve** issuing from u^-; we denote by $\sigma_i[u^-](s)$ the corresponding Rankine-Hugoniot **speed** of the *i*-th discontinuity $[u^-, S_i[u^-](s)]$: σ_i and S_i are defined by by the Implicit Function Theorem by the relation

$$\sigma_i[u^-](s) \left(S_i[u^-](s) - u^- \right) = f \left(S_i[u^-](s) \right) - f(u^-)$$

together with

$$S_i[u^-](0) = u^- \ , \qquad \sigma_i[u^-](0) = \lambda_i(u^-) \ , \qquad \frac{\mathrm{d}}{\mathrm{d}s} S_i[u^-](0) = r_i(0) \ .$$

One can suppose that $S_i[u^-]$ is parameterized by the *i*-th component relative to the basis $r_1(u), \ldots, r_N(u)$. If $u^+ = S_i[u^-](s)$, we denote also by $\sigma_i(u^-, u^+) = \sigma_i[u^-](s)$ the **speed** of the *i*-th discontinuity $[u^-, u^+]$. This *i*-th discontinuity is **admissible** when [25]

$$\forall 0 \leq |\tau| \leq |s| \qquad \sigma_i[u^-](\tau) \geq \sigma_i(u^-, u^+) \ .$$

Finally, if the *i*-th field is piecewise genuinely nonlinear we call [27] that an admissible *i*-jump $[u^-, u^+]$ is called **simple** if

$$\forall 0 < |\tau| < |s| \qquad \sigma_i[u^-](\tau) > \sigma_i(u^-, u^+) \qquad u^+ = S_i[u^-](s) \ .$$

If the admissible jump $[u^-, u^+]$ is not simple, we call it a **composition of the waves** $[u_0, u_1]$, $[u_1, u_2]$, ..., $[u_\ell, u_{\ell+1}]$ if

$$u_0 = u^- \ , \qquad u_{\ell+1} = u^+ \ , \qquad u_k = S_i[u^-](s_k) \ , \qquad \sigma_i[u^-](s_k) = \sigma_i(u^-, u^+)$$

for all $k = 0, \ldots, \ell + 1$ and for

$$0 = s_0 < s_1 < \cdots < s_\ell < s_{\ell+1} = s \quad \text{or} \quad s = s_{\ell+1} < s_\ell < \cdots < s_1 < s_0 = 0 \ .$$

Theorem 5 (Global structure of solutions). *Let u be the entropy solution of the Cauchy problem (1.1)-(1.2) under the assumption that each characteristic field is either linearly degenerate or piecewise genuinely nonlinear, and assuming that Tot.Var.$(\overline{u})$ is small enough. Assume that the source term $g = g(u)$ is Lipschitz continuous. Then there exists a countable set $\Theta = \{(t_\ell, x_\ell) : \ell \in \mathbb{N}\}$ and a countable family of Lipschitz continuous curves*

$$\mathcal{J} = \{y_m : (a_m, b_m) \to \mathbb{R} , \ m \in \mathbb{N}\} ,$$

whose graphs cover points of admissible shocks and points of contact discontinuities, such that u is continuous at least outside $\Theta \cup \mathrm{Graph}(\mathcal{J})$. Moreover, the following holds. For each curve $\overline{y} = y_m \in \mathcal{J}$ and each fixed $t \in (a_m, b_m)$ with $(t, \overline{y}(t)) \notin \Theta$ denote by

$$u^L := u^L(t, \overline{y}(t)) := u(t, \overline{y}(t)-) , \qquad u^R := u^R(t, \overline{y}(t)) := u(t, \overline{y}(t)+) .$$

Then there exist i and s such that $u^R = S_i[u^L](s)$ with $i \in \{1, \dots, N\}$. Moreover

- *If the i-th family is linearly degenerate or if $[u^L, u^R]$ is a simple jump of piecewise genuinely nonlinear family i then entropy conditions hold and*

$$u^L = \lim_{\substack{(r,x) \to (t, \overline{y}(t)) \\ x < \overline{y}(s)}} u(r, x) , \qquad u^R = \lim_{\substack{(r,x) \to (t, \overline{y}(t)) \\ x > \overline{y}(s)}} u(r, x) , \qquad \dot{\overline{y}}(t) = \sigma_i(u^L, u^R) .$$

  In case the i-th family is linearly degenerate it is also possible that $u^L = u^R$ for $s > 0$.

- *If $[u^L, u^R]$ is a composition of the waves $[u_0, u_1], [u_1, u_2], \dots, [u_\ell, u_{\ell+1}]$ then there exist*

$$\overline{y}_1, \dots, \overline{y}_p \in \mathcal{J}, \qquad \text{where } p \leq \ell + 1,$$

  depending on $(t, \overline{y}(t))$, such that there exists a neighborhood $\mathcal{U}(t)$ of t for which

$$\overline{y}_1(t) = \cdots = \overline{y}_p(t) , \qquad \dot{\overline{y}}_1(t) = \cdots = \dot{\overline{y}}_p(t) = \sigma(u^L, u^R) , \qquad \overline{y}_1(r) \leq \cdots \leq \overline{y}_p(r)$$

  for $r \in \mathcal{U}(t)$ and

$$u^L = \lim_{\substack{(r,x) \to (t, \overline{y}(t)) \\ x < \overline{y}_1(s)}} u(r, x) , \qquad u^R = \lim_{\substack{(r,x) \to (t, \overline{y}(t)) \\ x > \overline{y}_p(s)}} u(r, x) .$$

  Finally, one can also require that if $\overline{y}_j$ and $\overline{y}_{j+1}$ do not coincide in $\mathcal{U}(t)$, then

$$u_j = \lim_{\substack{(r,x) \to (t, \overline{y}(t)) \\ \overline{y}_j(s) < x < \overline{y}_{j+1}(s)}} u(r, x) .$$

As in [9, 8, 7], the above theorem is proved by approximation by means of a fine convergence result that will be precisely stated later in § 4. This is why we work under the hypothesis of the convergence Theorem 1.

2 Piecewise constant approximations

In this section we describe the main ingredients in order to construct a piecewise constant approximation of a solution u to (1.1)-(1.2).

2.1 The nonconservative Riemann problem

Since we deal with a system that, in general, it is not in conservation form, we briefly recall the construction of the solution to a Riemann problem in the homogeneous case, i.e.

$$u_t + A(u)u_x = 0 \tag{2.26a}$$

$$u(0, x) = \begin{cases} u^L & \text{if } x < 0, \\ u^R & \text{if } x > 0. \end{cases} \tag{2.26b}$$

We refer to [6, 5] for the details.

As in the Introduction, we let A be a smooth matrix-valued map, with eigenvalues given by (1.5), and right and left eigenvalues (1.6)-(1.7). Since we are interested in solutions to (1.1) with small total variation, it is not restrictive to assume that there exist constants $\widehat{\lambda}_0 < \cdots < \widehat{\lambda}_N$ such that

$$\widehat{\lambda}_{k-1} < \lambda_k(u) < \widehat{\lambda}_k , \qquad \forall\, u, \quad k = 1, \dots, N . \tag{2.27}$$

Given any continuous function $f : I \subset \mathbb{R} \to \mathbb{R}$, and any interval $[a, b] \subset I$, we will denote the *lower convex envelope* and the *upper concave envelope* of f on $[a, b]$, respectively, as

$$\text{conv}_{[a,b]} f(x) \doteq \inf \left\{ \theta f(y) + (1 - \theta) f(z) : \theta \in [0, 1], \ y, z \in [a, b], \ x = \theta y + (1 - \theta) z \right\}, \tag{2.28}$$

and

$$\text{conc}_{[a,b]} f(x) \doteq \sup \left\{ \theta f(y) + (1 - \theta) f(z) : \theta \in [0, 1], \ y, z \in [a, b], \ x = \theta y + (1 - \theta) z \right\}. \tag{2.29}$$

We will simply write $\text{conv}\, f$, $\text{conc}\, f$, whenever there is no ambiguity on the interval $[a, b]$ taken in consideration. As usual, in order to contruct a solution to (2.26), the basic step consists in constructing the *elementary curve of the k-th family* ($k = 1, \ldots, N$) for every given left state u^L, which is a one parameter curve of right states $s \mapsto T_k[u^L](s)$ with the property that the Riemann problem having initial data (u^L, u^R), $u^R \doteq T_k[u^L](s)$, admits a vanishing viscosity solution consisting only of waves of the k-th characteristic family. In order to construct such a curve, we look for travelling waves solutions to the parabolic system

$$u_t + A(u)u_x = u_{xx}, \tag{2.30}$$

solutions to (2.30) of the form $u(t, x) = \phi(x - \sigma\, t)$, for some constant σ. The profile ϕ satisfies the second order ODE

$$\big(A(\phi) - \sigma \big)\, \phi' = \phi'',$$

which can be written as a first order system of ODEs on the space $\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$:

$$\begin{cases} \dot{u} = v, \\ \dot{v} = \big(A(\phi) - \sigma \big) v, \\ \dot{\sigma} = 0. \end{cases} \tag{2.31}$$

Applying the Center Manifold Theorem, we get that in a neighborhood of a given equilibrium point $(u_0, 0, \lambda_k(u_0)) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$ for (2.31) there exists an $N + 2$-dimensional center manifold $\mathcal{M}_k$ which is locally invariant under the flow of (2.31). Introducing the coordinates

$$v_h \doteq \big\langle l_h(u_0), v \big\rangle, \qquad h = 1, \ldots, N,$$

of a vector $v \in \mathbb{R}^N$ relative to the basis $r_1(u_0), \ldots \ldots, r_N(u_0)$, one can parameterize $\mathcal{M}_k$ in terms of the variables u, v_k, σ, namely

$$\mathcal{M}_k = \big\{ (u, v, \sigma) \, ; \quad v = v_k\, \widetilde{r}_k(u, v_k, \sigma) \big\} \tag{2.32}$$

for suitable smooth vector functions $(u, v_k, \sigma) \mapsto \widetilde{r}_k(u, v_k, \sigma)$ defined on a neighborhood of $(u_0, 0, \lambda_k(u_0))$, that satisfy

$$\widetilde{r}_k\big(u_0, 0, \sigma\big) = r_k(u_0) \qquad \forall\, \sigma, \tag{2.33}$$

and are normalized so that

$$\big\langle l_k(u_0), \widetilde{r}_k(u, v_k, \sigma) \big\rangle = 1 \qquad \forall\, u, \, v_k, \, \sigma. \tag{2.34}$$

By construction, $\mathcal{M}_k$ contains all bounded viscous traveling profiles with speed close to $\lambda_k(u_0)$. Thus, we can rewrite the linearized equations for (2.31) at $(u_0, 0, \lambda_k(u_0))$ on the manifold $\mathcal{M}_k$, and obtain a system on the space $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$:

$$\begin{cases} u_x = v_k\widetilde{r}_k(u, v_k, \sigma), \\ v_{k,x} = v_k\big(\widetilde{\lambda}_k(u, v_k, \sigma) - \sigma \big), \\ \sigma_x = 0, \end{cases} \tag{2.35}$$

where

$$\widetilde{\lambda}_k(u, v_k, \sigma) \doteq \big\langle l_k(u), A(u)\, \widetilde{r}_k(u, v_k, \sigma) \big\rangle. \tag{2.36}$$

Because of the normalization (2.34), the smooth scalar function $(u, v_k, \sigma) \mapsto \widetilde{\lambda}_k(u, v_k, \sigma)$ satisfies the identity

$$\widetilde{\lambda}_k\big(u_0, v_k, \sigma\big) = \lambda_k(u_0) \qquad \forall\, v_k, \, \sigma. \tag{2.37}$$

Next, given a left state u^L in a neighborhood of u_0 and $0 < s << 1$, in connection with the equations (2.35) describing the evolution of traveling profiles on the manifold $\mathcal{M}_k$ we associate the integral system

$$\begin{cases} u(\tau) = u^L + \int_0^\tau \widetilde{r}_k\big(u(\xi), v_k(\xi), \sigma(\xi)\big)\, d\xi, \\ v_k(\tau) = \widetilde{F}_k\big(\tau;\, u, v_k, \sigma\big) - \text{conv}_{[0,s]}\, \widetilde{F}_k\big(\tau;\, u, v_k, \sigma\big), \qquad 0 \le \tau \le s, \\ \sigma(\tau) = \dfrac{d}{d\tau} \text{conv}_{[0,s]}\, \widetilde{F}_k\big(\tau;\, u, v_k, \sigma\big), \end{cases} \tag{2.38}$$

where $\tau \mapsto \widetilde{F}_k(\tau; u, v_k, \sigma)$ is the *"reduced flux function"* associated to (1.13) defined, by

$$\widetilde{F}_k(\tau; u, v_k, \sigma) \doteq \int_0^\tau \widetilde{\lambda}_k\big(u(\xi), v_k(\xi), \sigma(\xi)\big) \, d\xi \,. \tag{2.39}$$

In [6] it is shown that, for s sufficiently small, the transformation defined by the right-hand side of (2.38) maps a domain of continuous curves $\tau \mapsto (u(\tau), v_k(\tau), \sigma(\tau))$ into itself, and is a contraction w.r.t. a suitable weighted norm. Hence, for every u^L in a neighborhood $\mathcal{U}_0$ of u_0, the transformation defined by (2.38) admits a unique fixed point

$$\tau \mapsto \big(\overline{u}(\tau; u^L, s), \; \overline{v}_k(\tau; u^L, s), \; \overline{\sigma}(\tau; u^L, s)\big) \qquad \tau \in [0, s]\,, \tag{2.40}$$

which provides a Lipschitz continuous solution to the integral system (2.38). The elementary curve of right states of the k-th family issuing from u^L is then defined as the terminal value at $\tau = s$ of the u-component of the solution to the integral system (2.38), i.e. by setting

$$T_k[u^L](s) \doteq u(s; u^L, s)\,. \tag{2.41}$$

For the sake of convenience, we denote

$$\begin{aligned} \sigma_k[u^L](s, \tau) &\doteq \sigma(\tau; u^L, s)\,, \\ \widetilde{F}_k[u^L](s, \tau) &\doteq \widetilde{F}_k\big(\tau; u(\cdot\,; u^L, s), v_k(\cdot; u^L, s), \sigma(\cdot; u^L, s)\big) \end{aligned} \tag{2.42}$$

For negative values $s < 0$, $|s| \ll 1$, one replaces in (2.38) the lower convex envelope of $\widetilde{F}_k$ on the interval $[0, s]$ with its upper concave envelope on $[s, 0]$, and then constructs the curve $T_k[u^L]$ and the map $\sigma_k[u^L]$ exactly in the same way as above looking at the solution of the integral system (2.38) on the interval $[s, 0]$. In such a way, given any pair of states u^L, u^R with $|u^L - u_0|, |u^R - u_0| \ll 1$, if $u^R = T_k[u^L](s)$, for some wave size s, then the self-similar solution to the Riemann problem with initial data (u^L, u^R), determined by the vanishing viscosity approximation (1.3) as $\varepsilon \to 0+$, is given by the piecewise continuous function

$$u(t, x) = \begin{cases} u^L & \text{if} \quad x/t < \sigma_k[u^L](s, 0)\,, \\ T_k[u^L](\tau) & \text{if} \quad x/t = \sigma_k[u^L](s, \tau) \quad \text{for some } \tau \in \mathcal{I}\,, \\ u^R & \text{if} \quad x/t > \sigma_k[u^L](s, s)\,, \end{cases} \tag{2.43}$$

Remark 6. If the system (1.1) is in conservation form, i.e. in the case where $A(u) = DF(u)$ for some smooth flux function F, the general solution of the Riemann problem provided by (2.43) is a composed wave of the k-th family containing a countable number of rarefaction waves and contact-discontinuities or compressive shocks which satisfy the Liu admissibility condition [24, 25]. Namely, the regions where the v_k-component of the solution to (2.38) vanishes correspond to rarefaction waves if the σ-component is strictly increasing and to contact discontinuities if the σ-component is constant, while the regions where the v_k-component of the solution to (2.38) is different from zero correspond to compressive shocks.

In view of the considerations of Remark 6, we will extend the standard terminology adopted for the elementary waves that are present in the solution of an hyperbolic system of conservation laws to the general case of non conservative systems. Thus, we will say that any (vanishing viscosity) solution of the Riemann problem for (1.1) of the form (2.43) is a *centered rarefaction wave* of the k-th family whenever $u^R \in R_k[u^L](s)$ for some wave size s such that $\tau \mapsto \sigma_k[u^L](s, \tau)$ be strictly increasing on $[0, s]$, $s > 0$ (or strictly decreasing on $[s, 0]$ if $s < 0$), while we will say that any (vanishing viscosity) solution of a Riemann problem for (1.1) of the form

$$u(t, x) = \begin{cases} u^L & \text{if} \quad x < \lambda t\,, \\ u^R & \text{if} \quad x > \lambda t\,, \end{cases}$$

is an *admissible shock wave* of the k-th family when $u^R = T_k[u^L](s)$ and $\sigma_k[u^L](s, 0) = \sigma_k[u^L](s, s) = \lambda$. Once we have constructed the elementary curves T_k for each k-th characteristic family, the *vanishing viscosity solution* of a general Riemann problem for (1.1) is then obtained by a standard procedure observing that the composite mapping

$$\Phi(s_1, \dots, s_N)[u^L] \doteq T_N\Big[T_{N-1}\big[\cdots [T_1[u^L](s_1)] \cdots\big](s_{N-1})\Big](s_N) \doteq u^R\,, \tag{2.44}$$

is one-to-one from a neighborhood of the origin onto a neighborhood of u^L. This is a consequence of the fact that the curves $T_k[u]$ are tangent to $r_k(u)$ at zero $s = 0$ [6, 5]. Therefore, we can uniquely determine intermediate states $u^L \doteq \omega_0, \omega_1, \dots, \omega_N \doteq u^R$, and wave sizes $s_1, \dots, s_N$, such that there holds

$$\omega_k = T_k[\omega_{k-1}](s_k) \qquad k = 1, \dots, N\,, \tag{2.45}$$

provided that the left and right states u^L, u^R are sufficiently close to each other. Each Riemann problem with initial data

$$\overline{u}_k(x) = \begin{cases} \omega_{k-1} & \text{if } x < 0, \\ \omega_k & \text{if } x > 0, \end{cases} \tag{2.46}$$

admits a vanishing viscosity solution of *total size* s_k, containing a sequence of rarefactions and Liu admissible discontinuities of the k-th family. Then, because of the uniform strict hyperbolicity assumption (2.27), the general solution of the Riemann Problem with initial data $\left(u^L, u^R\right)$ is obtained by piecing together the vanishing viscosity solutions of the elementary Riemann problems (1.1) (2.46). Throughout the paper, with a slight abuse of notation, we shall often call s a wave of (total) size s, and, if $u^R = T_k[u^L](s)$, we will say that (u^L, u^R) is a wave of size s of the k-th characteristic family.

2.2 The algorithm

Now we briefly describe the algortihm we use in order to construct a piecewise constant approximate solution to (1.1)-(1.2). First of all let us recall what a front tracking solution to an homogeneous hyperbolic system is (see [3] for details).

Definition 7. *Let $\varepsilon > 0$ and an interval $I \subset \mathbb{R}$ be fixed, and let $A = A(u)$, $u \in \mathbb{R}^N$, be a smooth hyperbolic $N \times N$ matrix. We say that a continuous map $u : I \mapsto \mathbb{L}_{loc}^1(\mathbb{R}; \mathbb{R}^N)$, is an ε-approximate front tracking solution to (2.26a) if the following conditions hold:*

1. *As a function of two variables, $u = u(t, x)$ is piecewise constant with discontinuities occurring along finitely many straight lines in the t-x plane. Jumps can be of two types: elementary wave-fronts and non-physical wave-fronts, denoted, respectively, as $\mathcal{E}$ and $\mathcal{NP}$. Only finitely many wave-fronts interactions occur, each involving exactly two incoming fronts.*

2. *Along each elementary front $x = x_\alpha(t)$, $\alpha \in \mathcal{E}$, the values $u^L \doteq u(t, x_\alpha-)$ and $u^R \doteq u(t, x_\alpha+)$ satisfy the following properties. There exists some wave size s_α and some index $k_\alpha \in \{1, \ldots, N\}$ such that*

$$u^R = T_{k_\alpha}[u^L](s_\alpha). \tag{2.47}$$

Moreover, the speed $\dot{x}_\alpha$ of the wave-front satisfies

$$\left| \dot{x}_\alpha - \sigma_{k_\alpha}[u^L](s_\alpha, \tau) \right| \leq 2\varepsilon, \qquad \forall \tau \in [0, s_\alpha]. \tag{2.48}$$

3. *All non-physical fronts $x = x_\alpha(t)$, $\alpha \in \mathcal{N}$ have the same speed*

$$\dot{x}_\alpha \equiv \widehat{\lambda}, \tag{2.49}$$

where $\widehat{\lambda}$ is a fixed constant strictly greater than all characteristic speeds, i.e.

$$\widehat{\lambda} > \lambda_k(u) \qquad \forall u \in \Omega, \quad k = 1, \ldots, N. \tag{2.50}$$

Moreover, the total strength of all non-physical fronts in $u(t, \cdot)$ remains uniformly small, namely one has

$$\sum_{\alpha \in \mathcal{NP}} \left| u(t, x_\alpha+) - u(t, x_\alpha-) \right| \leq \varepsilon \qquad \forall t \geq 0. \tag{2.51}$$

In order to construct piecewise constant approximations to (1.1)-(1.2), we follow the approach of [13], and construct a local solution to (1.1)-(1.2) by means of a fractional step algorithm combined with a front tracking method. In order to do this we assume that assumption **(G)** at 2 holds. Hence, once two sequences

$$\{\tau_\nu\}_{\nu \in \mathbb{N}}, \ \{\varepsilon_\nu\}_{\nu \in \mathbb{N}}, \qquad 0 < \tau_\nu \leq \varepsilon_\nu \downarrow 0,$$

are given, we fix $\nu \in \mathbb{N}$ and we proceed in this way in order to construct and ε_ν-approximate fractional-step approximation $u_\nu = u_{\varepsilon_\nu}$ of the solution. Fist of all, we approximate the initial datum $\overline{u}$ by means of a piecewise constant function $\overline{u}_\nu$ such that

$$\text{Tot.Var.} \overline{u}_\nu \leq \text{Tot.Var.} \overline{u}, \qquad \|\overline{u}_\nu - \overline{u}\|_{L^1} \to 0 \quad \text{as } \nu \uparrow \infty.$$

Then, we take a suitable approximation g_ν of g piecewise contant w.r.t. x, i.e., following [13, § 3], we let

$$g_\nu(t, x, v) \doteq \sum_{j \in \mathbb{Z}} \chi_{[j\varepsilon_\nu, (j+1)\varepsilon_\nu[}(x) g_j(t, v), \tag{2.52}$$

where χ_I is the characteristic function of the set I, and

$$g_j(t,v) = \frac{1}{\varepsilon_\nu} \int_{j\varepsilon_\nu}^{(j+1)\varepsilon_\nu} g(t,x,v)\, dx\,. \tag{2.53}$$

Then, the algorithm that leads to the construction of the approximation $u_\nu = u_\nu(t,x)$ essentially consists of the following steps.

1. We apply a front tracking algorithm as described in [3], which we refer to, to construct an ε_ν-approximate front tracking solution in the sense of Definition 7 in the time interval $]0,\tau_\nu[$.

2. At $t=\tau_\nu$ we correct the term $u_\nu(\tau_\nu-,\cdot)$ by setting

$$u_\nu(\tau_\nu+,\cdot) = u_\nu(\tau_\nu-,\cdot) + \tau_\nu g_\nu\big(\tau_\nu,\cdot, u_\nu(\tau_\nu-,\cdot)\big)\,,$$

which turns out to be piecewise constant by construction.

3. In general, once $u_\nu(n\tau_\nu+,\cdot)$, $n \geq 1$, is given, we again use the algorithm in [3] to construct an ε_ν-approximate front tracking solution in the time interval $]n\tau_\nu,(n+1)\tau_\nu[$.

4. Similarly to what done above at $t=\tau_\nu$, at $t=(n+1)\tau_\nu$ we correct the term $u_\nu((n+1)\tau_\nu-,\cdot)$ by setting

$$u_\nu\big((n+1)\tau_\nu+,\cdot\big) = u_\nu\big((n+1)\tau_\nu-,\cdot\big) + \tau_\nu g_\nu\big((n+1)\tau_\nu,\cdot, u_\nu\big((n+1)\tau_\nu-,\cdot\big)\big)\,.$$

We stress that, in the construction described above, nonphysical waves are implicitly restarted at each time step: the corresponding jumps are solved using physical waves. As it is usual with such algorithms, the main difficulties we have to face are to

- bound uniformly the total variation of $u_\nu(t,\cdot)$ in order to get compactness of the approximating sequence;

- let the number of the fronts to remain bounded in any time interval $[0,t]$.

We will briefly discuss how to overcome the first difficulty in Subsection 2.3, taking advantage of the results contained in [3, 4, 13]. Regarding the second difficulty, using the arguments contained in [3, Subsection 6.2], it can be easily seen that the number of wave fronts stays bounded in each time interval $[k\tau_\nu,(k+1)\tau_\nu[$, and their number depends on the parameter ε_ν and on the total variation of $u_\nu(t,\cdot)$ which remains uniformly bounded.

2.3 Evolution / interaction estimates

In correspondence of a sequence $\{\varepsilon_\nu\}_{\nu\geq 1} \subset \mathbb{R}^{>0}$, $\varepsilon_\nu \to 0$, and following [22], in this subsection we will define the interaction potential and give the interaction estimates that will allow us to perform uniform bounds on the total variation of an ε_ν frotn tracking approximate solution. To this purpose, following [4, Definition 3.5], we first introduce a definition of quantity of interaction between wave-fronts of an approximate solution.

Definition 8. Consider two interacting wave-fronts of sizes s',s'' (s' located on the left of s''), belonging to the $k',k'' \in \{1,\ldots,N+1\}$-th characteristic family, respectively, and let u^L, u^M, u^R, denote the left, middle and right states before the interaction. We say that the *amount of interaction* $\mathcal{I}(s',s'')$ between s' and s'' is the quantity defined as follows.

1. If s' and s'' belong to different characteristic families, i.e. if $k'' < k' \leq N+1$, then set

$$\mathcal{I}(s',s'') \doteq |s's''|\,. \tag{2.54}$$

2. If s' and s'' belong to the same $k\,(\leq N)$-th characteristic family ($k \doteq k' = k''$), i.e. if $u^M = T_k[u^L](s')$, $u^R = T_k[u^M](s'')$, let $\widetilde{F}',^L \doteq \widetilde{F}_k[u^L](s',\cdot)$ and $\widetilde{F}'',^M \doteq \widetilde{F}_k[u^M](s'',\cdot)$ be the reduced flux with starting point u^L, u^M, evaluated along the solution of (2.38) on the interval $[0,s']$, and $[0,s'']$, respectively (cfr. def. (2.42)). Then, assuming that $s \geq 0$, we shall distinguish three cases.

   (a) if $s'' \geq 0$ set:

$$\mathcal{I}(s',s'') \doteq \int_0^{s'} \left| \mathrm{conv}_{[0,\,s']}\, \widetilde{F}',^L(\xi) - \mathrm{conv}_{[0,\,s'+s'']}\, \widetilde{F}',^L \cup \widetilde{F}'',^M(\xi) \right| d\xi$$

$$+ \int_{s'}^{s'+s''} \left| \widetilde{F}',^L(s') + \mathrm{conv}_{[0,\,s'']}\, \widetilde{F}'',^M(\xi - s') \right. \tag{2.55}$$

$$\left. - \mathrm{conv}_{[0,\,s'+s'']}\, \widetilde{F}',^L \cup \widetilde{F}'',^M(\xi) \right| d\xi\,,$$

where $\widetilde{F}',^L \cup \widetilde{F}'',^M$ is the function defined on $[0,\,s'+s'']$ as

$$\widetilde{F}',^L \cup \widetilde{F}'',^M(s) \doteq \begin{cases} \widetilde{F}',^L(s) & \text{if } s \in [0,s']\,, \\ \widetilde{F}',^L(s') + \widetilde{F}'',^M(s-s') & \text{if } s \in [s',s'+s'']\,. \end{cases} \tag{2.56}$$

(b) if $-s' \leq s'' < 0$ set:

$$\mathcal{I}(s',\, s'') \doteq \int_0^{s'+s''} \left| \mathrm{conv}_{[0,\, s']}\, \widetilde{F}^{',L}(\xi) - \mathrm{conv}_{[0,\, s'+s'']}\, \widetilde{F}^{',L}(\xi) \right| d\xi \tag{2.57}$$
$$+ \int_{s'+s''}^{s'} \left| \mathrm{conv}_{[0,\, s']}\, \widetilde{F}^{',L}(\xi) - \mathrm{conc}_{[s'+s'',\, s']}\, F^{',L}(\xi) \right| d\xi\,.$$

(c) if $s'' < -s'$ set:

$$\mathcal{I}(s',\, s'') \doteq \int_{s'+s''}^0 \left| \mathrm{conc}_{[s'',\, 0]}\, \widetilde{F}^{'',M}(\xi - s') - \mathrm{conc}_{[s'',\, -s']}\, \widetilde{F}^{'',M}(\xi - s') \right| d\xi \tag{2.58}$$
$$+ \int_0^{s'} \left| \mathrm{conc}_{[s'',\, 0]}\, \widetilde{F}^{'',M}(\xi - s') - \mathrm{conv}_{[-s',\, 0]}\, \widetilde{F}^{'',M}(\xi - s') \right| d\xi\,.$$

In the case where $s' < 0$, one replaces in (2.55)-(2.58) the convex envelope with the concave one, and vice-versa.

Remark 9. By Remark 6 one can easily verify that, in the conservative case, if s', s'' are both shocks of the k-th family that have the same sign, then the amount of interaction in (2.55) takes the form

$$\mathcal{I}(s',\, s'') = \left| s's'' \right| \left| \sigma_k[u^L, u^M] - \sigma_k[u^M, u^R] \right|,$$

i.e. it is precisely the product of the strength of the waves times the difference of their Rankine Hugoniot speeds.

Now, whenever a ε-approximate front tracking solution $u^\varepsilon = u^\varepsilon(t,x)$ to (2.26a) is given, we define the interaction potential (see [4, (4.2)])

$$\mathcal{Q}(u^\varepsilon(t,\cdot)) = \sum_{\substack{i<j \\ x'>x''}} \left| s'_{x',i} s''_{x'',j} \right| + \frac{1}{4} \sum_{x',x'',i} \int_0^{|s_{x',i}|} \int_0^{s_{x'',i}} \left| \sigma_{x',i}(\tau') - \sigma_{x'',i}(\tau'') \right| d\tau' d\tau''\,, \tag{2.59}$$

where $s_{x,k}$ is the size of the wave of the k-th characteristic family at x, and $\sigma_{x,k}(\tau)$ is its speed as it is defined at (2.38). Moreover we let

$$\mathcal{V}(u^\varepsilon(t,\cdot)) = \sum_{x,i} |s_{x,i}| \tag{2.60}$$

With these definitions, the following result holds (see [4, Proposition 4.1]):

Proposition 10. *There exists $\delta_H > 0$ such that, if $u^\varepsilon = u^\varepsilon(t,x)$ is an ε-approximate front tracking solution to (2.26a)-(1.2) with $\mathrm{Tot.Var.}\overline{u} < \delta_H$, the the following holds. There exists constants $c, C_1 > 0$ such that, whenever two wave fronts s', s'' interact, then*

$$\Delta \mathcal{Q} \leq -c\mathcal{I}(s',\, s'')\,,$$

and, moreover, the functional

$$t \mapsto \Upsilon(u^\varepsilon(t,\cdot)) = \mathcal{V}(u^\varepsilon(t,\cdot)) + C_1 \mathcal{Q}(u^\varepsilon(t,\cdot)) \tag{2.61}$$

is decreasing.

3 Existence and convergence of approximations

In this section we prove that the approximations constructed in § 2 converge to the entropy solution of the Cauchy problem (1.1)-(1.2). We first prove rough estimates that ensure the local-in-time convergence, as stated in Theorems 11-14 below, which yield Theorem 1. Uniqueness is proved roughly following the lines of [2].

3.1 Local in time existence of time-step approximations

Let Υ be the functional introduced in (2.61).

Theorem 11. *There exist $\overline{\delta}, T > 0$ such that for initial data $\overline{u}$ in the closed domain*

$$\mathfrak{D}_p(\overline{\delta}) := \left\{ u \in L^1(\mathbb{R}; \mathbb{R}^N) \cap \mathrm{BV}(\mathbb{R}; \mathbb{R}^N) \text{ piecewise constant s.t. } \Upsilon(u) \leq \overline{\delta} \right\}$$

the algorithm described in § 2 defines for $t \in [0,T]$ and for every ν an approximating function

$$w_\nu(t,\cdot) \in \mathfrak{D}_p\left(\overline{\delta} + Gt\right) \quad \text{where } C, G \text{ only depend on } A \text{ and } g. \tag{3.62}$$

This approximating function u_ν satisfies the following comparison estimate with the viscous semigroup [10] $P_{t,h}[\cdot]$ of the Cauchy problem (1.1)-(1.2) starting at time h: there is a function $o(s)$ depending only on A, g, $\overline{\delta}$, T such that $o(s)/s \to 0$ if $s \to 0$ and such that for $n \in \mathbb{N}$

$$\left\| w_\nu(n\tau_\nu+,\cdot) - P_{n\tau_\nu,(n-1)\tau_\nu}\left[w_\nu((n-1)\tau_\nu+,\cdot)\right] \right\|_{L^1} \leq \mathcal{O}(1)\left(o(\tau_\nu) + \varepsilon_\nu \tau_\nu\right)\,. \tag{3.63}$$

Introduction to the proof.   Before the proof, we briefly remind our notation and previous results that we need. We denote by $S_{t,h}^{FT}$ the wave-front tracking approximation of the semigroup $S_{t,h}[\cdot]$ relative to the homogeneous system, constructed by vanishing viscosity [6], where the 'initial datum' is fixed at time $h \leq t$ rather than at $h = 0$.

We exploit the definition in § 2 of the approximation

$$w_\nu(t,\cdot) = S_{t,(n-1)\tau_\nu}^{FT} \left[w_\nu((n-1)\tau_\nu+,\cdot) \right] \qquad \text{for } (n-1)\tau_\nu < t < n\tau_\nu, \ n \in \mathbb{N}$$

$$w_\nu(n\tau_\nu+,\cdot) \equiv S_{n\tau_\nu,(n-1)\tau_\nu}^{FT} \left[w_\nu((n-1)\tau_\nu+,\cdot) \right] + \tau_\nu g \left(n\tau_\nu, \cdot, S_{n\tau_\nu,(n-1)\tau_\nu}^{FT} \left[w_\nu(n-1)\tau_\nu+,\cdot \right] \right) \tag{3.64}$$

$$\equiv w_\nu(n\tau_\nu-,\cdot) + \tau_\nu g \left(n\tau_\nu, \cdot, w_\nu(n\tau_\nu-,\cdot) \right)$$

relative to the balance law with the initial condition $w_\nu(0+,\cdot) \equiv \bar{u}(\cdot)$. We recall that

$$\text{if } \overline{w} \in \mathfrak{D}_p(\overline{\delta}), \text{ for } \overline{\delta} \leq \delta_H \text{ small enough as in [3]},$$

then

$$\Upsilon\left(S_{t,h}^{FT}\overline{w}\right) \leq \Upsilon(\overline{w}) \qquad \forall 0 \leq h \leq t \leq \tau_\nu < T \qquad \text{see [3, (6.4)] or Proposition 10 above} \tag{3.65}$$

$$\|S_{n\tau_\nu,(n-1)\tau_\nu}^{FT}\overline{w} - S_{n\tau_\nu,(n-1)\tau_\nu}\overline{w}\|_{L^1} \lesssim (1+\overline{\delta})\varepsilon_\nu\tau_\nu \qquad \text{see [3, (3.5)]} \tag{3.66}$$

$$\|S_{t+s,t}^{FT}\overline{w} - \overline{w}\| \leq Ls \qquad \text{see [3, (1.23)]} \tag{3.67}$$

We also borrow the following lemma from [2, Lemmas 2.1-2], given in a similar setting. Of course we could state it similarly also localizing in space the estimates. We remind that $\Upsilon, \mathcal{Q}$ are the functionals introduced in (2.59)-(2.61) while ℓ_g and α are as in the assumption (G) on the source term at Page 2.

Lemma 12. *Let $t > 0$. If $0 < \overline{\delta} < \delta_H$ and $\overline{w}, \overline{u}$ are piecewise constant with $\Upsilon(\overline{u}) + \Upsilon(\overline{w}) \leq \overline{\delta}$ then*

$$v(x) := \overline{u}(x) + \tau g_\nu(t, x, \overline{w}(x))$$

satisfies for $G = \max\{\ell_g \text{Tot.Var.}(\overline{w}) + \text{Tot.Var.}(\overline{u}) + \|\alpha\|_{L^1}; 1\}$ the inequalities

$$|\text{Tot.Var.}(v) - \text{Tot.Var.}(\overline{u})| \lesssim G\tau \qquad |\mathcal{Q}(v) - \mathcal{Q}(\overline{u})| \lesssim G^2\tau \qquad |\Upsilon(v) - \Upsilon(\overline{u})| \lesssim G^2\tau . \tag{3.68}$$

Proof. We remind the idea of the proof from [2, Lemma 2.1] for completeness. Suppose either $\overline{u}(x)$ or $v(x)$ has a jump at x. Denoting by $\Phi(\cdot)[\cdot]$ the map defined at (2.44) for the Riemann problem, set $\widehat{\sigma}$ by the relation

$$\overline{w}(x+) = \Phi(\widehat{\sigma})[\overline{w}(x-)] .$$

Define then σ' so that the following diagram commutes:

$$
\begin{array}{ccc}
\overline{u}(x-) & \xrightarrow{\ \text{source}\ } & v(x-) := \overline{u}(x-) + \tau g_\nu(t, x-, \overline{w}(x-)) \\
\ \downarrow{\sigma} & & \Rightarrow \downarrow{\sigma'} \\
& & v(x+) := \overline{u}(x+) + \tau g_\nu(t, x+, \overline{w}(x+)) \\
\overline{u}(x+) = \Phi(\sigma)[\overline{u}(x-)] & \xrightarrow{\ \text{source}\ } & \\
& & \equiv \Phi(\sigma')[v(x-)]
\end{array}
$$

Estimate on the total variation   The first estimate immediately follows since

$$|\text{Tot.Var.}(v) - \text{Tot.Var.}(\overline{u})| \leq \text{Tot.Var.}(v - \overline{u}) = \tau \cdot \text{Tot.Var.}(g_\nu(t, x+, \overline{w}(x+)))$$

$$\leq (\ell_g \text{Tot.Var.}(\overline{w}) + \|\alpha\|_{L^1})\tau .$$

Estimate on $|\sigma' - \sigma|$ Set

$$d = g_\nu(t, x+, \overline{w}(x+)) - g_\nu(t, x-, \overline{w}(x-)) \quad \Rightarrow \quad |d| \leq \int_{(j-1)\varepsilon_\nu}^{(j+1)\varepsilon_\nu} |\alpha| + \ell_g \text{Lip}\Phi \, \widehat{\sigma} . \tag{3.69}$$

Notice that the difference $|\sigma' - \sigma|$ is a function of σ, τ, d which identically vanishes both when $\sigma = d = 0$ and when $\tau = 0$, as the two rows / columns of the commutative diagram above collapse. One can thus estimate $\Psi(\sigma,\tau,d) = |\sigma' - \sigma|$ by calculus similarly to [8, Lemma 2.5], since $\Psi_\sigma(\sigma,0,d) = \Psi_d(\sigma,0,d) = 0$ and $\Psi_\tau(0,\tau,0) = 0$:

$$|\sigma' - \sigma| = \left| \int_0^1 (\sigma \Psi_\sigma + d\Psi_d)(z\sigma, \tau, zd) \, dz \right| = \left| \int_0^1 \int_0^\tau (\sigma \Psi_{\sigma\tau} + d\Psi_{d\tau})(z\sigma, z', zd) \, dz' dz \right|$$

$$\lesssim (|\sigma| + |d|)\tau$$

$$\stackrel{(3.69)}{\lesssim} \left(|\sigma| + \ell_g|\widehat{\sigma}| + \int_{(j-1)\varepsilon_\nu}^{(j+1)\varepsilon_\nu} |\alpha| \right) \cdot \tau \tag{3.70}$$

since the derivatives $\partial_{\sigma\tau}|\sigma'-\sigma|$ and $\partial_{d\tau}|\sigma'-\sigma|$ are easily well defined for $\sigma \neq 0$ and locally bounded: notice that we differentiate only once the elementary curve of right states of the k-th family in its parameter and more times the strengths of the Riemann problem in the left / right states, thanks to the smoothness of the matrix A in (1.1).

Since (3.70) holds at each jump either of $\overline{u}(x)$ or $v(x)$, then by algebraic computations we get the thesis. $\quad\square$

Remark 13. *When $\overline{w} = \overline{u}$, then the proof of Lemma 12 states that where $\overline{u}$ has a jump of strength σ then the strength σ' of the corresponding jump in v by (3.70) satisfies*

$$0 < (1 - \mathcal{O}(1)\tau)\,\sigma \leq \sigma' \leq (1 + \mathcal{O}(1)\tau)\,\sigma \quad or \quad (1 + \mathcal{O}(1)\tau)\,\sigma \leq \sigma' \leq (1 - \mathcal{O}(1)\tau)\,\sigma < 0\,.$$

Moreover, if $\overline{u}$ does not any jump at $x = j\varepsilon_\nu$, then the new jump introduced because of the discontinuity of g_ν at $j\varepsilon_\nu$ satifies

$$\sum_{k=1}^{N} |\sigma_k''| \leq \tau \int_{(j-1)\varepsilon_\nu}^{(j+1)\varepsilon_\nu} |\alpha|\,,$$

where σ'' is the strenght of the new front of the k-th family emerging from $(\tau, j\varepsilon_\nu)$.

We are now able to present the proof of Theorem 11 above.

Proof. We first prove by induction that if

$$\overline{\delta} + G^2 T \leq \delta_H \tag{3.71}$$

then estimate (3.62) concerning $\Upsilon(w_\nu(t+, \cdot))$ holds when $0 < t < T$. We then prove the comparison with the exact viscous semigroup (3.63). For brevity, we denote $\Upsilon(t) := \Upsilon(w_\nu(t+, \cdot))$ all along this proof.

$\ominus$ STEP 1: Initial step of induction. We show, assuming (3.71), that

$$\Upsilon(\tau_\nu+) \;\leq\; \Upsilon(0) + G^2\tau_\nu \;\overset{(3.65)}{\leq}\; \Upsilon(\overline{u}(\cdot)) + G^2\tau_\nu\,. \tag{3.72}$$

In particular, this step shows that if $\Upsilon(\overline{u}(\cdot)) \leq \overline{\delta}$ then being $\tau_\nu < T$ one has

$$\Upsilon(\tau_\nu+) \leq \overline{\delta} + G^2\tau_\nu \leq \overline{\delta} + G^2 T \leq \delta_H\,.$$

In particular one can restart the iteration procedure for defining u_ν up to $2\tau_\nu+$. Observe first of all that estimate (3.65) allows to construct [3] the wave-front-tracking approximation $S_{t,0}^{FT}\overline{u}$ for all $t > 0$. By definition and by estimates (3.68)-(3.65) recalled above then

$$\Upsilon(\tau_\nu+) = \Upsilon(w_\nu(\tau_\nu+, \cdot)) \equiv \Upsilon\left(S_{\tau_\nu,0}^{FT}\overline{u}_\nu + \tau_\nu g\left(\tau_\nu, \cdot, S_{\tau_\nu,0}^{FT}\overline{u}_\nu\right)\right)$$

$$\overset{(3.68)}{\leq} \Upsilon\left(S_{\tau_\nu,0}^{FT}\overline{u}_\nu\right) + G^2\tau_\nu$$

$$\overset{(3.65)}{\leq} \Upsilon(\overline{u}_\nu) + G^2\tau_\nu$$

$$\leq \overline{\delta} + G^2\tau_\nu\,.$$

$\ominus$ STEP 2: Induction step I. Suppose that $\Upsilon((n-1)\tau_\nu+) \leq \delta_H$. We show that

$$\Upsilon(n\tau_\nu+) \leq \Upsilon((n-1)\tau_\nu+) + G^2\tau_\nu\,. \tag{3.73}$$

In this step we adopt the notation $w_\nu^{n-1}(\cdot) = w_\nu((n-1)\tau_\nu+, \cdot)$ for the approximation at time $(n-1)\tau_\nu+$. By definition and by the estimates (3.68)-(3.65) recalled above one has

$$\Upsilon(n\tau_\nu+) \equiv \Upsilon\left(S_{n\tau_\nu,(n-1)\tau_\nu}^{FT} w_\nu^{n-1} + \tau_\nu g\left(n\tau_\nu, \cdot, S_{n\tau_\nu,(n-1)\tau_\nu}^{FT} w_\nu^{n-1}\right)\right)$$

$$\overset{(3.68)}{\leq} \Upsilon\left(S_{n\tau_\nu,(n-1)\tau_\nu}^{FT} w_\nu^{n-1}\right) + G^2\tau_\nu$$

$$\overset{(3.65)}{\leq} \Upsilon\left(w_\nu^{n-1}\right) + G^2\tau_\nu = \Upsilon((n-1)\tau_\nu+) + G^2\tau_\nu\,.$$

$\ominus$ STEP 3: Conclusion of (3.62). We deduce that whenever (3.71) holds then

$$\Upsilon(t+) \leq \overline{\delta} + G^2 t \leq \delta_H \qquad \text{for } 0 < t < T. \tag{3.74}$$

In particular, we show that $w_\nu(k\tau_\nu+, \cdot)$ is well defined for all $0 \leq k\tau_\nu < T$.

By (3.65) and the definition of the approximation, it suffices to prove (3.74) at time-steps. Estimate (3.72) provides the thesis at the first time-step $t = \tau_\nu$. At later time-steps, the thesis follows by induction by (3.73).

⊖ STEP 4: Proof of (3.63). We recall [10, 11] that there exists a small enough $\bar{s} > 0$ for which one has the estimate

$$\forall s \leq \bar{s}, \forall \overline{w} \in \mathfrak{D}, \forall 0 \leq h \leq t - s \qquad \|P_{h+s,h}\overline{w} - S_{h+s,h}\overline{w} - sg(h, \cdot, \overline{w})\|_{L^1} \leq \mathcal{O}(1)s^2 . \tag{3.75}$$

By the triangular inequality

$$\|g(h, \cdot, \overline{w}) - g(h + s, \cdot, P_{h+s,h}\overline{w})\|_{L^1} \leq \|g(h, \cdot, \overline{w}) - g(h + s, \cdot, \overline{w})\|_{L^1}$$
$$+ \|g(h + s, \cdot, \overline{w}) - g(h, \cdot, P_{h+s,h}\overline{w})\|_{L^1}$$

Assumption (G) at Page 2 thus yields that for $s \to 0$, denoting by $o(s)$ a function such that $o(s)/s \to 0$,

$$\forall s \leq \bar{s}, \forall \overline{w} \in \mathfrak{D}, \forall 0 \leq h \leq t - s \qquad \|P_{h+s,h}\overline{w} - S_{h+s,h}\overline{w} - sg(h + s, \cdot, P_{h+s,h}\overline{w})\|_{L^1} \leq Co(s) . \tag{3.76}$$

Let's adopt the shortcut $w_\nu^n(\cdot)$ for $w_\nu(n\tau_\nu+, \cdot)$: then by (3.64) and the triangular inequality

$$\|w_\nu^n - P_{n\tau_\nu,(n-1)\tau_\nu}w_\nu^{n-1}\|_{L^1}$$
$$\equiv \|S_{n\tau_\nu,(n-1)\tau_\nu}^{FT}w_\nu^{n-1} + \tau_\nu g\left(n\tau_\nu, \cdot, S_{n\tau_\nu,(n-1)\tau_\nu}^{FT}w_\nu^{n-1}\right) - P_{n\tau_\nu,(n-1)\tau_\nu}w_\nu^{n-1}\|_{L^1}$$
$$\leq \|S_{n\tau_\nu,(n-1)\tau_\nu}w_\nu^{n-1} + \tau_\nu g\left(n\tau_\nu, \cdot, S_{n\tau_\nu,(n-1)\tau_\nu}^{FT}w_\nu^{n-1}\right) - P_{n\tau_\nu,(n-1)\tau_\nu}w_\nu^{n-1}\|_{L^1}$$
$$+ \|S_{n\tau_\nu,(n-1)\tau_\nu}^{FT}w_\nu^{n-1} - S_{n\tau_\nu,(n-1)\tau_\nu}w_\nu^{n-1}\|_{L^1}$$

We directly estimate the first addend by (3.76), the second addend by (3.66):

$$\|w_\nu^n - P_{n\tau_\nu,(n-1)\tau_\nu}w_\nu^{n-1}\|_{L^1} \leq \mathcal{O}(1)\left(o(\tau_\nu) + \varepsilon_\nu\tau_\nu\right)$$

The proof of (3.63) is thus concluded. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.2 Converge of time-step approximations to the viscous solution

Theorem 14. *Suppose there exists $0 < \delta < \delta_H$, $T > 0$ and a closed domain*

$$\mathfrak{D} := \mathfrak{D}(\delta) := \left\{u \in L^1(\mathbb{R}; \mathbb{R}^N) \cap BV(\mathbb{R}; \mathbb{R}^N) \ : \ \Upsilon(u) \leq \delta\right\}$$

such that

- *for all ν and for every piecewise-constant initial data $\overline{u} \in \mathfrak{D}$ the approximations w_ν constructed in § 2 satisfy estimates (3.62)-(3.63) in $[0, T]$ and*

- *for all $\overline{u} \in \mathfrak{D}$ and for $0 \leq t \leq T$ [10] provides a vanishing viscosity solution $P_{t,0}\overline{u}$ of the Cauchy problem (1.1)-(1.2).*

Then for every $\overline{u} \in \mathfrak{D}$ one can choose a suitable piecewise-constant approximation $\overline{w}_\nu \in \mathfrak{D}$ of $\overline{u}$ such that, denoting by w_ν the ν-approximation as in § 2 with initial datum $\overline{w}_\nu$, for a.e. $t \in [0, T]$ the sequence $w_\nu(t, \cdot)$ converges in $L^1(\mathbb{R}; \mathbb{R}^N)$ to $P_{t,0}\overline{w}$.

Proof. ⊖ STEP 1: Introduction. Let $S_{t,h}w$ denote the semigroup of the homogeneous system constructed by vanishing viscosity [6] where the 'initial datum' is fixed at time h rather than at $h = 0$. We recall [10, 11] that there exists $L > 0$ s.t. for $\overline{w}_1, \overline{w}_2 \in \mathfrak{D}$, $h \in [0, T]$, $t_1, t_2 \in [h, T]$ then

$$\|P_{t_1,h}\overline{w}_1(\cdot) - P_{t_2,h}\overline{w}_2(\cdot)\|_{L^1} \leq L\left(\|\overline{w}_1 - \overline{w}_2\|_{L^1} + |t_2 - t_1|\right) \tag{3.77}$$

and for $\overline{w} \in \mathfrak{D}$, $0 \leq t_0 \leq t_1 \leq t_2 \leq T$, one has the semigroup property

$$P_{t_2,t_0}\overline{w} \equiv P_{t_2,t_1}\left[P_{t_1,t_0}\overline{w}\right] \tag{3.78}$$

⊖ STEP 2: Strategy. In the spirit of [8, Theorem 2.9], fix any $0 \leq \bar{t} \leq T$ and define the auxiliary function

$$\Psi_\nu(t, \cdot) = P_{\bar{t},t}\left[w_\nu(t+, \cdot)\right] - P_{\bar{t},0}\overline{w}_\nu(\cdot) \qquad t \in [0, \bar{t}]$$

We now prove that $w_\nu(\bar{t}, \cdot)$ converges in L^1 to $P_{\bar{t},0}\overline{w}$ by showing that the following limit vanishes:

$$\lim_\nu\|w_\nu(\bar{t}, \cdot) - P_{\bar{t},0}\overline{w}(\cdot)\|_{L^1} \leq \lim_\nu\|w_\nu(\bar{t}, \cdot) - P_{\bar{t},0}\overline{w}_\nu(\cdot)\|_{L^1} + \lim_\nu\|P_{\bar{t},0}\overline{w}(\cdot) - P_{\bar{t},0}\overline{w}_\nu(\cdot)\|_{L^1}$$
$$\equiv \lim_\nu\|\Psi_\nu(\bar{t}, \cdot)\|_{L^1} + 0 .$$

The second addend indeed is trivially converging to 0 by (3.77) as $\overline{u}_\nu$ converges to $\overline{w}$ in L^1.

⊖ STEP 3: Estimates. Let $\bar{t} = n\tau_\nu + \hat{t}$ with $n \in \mathbb{N} \cup \{0\}$ and $\hat{t} \in [0, \tau_\nu)$. By the triangular inequality

$$\|\Psi_\nu(\bar{t}, \cdot)\|_{L^1} \equiv \|\Psi_\nu(\bar{t}, \cdot) - \Psi_\nu(0, \cdot)\|_{L^1}$$
$$\leq \sum_{k=0}^{n-1} \|\Psi_\nu((k+1)\tau_\nu, \cdot) - \Psi_\nu(k\tau_\nu, \cdot)\|_{L^1} + \|\Psi_\nu(\bar{t}, \cdot) - \Psi_\nu(n\tau_\nu, \cdot)\|_{L^1}$$
$$\leq \sum_{k=0}^{n-1} \|\Psi_\nu((k+1)\tau_\nu, \cdot) - \Psi_\nu(k\tau_\nu, \cdot)\|_{L^1} + \mathcal{O}(1)\tau_\nu . \tag{3.79}$$

In the last step we estimated the norm of $\Psi_\nu(\bar{t}, \cdot) - \Psi_\nu(n\tau_\nu, \cdot) \equiv w_\nu(\bar{t}+, \cdot) - P_{\bar{t}, n\tau_\nu}[w_\nu(n\tau_\nu+, \cdot)]$ by the Lipschitz continuity (3.75)-(3.66) exploiting the fact that $n\tau_\nu \leq \bar{t} < (n+1)\tau_\nu$ and thus by definition $w_\nu(\bar{t}, \cdot) \equiv S_{\bar{t}, n\tau_\nu}^{FT} w_\nu(n\tau_\nu+, \cdot)$. Moreover, by definition and the semigroup property (3.78)

$$\|\Psi_\nu((k+1)\tau_\nu, \cdot) - \Psi_\nu(k\tau_\nu, \cdot)\|_{L^1} \equiv \|P_{\bar{t}, (k+1)\tau_\nu}[w_\nu((k+1)\tau_\nu+, \cdot)] - P_{\bar{t}, k\tau_\nu}[w_\nu(k\tau_\nu+, \cdot)]\|_{L^1}$$
$$\equiv \|P_{\bar{t}, (k+1)\tau_\nu}[w_\nu((k+1)\tau_\nu+, \cdot)] - P_{\bar{t}, (k+1)\tau_\nu}[P_{(k+1)\tau_\nu, k\tau_\nu}[w_\nu(k\tau_\nu+, \cdot)]]\|_{L^1}$$
$$\overset{(3.77)}{\leq} L\|w_\nu((k+1)\tau_\nu+, \cdot) - P_{(k+1)\tau_\nu, k\tau_\nu}[w_\nu(k\tau_\nu+, \cdot)]\|_{L^1} .$$

Estimating this term by (3.63) and plugging this into (3.79) we finally deduce, being $n\tau_\nu = T$, that

$$\|\Psi_\nu(\bar{t})\|_{L^1} \leq \mathcal{O}(1)\tau_\nu + \sum_{k=0}^{n-1} \mathcal{O}(1)(o(\tau_\nu) + \varepsilon_\nu\tau_\nu) \leq \mathcal{O}(1)\tau_\nu + T\left(\frac{o(\tau_\nu)}{\tau_\nu} + \varepsilon_\nu\right) \xrightarrow{\nu\uparrow\infty} 0 .$$

This concludes the proof of the L^1-convergence.   □

4   Qualitative properties of the entropy solution

This section is devoted to the proof of Theorem 5 concerning the structure of solutions to balance laws when the characteristic fields are

- either linearly degenerate in the sense of Definition 3,

- or piecewise-genuinely nonlinear in the sense of Definition 4.

We work under the standard Lipschitz regularity assumption **(G)** at Page 2 on the source term, and we assume furthermore in this section that g only depends on the state variable:

$$g = g(u) .$$

The proof is by approximation, following ideas already in [9, 8, 7]. We construct suitable objects, estimates and arguments on the approximate solutions defined in § 2. Owing to the convergence result proved in § 3, we are then able to obtain our thesis in the limit. The section is organized as follows:

§ 4.1 Establishes balances for the positive/negative amount of i-waves in a space-time region.

§ 4.2 Defines sub-discontinuities of shocks for each piecewise genuinely nonlinear families.

§ 4.3.1 Defines the fractional-step approximations of i-shocks and i-contact discontinuities.

§ 4.3.2 Proves the limits in Theorem 5 at shocks of piecewise genuinely nonlinear families.

§ 4.3.3 Proves the limits in Theorem 5 at contact discontinuities of linearly degenerate families.

§ 4.3.4 proves the limits in Theorem 5 at continuity points.

§ 4.3.5 Contains elementary geometric lemmas on piecewise genuinely nonlinear families.

§ 4.3.6 Contains the proof of intuitive auxiliary lemmas.

4.1 Preliminary estimates: balances on characteristic regions

In this section we generalize balances for the flux of positive and negative waves of a fixed approximation u_ν which was constructed in §§ 2-3. These balances reduce to well known ones for the homogeneous system which are for example in [8, § 7.6].

We remind [8, § 7.6] in particular the definition of the interaction measure μ_ν^I and interaction-cancellation measure μ_ν^{IC}: they are purely atomic measures which are concentrated at interaction points of physical fronts belonging to two characteristic families i, j. If σ' and σ'' are the incoming strengths of the fronts interacting at a point P then, using the Definition 8 of amount of interaction, one has

$$\mu_\nu^I(\{P\}) := \mathcal{I}(\sigma', \sigma'') \tag{4.80a}$$

$$\mu_\nu^{IC}(\{P\}) := \mathcal{I}(\sigma', \sigma'') + \begin{cases} |\sigma'| + |\sigma''| - |\sigma' + \sigma''| & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{4.80b}$$

Of course $0 \leq \mu_\nu^I \leq \mu_\nu^{IC}$. We state that the interaction-cancellation measure can be controlled by $\mathcal{Q}$ even when a source term is present.

Lemma 15. *The interaction-cancellation measure satisfies the estimates:*

$$\mu_\nu^{IC}((t_1, t_2) \times \mathbb{R}) \lesssim \text{Tot.Var.}^-(\mathcal{Q}(u_\nu); (t_1, t_2)) \qquad \forall \, 0 \leq t_1 < t_2 \,,$$

where Tot.Var.$^-$ *is the negative total variation, and $\mathcal{Q}$ is the interaction potential defined at (2.59). In particular, μ_ν^{IC} is a locally bounded Radon measure.*

Proof. At an interaction point $P = (t, x)$ by classical interaction estimates, as recalled in Proposition 10

$$0 < \mu_\nu^{IC}(P) \lesssim |\mathcal{Q}(u_\nu(t-)) - \mathcal{Q}(u_\nu(t+))| \,.$$

At time updates $k\tau_\nu$ the interaction cancellation measure is null by construction, thus the thesis holds trivially. We also recall that we stay in a domain with small total variation. □

Consider a polygonal region Γ with edges transversal to the waves it encounters. Consider the total amount $W_{\text{in}}^{\nu i \pm}$, $W_{\text{out}}^{\nu i \pm}$ of positive and negative i-waves entering the region:

$$W_{\text{in}}^{\nu i \pm}(\Gamma) := \sum_{\text{entering } \Gamma} s_i^\pm \,, \quad W_{\text{out}}^{\nu i \pm}(\Gamma) := \sum_{\text{exiting } \Gamma} s_i^\pm \,, \quad s_i^\pm = \max\{\pm s_i, 0\} \,.$$

Define the incoming and outgoing flux of the i-th wave through the boundary of the region as

$$W_{\text{in}}^{\nu i} = W_{\text{in}}^{\nu i+} - W_{\text{in}}^{\nu i-}, \qquad W_{\text{out}}^{\nu i} = W_{\text{out}}^{\nu i+} - W_{\text{out}}^{\nu i-} \qquad i = 1, \dots, N.$$

Lemma 16. *There exists a positive constant C depending only on A, g, $\overline{\delta}$, T such that the following holds. If $\Gamma \subset [h\tau_\nu, k\tau_\nu] \times \mathbb{R}$, for some $h, k \in \{0, 1, \dots, \text{floor}(T/\tau_\nu)\}$, then for $i = 1, \dots, N$ one has the estimate*

$$e^{-(k-h)\tau_\nu C} \left[W_{\text{in}}^{\nu i+} - C\mu_\nu^{IC}(\overline{\Gamma}) \right] \leq W_{\text{out}}^{\nu i+} \leq e^{(k-h)\tau_\nu C} \left[W_{\text{in}}^{\nu i+} + C\mu_\nu^{IC}(\overline{\Gamma}) \right] \,, \tag{4.81a}$$

$$e^{-(k-h)\tau_\nu C} \left[W_{\text{in}}^{\nu i-} - C\mu_\nu^{IC}(\overline{\Gamma}) \right] \leq W_{\text{out}}^{\nu i-} \leq e^{(k-h)\tau_\nu C} \left[W_{\text{in}}^{\nu i-} + C\mu_\nu^{IC}(\overline{\Gamma}) \right] \,. \tag{4.81b}$$

Proof. Waves might change only at interaction times and at update times. Denote by

$$W^{\nu i+}(t), \quad W^{\nu i-}(t)$$

respectively the positive and negative i-waves of u_ν present in Γ at time t.

⊖ STEP 1: Interaction times t. Denote by σ' and σ'' the incoming strengths of the fronts interacting at a point $P \in \Gamma$. By interaction estimates [3, Lemma 1], as in [8, (7.98)], one has

$$|W^{\nu i+}(t+) - W^{\nu i+}(t-)| + |W^{\nu i-}(t+) - W^{\nu i-}(t-)| \lesssim \mu_\nu^{IC}(\{P\}) \,.$$

⊖ STEP 2: Update time t. We can assume that no front enters / exits Γ at the update time. Since we are considering $g = g(u)$, then denoting by τ_ν the time-step, by Remark 13 in the construction of the approximation one has precisely as in [2, (2.8)] that

$$\left| W^{\nu i \pm}(t+) - W^{\nu i \pm}(t-) \right| \lesssim \tau_\nu W^{\nu i \pm}(t-) \,.$$

In particular, if in the interval $[t_1, t_2)$ there is no interaction and no wave enters / exits Γ, then

$$e^{-C(k-h)\tau_\nu} W^{\nu i \pm}(t_1-) \leq W^{\nu i \pm}(t_2-) \leq e^{C(k-h)\tau_\nu} W^{\nu i \pm}(t_1-)$$

where $h\tau_\nu \leq t_1 < t_2 \leq k\tau_\nu$ for some $h, k \in \mathbb{N}$.

⊖ STEP 3: Conclusion. Combining in a rough way the estimates in the previous steps, and since taking into account that waves might enter later than $h\tau_\nu$ or might exit before $k\tau_\nu$ the estimate would just be finer, we get the thesis by standard calculus. □
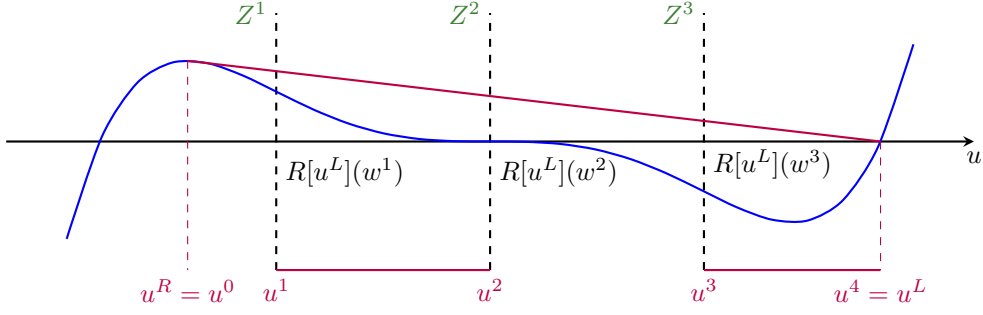
Figure 2: We consider a jump $[u^L, u^R]$ where $u^R = T_i[u^L](s)$ with $s < 0$. We highlight different points belonging to the hyper-surfaces Z_i^1, Z_i^2 and Z_i^3: • the points $R_i[u^L](w^1)$, $R_i[u^L](w^2)$, $R_i[u^L](w^3)$ of intersection with the rarefaction curve through u^L and • the points u^1, u^2, u^3 in (4.82) which identify the $(i, 3)$ and $(i, 1)$ sub-discontinuity fronts $[u^4, u^3]$, $[u^2, u^1]$.
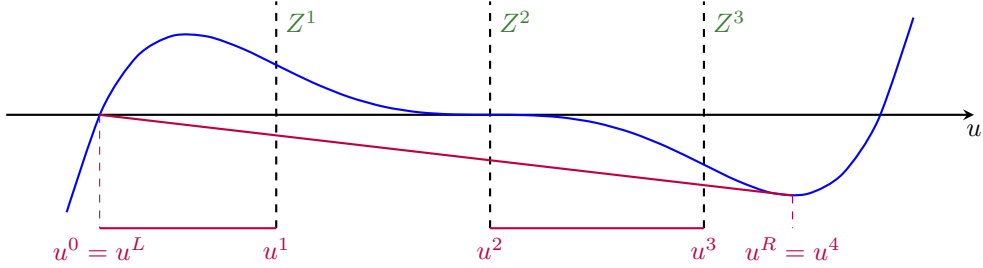

Figure 3: We consider a jump $[u^L, u^R]$ where $u^R = T_i[u^L](s)$ with $s > 0$. The $(i, 0)$ and $(i, 2)$ sub-discontinuities in this case are $[u^0, u^1]$ and $[u^2, u^3]$, where u^1, u^2, u^3 are defined in (4.82).

4.2   Definition of approximate sub-discontinuity curves

Assume that the ith-characteristic field is piecewise genuinely nonlinear as in Definition 4. Let us directly assume that $\omega_i^1[u^-], \ldots, \omega_i^{J_i}[u^-]$, defined as follows, are monotone increasing:

$$\omega_i^j[u^-] \ : \ R_i[u^-](\omega_i^j[u^-]) \in Z_i^j, \quad j = 1, \ldots, J_i, \qquad \omega_i^0[u^-] := -\infty, \ \omega_i^{J_i+1}[u^-] := +\infty$$

where we remind that the hyper-surfaces Z_i^j are the connected components of

$$Z_i = \{u : \nabla \lambda_i(u) \cdot r_i(u) = 0\} = \bigcup_{j=1}^{J_i} Z_i^j \ .$$

We directly assume, since the analysis of the other case is perfectly analogous, that

$$\begin{cases} \nabla \lambda_i(u) \cdot r_i(u) < 0 & \text{if } j \text{ is even and } \omega_i^j[u] < 0 < \omega_i^{j+1}[u], \\ \nabla \lambda_i(u) \cdot r_i(u) > 0 & \text{if } j \text{ is odd and } \omega_i^j[u] < 0 < \omega_i^{j+1}[u]. \end{cases}$$

For these piecewise genuinely nonlinear fields, following [9, 8, 25, 7] we now define approximate sub-discontinuities of a fixed approximation u_ν which was constructed in §§ 2-3. In particular, we extend [7] in the presence of a Lipschitz continuous source term $g = g(u)$.

Let $[u^L, u^R]$ be a wavefront of u_ν belonging to the i-th family, where $u^R = T_i[u^L](s)$. Suppose for instance that $s > 0$, which means $u_i^R > u_i^L$. When the i-th field is piecewise genuinely nonlinear one can split $[u^L, u^R]$ into sub-discontinuities: if $\bar{u}$ is the function defined in (2.40) for the construction of the Rieman solver, then since $\tau \mapsto \bar{u}(\tau; u^L, s)$ is transversal to Z_i^j there are

$$0 \leq \tau^{j_1} < \cdots < \tau^{j_2} \leq s \quad : \quad u^{j_1+k} := \bar{u}(\tau^{j_1+k}; u^L, s) \in Z_i^{j_1+k} \qquad k = 0, \ldots, j_2 - j_1. \tag{4.82}$$

When u^L, u^R do not belong to any Z_i^j we still need to include the extremal points: set

- $\tau^{j_1-1} = 0$ and $u^{j_1-1} = u^L$ in case $\tau^{j_1} > 0$;

- $\tau^{j_2+1} = s$ and $u^{j_2+1} = u^R$ in case $\tau^{j_2} < s$.

If instead $s < 0$ the definition is analogous with $0 \geq \tau^{j_2} > \cdots > \tau^{j_1} \geq s$, where $j_2 > \cdots > j_1$.

Definition 17. *Suppose that $s_i^j = \tau^{j+1} - \tau^j \neq 0$: then we call an (i,j)-sub-discontinuity of strength s_i^j of the i-th wavefront $[u^L, u^R]$ of u_ν*

- $[u^j, u^{j+1}]$, *if $s > 0$ and j is even, or*

- $[u^{j+1}, u^j]$, *if $s < 0$ and j is odd.*

Notice that, by definition, the state vector of (i,j)-sub-discontinuities belongs to the part of the wavefront where the i-th eigenvalue is decreasing. Rarefaction fronts are instead contained in regions where the i-th eigenvalue is increasing across the discontinuities.

One of the reasons to introduce sub-discontinuities when the flux is not genuinely nonlinear, but only piecewise genuinely nonlinear, is that discontinuities might split either at interaction times or at update times. Since the approximate solution of a Riemann problem contains at most one sub discontinuity s_i^j for $j \in \{0, 1, \ldots, J_i\}$ [7, Lemma 4.3], sub-discontinuities do not.

The next step is to identify which sub-discontinuities in the approximation u_ν are in the limit converging to a sub-discontinuity of the entropy solution u: we call these 'surviving' discontinuities "approximate discontinuities". We fix for this purpose thresholds β and $\beta/2$.

Definition 18. *Let $\beta > 0$. A maximal, leftmost (β, i, j)-approximate sub-discontinuity curve is any maximal (concerning set inclusion) closed polygonal line—parametrized with time in the (t, x)-plane—with nodes (t_0, x_0), $(t_1, x_1), \ldots, (t_n, x_n)$, where $t_0 \leq \cdots \leq t_n$, such that*

1. *each node (t_k, x_k), $k = 1, \ldots, n$ is an interaction point or an update time;*

2. *the segment $[(t_{k-1}, x_{k-1}), (t_k, x_k)]$ is the support of an (i,j)-sub-discontinuity front with strength $|s_i^j| \geq \beta/4$ and there is at least one time $t \in [t_0, t_n]$ such that $|s_i^j| \geq \beta$; the index j must be either odd if the strength of the i-th jump $s_i > 0$ or j must be even if $s_i < 0$;*

3. *it stays on the left of any other polygonal line it intersects and having the above properties.*

We write an interaction estimate for sub-discontinuities in order to familiarize with them.

Lemma 19. *For any compact $K \subset \Omega \subseteq \mathbb{R}^N$ there exist constants C_1, C_2 and χ_1 so that: Consider an interaction between a i-front strength $|s_i|$ and a j-front of strength $|s_j|$ for $i \neq j$ and $i, j \in \{1, \ldots, N\}$. Let u^L / u^R denote the left / right states of u_ν at that interaction, which belong to K, and let $s_1^+, \ldots, s_N^+$ be the outgoing strengths, so that*

$$u^R = T_N[T_{N-1}[\ldots T_2[T_1[u^L](s_1^+)](s_2^+)\ldots](s_{N-1}^+)](s_N^+).$$

Then, calling $\overline{P}$ the point of interaction, there holds

$$|s_i^+ - s_i| + |s_j^+ - s_j| + \sum_{\ell \neq i,j} |s_\ell^+| \leq C_1 \mathcal{I}(s_i, s_j) \leq C_1 |s_i s_j| = C_1 \mu_\nu^I(\overline{P}).$$

Moreover, consider at each node $\overline{P}$ which is a point of interaction the strengths s_i^{k-} / s_i^{k+} of each incoming / outgoing (i, k)-sub-discontinuity, possibly except for the first one (t_0, x_0): they satisfy

$$|s_i^{k+} - s_i^{k-}| \leq \frac{C_2}{\beta} \mu_\nu^I(\overline{P}). \tag{4.83}$$

Proof. By classical interaction estimates, and the definition of the interaction measure, we just need to prove the last inequality concerning sub-discontinuities: the first part of the statement indeed is just by construction. The proof of (4.83) is a consequence of the fact that the strength s_i^k of any (i, k) sub-discontinuity is Lipschitz continuous in the left and right states of the i-front, together with the estimates below. If u'^L and u'^R are the left / right states of the i-front after the interaction then

$$|u'^L - u^L| + |u'^R - u'^R| \lesssim |s_j|.$$

Since $|s_i| \geq |s_i^k| \geq \beta/4$ by Definition 18 of (i, k)-sub-discontinuity, and since $\mu_\nu^I(\overline{P}) = |s_i s_j|$ due to the fact that $i \neq j$, then $|s_j| \leq \frac{|s_i|}{\beta/4}|s_j| \leq \frac{4\mu_\nu^I(\overline{P})}{\beta}$, from which we get the thesis. $\square$

We are now able to determine a countable family $\mathcal{J}_{\beta,i}^j(\nu)$ of maximal, leftmost (β, i, j)-approximate sub-discontinuity curves of u_ν which will in the limit define the family of curves $\mathcal{J}$ in the statement of Theorem 5. This is due to the fact that when β is fixed then the cardinality of maximal, leftmost (β, i, j)-approximate sub-discontinuity curves of u_ν is, definitively as $\nu \uparrow \infty$, bounded by a constant $M_{\beta,i,j}$ independent of ν thanks again to the bounds on the total variation—and of course up to a fixed finite time. Notice that the set of curves $\mathcal{J}_{\beta,i}^j(\nu)$ enriches as $\beta \downarrow 0$.

Lemma 20. *When the threshold β is fixed, then the cardinality $\sharp \mathcal{J}_{\beta,i}^{j}(\nu) =: M_{\beta,i,j}(\nu)$ of maximal, leftmost (β, i, j)-approximate sub-discontinuity curves—up to any fixed positive time—is uniformly bounded in ν, and thus also in $i = 1, \ldots, N$ and j: it is of order $M_{\beta,i,j}(\nu) \lesssim \beta^{-2}$.*

Proof. Suppose that—up to the fixed time $\overline{T}$ we are considering—the total variation of every u_ν is less than $\overline{V}$, which is possible by Theorem 11.

We begin fixing notations. Fix any admissible triple of indices β, i, j. Consider a maximal, leftmost (β, i, j)-approximate sub-discontinuity curve $\overline{\gamma}$ of u_ν for ν large enough: denote

- by $|s_i^j(t)|$ the strength of the (β, i, j)-approximate sub-discontinuity $\overline{\gamma}$ of u_ν,

- by $\overline{P}_k = (\overline{t}_k, \overline{x}_k)$ the nodes of the (β, i, j)-approximate sub-discontinuity $\overline{\gamma}$ of u_ν, and

- by $|s_i(t)|$ the strength of the whole discontinuity $\overline{\gamma}$ of u_ν, for $\overline{t}_0 < t < \overline{t}_n$.

Suppose $\beta < \frac{1}{1000} \text{dist}(Z_i^j, Z_i^{j+1})$ and $\varepsilon_\nu \ll 10\beta \ll 1$: this is allowed since $\varepsilon_\nu \downarrow 0$ and since decreasing β increases the number of maximal, leftmost (β, i, j)-approximate sub-discontinuity curves. Fix for example $s_i^j > 0$ for notational convenience, the other case being similar.

Before proving Lemma 20, we remind relevant estimates at nodes:

- By Lemma 19 if $\overline{\gamma}$ interacts with an i-front of a different characteristic family

$$|s_i^j(\overline{t}_k+) - s_i^j(\overline{t}_k-)| \leq \frac{C_2}{\beta} \mu_\nu^{IC}(\overline{P}_k) . \tag{4.84}$$

- At any update time $\overline{t}_k$, denoting by C the constant given by Remark 13, one has

$$|s_i^j(\overline{t}_k+) - s_i^j(\overline{t}_k-)| < C\tau_\nu |s_i(\overline{t}_k-)| < C\tau_\nu \overline{V} . \tag{4.85}$$

- Let u^L / u^R denote the left and right value of the maximal, leftmost (β, i, j)-approximate sub-discontinuity curve $\overline{\gamma}$ that we are considering. We make preliminary observations concerning interactions among i-waves, since we are interested only in the strength of the j-th component, before providing complete estimates:

  1. Suppose that $u^R \in Z_i^{j+1}$ and $u^L \in Z_i^j$ between $\overline{t}_{k-1}$ and $\overline{t}_k$. Then: • In case $\overline{\gamma}$ interacts at $\overline{P}_k$ with another i-shock then we have that $u^R \in Z_i^{j+1}$ and $u^L \in Z_i^j$ also for $\overline{t}_k < t < \overline{t}_{k+1}$. If the fixed β is smaller than $\frac{1}{1000} \text{dist}(Z_i^j, Z_i^{j+1})$, this proves that the strength of the (β, i, j)-approximate sub-discontinuity is more than β at both times $\overline{t}_k^-$ and $\overline{t}_k^+$. • In case $\overline{\gamma}$ interacts at $\overline{P}_k$ with an i-rarefaction, since the strength of rarefactions is vanishingly small again the strength of the (β, i, j)-approximate sub-discontinuity is more than β at both times $\overline{t}_k^-$ and $\overline{t}_k^+$ owing to the condition $\beta < \frac{1}{1000} \text{dist}(Z_i^j, Z_i^{j+1})$.

  2. If $u^R \notin Z_i^{j+1}$ for $\overline{t}_{k-1} < t < \overline{t}_k$, then $u^R = u_\nu(t, \overline{\gamma}(t)+)$ is the terminal value of the i-jump in $\overline{\gamma}(\overline{t}_k-)$. In particular, if $\overline{\gamma}$ interacts at $\overline{P}_k$ with another i-front, by the classical analysis of interactions roughly

$$s_i^j(\overline{t}_k+) \geq s_i^j(\overline{t}_k-) \qquad \text{(interactions with shocks)}$$
$$s_i^j(\overline{t}_k-) - \mu^{IC}(\overline{P}_k) \leq s_i^j(\overline{t}_k+) \leq s_i^j(\overline{t}_k-) \qquad \text{(interactions with rarefactions)}$$

  which more precisely becomes

$$s_i^j(\overline{t}_k-) - s_i^j(\overline{t}_k+) \lesssim \mu^{IC}(\overline{P}_k) . \tag{4.87}$$

We can start now with the principal argument. Collecting estimates (4.84), (4.85), (4.87) at nodes, if $u_\nu^\pm(\overline{\gamma})$ between times q and $t > q$ is valued strictly between Z_i^j and Z_i^{j+1} we obtain

$$s_i^j(q) - s_i^j(t+) \leq \frac{C_2 \mu_\nu^{IC}(\{(r, z) \; : \; r \in (q, t] \; , \; z = \overline{\gamma}(r)\})}{\beta} + C\overline{V}(t - q + \tau_\nu) . \tag{4.88}$$

We are now able to estimate the number of maximal, leftmost (β, i, j)-approximate sub-discontinuity curves. Fix before intermediate times $k\overline{t}$ where

$$k = 0, \ldots, K \; , \quad K := \text{ceil}\left(\frac{8\overline{T}C\overline{V}}{\beta}\right) \; , \qquad \overline{t} = \frac{\overline{T}}{K} \quad \Rightarrow \quad \overline{t}C\overline{V} \leq \frac{\beta}{8} . \tag{4.89}$$

Point 3 in Definition 18 requires that different maximal, leftmost (β, i, j)-approximate sub-discontinuity curves—when the triple is fixed—are disjoint. This disjointness yields that:

- At time $k\overline{t}$ by Point 2 in Definition 18 there are at most $\text{ceil}(4\overline{V}/\beta)$ many of them.

- Those (β, i, j)-approximate sub-discontinuity curves whose interaction-cancellation measure is more than $\frac{\beta^2}{8C_2}$ are at most ceil $\left(\frac{8C_2\mu^{IC}([0,T])\times\mathbb{R}}{\beta^2}\right)$ by sub-additivity of measures.

- The strength of the j-th component of those (β, i, j)-approximate sub-discontinuity curves $\overline{\gamma}$ which are defined strictly between times $k\overline{t}$ and $(k+1)\overline{t}$ increases from a value less than $\beta/4$ at $k\overline{t}$ to a value at least β at some $\hat{t} \in (k\overline{t}, (k+1)\overline{t})$, and decreases to a value less than $\beta/4$ at $(k+1)\overline{t}$: then in some subinterval of $(\hat{t}, (k+1)\overline{t})$ estimate (4.88) yields

$$\frac{3\beta}{4} < \frac{C_2\mu_\nu^{IC}(\{(r,z)\ :\ z = \overline{\gamma}(r)\})}{\beta} + C\overline{V}t \overset{(4.89)}{\Rightarrow} \mu_\nu^{IC}(\{(r,z)\ :\ z = \overline{\gamma}(r)\}) > \frac{\beta^2}{8C_2}\ .$$

We thus estimate the number of (β, i, j)-approximate sub-discontinuity curves up to time $\overline{T}$ by

$$\text{ceil}\left(\frac{4\overline{V}}{\beta}\right) \cdot \text{ceil}\left(\overline{T}\frac{8C\overline{V}}{\beta}\right) + \text{ceil}\left(\frac{8C_2\mu^{IC}([0,T])\times\mathbb{R})}{\beta^2}\right) \qquad\qquad \square$$

4.3 Proof of the global structure of solutions

The proof of the global structure of solutions stated in Theorem 5 above proceeds distinguishing the case of piecewise-genuinely nonlinear fields and the case of linearly degenerate fields. The reason is not only that estimates are different in the two cases, but really the geometry of the approximation of shocks and of contact discontinuities with a family of jumps in u_ν is qualitatively different. For each case, we will then have sub-cases extending the analysis in [8, 7].

We proceed in the next subsections with the core of the proof.

4.3.1 Constructing the exceptional sets and limit discontinuity curves

Fix a sequence $\varepsilon_\nu \downarrow 0$ and let u_ν be the approximate solution of the Cauchy problem (1.1)-(1.2) constructed in §§ 2-3. By possibly extracting a subsequence, we can assume that the interaction and interaction-cancellation measures (4.80) converge weakly* to some nonnegative measures μ^I and μ^{IC}:

$$\mu_\nu^I \rightharpoonup \mu^I \geq 0\ , \qquad \mu_\nu^{IC} \rightharpoonup \mu^{IC} \geq 0\ .$$

Of course μ^{IC} might change changing the sequence $\{u_\nu\}_\nu$. Define now the exceptional sets

$$\Theta_0 := \{(0, x)\ :\ \overline{u}(x+) \neq \overline{u}(x-)\}\ , \qquad \Theta_1 := \{(t, x)\ :\ \mu^{IC}(\{(t, x)\}) > 0\} \qquad (4.90)$$

Piecewise genuinely nonlinear fields Fix a threshold $\beta > 0$. Suppose the i-th field is piecewise genuinely nonlinear. Let

$$\mathcal{J}_{\beta,i}(\nu) := \{\gamma_{\nu,ik}\}_{k=1}^{M_\beta}$$

be the family of all maximal, leftmost (β, i, j)-approximate sub-discontinuity curves in the approximate solution u_ν defined in § 4.2, if needed with repetitions of the curves. This enumeration is possible as the number these curves is uniformly bounded in i, j, ν by Lemma 20.

Suppose $\gamma_{\nu,ik} : (t_{\nu,ik}^-, t_{\nu,ik}^+) \to \mathbb{R}$ belong to some $\mathcal{J}_{\beta,i}(\nu)$ where i, k are fixed, with the i-th family piecewise genuinely nonlinear. One can then assume that $t_{\nu,ik}^-$ and $t_{\nu,ik}^+$ converge to t_{ik}^- and t_{ik}^+, respectively, and that the curves $\gamma_{\nu,ik}$ converge locally uniformly on (t_{ik}^-, t_{ik}^+) to some curve γ_{ik} by Ascoli-Arzela theorem as $\nu \uparrow \infty$. Denote the family of such limit curves, which possibly contain repetitions, by

$$\mathcal{J}_{\beta,i} := \{\gamma_{ik}\}_{k=1,\ldots,M_\beta}\ .$$

Linearly degenerate fields Suppose the i-th field is linearly degenerate: we follow the construction in [8, Page 221], that we repeat for completeness. Call $x_{i,\nu}(t; \overline{t}, \overline{x})$ the i-th characteristic curve through the point $(\overline{t}, \overline{x})$, defined by

$$\dot{x}_{i,\nu}(t; \overline{t}, \overline{x}) = \lambda_i(u_\nu(t, x_{i,\nu}(t; \overline{t}, \overline{x})))\ , \quad x_{i,\nu}(\overline{t}; \overline{t}, \overline{x}) = \overline{x}\ .$$

This is allowed by linear degeneracy of the i-th field. Up to extracting a subsequence, one can assume that

$$x_{i,\nu}(t; \overline{t}, \overline{x}) \quad \to \quad x_i(t; \overline{t}, \overline{x}) \qquad \text{as } \nu \uparrow \infty.$$

Denote by $\overline{\mu}_\nu^{i\pm}$ the measures of the positive and negative i-waves in the approximation $u_\nu(0+, \cdot)$ of the initial datum $\overline{u}$ and by $\overline{\mu}^{i\pm}$ the measures of the positive and negative i-waves in the initial datum $\overline{u}$. As $\nu \uparrow \infty$, by suitably choosing $\overline{u}_\nu$ we can assume the weak*-convergence

$$\overline{\mu}_\nu^{i\pm} \rightharpoonup \overline{\mu}^{i\pm}\ .$$

Define the measure of the total amount of positive and negative i-waves on $u(0, \cdot)$ present in an interval (a, b) plus the total amount of interaction and cancellation occurring in the corresponding forward strip as

$$\mu_\nu^{i*}((a,b)) := \overline{\mu}^{i+}((a,b)) + \overline{\mu}^{i-}((a,b)) + \mu^{IC}\left(\{(t,x) \; : \; t > 0, \; x_{i,\nu}(t;0,a) < x < x_{i,\nu}(t;0,b)\}\right) \; .$$

We finally define for positive times the family of *at most* countably many curves

$$\mathcal{J}_i = \{\gamma_{im} := x_i(\cdot;0,\xi_{im}) \quad : \quad \mu^{i*}(\{\xi_{im}\}) > 0 \; , \quad m \in \mathbb{N}\} \tag{4.91a}$$

where μ^{i*} are constructed as follows. By upper semicontinuity properties of w^*-convergence of nonnegative measures, we define points

$$\xi_{\nu,im}^\pm \to \xi_{\nu,im} \quad \text{such that} \quad \mu_\nu^{i*}\left([\xi_{\nu,im}^-, \xi_{\nu,im}^+]\right) \xrightarrow{\nu\uparrow\infty} \mu^{i*}(\{\xi_{im}\}) \quad \text{as } \nu\uparrow\infty \tag{4.91b}$$

and consequently approximating regions $\{\gamma_{\nu,im}^-(t) \le x \le \gamma_{\nu,im}^+(t)\}$ where the curves $\gamma_{\nu,im}^\pm$ are

$$\mathcal{J}_i^-(\nu) := \left\{\gamma_{\nu,im}^- := x_{i,\nu}\left(\cdot;0,\xi_{\nu,im}^-\right)\right\} \; , \quad \mathcal{J}_i^+(\nu) := \left\{\gamma_{\nu,im}^- := x_{i,\nu}\left(\cdot;0,\xi_{\nu,im}^+\right)\right\} \; . \tag{4.91c}$$

Notice that we can also assume that $\gamma_{\nu,im}^-$ and $\gamma_{\nu,im}^+$ converge to γ_{im} locally uniformly. We stress that, differently from the case of shocks for piecewise genuinely nonlinear fields, contact discontinuities in u are not necessarily approximated by discontinuities in u_ν with strength definitively above a fixed threshold: they might be approximated by an increasing number of small discontinuities of u_ν in a region shrinking to the limit curve.

Exceptional set and family of limit curves   Denote by Θ_2 the subset of the plane where two limit curves $\gamma_{ik}, \gamma_{i'k'}$ belonging to different characteristic families $i \ne i'$ cross each-other:

$$\Theta_2 = \left\{\left(\overline{t}, \overline{x}\right) \; : \; \exists i, i', k, k' \quad i \ne i' \mid \overline{x} = \gamma_{ik}\left(\overline{t}\right) = \gamma_{i'k'}\left(\overline{t}\right)\right\}$$

We list Θ_2 separately for being more explicit, but by Lemma 23 below one proves $\Theta_2 \subset \Theta_1$. Define the exceptional set Θ and the family of curves $\mathcal{J}$ in the statement of Theorem 5 as

$$\Theta := \Theta_0 \cup \Theta_1 \cup \Theta_2 \; , \qquad \mathcal{J} := \bigcup_{\substack{i \text{ linearly} \\ \text{degenerate}}} \mathcal{J}_i \cup \bigcup_{\substack{i \text{ pw genuinely} \\ \text{nonlinear}}} \cup_{n\in\mathbb{N}}\mathcal{J}_{\frac{1}{n},i} \; ,$$

where Θ_0, Θ_1, Θ_2 and $\mathcal{J}_i$, $\mathcal{J}_{\beta,i}$ are defined just above.

4.3.2   Jumps: Piecewise-genuinely nonlinear fields

Let $\overline{P} = (\overline{t}, \overline{x}) \notin \Theta$ be a point along a curve $\overline{\gamma} \in \mathcal{J}_\beta$ of a piecewise-genuinely nonlinear family i: since u_ν is converging to u in L^1 and by construction of these limit curves, in $\overline{P}$ there must be a jump of the limit function u of strength at least $\beta/4$—see Definition 18. We prove below the sided limits of u at $\overline{P}$ stated in Theorem 5, while instead the slope and entropy condition of the jump can be then deduced precisely following [8, Step 7, Page 227], therefore we omit the proof here.

Simple jumps   Let $\overline{P} = (\overline{t}, \overline{x})$ be a point along a simple i-shock curve $\overline{\gamma} \in \mathcal{J}_\beta$ of a piecewise-genuinely nonlinear family such that $\mu^{IC}(\{\overline{P}\}) = 0$. Even if we do not need it, we remind that by the tame oscillation condition—see [11, Lemma 2.3] jointly with Theorem 14—one can define

$$u^L = \lim_{\substack{(t,x)\to(\overline{t},\overline{x}) \\ \overline{t}\le t\le\overline{t}+(\overline{x}-x)/\hat{\lambda}}} u(t,x) \; .$$

Let $\overline{\gamma}_\nu$ be a maximal, leftmost (β, i, j)-approximate sub-discontinuity curve converging to $\overline{\gamma}$.

Suppose that Theorem 5 fails: equivalently, suppose by contradiction that

$$\lim_{r\downarrow 0}\limsup_{\nu\uparrow\infty}\left(\sup_{\substack{|t-\overline{t}|+|x-\overline{x}|\le r \\ x<\overline{\gamma}_\nu}}\left|u_\nu(t,x) - u^L\right|\right) > \varepsilon > 0 \; .$$

By possibly extracting a subsequence and supposing that we suitably normalize the pointwise representative of the solution, we directly assume there are points $P_\nu = (\overline{t}_\nu, \overline{p}_\nu)$ and $Q_\nu = (\overline{t}_\nu, \overline{q}_\nu)$ on the left of $\overline{P}$, i.e. satisfying $\overline{p}_\nu < \overline{q}_\nu < \overline{\gamma}_\nu(\overline{t}_\nu)$), such that

$$Q_\nu \xrightarrow{\nu\uparrow\infty} \overline{P} \; , \qquad |u_\nu(Q_\nu) - u^L| \ge \varepsilon \quad \forall\nu,$$

$$P_\nu \xrightarrow{\nu\uparrow\infty} \overline{P} \; , \qquad |u_\nu(P_\nu) - u^L| \xrightarrow{\nu\uparrow\infty} 0 \; .$$

The segment $P_\nu Q_\nu$ must then be crossed by a large amount of waves. As in [8, 7] we show below that in any neighborhood of $\overline{P}$ these waves either interact among themselves or with $\overline{\gamma}_\nu$. As a result, $\mu^{IC}(\{\overline{P}\})$ cannot vanish and therefore we reach an absurd. Two cases are possible:
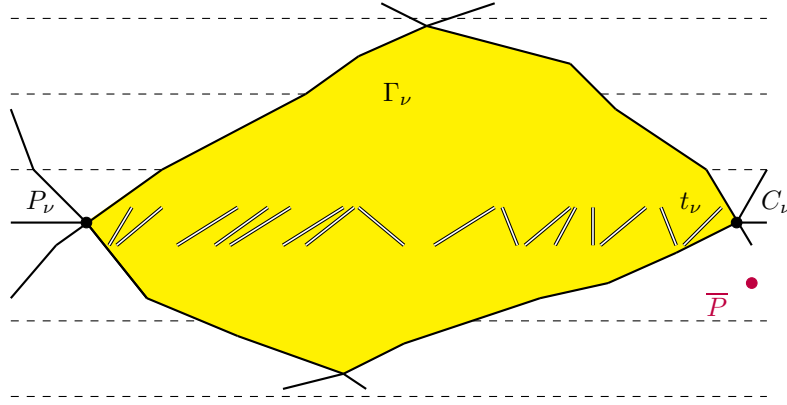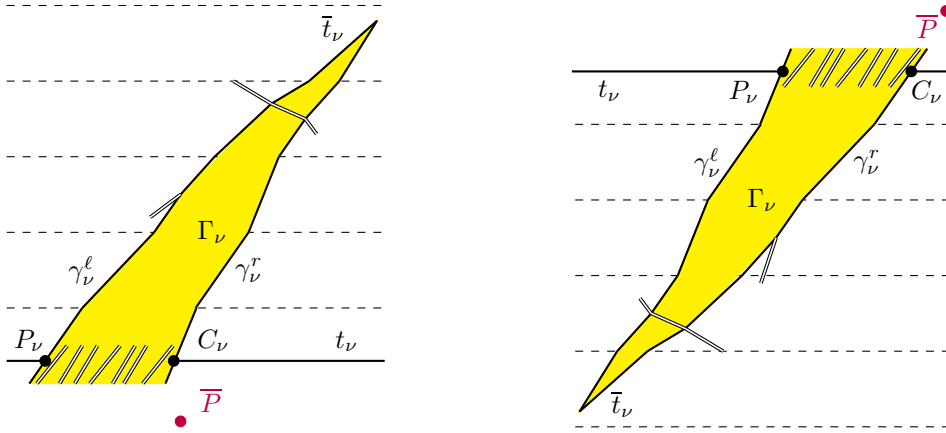
Figure 4: Lemma 23: there is a large amount of waves in at least two distinct families.

Figure 5: Lemma 24. There might be one (β, i, j)-approximate sub-discontinuity curve merging.

Case 1 (Figure 4) Each segment $P_\nu Q_\nu$ is crossed by a fixed amount of k-waves in u_ν for some fixed $k \neq i$. Replacing Q_ν by the point $(\bar{t}_\nu, \overline{\gamma}_\nu(\bar{t}_\nu) + c_\nu)$, for a suitable $c_\nu \downarrow 0$, Lemma 23 below applies with the families i and k and it contradicts the assumption $\mu^{IC}(\{\overline{P}\}) = 0$, i.e. $\overline{P} \notin \Theta_2$, because there is a positive amount of interactions among different families.

Case 2 (Figure 5) Suppose that

- the j-th component of i-fronts in u_ν other than $\overline{\gamma}_\nu$, crossing each segment $P_\nu Q_\nu$ has total strength more than ε but

- in any region $\overline{\Gamma}_\nu$ shrinking to $\overline{P}$ when $\nu \uparrow \infty$ for all $k \neq i$ the amount of k-waves converges to 0 and

- in any region $\overline{\Gamma}_\nu$ shrinking to $\overline{P}$ also the maximum strength of the j-th component of i-fronts other than $\overline{\gamma}_\nu$ is vanishingly small.

Lemma 24 below then applies yielding that there is a uniformly positive amount of interaction-cancellation in the i-th family in $\overline{P}$: it contradicts the assumption $\overline{P} \notin \Theta_2$, since this would imply $\mu^{IC}(\{\overline{P}\}) = 0$, and therefore it ends the proof.

Composite waves If the limit wave $\overline{\gamma}$ is composite, then one can apply the argument above to each sub-component of the jump, as in [7], since Lemma 23 and Lemma 24 below still apply. Of course, the limiting value of u depends on the j-th component we are considering and therefore on the respective (t, x)-region where the limit is taken.

4.3.3 Jump points: Linearly degenerate fields

Suppose now the i-th characteristic field is linearly degenerate. Let $\overline{P} = (\bar{t}, \overline{x})$ be a point along a curve $\overline{\gamma} \in \mathcal{J}$ as constructed in (4.91) and such that $\mu^{IC}(\{\overline{P}\}) = 0$. In particular, thus, $\overline{P} \notin \Theta$. Let moreover $\overline{\gamma}_\nu^- \in \mathcal{J}^-(\nu)$ and $\overline{\gamma}_\nu^+ \in \mathcal{J}^+(\nu)$ be corresponding curves delimitating the approximating region which converge locally uniformly to $\bar{\gamma}$,

again as constructed in (4.91). Even if we do not need it, we remind that by the tame oscillation condition—see [11, Lemma 2.3] jointly with Theorem 14—one can define

$$u^L = \lim_{\substack{(t,x)\to(\bar{t},\bar{x}) \\ \bar{t}\leq t\leq\bar{t}+(\bar{x}-x)/\hat{\lambda}}} u(t,x) \ .$$

We prove below the sided limits of u at $\overline{P}$ stated in Theorem 5. We omit here, instead, the proof of the relation $\dot{\overline{y}}(t) = \sigma_i(u^L, u^R)$ because it can be deduced as in [8, Step 7, Page 227].

Suppose that Theorem 5 fails: equivalently, suppose by contradiction that

$$\lim_{r\downarrow 0}\limsup_{\nu\uparrow\infty}\left(\sup_{\substack{|t-\bar{t}|+|x-\bar{x}|\leq r \\ x<\overline{\gamma}_\nu^-}}\left|u_\nu(t,x)-u^L\right|\right) > \varepsilon > 0 \ .$$

By possibly taking a subsequence and supposing that we suitably normalize the pointwise representative of the solution, we assume there are points $Q_\nu = (\bar{t}_\nu, \overline{q}_\nu)$ and $P_\nu = (\bar{t}_\nu, \overline{p}_\nu)$ on the left of $\overline{\gamma}_\nu^-$, i.e. satisfying $\overline{p}_\nu < \overline{q}_\nu < \overline{\gamma}_\nu^-(\bar{t}_\nu)$, such that $\bar{t}_\nu \to \bar{t}$ when $\nu\uparrow\infty$ and

$$Q_\nu \xrightarrow{\nu\uparrow\infty} \overline{P} \ , \qquad |u_\nu(Q_\nu) - u^L| \geq \varepsilon \quad \forall\nu,$$

$$P_\nu \xrightarrow{\nu\uparrow\infty} \overline{P} \ , \qquad |u_\nu(P_\nu) - u^L| \xrightarrow{\nu\uparrow\infty} 0 \ .$$

By the construction of the limit curve for linearly degenerate families of § 4.3.1:

- For every linearly degenerate field $j \neq i$ the total strength of j-waves crossing $P_\nu Q_\nu$ converges to 0 as $\nu\uparrow\infty$: see [8, (10.76)] for a full proof.

- The amount of waves of the i-th family crossing $P_\nu Q_\nu$ is vanishingly small, as a consequence of (4.91b), because we fixed P_ν and Q_ν converging to $\overline{P}$ but on the left of $\overline{\gamma}_\nu^-$.

Nevertheless, the segment $P_\nu Q_\nu$ must be crossed by a large amount of waves, which must therefore be of piecewise genuinely nonlinear families. We show that in any neighborhood of $\overline{P}$ these waves must interact among themselves. As a result, $\mu^{IC}(\{\overline{P}\})$ cannot vanish and therefore we reach an absurd. Two cases are possible:

Case 1 (Figure 4)   There are two distinct indexes $j \neq k$ such that each segment $P_\nu Q_\nu$ is crossed by a fixed amount of both j-waves and k-waves in u_ν. Lemma 23 below applies with the families k and j and it contradicts the assumption $\mu^{IC}(\{\overline{P}\}) = 0$ because of interactions among different families.

Case 2 (Figure 5)   For a singe index $j \in \{1, \ldots, n\}$ each segment $P_\nu Q_\nu$ is crossed by an amount $\geq \varepsilon > 0$ of waves of a piecewise genuinely nonlinear family j, but for $k \neq j$ the total amount of k-waves crossing the segment $P_\nu Q_\nu$ vanishes as $\nu \to \infty$. As in the limit there is no (β, j, ℓ)-shock front at $\overline{P}$, then one must have that the maximum strength of j-waves crossing the segment $P_\nu Q_\nu$ vanishes as $\nu \to \infty$ and therefore that the segment $P_\nu Q_\nu$ is crossed by a large number of small j-waves. Lemma 24 below applies and it contradicts the assumption $\mu^{IC}(\{\overline{P}\}) = 0$.

4.3.4   Continuity points

Consider now a point $\overline{P} \notin \Theta$ which does not belong to any of the curves in $\mathcal{J}$ constructed in § 4.3.1. We prove now that u is continuous at $\overline{P}$, concluding thus the statement of Theorem 5.

Assume by contradiction that u is discontinuous at $\overline{P}$: there exists $\varepsilon > 0$ and a space-like segment $P_\nu Q_\nu$ degenerating to the singe point $\overline{P}$ for which

$$u_\nu(P_\nu) \to u(\overline{P}) \ , \qquad |u_\nu(Q_\nu) - u(\overline{P})| \geq \varepsilon \quad \forall\nu \ .$$

Two cases are possible.

Case 1 (Figure 4)   There are two distinct indexes $j \neq k$ such that each segment $P_\nu Q_\nu$ is crossed by a fixed amount of both j-waves and k-waves in u_ν. Lemma 23 below applies with the families k and j and it contradicts the assumption $\mu^{IC}(\{\overline{P}\}) = 0$ because of interactions among different families.

Case 2 (Figure 5) For a singe index $j \in \{1, \ldots, n\}$ each segment $P_\nu Q_\nu$ is crossed by an amount $\geq \varepsilon > 0$ of j-waves, but for $k \neq j$ the total amount of k-waves crossing the segment $P_\nu Q_\nu$ vanishes.

Suppose the family j is piecewise genuinely nonlinear. As in the limit there is no (β, j, ℓ)-shock front, then one must have that the maximum strength of j-waves crossing the segment $P_\nu Q_\nu$ vanishes as $\nu \to \infty$ and therefore that the segment $P_\nu Q_\nu$ is crossed by a large number of small j-waves. Lemma 24 below applies and it contradicts the assumption $\mu^{IC}(\{\overline{P}\}) = 0$.

It is moreover not possible that the family j is linearly degenerate unless $\mu_\nu^{j*}(\overline{P}) > 0$ see [8, (10.76)] for a full proof. This contradicts the assumption that $\overline{P}$ is not covered by any curve in $\mathcal{J}$.

4.3.5 Geometric lemmas on piecewise genuinely nonlinear characteristic fields

We now formalize the following intuitive but nontrivial fact. The strength of the i-th wave in the solution of the Riemann problem among $v(0)$ (left value) and $v(1)$, for a piecewise constant function $v : [0, 1] \to \mathbb{R}^N$ whose jumps are mostly in the i-th family and small except possibly for one of them, is approximatively the sum of the strengths of the i-th waves in its jumps. For example, jumping many times almost along the i-th elementary curve, and staying "close", if measured along the i-th elementary curve, to the initial point, one does not move transversally too much.

Lemma 21. *There exists $C > 0$ which satisfies the following. Let $\rho > 0$ and let $v : [0,1] \to \mathbb{R}^N$ be a piecewise constant function such that*

$$Ce^{C\text{Tot.Var.}(v)}\text{Tot.Var.}(v) < \delta_H$$

and the following conditions hold:

- *the total strength of waves of families different from i, present at any jump, is less than ρ;*

- *the strength of the single i-th wave in the solution of the Riemann problem at each jump, apart from at most one of them, is less than ρ.*

Denote by

- *s_i^k the strength of the i-th wave present in the solution of the Riemann problem relative to the k-th jump of v;*

- *$\sigma_1, \ldots, \sigma_N$ the strengths of the outgoing waves relative to the jump among $v(0)$ (left value) and $v(1)$ (right value).*

We conclude then that

$$\left| \sigma_i - \sum_k s_i^k \right| + \sum_{j \neq i} |\sigma_j| \leq Ce^{C\,\text{Tot.Var.}v} \cdot (1 + \text{Tot.Var.}v) \cdot \rho .$$

Proof. The thesis holds if we prove it when all jumps of v belong to the i-th characteristic family, by classical interaction estimates, since the total strength of waves of all families different from i is assumed to be less than ρ.

Suppose therefore that all jumps of v are along the i-th elementary waves. In this case, s_i^k already denotes the strength of the k-th jump. We argue by induction on the number K of jumps of v that, in the absence of waves of other families,

$$\left| \sigma_i - \sum_k s_i^k \right| + \sum_{j \neq i} |\sigma_j| \leq Ce^{C\,(|s_i^1| + \cdots + |s_i^K|)} \cdot (|s_i^1| + \cdots + |s_i^K|) \cdot \rho . \tag{4.92}$$

Of course when $K = 1$ the thesis is trivial while if $K = 2$ by classical interaction estimates [3, Lemma 1], and since either $|s_i^1| \leq \rho$ or $|s_i^2| \leq \rho$ by assumption, one has that the thesis holds with $C > C_2$ where

$$\left| \sigma_i - (s_i^1 + s_i^2) \right| + \sum_{j \neq i} |\sigma_j| \leq C_1 \mathcal{I}(s_i^1, s_i^2) \leq C_1 |s_i^1 s_i^2| \leq C_2 e^{C_2\,(|s_i^1| + |s_i^2|)} \cdot (|s_i^1| + |s_i^2|) \cdot \rho .$$

Suppose now that estimate (4.92) holds if v has K jumps. Denote by $\sigma_1^*, \ldots, \sigma_N^*$ the strengths of the outgoing waves relative to the Riemann problem among $v(0)$ (left value) and the second-to-last value v^* of v (right value). By the induction hypothesis we have the estimate

$$\left| \sigma_i^* - \sum_{k=1}^{K-1} s_i^k \right| + \sum_{j \neq i} |\sigma_j^*| \leq Ce^{C\,(|s_i^1| + \cdots + |s_i^{K-1}|)} \cdot (|s_i^1| + \cdots + |s_i^{K-1}|) \cdot \rho . \tag{4.93}$$

Moreover, by the estimates on interactions among two consecutive Riemann problems [4, Theorem 3.7] applied to the states $v(0)$ (left), v^* (middle) and $v(1)$ (right) one has that the strengths $\sigma_1, \ldots, \sigma_N$ in the outgoing waves of the Riemann problem among $v(0)$ and $v(1)$ satisfy

$$\left|\sigma_i - (\sigma_i^* + s_i^K)\right| + \sum_{j \neq i} \left|\sigma_j - \sigma_j^*\right| \leq \overline{C}|s_i^K| \sum_{j \geq i} |\sigma_j^*| \ .$$

By the last estimate and by the triangular inequalities

$$\left|\sigma_i - \sum_{k=1}^K s_i^k\right| \leq \left|\sigma_i - (\sigma_i^* + s_i^K)\right| + \left|\sigma_i^* + s_i^K - \sum_{k=1}^K s_i^k\right| \qquad \sum_{j \neq i} |\sigma_j| = \sum_{j \neq i} |\sigma_j - \sigma_j^*| + \sum_{j \neq i} |\sigma_j^*|$$

we get

$$\left|\sigma_i - \sum_k s_i^k\right| + \sum_{j \neq i} |\sigma_j| \leq \left|\sigma_i - (\sigma_i^* + s_i^K)\right| + \left|\sigma_i^* - \sum_{k=1}^{K-1} s_i^k\right| + \sum_{j \neq i} |\sigma_j - \sigma_j^*| + \sum_{j \neq i} |\sigma_j^*|$$

$$\leq \left|\sigma_i^* - \sum_{k=1}^{K-1} s_i^k\right| + \sum_{j \neq i} |\sigma_j^*| + \overline{C}|s_i^K| \sum_{j > i} |\sigma_j^*| \ .$$

Since $Ce^C \mathrm{Tot.Var.}(v) \mathrm{Tot.Var.}(v) \leq 1$ implies jointly with (4.93) that

$$\sum_{j > i} |\sigma_j^*| \leq \sum_{j \neq i} |\sigma_j^*| \leq \rho \ ,$$

then the induction hypothesis (4.93) yields

$$\left|\sigma_i - \sum_k s_i^k\right| + \sum_{j \neq i} |\sigma_j| \leq \overline{C}|s_i^K|\rho + Ce^{C\left(|s_i^1| + \cdots + |s_i^{K-1}|\right)} \cdot (|s_i^1| + \cdots + |s_i^{K-1}|) \cdot \rho$$

$$\leq Ce^{C\left(|s_i^1| + \cdots + |s_i^K|\right)} \cdot (|s_i^1| + \cdots + |s_i^K|) \cdot \rho \ . \qquad \square$$

We state an elementary lemma on the geometric structure of piecewise-genuinely nonlinear characteristic fields. We recall that by piecewise-genuine-nonlinearity $\nabla\lambda_i(u) \cdot r_i(u)$ vanishes only on Z_i^k for $k = 1, \ldots, J_i$, which are hypersurfaces transversal to the i-rarefaction curves.

Lemma 22. *Let $\rho, M > 0$. Suppose the i-th field is piecewise genuinely nonlinear. Then there exists a positive constant $\overline{C}$ such that the following holds. Suppose v is a piecewise constant function as in Lemma 21 and whose image lies in a connected component of the compact set*

$$D := [-M, M]^N \setminus \bigcup_{k=1}^{J_i} B_\rho(Z_k^i) \ .$$

Define t_1, t_2 and $\ell, L > 0$ by the relations

$$L = \mathrm{diam\,Image}\, v_i = |v_i(t_1) - v_i(t_2)| \ , \qquad \ell = \min\left\{|\nabla\lambda_i(u) \cdot r_i(u)| \ : \ u \in D\right\} \ .$$

One has then that $\quad |\lambda_i(v(t_1)) - \lambda_i(v(t_2))| \geq \ell L - \overline{C}\rho \ .$

Proof. The assumption that v is valued in a connected component of D is of course crucial. The proof is a direct consequence of the fact that, in the region where v is valued, the system is actually genuinely nonlinear with $|\nabla\lambda_i(u) \cdot r_i(u)| \geq \ell$. Indeed, due to the choice of the parameterization jointly with Lemma 21, the Riemann problem having $v(t_1)$, $v(t_2)$ as left / right states, or viceversa, contains an i-rarefaction curve of strength at least $L - \delta_H\rho$ while the total strengths of other waves are less than $\delta_H\rho$. Along that i rarefaction curve λ_i varies at least of ℓL since $|\nabla\lambda_i(u) \cdot r_i(u)| \geq \ell$, and then one hast the thesis by the Lipschitz continuity of λ_i. $\qquad \square$

4.3.6   Auxiliary lemmas

Lemma 23. *Let $\varepsilon > 0$. Consider a space-like segment $P_\nu Q_\nu$ for which there are two distinct indexes $j < k$ such that each segment $P_\nu Q_\nu$ is crossed by an amount $\geq \varepsilon > 0$ of both j-waves in u_ν and k-waves in u_ν. If $P_\nu \to \overline{P}$ and $Q_\nu \to \overline{P}$ then necessarily $\mu^{IC}(\{\overline{P}\}) > 0$—see Figure 4.*

Proof. As the segment $P_\nu Q_\nu$ is space like, suppose for example that P_ν is on the left of Q_ν. Consider the region Γ_ν delimited by the leftmost k forward and backward k-characteristics through P_ν and by the rightmost forward and backward j-characteristics through Q_ν. By strict hyperbolicity (1.5) they intersect at points R_ν^*, S_ν^* which converge to $\overline{P}$ and therefore the region Γ_ν shrinks to the single point $\overline{P}$. By interaction estimates, there is an amount of interaction $\gtrsim \varepsilon^2$ in the region $\overline{\Gamma}_\nu$, which yields in the ν-limit that $\mu^{IC}(\{\overline{P}\}) > 0$. $\quad\square$

Lemma 24. *Assume that the i-th family is piecewise genuinely nonlinear. Let $\varepsilon > 0$. If in some open region Γ_ν shrinking to a given point $\overline{P} = (\overline{t}, \overline{x})$, as $\nu \uparrow \infty$, in u_ν*

1. *there is at most one i-jump $\overline{\gamma}_\nu$ whose j-th component has strength more than ε while the strengths of the j-th component of other i-waves are vanishingly small but*

2. *the total amount of strengths of the j-th component of i-waves different from $\overline{\gamma}_\nu$ is more than 3ε and*

3. *for all $k \neq i$ the amount of k-waves of u_ν is vanishingly small as $\nu \uparrow \infty$,*

then $\mu^{IC}(\{\overline{P}\}) > 0$—see Figure 5.

Proof. $\ominus$ STEP 1: Finding two i-fronts γ_ν^ℓ, γ_ν^r of u_ν whose slopes at some time $t_\nu \to \overline{t}$ remain distant without having any (β, i, j)-front in between. In the hypothesis of the lemma, we can fix a space segment $P_\nu Q_\nu \subset \{t = t_\nu\}$, for $P_\nu Q_\nu \in \Gamma_\nu$, which is shrinking to the given point $\overline{P} = (\overline{t}, \overline{x})$ as $\nu \to 0$, such that the total amount the j-th component of i-waves in u_ν along $P_\nu Q_\nu$ is more than ε, but each one is vanishingly small as $\nu \uparrow \infty$. Assume also that, if present, $\overline{\gamma}_\nu$ lies, for example, on the right of $P_\nu Q_\nu$, Q_ν on the right of P_ν and

$$u_\nu(P_\nu) \to u^L , \qquad |u_\nu(Q_\nu) - u^L| > \varepsilon$$

where u^L denotes the left limit of $u(\overline{t}, \cdot)$ at $\overline{x}$, if suitably normalized. We distinguish cases:

1. If $u^L \notin Z_i^j$ and $u^L \notin Z_i^{j+1}$ then Lemma 22 applies to u_ν restricted to an initial part of the interval $P_\nu Q_\nu$ and thus for some $c > 0$ we can pick up a point C_ν belonging to the segment $P_\nu Q_\nu$ such that $\bullet$ as $\nu \uparrow \infty$

$$|\lambda_i(u_\nu(C_\nu)) - \lambda_i(u_\nu(P_\nu))| > c > 0 ,$$

   such that $\bullet$ the total strength of j-components of i-fronts crossing $P_\nu C_\nu$ is at least c and such that $\bullet$ along the segment $P_\nu C_\nu$ the distance of u from Z_i^j and Z_i^{j+1} is at least c, so that u takes values in a region where the i-characteristic field is genuinely nonlinear.

2. Even if $u^L \in Z_i^j$ or $u^L \in Z_i^{j+1}$, the maximum distance of $u_\nu(C)$ from Z_i^j and Z_i^{j+1}, for C varying in the interval $P_\nu Q_\nu$, cannot be vanishingly small: we could otherwise apply Lemma 21 to u_ν restricted on the whole segment $P_\nu Q_\nu$ and we would reach a contradiction with the assumption that the total amount of i-waves is not vanishingly small. As the maximum distance of $u_\nu(C)$ from Z_i^j and Z_i^{j+1}, for $C \in P_\nu Q_\nu$, is not vanishingly small, we can apply Lemma 22 to u_ν restricted on some sub-segment of $P_\nu Q_\nu$: for some $c > 0$ we can thus pick up a point C_ν belonging to the segment $P_\nu Q_\nu$ such that $\bullet$ as $\nu \uparrow \infty$

$$|\lambda_i(u_\nu(C_\nu)) - \lambda_i(u_\nu(P_\nu))| > c > 0 ,$$

   such that $\bullet$ the total strength of j-components of i-fronts crossing $P_\nu C_\nu$ is at least c and such that $\bullet$ along the segment $P_\nu C_\nu$ the distance of u from Z_i^j and Z_i^{j+1} is at least c, so that u takes values in a region where the i-characteristic field is genuinely nonlinear.

3. If any limit point of $u_\nu(Q_\nu)$, up to subsequence, belongs to the closed region between Z_i^j and Z_i^{j+1} then the previous points still apply similarly. In case not, by the assumptions one can replace Q_ν with another point $\overline{Q_\nu}$ having the same properties above and such that a limit point of the sequence $\overline{Q_\nu}$ falls in the closed region between Z_i^j and Z_i^{j+1}.

Consider now the leftmost i-discontinuity curve γ_ν^ℓ of u_ν, through P_ν, and the rightmost one γ_ν^r, through the point C_ν just determined. Since the maximum size of jumps is vanishingly small we can also assume that

$$|\dot\gamma_\nu^\ell(t_\nu) - \dot\gamma_\nu^r(t_\nu)| > c > 0 \tag{4.94}$$

by the construction above of the point C_ν.

$\ominus$ STEP 2: Conclusion when γ_ν^ℓ and γ_ν^r meet at some time $\overline{t}_\nu \to \overline{t}$. In case γ_ν^ℓ and γ_ν^r meet at time $\overline{t}_\nu$ with

$$|\overline{t}_\nu - t_\nu| \leq \Delta_\nu \qquad \text{where} \qquad \Delta_\nu := 4\frac{|P_\nu C_\nu|}{c} ,$$

then denote by Γ_ν the region delimited by the segment $P_\nu C_\nu$, by γ_ν^ℓ and by γ_ν^r between times $\overline{t}_\nu$ and $\overline{t}$. The region Γ_ν shrinks to $\overline{P}$ and, by construction, one can prove that $\mu_\nu^{IC}(\overline{\Gamma}_\nu)$ is uniformly positive: this yields the thesis $\mu^{IC}(\overline{P}) > 0$. Of course, γ_ν^ℓ and γ_ν^r necessarily meet by one such time $\overline{t}_\nu$ if the slopes $\dot{\gamma}_\nu^\ell$, $\dot{\gamma}_\nu^r$ satisfy

$$|\dot{\gamma}_\nu^\ell(q) - \dot{\gamma}_\nu^r(q)| > \frac{c}{4} \qquad \text{for all } q \text{ such that } |q - t_\nu| \leq \Delta_\nu.$$

$\ominus$ STEP 3: Claim when $\gamma_\nu^\ell(q)$ and $\gamma_\nu^r(q)$ do not meet for $|q - t_\nu| \leq \Delta_\nu$. In case γ_ν^ℓ and γ_ν^r do not meet in the time interval $[t_\nu - \Delta_\nu, t_\nu + \Delta_\nu]$, we claim that

$$\mu_\nu^{IC}\left(i_{\gamma_\nu^\ell}\left([t_\nu - \Delta_\nu, t_\nu + \Delta_\nu]\right) \cup i_{\gamma_\nu^r}\left([t_\nu - \Delta_\nu, t_\nu + \Delta_\nu]\right)\right) \not\rightarrow 0 \qquad \nu \uparrow \infty. \tag{4.95}$$

The symbol i_γ denotes the map $s \mapsto (s, \gamma(s))$ for s in the domain of γ. We now prove (4.95).

$\ominus$ STEP 4: Necessary condition if $\gamma_\nu^\ell(q)$ and $\gamma_\nu^r(q)$ do not meet of $|q - t_\nu| \leq \Delta_\nu$. If γ_ν^ℓ and γ_ν^r do not meet for $|q - t_\nu| \leq \Delta_\nu$ then by (4.94) and by the triangular inequality necessarily:

1. In case γ_ν^ℓ and γ_ν^r are approaching at time t_ν—namely if

$$\lambda_i(u_\nu(P_\nu)) = \dot{\gamma}_\nu^\ell(\overline{t}_\nu) > \dot{\gamma}_\nu^r(\overline{t}_\nu) + c = \lambda_i(u_\nu(C_\nu)) + c$$

then at some $\overline{t}_\nu$ with $t_\nu < \overline{t}_\nu < t_\nu + \Delta_\nu$

$$\text{either } \dot{\gamma}_\nu^\ell(\overline{t}_\nu) < \dot{\gamma}_\nu^\ell(t_\nu) - \frac{c}{4} \qquad \text{or} \qquad \dot{\gamma}_\nu^r(\overline{t}_\nu) > \dot{\gamma}_\nu^r(t_\nu) + \frac{c}{4}. \tag{4.96}$$

2. If at time t_ν the fronts γ_ν^ℓ and γ_ν^r are getting far apart—i.e. $\dot{\gamma}_\nu^\ell(\overline{t}_\nu) < \dot{\gamma}_\nu^r(\overline{t}_\nu)$—then

$$\text{either } \dot{\gamma}_\nu^\ell(\overline{t}_\nu) > \dot{\gamma}_\nu^\ell(t_\nu) - \frac{c}{4} \qquad \text{or} \qquad \dot{\gamma}_\nu^r(\overline{t}_\nu) < \dot{\gamma}_\nu^r(t_\nu) + \frac{c}{4} \tag{4.97}$$

at some $\overline{t}_\nu$ with $t_\nu - \Delta_\nu < \overline{t}_\nu < t_\nu$.

$\ominus$ STEP 5: Proof of the claim. Suppose e.g $\gamma_\nu^\ell(q)$ and $\gamma_\nu^r(q)$ do not meet for $t_\nu - \Delta_\nu < q < t_\nu$, as in the other case the analysis is analogous. Let $\overline{t}_\nu$ be as in (4.97). We first estimate how the slope of an i-front γ might vary. Collecting the estimates in Lemma 12 at each update time in $[\overline{t}_\nu, t_\nu]$ jointly with interaction estimates in [3, Lemma 1], we find the rough estimate

$$\left|\dot{\gamma}(\overline{t}_\nu) - \dot{\gamma}(t_\nu)\right| \lesssim \Delta_\nu + \tau_\nu + M_i^\nu(\gamma, \overline{t}_\nu, t_\nu) + M_*^\nu(\gamma, \overline{t}_\nu, t_\nu) + \mu_\nu^{IC}\left(i_\gamma([\overline{t}_\nu, t_\nu])\right) \tag{4.98}$$

where we adopted the following notation:

- $M_*^\nu(\gamma, r, t)$ is the strength of all k-waves, $k \neq i$, interacting with γ between times r and t;

- $M_i^\nu(\gamma, r, t)$ is the strength of i-waves interacting with γ between times r and t.

By assumption $M_*^\nu(\gamma_\nu^\ell, \overline{t}_\nu, t_\nu)$, $M_*^\nu(\gamma_\nu^r, \overline{t}_\nu, t_\nu)$, Δ_ν, τ_ν are vanishingly small as $\nu \uparrow \infty$. If also $M_i^\nu(\gamma_\nu^\ell, \overline{t}_\nu, t_\nu)$ and $M_i^\nu(\gamma_\nu^r, \overline{t}_\nu, t_\nu)$ are vanishingly small, then (4.98) directly implies (4.95) for ν large enough, so that $\mu^{IC}(\overline{P}) > 0$.

Suppose instead that one among $M_i^\nu(\gamma_\nu^\ell, \overline{t}_\nu, t_\nu)$ and $M_i^\nu(\gamma_\nu^r, \overline{t}_\nu, t_\nu)$ is not vanishingly small. Denote simply by γ such front. Assume also that j is even, so that (β, i, j)-sub-discontinuity fronts correspond to positive parameters s in Definition 17, for notational convenience.

By Lemma 16 applied with Γ a neighborhood of $i_\gamma([\overline{t}_\nu, t_\nu])$, one has

$$|(s_i)^+(t_\nu+) - (s_i)^+(\overline{t}_\nu-) - M_i^{+\nu}(\gamma, \overline{t}_\nu, t_\nu)| \lesssim \Delta_\nu + \tau_\nu + \mu_\nu^{IC}\left(i_\gamma([\overline{t}_\nu, t_\nu])\right) \tag{4.99}$$

$$|(s_i)^-(t_\nu+) - (s_i)^-(\overline{t}_\nu-) - M_i^{-\nu}(\gamma, \overline{t}_\nu, t_\nu)| \lesssim \Delta_\nu + \tau_\nu + \mu_\nu^{IC}\left(i_\gamma([\overline{t}_\nu, t_\nu])\right) \tag{4.100}$$

where $|s_i|$ is the strength of the i-front γ and the apex $^+$ (resp. $^-$) means that we are taking into account only strengths of positive (resp. negative) i-waves. Since by construction

$$(s_i)^+(t_\nu+) = s_i^j(t_\nu+), \qquad (s_i)^-(t_\nu+), \qquad (s_i)^-(\overline{t}_\nu-)$$

are vanishingly small—the first one by the choice of the point C_ν / P_ν while the second one since single rarefaction fronts have vanishingly small strengths—and since $(s_i)^+(\overline{t}_\nu-)$, $M_i^{+\nu}(\gamma, \overline{t}_\nu, t_\nu)$ are both nonnegative by definition, then

- either $(s_i)^+(\overline{t}_\nu-) + M_i^{+\nu}(\gamma, \overline{t}_\nu, t_\nu)$ is not vanishingly small, so that (4.99) implies (4.95),

- or $M_i^{-\nu}(\gamma, \overline{t}_\nu, t_\nu)$ is not vanishingly small, from which (4.100) implies (4.95),

or both of the cases happen. This concludes the proof of the theorem. $\qquad\square$

Remark 25. *Lemma 24 extends with little modification to the case of two or more i-jumps converging to $\overline{P}$ whose j-th component has strength more than ε, rather than a single one, both with or without any uniformly positive amount of i-waves of vanishingly small strength.*

5 Essential nomenclature

ceil: Smallest integer bigger than a given real number, i.e. integer part of the number plus one.

δ_{hk}: Delta di Kronecker, which is equal to 1 if $h = k$ and it vanishes otherwise.

$\hat{\lambda}$: Uniform bound for the characteristic speeds of very family.

$P_{t,h}\bar{u}$: The viscous semigroup of the Cauchy problem (1.1)-(1.2) starting at time h, rather than fixing the initial time $h = 0$. See [10].

$S_{t,h}$: The semigroup of the Cauchy problem for the homogeneous system (1.1)-(1.2) when $g \equiv 0$ starting at time h, rather than fixing the initial time $h = 0$. See [6].

$S_{t,h}^{FT}$: The ε_ν-wave-front tracking approximation of $S_{t,h}$ by [3].

Ω: Open, bounded, connected subset of $\mathbb{R}^N$ where u is valued.

$\overline{\delta}$: Smallness parameter for initial datum in the Cauchy problem of the viscous system (1.1) as in Theorem 11

δ_H: Smallness parameter for initial datum in the Cauchy problem of the homogeneous system as at Page 13 which provides a threshold for global existence and convergence of wave-front tracking approximations. See [3], where it is denoted by δ_0.

ε: Positive parameter.

ν: Positive integer parameter relative to subsequences of ε_ν-wave-front tracking approximations or of $(\varepsilon_\nu, \tau_\nu)$-fractional-step approximations. See § 2.

ε_ν: Positive vanishing constant, as $\nu \uparrow \infty$, in ε_ν-wave-front tracking approximation and in $(\varepsilon_\nu, \tau_\nu)$-fractional-step approximation. It correspond also to maximum size of rarefactions.

τ_ν: Size of the time-step in $(\varepsilon_\nu, \tau_\nu)$-fractional-step approximations. See § 2.

u_ν, u: Often, ε_ν-wave front tracking approximation of the Cauchy problem (1.1)-(1.2), either homogeneous or not, and its limit entropy solution, as constructed in [3] and recalled in § 2.2.

w_ν, w: Fractional step approximation, and its limit as $\nu \uparrow \infty$, of the Cauchy problem (1.1)-(1.2) that we construct in §§ 2-3. We fix the right-continuous representative in time and space.

ℓ_g: The Lipschitz constant of $g(\cdot, x, u)$ in x, u. See the assumption (G) at Page 2.

α: See the assumption (G) at Page 2.

Φ: Functional defined at (2.44).

$\mathcal{Q}, \mathcal{V}, \Upsilon$: Functionals defined in (2.59)-(2.61).

$\lesssim$ Less or equal up to a constant which only depends only on the flux of (1.1) and on δ or δ_H.

$\mu_{(\nu)}^I, \mu_{(\nu)}^{IC}$: Interaction and interaction-cancellation measures, either on a sequence of $(\varepsilon_\nu$-$\tau_\nu)$-fractional step approximations or a fixed limit of them. See (4.80).

(β, i, k)-approximate sub-discontinuity as defined in § 4.2.

$\mu_{(\nu)}^{i*}, \overline{\mu}_{(\nu)}^{i\pm}$: Real measures detecting possible initial points of i-contact discontinuities, § 4.3.1.

$\mathcal{J}_i \ \mathcal{J}_{\beta,i} \ \mathcal{J}$: i-Shock-fronts and i-contact discontinuity fronts,e § 4.3.1.

Θ: At most countable set containing interaction points, § 4.3.1.

Tot.Var.(v): The total variation of $v : \mathbb{R} \to \mathbb{R}^m$, $m \in \mathbb{N}$, which is $\sup_{K \in \mathbb{N}} \sup_{x_1 < \cdots < x_K} \sum_j |v(x_{j+1}) - v(x_j)|$.

References

[1] Debora Amadori, Laurent Gosse, and Graziano Guerra. Global BV entropy solutions and uniqueness for hyperbolic systems of balance laws. *Arch. Ration. Mech. Anal.*, 162(4):327–366, 2002.

[2] Debora Amadori and Graziano Guerra. Uniqueness and continuous dependence for systems of balance laws with dissipation. *Nonlinear Anal.*, 49(7, Ser. A: Theory Methods):987–1014, 2002.

[3] Fabio Ancona and Andrea Marson. Existence theory by front tracking for general nonlinear hyperbolic systems. *Arch. Ration. Mech. Anal.*, 185(2):287–340, 2007.

[4] Stefano Bianchini. Interaction estimates and Glimm functional for general hyperbolic systems. *Discrete Contin. Dyn. Syst.*, 9(1):133–166, 2003.

[5] Stefano Bianchini. On the Riemann problem for non-conservative hyperbolic systems. *Arch. Ration. Mech. Anal.*, 166(1):1–26, 2003.

[6] Stefano Bianchini and Alberto Bressan. Vanishing viscosity solutions of nonlinear hyperbolic systems. *Ann. of Math. (2)*, 161(1):223–342, 2005.

[7] Stefano Bianchini and Lei Yu. Global structure of admissible BV solutions to piecewise genuinely nonlinear, strictly hyperbolic conservation laws in one space dimension. *Comm. Partial Differential Equations*, 39(2):244–273, 2014.

[8] Alberto Bressan. *Hyperbolic systems of conservation laws*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.

[9] Alberto Bressan and Philippe G. LeFloch. Structural stability and regularity of entropy solutions to hyperbolic systems of conservation laws. *Indiana Univ. Math. J.*, 48(1):43–84, 1999.

[10] Cleopatra C. Christoforou. Hyperbolic systems of balance laws via vanishing viscosity. *J. Differential Equations*, 221(2):470–541, 2006.

[11] Cleopatra C. Christoforou. Uniqueness and sharp estimates on solutions to hyperbolic systems with dissipative source. *Comm. Partial Differential Equations*, 31(10-12):1825–1839, 2006.

[12] Bernard D. Coleman and Daniel C. Newman. Implications of a nonlinearity in the theory of second sound in solids. *Phys. Rev. B*, 37:1492–1498, Feb 1988.

[13] Graziano Crasta and Benedetto Piccoli. Viscosity solutions and uniqueness for systems of inhomogeneous balance laws. *Discrete Contin. Dynam. Systems*, 3(4):477–502, 1997.

[14] C. M. Dafermos. A system of hyperbolic conservation laws with frictional damping. *Z. Angew. Math. Phys.*, 46(Special Issue):S294–S307, 1995. Theoretical, experimental, and numerical contributions to the mechanics of fluids and solids.

[15] C. M. Dafermos. Hyperbolic systems of balance laws with weak dissipation. *J. Hyperbolic Differ. Equ.*, 3(3):505–527, 2006.

[16] C. M. Dafermos and L. Hsiao. Hyperbolic systems and balance laws with inhomogeneity and dissipation. *Indiana Univ. Math. J.*, 31(4):471–491, 1982.

[17] Constantine M. Dafermos. BV solutions for hyperbolic systems of balance laws with relaxation. *J. Differential Equations*, 255(8):2521–2533, 2013.

[18] Constantine M. Dafermos. Hyperbolic systems of balance laws with weak dissipation II. *J. Hyperbolic Differ. Equ.*, 10(1):173–179, 2013.

[19] Constantine M. Dafermos. Hyperbolic balance laws with relaxation. *Discrete Contin. Dyn. Syst.*, 36(8):4271–4285, 2016.

[20] Constantine M. Dafermos and Ronghua Pan. Global *BV* solutions for the *p*-system with frictional damping. *SIAM J. Math. Anal.*, 41(3):1190–1205, 2009.

[21] R. J. DiPerna. Convergence of approximate solutions to conservation laws. *Arch. Rational Mech. Anal.*, 82(1):27–70, 1983.

[22] James Glimm. Solutions in the large for nonlinear hyperbolic systems of equations. *Comm. Pure Appl. Math.*, 18:697–715, 1965.

[23] P. D. Lax. Hyperbolic systems of conservation laws. II. *Comm. Pure Appl. Math.*, 10:537–566, 1957.

[24] Tai Ping Liu. The Riemann problem for general 2×2 conservation laws. *Trans. Amer. Math. Soc.*, 199:89–112, 1974.

[25] Tai Ping Liu. The Riemann problem for general systems of conservation laws. *J. Differential Equations*, 18:218–234, 1975.

[26] Tai Ping Liu. Quasilinear hyperbolic systems. *Comm. Math. Phys.*, 68(2):141–172, 1979.

[27] Tai Ping Liu. Admissible solutions of hyperbolic conservation laws. *Mem. Amer. Math. Soc.*, 30(240):iv+78, 1981.

[28] T. Ruggeri, A. Muracchini, and L. Seccia. Continuum approach to phonon gas and shape changes of second sound via shock waves theory. *Il Nuovo Cimento D*, 16(1):15–44, 1994.

[29] T. Ruggeri, A. Muracchini, and L. Seccia. Second sound and characteristic temperature in solids. *Phys. Rev. B*, 54:332–339, Jul 1996.

[30] L. Seccia, T. Ruggeri, and A. Muracchini. Second sound and multiple shocks in superfluid helium. *Z. Angew. Math. Phys.*, 60(6):1074–1094, 2009.

[31] Yasushi Shizuta and Shuichi Kawashima. Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation. *Hokkaido Math. J.*, 14(2):249–275, 1985.