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OGUS REALIZATION OF 1-MOTIVES

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ABSTRACT. After introducing the Ogus realization of 1-motives we prove that it is a fully faithful functor. More precisely, following a framework introduced by Ogus, considering an enriched structure on the de Rham realization of 1-motives over a number field, we show that it yields a full functor by making use of an algebraicity theorem of Bost.

INTRODUCTION

The Ogus realization of motives over a number field is considered as an analogue of the Hodge realization over the complex numbers and the ℓ-adic realization over fields which are finitely generated over the prime field. The fullness of these realizations, along with the semi-simplicity of the essential image of pure motives, is a longstanding conjecture which implies the Grothendieck standard conjectures (*e.g.* see [1, §7.1]).

The named conjecture on fullness is actually a theorem if we restrict to the category of abelian varieties up to isogenies regarded as the semi-simple abelian $\mathbb{Q}$-linear category of *pure* 1-motives (*e.g.* see [1, Prop. 4.3.4.1 & Thm. 7.1.7.5]). A natural task is then to extend this theorem to *mixed* 1-motives up to isogenies. For the Hodge realization this goes through Deligne's result on the algebraicity of the effective mixed polarizable Hodge structures of level ≤ 1 (see [15, §10.1.3]). For the ℓ-adic realization, the fullness follows from the Tate conjectures for abelian varieties (proven by Faltings) and the fullness for 1-motives is proven by Jannsen (see [18, §4]).

The main task of this paper is to show that there is a suitable version of Ogus realization for 1-motives such that the fullness can be achieved: this is Theorem 3.3.5 below. This result for abelian varieties relies on a theorem of Bost as explained by André (see [1, §7.4.2]). For pure 0-motives is our Lemma 3.3.4. However, in the mixed case, our theorem doesn't follow directly from Bost's theorem nor André's arguments (see Example 3.3.6).

For a number field K, recall that the Ogus category $\mathbf{Og}(K)$ is the $\mathbb{Q}$-linear abelian category whose objects are finite dimensional K-vector spaces V such that the v-adic completion V_v is endowed, for almost every unramified place v of K, of a bijective semilinear endomorphism F_v. Actually, we here introduce an enriched version of the Ogus category denoted by $\mathbf{FOg}(K)$ whose objects $\mathcal{V} \in \mathbf{Og}(K)$ are endowed with an increasing finite exhaustive weight filtration $W_\bullet \mathcal{V}$ (see Section 1 for details). We provide a realization functor

$$T_{\mathrm{Og}} \colon \mathcal{M}_{1,\mathbb{Q}} \to \mathbf{FOg}(K)$$

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where $\mathcal{M}_{1,\mathbb{Q}}$ is the abelian $\mathbb{Q}$-linear category of 1-motives up to isogenies (see Proposition 3.2.3). With the notation adopted below, this $T_{\mathrm{Og}}(\mathsf{M}_K)$ of a 1-motive M_K (see Definition 3.2.1), is given by $V := T_{\mathrm{dR}}(\mathsf{M}_K)$ the de Rham realization (see Definition 2.3.1), as a K-vector space, so that $V_v \simeq T_{\mathrm{dR}}(\mathsf{M}_{\mathcal{O}_{K_v}}) \otimes_{\mathcal{O}_{K_v}} K_v$ and $F_v := (\Phi_v \otimes \mathrm{id})^{-1}$ where the σ_v^{-1}-semilinear endomorphism Φ_v (Verschiebung) on $T_{\mathrm{dR}}(\mathsf{M}_{\mathcal{O}_{K_v}})$ is obtained via the canonical isomorphism $T_{\mathrm{dR}}(\mathsf{M}_{\mathcal{O}_{K_v}}) \simeq T_{\mathrm{cris}}(\mathsf{M}_{k_v})$ given by the comparison with the crystalline realization of the special fiber M_{k_v} for every unramified place v of good reduction for M_K (see [2, §4, Cor. 4.2.1]).

In the proof of the fullness of T_{Og} for arbitrary 1-motives, with nontrivial discrete part, the main ingredient is the fact that the decomposition of V_v as a sum of pure F-K_v-isocrystals is realized geometrically via the p-adic logarithm (see Section 2, in particular Lemma 2.2.4, and the key Lemma 3.3.1).

Note that the essential image of T_{Og} is contained in the category $\mathbf{FOg}(K)_{(1)}$ of effective objects of level ≤ 1 (see Definition 1.4.4). After Theorem 3.3.5 it is clear that the Ogus realization of pure 1-motives in $\mathcal{M}_{1,\mathbb{Q}}$ is semi-simple. However, the characterization of the essential image is an open question (unless we are in the case of Artin motives, *cf.* [20]).

Further directions of investigation are related to Voevodsky motives. Recall that the derived category of 1-motives $D^b(\mathcal{M}_{1,\mathbb{Q}})$ can be regarded as a reflective triangulated full subcategory of effective Voevodsky motives $\mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}}$ (see [6, Thm. 6.2.1 & Cor. 6.2.2]). In fact, there is a functor $\mathrm{LAlb}^{\mathbb{Q}}\colon \mathrm{DM}^{\mathrm{eff}}_{\mathrm{gm}} \to D^b(\mathcal{M}_{1,\mathbb{Q}})$ which is a left adjoint to the inclusion. Thus, associated to any algebraic K-scheme X, we get a complex of 1-motives $\mathrm{LAlb}^{\mathbb{Q}}(X) \in D^b(\mathcal{M}_{1,\mathbb{Q}})$. Whence, by taking i-th homology, we get $\mathrm{L}_i\mathrm{Alb}^{\mathbb{Q}}(X)_K \in \mathcal{M}_{1,\mathbb{Q}}$ which, most likely, is the geometric avatar of level ≤ 1 Ogus i-th homology of X, *i.e.*, the Ogus realization $T_{\mathrm{Og}}(\mathrm{L}_i\mathrm{Alb}^{\mathbb{Q}}(X)) \in \mathbf{FOg}(K)_{(1)}$ is the largest quotient of level ≤ 1 of the i-th Ogus homology of X, according with the framework of Deligne's conjecture (see [6, §14] and compare with [2, Conj. C]). This is actually the case for the underlying K-vector spaces by the corresponding result for the mixed realization but Ogus realization (see [6, Thm. 16.3.1]).

Notation. We here denote by K a number field. For any finite place v of K, *i.e.*, any prime ideal of the ring of integers $\mathcal{O}_K$ of K, we let K_v be the completion of K with respect to the valuation v and $\mathcal{O}_{K_v}$ its ring of integers. Let $\mathfrak{p}_v$ be the unique maximal ideal of $\mathcal{O}_{K_v}$, k_v the residue field of $\mathcal{O}_{K_v}$, p_v its characteristic and $n_v := [k_v : \mathbb{F}_{p_v}]$. If v is unramified, we let σ_v be the canonical Frobenius map on K_v and σ_v will also denote the Frobenius maps on $\mathcal{O}_{K_v}$ and on k_v.

1. Ogus categories

1.1. **The plain Ogus category $\mathbf{Og}(K)$.** As in [1, §7.1.5] let $\mathbf{Og}(K)$ be the $\mathbb{Q}$-linear abelian category whose objects are finite dimensional K-vector spaces V such that the v-adic completion $V_v = V \otimes_K K_v$ is equipped for almost every unramified place v of K, of a bijective semilinear endomorphism F_v, *i.e.*, for any $\alpha \in K_v, x, y \in V_v$ we have $F_v(\alpha x + y) = \sigma_v(\alpha)F_v(x) + F_v(y)$. Morphisms in $\mathbf{Og}(K)$ are K-linear maps compatible with the F_v's for almost all v.

To give a more precise definition of the above category we have to present it as a 2-colimit category. Let $\mathcal{P}$ denote the set of unramified places of K. For any cofinite subset $\mathcal{P}'$ of $\mathcal{P}$, let $\mathcal{C}_{\mathcal{P}'}$ denote the $\mathbb{Q}$-linear category whose objects are of the type $(V,(V_v,F_v)_{v\in\mathcal{P}'},(g_v)_{v\in\mathcal{P}'})$ where V is a finite dimensional K-vector space, V_v are finite dimensional K_v-vector spaces, F_v is a bijective σ_v-semilinear endomorphism on V_v, and $g_v\colon V\otimes_K K_v\to V_v$ is an isomorphism of K_v-vector spaces. Morphisms are morphisms of vector spaces (on K and on K_v) which respect the given structures (F_v and g_v). Clearly, for any cofinite subset $\mathcal{P}''$ of $\mathcal{P}'$, we have a canonical restriction functor $i_{\mathcal{P}',\mathcal{P}''}\colon \mathcal{C}_{\mathcal{P}'}\to\mathcal{C}_{\mathcal{P}''}$ which simply forgets the data for $v\in\mathcal{P}'\smallsetminus\mathcal{P}''$.

1.1.1. **Definition.** Let

$$\mathbf{Og}(K) := 2\text{-}\varinjlim_{\mathcal{P}'\subseteq\mathcal{P}}\ \mathcal{C}_{\mathcal{P}'}$$

be the 2-colimit category. If $\mathcal{V}_1\in\mathcal{C}_{\mathcal{P}_1}$ and $\mathcal{V}_2\in\mathcal{C}_{\mathcal{P}_2}$ then

$$\mathrm{Hom}_{\mathbf{Og}(K)}(\mathcal{V}_1,\mathcal{V}_2) = \varinjlim_{\mathcal{P}_3\subset\mathcal{P}_1\cap\mathcal{P}_2}\mathrm{Hom}_{\mathcal{C}_{\mathcal{P}_3}}(i_{\mathcal{P}_1,\mathcal{P}_3}(\mathcal{V}_1), i_{\mathcal{P}_2,\mathcal{P}_3}(\mathcal{V}_2)).$$

1.1.2. **Remark.** Given a positive integer n we denote by $\mathcal{P}_n\subset\mathcal{P}$ the subset of places v such that n is invertible in $\mathcal{O}_{K_v}$. Since any cofinite subset $\mathcal{P}'$ in $\mathcal{P}$ contains $\mathcal{P}_n$, for any n divisible by p_v with $v\in\mathcal{P}\smallsetminus\mathcal{P}'$, we may equivalently get $\mathbf{Og}(K)$ as $2\text{-}\varinjlim_n \mathcal{C}_{\mathcal{P}_n}$.

For a given $\mathcal{V} := (V,(V_v,F_v)_{v\in\mathcal{P}'},(g_v)_{v\in\mathcal{P}'})$ in $\mathcal{C}_{\mathcal{P}'}$ and an integer $n\in\mathbb{Z}$ we define the twist $\mathcal{V}(n) := (V,(V_v,p_v^{-n}F_v)_{v\in\mathcal{P}'},(g_v)_{v\in\mathcal{P}'})$.

1.2. F-K_v**-isocrystals.** Recall that a *pure* F-K_v-isocrystal (H,ψ) of integral weight i relative to k_v in the sense of [11, II, §2.0] is a K_v-vector space H together with a K_v-linear endomorphism ψ such that the eigenvalues of ψ are Weil numbers of weight i relative to k_v. Namely, they are algebraic numbers such that they and all their conjugates have archimedean absolute value equal to $p_v^{in_v/2}$. In particular, ψ is an isomorphism. The trivial space $H=0$ is assumed to be a pure F-K_v-isocrystal of any weight. We say that (H,ψ) is a *mixed* F-K_v-isocrystal with integral weights relative to k_v if it admits a finite increasing K_v-filtration

$$0 = W_mH\subseteq\cdots\subseteq W_iH\subseteq W_{i+1}H\subseteq\cdots\subseteq W_nH = H$$

respected by ψ such that $\mathrm{gr}_i^W H_\bullet := W_iH/W_{i-1}H$ is pure of weight i for any $m<i\le n$. A morphism $(H,\psi)\to(H',\psi')$ of F-K_v-isocrystals is a homomorphism of K_v-vector spaces $f\colon H\to H'$ such that $f\circ\psi' = \psi\circ f$.

1.2.1. **Lemma.** *Let H be a K_v-vector space and $\psi\colon H\to H$ an endomorphism. The following are equivalent:*

- (i) *(H,ψ) is a mixed F-K_v-isocrystal.*
- (i') *(H,ψ) admits a unique finite increasing K_v-filtration $0 = W_mH\subseteq\cdots\subseteq W_nH = H$ respected by ψ such that $\mathrm{gr}_i^W H := W_iH/W_{i-1}H$ is pure of weight i for each $m<i\le n$.*
- (ii) *All eigenvalues of ψ are Weil numbers of integral weight relative to k_v.*
- (iii) *H has a unique decomposition $\oplus_{i=m+1}^n H_i$ by ψ-stable vector subspaces so that $(H_i,\psi_{|H_i})$ is a pure F-K_v-isocrystal of weight i.*

Proof. (*i*) $\Rightarrow$ (*ii*). Let $W_\bullet H$ be the filtration of H and let $\overline{\psi}_i$ denote the endomorphism of $\mathrm{gr}_i^W H$ induced by ψ. The eigenvalues of $\overline{\psi}_i$ are Weil numbers of integral weight by hypothesis. Since the characteristic polynomial of ψ is the product of the characteristic polynomials of $\psi_{|W_{n-1}H}$ and $\overline{\psi}_n$, one proves recursively that all eigenvalues of ψ are

Weil numbers (of integral weight).
(ii) ⇒ *(iii)*. If all eigenvalues of ψ are in K_v, then H is the direct sum of its generalized eigenspaces. Let H_i be the direct sum of generalized eigenspaces associated to eigenvalues of weight i. Then $(H_i, \psi_{|H_i})$ is a pure F-K_v-isocrystal of weight i and $H = \oplus_i H_i$. In the general case, let L/K_v be a finite Galois extension containing all the eigenvalues of ψ and let k_L be its residue field. Observe that $(H \otimes_{K_v} L, \psi \otimes \mathrm{id})$ is not an F-L-isocrystal in general. Indeed if an eigenvalue α of ψ has weight i (relative to k_v), *i.e.*, $|\alpha| = p_v^{n_v i/2}$ then $|\alpha| = p_v^{n_v ri/(2r)}$, with $p_v^{n_v r} = |k_L|$, would have weight i/r relative to k_L and i/r might not be an integer. However, the decomposition result works the same if we consider rational weights. Hence $H \otimes_{K_v} L = \oplus_i (H \otimes_{K_v} L)_{i/r}$ where the L-linear subspace $(H \otimes_{K_v} L)_{i/r}$ is the direct sum of generalized eigenspaces associated to eigenvalues of modulus $p_v^{in_v/2} = p_v^{in_v r/(2r)}$. Since all conjugates of a Weil number of weight i have the same weight, the conjugate of any eigenvalue α of ψ has the same weight. Hence the action of $\mathrm{Gal}(L/K_v)$ on $H \otimes_{K_v} L$ respects the decomposition and $(H \otimes_{K_v} L)_{i/r}$ descends to a K_v-linear subspace H_i of H which is a pure F-K_v-isocrystal of weight i.

Assume now $H = \oplus_{i=m+1}^n H'_i$ is another decomposition as in (iii). In order to prove that $H'_i = H_i$ we may assume $L = K_v$. Now H_i is sum of the generalized eigenspaces associated to eigenvalues of weight i. Since the eigenvalues of $\psi_{|H'_i}$ have weight i, it is $H'_i \subseteq H_i$. One concludes then by dimension reason.
(iii) ⇒ *(i)*. Let $\oplus_{i=m+1}^n H_i$ be a decomposition as in (iii) and set $W_m H = 0$ and $W_i H = \oplus_{j=m+1}^i H_j$ for $m < i \le n$. This filtration makes H a mixed F-K_v-isocrystal. Let $W'_\bullet H$ be another filtration making (H, ψ) a F-K_v-isocrystal. One proves recursively that the eigenvalues of $\psi_{|W'_i H}$ have weights $\le i$. In particular, the image of $W'_i H$ in $\mathrm{gr}_{i+1}^W H_\bullet$ is trivial. Hence $W'_i H \subseteq W_i H$. The reverse inclusion is proved analogously. Hence $W_\bullet H = W'_\bullet H$. □

It follows from the lemma above that any morphism of F-K_v-isocrystals respects the decompositions in pure F-K_v-isocrystals, thus the filtrations.

1.3. **The enriched Ogus category FOg(K).** We say that an object $\mathcal{V}$ of $\mathbf{Og}(K)$ is pure of weight i if for almost all unramified places v the K_v-vector space V_v and the K_v-linear operator $F_v^{n_v}$ on V_v define a pure F-K_v-isocrystal of weight i relative to k_v.

1.3.1. **Definition.** Let $\mathbf{FOg}(K)$ be the category whose objects are objects $\mathcal{V}$ in $\mathbf{Og}(K)$ endowed with an increasing finite exhaustive filtration

$$0 = W_m\mathcal{V} \subseteq \cdots \subseteq W_i\mathcal{V} \subseteq W_{i+1}\mathcal{V} \subseteq \cdots \subseteq W_n\mathcal{V} = \mathcal{V}$$

in $\mathbf{Og}(K)$ such that for every $i > m$ the graded $\mathrm{gr}_i^W \mathcal{V} := W_i\mathcal{V}/W_{i-1}\mathcal{V}$ is pure of weight i. In particular, $(V_v, F_v^{n_v})$ is a mixed F-K_v-isocrystal for almost all v. Morphisms are morphisms in $\mathbf{Og}(K)$.

Morphisms in $\mathbf{FOg}(K)$ actually respect the filtration by the following:

1.3.2. **Lemma.** *With the notation above we have the following properties.*

(1) *Given an object $\mathcal{V}$ of $\mathbf{Og}(K)$ there is at most one filtration $W_\bullet\mathcal{V}$ on $\mathcal{V}$ such that $(\mathcal{V}, W_\bullet\mathcal{V})$ is an object of $\mathbf{FOg}(K)$.*

(2) *Let* $(\mathcal{V}, W_\bullet\mathcal{V})$, $(\mathcal{V}', W_\bullet\mathcal{V}')$ *be objects of* $\mathbf{FOg}(K)$. *Then any morphism* $\mathcal{V} \to \mathcal{V}'$ *in* $\mathbf{Og}(K)$ *respects the filtration and is strict.*
(3) $\mathbf{FOg}(K)$ *is a* $\mathbb{Q}$*-linear abelian category.*
(4) *Given an object* $(\mathcal{V}, W_\bullet\mathcal{V})$ *of* $\mathbf{FOg}(K)$ *for almost all* v *we have that* $\mathcal{V}_v$ *has a unique decomposition as a direct sum of pure* F-K_v*-isocrystal of different weights.*

Proof. (1) and (4) follow from Lemma 1.2.1. Since the morphisms between pure F-K_v-isocrystals of different weights are trivial, it follows that morphisms between mixed F-K_v-isocrystal with integral weights respect the filtrations and are strict. Using the fact that g_v induces an isomorphism $W_\bullet V_v \simeq W_\bullet V \otimes_K K_v$ and that the map $K \to K_v$ is faithfully flat, we get (2). Assertion (3) follows easily from (2). □

Note that for $(\mathcal{V}, W_\bullet\mathcal{V}) \in \mathbf{FOg}(K)$ and an integer $n \in \mathbb{Z}$ we have that

$$(\mathcal{V}, W_\bullet\mathcal{V})(n) := (\mathcal{V}(n), W_{\bullet+2n}\mathcal{V}(n)) \in \mathbf{FOg}(K)$$

such that $\mathrm{gr}_i^W \mathcal{V}(n) := W_{i+2n}\mathcal{V}(n)/W_{i+2n-1}\mathcal{V}(n)$ is pure of weight i (*cf.* Definition 1.1.1).

1.4. **The weight filtration on $\mathbf{FOg}(K)$.** Consider the Serre subcategories $\mathbf{FOg}(K)_{\leq n}$ of $\mathbf{FOg}(K)$ given by objects $(\mathcal{V}, W_\bullet\mathcal{V})$ of $\mathbf{FOg}(K)$ with $W_n\mathcal{V} = \mathcal{V}$. We get a filtration

$$\cdots \to \mathbf{FOg}(K)_{\leq n} \xrightarrow{\iota_n} \mathbf{FOg}(K)_{\leq n+1} \xrightarrow{\iota_{n+1}} \cdots$$

Recall that a filtration of an abelian category by Serre subcategories is a weight filtration (in the sense of [6, Def. D.1.14]) if it is separated, exhaustive and split, *i.e.*, all the inclusion functors ι_n have exact right adjoints.

1.4.1. **Lemma.** $\mathbf{FOg}(K)_{\leq n} \subset \mathbf{FOg}(K)$ *is a weight filtration.*

Proof. In fact, the filtration is clearly separated, *i.e.*, $\cap\mathbf{FOg}(K)_{\leq n} = 0$, and exhaustive, *i.e.*, $\cup\mathbf{FOg}(K)_{\leq n} = \mathbf{FOg}(K)$. The claimed adjoints are given by $(\mathcal{V}, W_\bullet\mathcal{V}) \mapsto (W_n\mathcal{V}, W_\bullet^{\leq n}\mathcal{V})$ where $W_i^{\leq n}\mathcal{V} = W_i\mathcal{V}$ for $i < n$ and $W_i^{\leq n}\mathcal{V} = W_n\mathcal{V}$ for $i \geq n$ and they are exact by Lemma 1.3.2 (2). □

We have that

$$\mathrm{gr}_i^W \mathcal{V} = W_i\mathcal{V}/W_{i-1}\mathcal{V} \in \mathbf{FOg}(K)_i := \mathbf{FOg}(K)_{\leq i}/\mathbf{FOg}(K)_{\leq i-1}.$$

Note that these categories $\mathbf{FOg}(K)_i$ are not necessarily semi-simple.

We may introduce a notion of effectivity following [1, 7.4.2] or [6, 17.4.4].

1.4.2. **Definition.** An object $(V, (V_v, F_v)_{v\in\mathcal{P}'}, (g_v)_{v\in\mathcal{P}'})$ of $\mathbf{Og}(K)$ is said to be *l-effective* if there exists a $\mathcal{O}_K$-lattice L of V such that the image under g_v of $L \otimes_{\mathcal{O}_K} \mathcal{O}_{K_v}$ in V_v is preserved by the F_v for almost all $v \in \mathcal{P}'$. Denote $\mathbf{Og}(K)^{\mathrm{eff}} \subset \mathbf{Og}(K)$ the full subcategory of l-effective objects.

Similarly define $\mathbf{FOg}(K)^{\mathrm{eff}} \subset \mathbf{FOg}(K)$ as the full subcategory given by objects $(\mathcal{V}, W_\bullet\mathcal{V})$ such that $\mathcal{V}$ is in $\mathbf{Og}(K)^{\mathrm{eff}}$.

Moreover, we say that an object $(\mathcal{V}, W_\bullet\mathcal{V})$ of $\mathbf{FOg}(K)$ is *e-effective* if the eigenvalues of $F_v^{n_v}$ on V_v are algebraic integers for almost all v.

1.4.3. **Remarks.** (a) An object $(V,(V_v,F_v)_{v\in\mathcal{P}'},(g_v)_{v\in\mathcal{P}'})$ of $\mathbf{Og}(K)$ is l-effective if, and only if, *every* $\mathcal{O}_K$-lattice L of V satisfies the condition in Definition 1.4.2. Indeed any two $\mathcal{O}_K$-lattices L, L' of V coincide over $\mathcal{O}_K[1/n]$ for n sufficiently divisible. Hence if L satisfies the condition in Definition 1.4.2 for any $v\in\mathcal{P}''\subset\mathcal{P}'$, the same does L' for any $v\in\mathcal{P}''\cap\mathcal{P}_n$.

(b) Since any $\mathcal{O}_K[1/n]$-lattice L of V is isomorphic to the base change along $\mathcal{O}_K\to\mathcal{O}_K[1/n]$ of a $\mathcal{O}_K$-lattice, an object $(V,(V_v,F_v)_{v\in\mathcal{P}'},(g_v)_{v\in\mathcal{P}'})$ of $\mathbf{Og}(K)$ is l-effective if, and only if, there exists a positive integer n and a $\mathcal{O}_K[1/n]$-lattice L of V such that the image under g_v of $L\otimes_{\mathcal{O}_K}\mathcal{O}_{K_v}$ in V_v is preserved by the F_v for all $v\in\mathcal{P}_n$.

(c) If $\mathcal{V}$ in $\mathbf{Og}(K)$ is l-effective, the same is any subobject $\mathcal{V}'\subset\mathcal{V}$ in $\mathbf{Og}(K)$. As a consequence given $(\mathcal{V},W_\bullet\mathcal{V})\in\mathbf{FOg}(K)^{\mathrm{eff}}$ all $W_i\mathcal{V}$ and $\mathrm{gr}_i^W\mathcal{V}$ are in $\mathbf{FOg}(K)^{\mathrm{eff}}$ as well.

According with [6, Def. 14.3.2] we also set a subcategory of level ≤ 1 objects:

1.4.4. **Definition.** Call Artin-Lefschetz objects those objects of $\mathbf{FOg}(K)_{-2}$ which are $(\mathcal{V},W_\bullet\mathcal{V})(1)$ for $(\mathcal{V},W_\bullet\mathcal{V})$ both l-effective and e-effective of weight zero. Denote $\mathbf{FOg}(K)_{\mathbb{L}}$ the Serre subcategory of $\mathbf{FOg}(K)_{-2}$ given by Artin-Lefschetz objects. Denote $\mathbf{FOg}(K)_{(1)}$ the full subcategory of $\mathbf{FOg}(K)$ given by those $(\mathcal{V},W_\bullet\mathcal{V})$ that are e-effective of weights $\{-2,-1,0\}$ such that $W_{-2}\mathcal{V}$ is an Artin-Lefschetz object and $(\mathcal{V},W_\bullet\mathcal{V})(-1)$ is l-effective.

1.5. **The Bost-Ogus category $\mathbf{BOg}(K)$.** Let $\mathbf{BOg}(K)$ denote the $\mathbb{Q}$-linear category whose objects are finite dimensional K-vector spaces V such that the reduction modulo $\mathfrak{p}_v$ of V is equipped with a σ_v-semilinear endomorphism for almost every unramified place v of K. Morphisms are morphisms of K-vector spaces which respect the extra structure. The category $\mathbf{BOg}(K)$ is the category denoted $\mathbf{Frob}_{\mathbf{ae}}(K)$ in [9, 2.3.2].

Also this category can be better described as a 2-colimit category. Recall that, given a positive integer n, $\mathcal{P}_n$ denotes the subset of $\mathcal{P}$ consisting of those v such that n is invertible in $\mathcal{O}_{K_v}$ (see Remark 1.1.2). Let $\mathcal{L}_{\mathcal{P}_n}$ be the category whose objects are of the type $\big(V,L,({}^\flat F_v)_{v\in\mathcal{P}_n}\big)$ where V is a finite dimensional K-vector space, L is an $\mathcal{O}_K[1/n]$-lattice in V and ${}^\flat F_v$ is a σ_v-semilinear endomorphism on $L\otimes_{\mathcal{O}_K}k_v$. A morphism in $\mathcal{L}_{\mathcal{P}_n}$ is the data of a homomorphism of lattices (and by K-linearization of vector spaces) which respects the given ${}^\flat F_v$ for all v. We then define

$$\mathbf{BOg}(K):=2\text{-}\varinjlim_{\mathcal{P}_n\subseteq\mathcal{P}}\mathcal{L}_{\mathcal{P}_n}.$$

Note that there is a functor (see also [1, 7.4.2])

$$\Psi\colon\mathbf{FOg}(K)^{\mathrm{eff}}\to\mathbf{BOg}(K)\tag{1.1}$$

defined as follows. Consider an object $(\mathcal{V},W_\bullet\mathcal{V})$ in $\mathbf{FOg}(K)^{\mathrm{eff}}$ and assume $\mathcal{V}$ in $\mathbf{Og}(K)^{\mathrm{eff}}$ is represented by an object $(V,(V_v,F_v)_{v\in\mathcal{P}'},(g_v)_{v\in\mathcal{P}'})$ of $\mathcal{C}_{\mathcal{P}'}$. Let L be any $\mathcal{O}_K$-lattice of V. By Remarks 1.4.3 (a) and 1.1.2 there exists a positive integer n such that $\mathcal{P}_n\subset\mathcal{P}'$ and the image under g_v of $L\otimes_{\mathcal{O}_K}\mathcal{O}_{K_v}$ in V_v is preserved by F_v for all $v\in\mathcal{P}_n$. Let $\Psi(\mathcal{V},W_\bullet\mathcal{V})$ be represented by $\big(V,L\otimes_{\mathcal{O}_K}\mathcal{O}_K[1/n],({}^\flat F_v)_{v\in\mathcal{P}_n}\big)$ in $\mathcal{L}_{\mathcal{P}_n}$ where ${}^\flat F_v$ is the reduction modulo $\mathfrak{p}_v$ of the mapping on $L\otimes_{\mathcal{O}_K}\mathcal{O}_{K_v}$ induced by F_v. Note that a different choice of a lattice of V will provide the same lattice over $\mathcal{O}_K[1/n]$ for n sufficiently divisible and hence the functor is well defined. Furthermore, remark that the functor Ψ is not full (*cf.* Remark 3.3.7).

2. Logarithms and universal extensions

2.1. **The pth power operation.** Recall from [13, II §7 n.2 p. 273] and [14, Exp. $\mathrm{VII_A}$, §6] that given a field k of characteristic $p>0$ and a k-group scheme G one can define a pth power operation $x \mapsto x^{[p]}$ on $\mathrm{Lie}(G)$ as follows. Recall that

$$\mathrm{Lie}(G)=\mathrm{Ker}(G(k[\varepsilon]/(\varepsilon^2))\to G(k))$$

and for any $x\in \mathrm{Lie}(G)$ write $e^{\varepsilon x}$ for the corresponding element in $G(k[\varepsilon]/(\varepsilon^2))$. Let $k[\varsigma,\pi]\subseteq k[\varepsilon_1,\dots,\varepsilon_p]/(\varepsilon_1^2,\dots,\varepsilon_p^2)$, with $\varsigma=\sum_{i=1}^p \varepsilon_i$, $\pi=\prod_{i=1}^p\varepsilon_i$, be the subalgebra generated by the elementary symmetric polynomials in ε_i. Observe that $\varsigma^p=0$, $\varsigma\pi=0$ and $\pi^2=0$. Then $e^{\varepsilon_1 x}e^{\varepsilon_2 x}\cdots e^{\varepsilon_p x}$ makes sense as element in

$$\mathrm{Ker}(G(k[\varepsilon_1,\dots,\varepsilon_p]/(\varepsilon_1^2,\dots,\varepsilon_p^2))\to G(k))$$

where we use the multiplicative notation for the group law on G. Since $e^{\varepsilon_1 x}e^{\varepsilon_2 x}\cdots e^{\varepsilon_p x}$ is invariant by permutations of the ε_i's (see [13, II §4, 4.2 (6) p. 210]), it is indeed an element of $\mathrm{Ker}(G(k[\varsigma,\pi])\to G(k))$. Consider now the canonical projection $k[\varsigma,\pi]\to k[\pi], \varsigma\mapsto 0$. It induces a map

$$G(k[\varsigma,\pi])\to G(k[\pi]).$$

Let $e^{\pi y}$ be the image of $e^{\varepsilon_1 x}e^{\varepsilon_2 x}\cdots e^{\varepsilon_p x}$ via this map. We further have that $e^{\pi y}$ is mapped to the unit section via the map $G(k[\pi])\to G(k)$ induced by $\pi\mapsto 0$. We have

$$\begin{array}{ccc} G(k[\varsigma,\pi]) & \longrightarrow & G(k) \\ \downarrow & \nearrow & \\ G(k[\pi]) & & \end{array} \qquad \begin{array}{ccc} e^{\varepsilon_1 x}e^{\varepsilon_2 x}\cdots e^{\varepsilon_p x} & \longrightarrow & 1 \\ \downarrow & \nearrow & \\ e^{\pi y} & & \end{array}$$

Hence $y\in\mathrm{Lie}(G)=\mathrm{Ker}(G(k[\pi])\to G(k))$ and we define $x^{[p]}:=y$. The map

$$^{[p]}\colon \mathrm{Lie}(G)\to\mathrm{Lie}(G)\quad x\mapsto x^{[p]} \tag{2.1}$$

endows $\mathrm{Lie}(G)$ with a structure of Lie p-algebra over k (see [13, II, §7 Prop. 3.4 p. 277]). If G is commutative, then $[x,y]=0$ in $\mathrm{Lie}(G)$ (see [14, Exp. II, Def. 4.7.2]) and hence $^{[p]}$ is p-linear, *i.e.*, $(x+y)^{[p]}=x^{[p]}+y^{[p]}, (\lambda x)^{[p]}=\lambda^p x^{[p]}$ for $\lambda\in k$ and $x,y\in\mathrm{Lie}(G)$. Up to the usual identification of $\mathrm{Lie}(G)$ with the invariant derivations of G, the pth power operation maps a derivation D to D^p (*cf.* [14, Exp. $\mathrm{VII_A}$, §6.1], [13, II, §7, Prop. 3.4 p. 277]).

Let σ denote the Frobenius map on k. For the sake of exposition we provide a proof of the following well known fact (*cf.* [14, Exp. VII_A §4]).

2.1.1. **Lemma.** *Let G be a commutative algebraic k-group. Then the pth power operation (2.1) is a σ-semilinear map and coincides with the map on Lie algebras associated to the Verschiebung $\mathrm{Ver}_G\colon G^{(p)}\to G$.*

Proof. The first claim is obvious. By functoriality of the Veschiebung (see [14, Exp. $\mathrm{VII_A}$, 4.3]) and the fact that $\mathrm{Lie}(\mathrm{Fr}_G)=0$ with $\mathrm{Fr}_G\colon G\to G^{(p)}$ the (relative) Frobenius, it suffices to show the second claim replacing G by the kernel of Fr_G: we thus assume G to be finite (infinitesimal). By [7, Thm. 3.1.1] we can embed G as a closed subgroup-scheme of an abelian variety. By functoriality of the Verschiebung and of the pth power operation we can further reduce to the case of abelian varieties. The latter follows from Example 2.1.2 (b) below as duality on abelian varieties exchanges Frobenius with Verschiebung (*cf.* [16, Prop. 7.34]). □

2.1.2. **Examples.** (a) It follows from [13, II, §7 Exemples 2.2 p. 273] that the pth power operation on $\mathrm{Lie}(\mathbb{G}_a)$ is the zero map $x \mapsto 0$, while the pth power operation on $\mathrm{Lie}(\mathbb{G}_m)$ is given by $x \mapsto x^p$.

(b) When $G = A$ is an abelian variety and A^* is the dual abelian variety, it is known that there is a natural isomorphism $\mathrm{Lie}(A) \simeq \mathrm{H}^1(A^*, \mathcal{O}_{A^*})$ and the pth power operation on $\mathrm{Lie}(A)$ corresponds to the Frobenius map on $\mathrm{H}^1(A^*, \mathcal{O}_{A^*})$, *i.e.*, the σ-semilinear map induced by the Frobenius homomorphism $\alpha \mapsto \alpha^p$ on $\mathcal{O}_{A^*}$ (see [22, §15, Thm. 3]).

2.2. **The logarithms.** Let k be a finite field of characteristic p. Let $W(k)$ be the ring of Witt vectors over k, K_0 its quotient field, and set $W_n(k) := W(k)/(p^n)$. Let G be a group scheme of finite type over $W(k)$. As claimed in [17, III, 5.4.1] we have that

$$G(W(k)) = \varprojlim G(W_n(k)).$$

(The separatedness hypothesis in *loc. cit.* can be ignored since $W(k)$ is local.)

Consider the canonical homomomorphism of groups

$$\rho\colon G(W(k)) \to G(k) \tag{2.2}$$

induced by the closed immersion $\mathrm{Spec}(k) \to \mathrm{Spec}(W(k))$. Note that as k is a finite field, the k-valued points $G(k)$ of G form a finite, and hence torsion, group. In particular, if we denote Γ the kernel of ρ in (2.2), we obtain an isomorphism of $\mathbb{Q}$-vector spaces

$$G(W(k)) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{2.3}$$

Assume from now on that G is a *smooth* and *commutative* $W(k)$-group scheme and let G_n be the base change of G to $S_n = \mathrm{Spec}(W_n(k))$; in particular G_1 denotes the special fiber of G. Let $\mathcal{J}$ be the ideal sheaf of the unit section of G_n and recall that $\mathcal{O}_{G_n}/\mathcal{J}^N$ is a finite and free $W_n(k)$-module for any $N > 0$. Hence $D = \varinjlim_N \mathrm{Hom}_{W_n(k)\text{-mod}}(\mathcal{O}_{G_n}/\mathcal{J}^N, W_n(k))$ is a coalgebra with a $W_n(k)$-algebra structure induced by the group structure on G_n. The flatness of D over $W_n(k)$ ensures that the PD-structure on $pW_n(k)$ extends uniquely to a PD-structure (γ_m) on pD and hence there exist two mutually inverse maps

$$\exp\colon pD \to (1 + pD)^*, \qquad \log\colon (1 + pD)^* \to pD,$$

defined by $\exp(x) = \sum_{m\geq 0} \gamma_m(x)$ and $\log(1 + x) = \sum_{m\geq 1}(-1)^{m-1}(m-1)!\gamma_m(x)$ ([21, III, 1.6]). Let $\mathrm{Cospec}(D)(W(k)) \subset D$ denote the subgroup of $W_n(k)$-algebra homomorphisms. Then $p\mathrm{Cospec}(D)(W(k)) = \mathrm{Cospec}(D)(W(k)) \cap pD$ consists of those homomorphisms whose reduction modulo p is the homomorphism associated to the unit section of the special fiber of G. Let $\mathrm{Prim}(D) \subset D$ consists of the primitive elements of the coalgebra D, *i.e.*, those $x \in D$ such that $\Delta(x) = x \otimes 1 + 1 \otimes x$ with Δ the comultiplication of D, and let $p\mathrm{Prim}(D) = \mathrm{Prim}(D) \cap pD$. Then by [21, III, 2.2.5] (see also [3, §5.2]) there is an isomorphism of groups $\exp_{G,n}$ which makes the following diagram

$$\begin{array}{ccc} \mathrm{Ker}\big(\mathrm{Lie}(G_n) \to \mathrm{Lie}(G_1)\big) & \xrightarrow{\simeq} & p\mathrm{Prim}(D) \subseteq pD \\ \Big\downarrow{\scriptstyle \exp_{G,n}} & & \Big\downarrow{\scriptstyle \exp} \\ \mathrm{Ker}\big(G_n(W_n(k)) \to G_n(k)\big) & \xrightarrow{\simeq} & p(\mathrm{Cospec}(D)(W(k)) \subset (1+pD)^* \end{array} \tag{2.4}$$

commute. The vertical arrow on the left can also be written as an isomorphism

$$\exp_{G,n} \colon p\mathrm{Lie}(G_n) \xrightarrow{\simeq} \mathrm{Ker}\big(G(W_n(k)) \to G(k)\big).$$

Finally, taking the limit over n, one gets the exponential isomorphism for G

$$\exp_G \colon p\mathrm{Lie}(G) \xrightarrow{\simeq} \Gamma.$$

Let $\log_G \colon \Gamma \xrightarrow{\simeq} p\mathrm{Lie}(G)$ denote the inverse of $\exp_G$. We set (*cf.* [23, §2.4, p. 169]):

2.2.1. **Definition.** The logarithm is the isomorphism of $\mathbb{Q}$-vector spaces

$$\log_{G,\mathbb{Q}} \colon G(W(k)) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\simeq} \mathrm{Lie}(G) \otimes_{W(k)} K_0$$

obtained by composing (2.3) with $\log_G \otimes \mathrm{id}_{\mathbb{Q}}$ and recalling that $p\mathrm{Lie}(G) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathrm{Lie}(G) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathrm{Lie}(G) \otimes_{W(k)} K_0$ where the isomorphism $p\mathrm{Lie}(G) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\simeq} \mathrm{Lie}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the $\mathbb{Q}$-linearization of the inclusion $p\mathrm{Lie}(G) \to \mathrm{Lie}(G)$.

2.2.2. **Examples.** (a) Let $G = \mathbb{G}_{a,W(k)}$. Then (2.2) is the reduction map $W(k) \to k$ and hence $\Gamma = pW(k)$. Recall that $\mathrm{Lie}(\mathbb{G}_{a,W_n(k)}) = \mathrm{Ker}(W_n(k) + W_n(k)\varepsilon \to W_n(k), a + b\varepsilon \mapsto a) \simeq W_n(k)\varepsilon$ with $\varepsilon^2 = 0$ and $D = \mathrm{Hom}_{\mathrm{cont}}(W_n(k)[[Z]], W_n(k))$. Diagram (2.4) becomes

$$\begin{array}{ccc} pW_n(k)\varepsilon & \xrightarrow{\simeq} & p\mathrm{Prim}(D) \\ \Big\downarrow{\scriptstyle \exp_{\mathbb{G}_a,n}} & & \Big\downarrow{\scriptstyle \exp} \\ pW_n(k) & \xrightarrow{\simeq} & p(\mathrm{Cospec}(D)(W(k)) \end{array} \qquad \begin{array}{ccc} b\varepsilon & \longmapsto & f_b \\ & & \Big\downarrow \\ \exp(f_b)(Z) & \longleftarrow & \exp(f_b) \end{array}$$

where $f_b(1) = 0$, $f_b(Z^r) = rb$ for $r \geq 1$. Since $\gamma_m(f_b)(Z) = 0$ for $m \neq 1$ and $\gamma_1(f_b) = f_b$, one gets that $\exp(f_b)(Z) = b$ and hence, up to the obvious identifications, we may consider $\exp_{\mathbb{G}_a}$ and $\log_{\mathbb{G}_a}$ as the identity maps on $W_n(k)$. Finally $\mathrm{Lie}_{\mathbb{G}_a,\mathbb{Q}}$ is the identity of K_0, up to the usual identifications of $G(W(k))$ and $\mathrm{Lie}(\mathbb{G}_a)$ with $W(k)$, *i.e.*, $\log_{\mathbb{G}_a,\mathbb{Q}} \colon W(k) \otimes_{\mathbb{Z}} \mathbb{Q} \to \varepsilon W(k) \otimes_{W(k)} K_0$ in Definition 2.2.1 is $x \mapsto \varepsilon x$.

(b) Let $G = \mathbb{G}_{m,W(k)} = \mathrm{Spec}(W(k)[X^{\pm 1}])$. Then $\Gamma = 1 + pW(k) \subset W(k)^*$, $\mathrm{Lie}(\mathbb{G}_{m,W_n(k)}) \simeq 1 + pW_n(k)\varepsilon$ with $\varepsilon^2 = 0$, and D is as in (a) as a co-algebra (with $Z = X - 1$). Diagram (2.4) becomes

$$\begin{array}{ccc} 1 + pW_n(k)\varepsilon & \xrightarrow{\simeq} & p\mathrm{Prim}(D) \\ \Big\downarrow{\scriptstyle \exp_{\mathbb{G}_m,n}} & & \Big\downarrow{\scriptstyle \exp} \\ 1 + pW_n(k) & \xrightarrow{\simeq} & p(\mathrm{Cospec}(D)(W(k)) \end{array} \qquad \begin{array}{ccc} 1 + b\varepsilon & \longmapsto & f_b \\ & & \Big\downarrow \\ 1 + \exp(f_b)(Z) & \longleftarrow & \exp(f_b) \end{array}$$

where $f_b(1) = 0$, $f_b(Z^r) = rb$ for $r \geq 1$. Since $\gamma_m(f_b)$ maps Z to 0 if $m = 0$ and to $b^m/m!$ if $m \geq 1$ then $\exp(f_b)(Z) = \sum_{m \geq 1} b^m/m!$ and $\exp_{\mathbb{G}_m,n}(1 + b\varepsilon) = \sum_{m \geq 0} b^m/m!$. Hence, up to the obvious identifications, $\exp_{\mathbb{G}_m}$ is the exponential map $pW(k) \to 1 + pW(k)$ and

$\log_{\mathbb{G}_m}$ is the usual p-adic logarithm. Finally the isomorphism $\log_{\mathbb{G}_m,\mathbb{Q}}\colon W(k)^* \otimes_{\mathbb{Z}} \mathbb{Q} \to 1 + (W(k)\varepsilon \otimes_{W(k)} K_0)$ in Definition 2.2.1 is given by

$$x \otimes 1 \mapsto 1 + \frac{\log(1+y)\varepsilon}{(p_v^{n_v} - 1)}$$

where $x^{p_v^{n_v}-1} = 1 + y$, $y \in pW(k)$.

2.2.3. **Remark.** Note that for any $W_n(k)$-scheme T, one can define a functorial (in T and G) isomorphism $\exp_{G,n}\colon p\underline{\mathrm{Lie}}(G)(T) \xrightarrow{\simeq} \mathrm{Ker}(G(T) \to G(T_0))$ where T_0 is the reduction of T modulo p and $\underline{\mathrm{Lie}}(G)$ is the Lie algebra scheme of G (see [3, §5.2]). In particular any map $\exp_{G,n}$ (and thus $\exp_G$) behaves well with respect to finite unramified extension of $W(k)$. Further, for any finite Galois extension k'/k the map $\log_{G,\mathbb{Q}}$ in Definition 2.2.1 can be obtained by descent from the analogous isomorphism over $W(k')$.

Let $u\colon L \to G$ be a morphism of $W(k)$-group schemes where $L = \mathbb{Z}^r$ and G is a smooth and commutative $W(k)$-group scheme with connected fibers. Define δ_u as the K_0-linear extension of the composition

$$L(W(k)) \xrightarrow{u\otimes 1} G(W(k)) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\log_{G,\mathbb{Q}}} \mathrm{Lie}(G) \otimes_{W(k)} K_0 \tag{2.5}$$

2.2.4. **Lemma.** *Let $u\colon L \to G$ be a morphism of $W(k)$-group schemes where L is a lattice (i.e., isomorphic to $\mathbb{Z}^r$ over some finite unramified extension of $W(k)$) and G is a smooth, connected and commutative $W(k)$-group scheme. Then there is a unique morphism of K_0-vector spaces*

$$\delta_u\colon \mathrm{Lie}(L \otimes \mathbb{G}_a) \otimes_{W(k)} K_0 \to \mathrm{Lie}(G) \otimes_{W(k)} K_0$$

which is functorial in u and in $W(k)$ and agrees with the one in (2.5) for L constant. Moreover, if u is given by the obvious inclusion $\mathrm{id} \otimes 1\colon L \to L \otimes \mathbb{G}_a$, *then δ_u is the identity of* $\mathrm{Lie}(L \otimes \mathbb{G}_a) \otimes_{W(k)} K_0$.

Proof. It follows by Remark 2.2.3. The last assertion follows from the fact that, up to the usual identifications, $\log_{\mathbb{G}_a,\mathbb{Q}}$ in Definition 2.2.1 is the identity map as computed in Example 2.2.2 (a). □

2.2.5. **Remark.** For any formal $W(k)$-group scheme $\widehat{G} = \varprojlim_n G_n$ where G_n is a smooth commutative group scheme of finite type over $W_n(k)$, one can define in a similar way the logarithm $\log_{\widehat{G},\mathbb{Q}}\colon \widehat{G}(W(k)) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\simeq} \mathrm{Lie}(\widehat{G}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with $\mathrm{Lie}(\widehat{G}) := \varprojlim_n \mathrm{Lie}(G_n)$. This construction is again functorial, *i.e.*, given a morphism $g = \varprojlim_n g_n\colon \widehat{G} \to \widehat{H}$ then $\log_{\widehat{H},\mathbb{Q}} \circ (g \otimes 1) = (\mathrm{Lie}(g) \otimes \mathrm{id}) \circ \log_{\widehat{G},\mathbb{Q}}\colon \widehat{G}(W(k)) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathrm{Lie}(\widehat{H}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Further, $\log_{G,\mathbb{Q}}$ in Definition 2.2.1 equals $\log_{\widehat{G},\mathbb{Q}}$ with $\widehat{G}$ the p-adic completion of G.

As in Lemma 2.2.4, given a lattice L over $W(k)$ with base change L_n to $W_n(k)$ and a compatible system of morphisms of $W_n(k)$-group schemes $\{u_n\colon L_n \to G_n\}_{n\in\mathbb{N}}$ over $W_n(k)$ we get a natural morphism of K_0-vector spaces $\delta_u\colon \mathrm{Lie}(L \otimes \mathbb{G}_a) \otimes_{W(k)} K_0 \to \mathrm{Lie}(\widehat{G}) \otimes_{W(k)} K_0$.

2.3. **Universal extensions.** Let $\mathcal{M}_1(S)$ be the category of (Deligne) 1-motives over a scheme S (*cf.* [6, §1.2 & App. C]). If $S = \mathrm{Spec}(K)$, let $\mathcal{M}_{1,\mathbb{Q}}$ denote the $\mathbb{Q}$-linear category of 1-motives up to *isogenies* over K, *i.e.*, the category whose objects are

$$\mathsf{M}_K = [\mathsf{u}\colon \mathsf{L}_K \to \mathsf{G}_K]$$

in $\mathcal{M}_1 := \mathcal{M}_1(\mathrm{Spec}(K))$ and whose morphisms are given by $\mathrm{Hom}_{\mathcal{M}_1}(\mathsf{M}_K, \mathsf{N}_K) \otimes_{\mathbb{Z}} \mathbb{Q}$. See [6, Prop. 1.2.6] for a proof that $\mathcal{M}_{1,\mathbb{Q}}$ is an abelian category. A morphism (h_{-1}, h_0) in $\mathcal{M}_1$ becomes an isomorphism in $\mathcal{M}_{1,\mathbb{Q}}$ if and only if it is an isogeny, *i.e.*, h_{-1} is injective with finite cokernel and h_0 is an isogeny (see [6, Lemma 1.2.7]). Recall that the canonical weight filtration of 1-motives yields a weight filtration on $\mathcal{M}_{1,\mathbb{Q}}$ (see [6, Prop. 14.2.1]). We let A_K be the maximal abelian quotient of G_K and let

$$\mathsf{M}_{\mathrm{ab},K} := [\mathsf{u}_{\mathsf{A}_K}\colon \mathsf{L}_K \to \mathsf{A}_K]$$

denote the (Deligne) 1-motive induced by M_K via $\mathsf{L}_K \to \mathsf{G}_K \to \mathsf{A}_K$.

Let $\mathcal{M}_{0,\mathbb{Q}}$ be the abelian category of Artin motives identified with the full subcategory of $\mathcal{M}_{1,\mathbb{Q}}$ whose objects are of the type $[\mathsf{L}_K \to 0]$, *i.e.*, 1-motives which are pure of weight zero.

Let $\mathcal{M}_1^{\mathrm{a},l}$ be the larger category of generalized 1-motives with *additive factors* over K whose objects are two terms complexes (in degree -1, 0) $\mathsf{M}_K = [\mathsf{u}_K\colon \mathsf{L}_K \to \mathsf{E}_K]$ where L_K is a a lattice, E_K is a commutative connected algebraic K-group and u_K is a morphism of algebraic K-groups. Note that $\mathcal{M}_1^{\mathrm{a},l}$ is a full subcategory of the category $\mathcal{M}_1^{\mathrm{a}}$ considered in [4, §1].

Let $\mathsf{M} = [\mathsf{u}\colon \mathsf{L} \to \mathsf{G}] \in \mathcal{M}_1(S)$ be a 1-motive over S. Recall (*e.g.* see [2, §2]) that there exists the *universal* $\mathbb{G}_a$-extension of M which we denote by

$$\mathsf{M}^\natural := [\mathsf{u}^\natural\colon \mathsf{L} \to \mathsf{G}^\natural].$$

It is an extension of M by a vector group $\mathbb{V}(\mathsf{M})$ such that the homomorphism of push-out

$$\mathrm{Hom}_{\mathcal{O}_S}(\mathbb{V}(\mathsf{M}), W) \longrightarrow \mathrm{Ext}(\mathsf{M}, W)$$

is an isomorphism for all vector groups W over S (where $\mathrm{Hom}_{\mathcal{O}_S}(-,-)$ means homomorphisms of vector groups). If $S = \mathrm{Spec}\, K$ then $\mathsf{M}_K^\natural \in \mathcal{M}_1^{\mathrm{a},l}$ is a 1-motive with additive factors over K (and Ext is taken in $\mathcal{M}_1^{\mathrm{a},l}$).

2.3.1. **Definition.** The de Rham realization of M is

$$T_{\mathrm{dR}}(\mathsf{M}) := \mathrm{Lie}(\mathsf{G}^\natural).$$

It is easy to check that $\mathbb{V}(\mathsf{M})$ has to be the vector group associated to $\mathrm{Ext}(\mathsf{M}, \mathbb{G}_a)^\vee$. Furthermore $\mathbb{V}(\mathsf{M})$ is canonically isomorphic to the vector group associated to the sheaf ω_{G^*} of invariant differentials of the semiabelian scheme G^* Cartier dual of M_{ab}. We have a push-out diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{V}(\mathsf{G}) = \omega_{\mathsf{A}^*} & \longrightarrow & \mathsf{A}^\natural \times_{\mathsf{A}} \mathsf{G} & \longrightarrow & \mathsf{G} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle i} & & \big\downarrow & & \Vert & & \\
0 & \longrightarrow & \mathbb{V}(\mathsf{M}) = \omega_{\mathsf{G}^*} & \longrightarrow & \mathsf{G}^\natural & \xrightarrow{\rho} & \mathsf{G} & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \overline{\tau}} & & \big\downarrow{\scriptstyle \tau} & & & & \\
 & & \mathsf{L} \otimes \mathbb{G}_a & = & \mathsf{L} \otimes \mathbb{G}_a & & & &
\end{array} \tag{2.6}$$

with $\mathsf{A}^\natural := \mathrm{Pic}^{\natural,0}(\mathsf{A}^*)$ the universal extension of A. Note that the lifting $\mathsf{u}^\natural$ of u composed with τ gives the map $\mathsf{L} \to \mathsf{L} \otimes \mathbb{G}_a, x \mapsto x \otimes 1$; see [5, §2] for details.

Any morphism of 1-motives $\varphi\colon \mathsf{M} \to \mathsf{N}$ provides a morphism $\varphi^\natural\colon \mathsf{M}^\natural \to \mathsf{N}^\natural$ that maps term by term the elements of the corresponding diagrams (2.6); it provides a morphism such that the induced morphism $\mathbb{V}(\mathsf{M}) \to \mathbb{V}(\mathsf{N})$ corresponds to the pull-back on invariant differentials along the induced morphism obtained via Cartier duality from $\mathsf{M}_{\mathrm{ab}} \to \mathsf{N}_{\mathrm{ab}}$.

2.3.2. **Remark.** Assume $S = \mathrm{Spec}(W(k))$, and recall the morphism $\mathsf{u}^\natural\colon \mathsf{L} \to \mathsf{G}^\natural$ defining $\mathsf{M}^\natural$. By Lemma 2.2.4 we have a morphism of K_0-vector spaces

$$\delta_{\mathsf{u}^\natural}\colon \mathrm{Lie}(\mathsf{L} \otimes \mathbb{G}_a) \otimes_{W(k)} K_0 \to \mathrm{Lie}(\mathsf{G}^\natural) \otimes_{W(k)} K_0$$

which is a section of $\mathrm{Lie}(\tau) \otimes \mathrm{id}_{K_0}$ with τ as in (2.6). We will show next that these K_0-vector spaces are endowed with a structure of an F-K_0-isocrystal and that $\delta_{\mathsf{u}^\natural}$ commutes with the Frobenius, see Lemma 3.3.1.

3. Fullness of the Ogus realization

3.1. **Models.** By writing $K = \varinjlim_n \mathcal{O}_K[1/n]$ any scheme X_K of finite type over K admits a model $X[1/n]$ of finite presentation over $\mathcal{O}_K[1/n]$ for n sufficiently divisible, *i.e.*, large enough in the preorder given by divisibility (see [17, IV, 8.8.2(ii) p. 28]). Furthermore this model is essentially unique, *i.e.*, models $X[1/n]$ and $X[1/m]$ of X_K become isomorphic on $\mathcal{O}_K[1/N]$, with N a suitable multiple of m and n (see [17, IV, 8.8.2.5 p. 32]).

Any algebraic K-group G_K thus admits a model G which is a group scheme of finite presentation over $S = \mathrm{Spec}(\mathcal{O}_K[1/n])$, for n sufficiently divisible, and G is essentially unique. Note that G may be assumed to be smooth, see [17, IV, Proposition 17.7.8(ii)]. Let $G_{\mathcal{O}_{K_v}}$ denote the base change of G to $\mathcal{O}_{K_v}$, when it makes sense, and let G_{k_v} be its special fiber.

A morphism of algebraic K-groups $f_K\colon G_K \to G'_K$ extends to an S-morphism of *schemes* between the models (see [17, IV, 8.8.1.1 p. 28]). Up to inverting finitely many primes, one can assume that this is indeed a morphism of S-group schemes.

Let $\mathbf{GCC}(K)$ be the category of commutative connected algebraic K-groups and $\mathbf{GCC}(K)_{\mathbb{Q}}$ its localization at the class of isogenies. One can define a functor

$$\mathbf{Lie}\colon \mathbf{GCC}(K) \to \mathbf{BOg}(K) \tag{3.1}$$

which associates to any commutative connected algebraic K-group G_K the object in $\mathbf{BOg}(K)$ (see §1.5) represented by the triple $(V, L, ({}^\flat F_v)_{v\in\mathcal{P}_n})$ in $\mathcal{L}_{\mathcal{P}_n}$ defined as follows. Set $V := \mathrm{Lie}(G_K)$, $L := \mathrm{Lie}(G)$ and ${}^\flat F_v$ on $\mathrm{Lie}(G_{k_v}) \cong L \otimes_{\mathcal{O}_K} k_v$ the canonical σ_v-semilinear homomorphism described in (2.1) and induced by the Verschiebung $\mathrm{Ver}_{G_{k_v}}$ (see Lemma 2.1.1).

Note that L is a $\mathcal{O}_K[1/n]$-lattice in $\mathrm{Lie}(G_K)$ via the isomorphism $L \otimes_{\mathcal{O}_K} K \cong \mathrm{Lie}(G_K)$ and that $\mathrm{Lie}(G_{k_v}) \cong L \otimes_{\mathcal{O}_K} k_v$ is the canonical isomorphism (see [14, Exp. II, Prop. 3.4 and §3.9.0]).

We have:

3.1.1. **Theorem** (Bost)**.** *The functor* $\mathbf{Lie}\colon \mathbf{GCC}(K)_{\mathbb{Q}} \to \mathbf{BOg}(K)$ *is fully faithful.*

Proof. The difficult part in the proof is the fullness. See [9, Thm. 2.3 & Cor. 2.6] and [10] for details. On the other hand, the proof of the faithfulness is immediate. Let $f_K\colon G_K \to G'_K$ be a morphism of (commutative connected) algebraic K-groups and let $f\colon G \to G'$ be a morphism of $\mathcal{O}_K[1/n]$-group schemes which extends it, for n sufficiently divisible. If $\mathrm{Lie}(f_K) = 0$ then $f_K = 0$, see [13, II, §6, n. 2, Prop. 2.1 (b)]. □

Now consider a 1-motive M_K over K and models M over $S = \mathrm{Spec}(\mathcal{O}_K[1/n])$, *i.e.*, 1-motives M over S such that the base change to K is isomorphic to M_K, for n sufficiently divisible. We have:

3.1.2. **Lemma.** *Any* 1*-motive* $\mathsf{M}_K = [\mathsf{u}_K\colon \mathsf{L}_K \to \mathsf{G}_K]$ *over* K *admits a model* $\mathsf{M} = [\mathsf{u}\colon \mathsf{L} \to \mathsf{G}]$ *over an* $S = \mathrm{Spec}(\mathcal{O}_K[1/n])$ *for* n *sufficiently divisible. This model is essentially unique, i.e., any two such models are isomorphic over a* $\mathrm{Spec}(\mathcal{O}_K[1/m])$ *with* $n|m$.

Proof. Let T_K be the maximal torus in G_K and let A_K be the maximal abelian quotient of G_K. Let T be the model of T_K over S. We may assume that it is a torus. Indeed T_K becomes split over a finite Galois extension K' of K and, up to enlarging n, we may assume that $S' = \mathrm{Spec}(\mathcal{O}_{K'}[1/n])$ is étale over $S = \mathrm{Spec}(\mathcal{O}_K[1/n])$. Now, T is a torus since its base change to S' is a split torus by the essential unicity of the model. Further, by Cartier duality, the group of characters of any torus over K admits a model over an S which is étale locally a split lattice of fixed rank. In particular L_K extends to such a model L over S. On the other side, we may assume that the model A of A_K is an abelian scheme over S [8, §1.2, Thm. 3], hence A is the global Néron model of A_K over S. Finally, by [8, §10.1, Prop. 4 & 7] G_K admits a global Néron lft-model $\mathcal{G}$ over S and the identity component $\mathsf{G} := \mathcal{G}^0$ is a smooth model of $\mathsf{G_K}$ of finite type over S.

Recall that $\mathsf{G_K}$ is the Cartier dual of the 1-motive $\mathsf{M}^*_{\mathrm{ab},K} = [\mathsf{u}^*_{\mathsf{A}_K}\colon \mathsf{Y}_K \to \mathsf{A}^*_K]$ where Y_K is the group of characters of T_K. By the above discussion on models of abelian varieties and lattices and by the universal property of Néron models [8, §1.2, Definition 1], the 1-motive $\mathsf{M}^*_{\mathrm{ab},K}$ extends uniquely to a 1-motive $\mathsf{M}^*_{\mathrm{ab}} = [\mathsf{u}^*_{\mathsf{A}}\colon \mathsf{Y} \to \mathsf{A}^*]$ over S and, by the essential unicity of the model of $\mathsf{G_K}$, G is the Cartier dual of $\mathsf{M}^*_{\mathrm{ab}}$, in particular it is extension of the abelian scheme A by the torus T.

Clearly u_K extends uniquely to a morphism $\mathsf{u}\colon \mathsf{L} \to \mathcal{G}$. It remains to check that up to enlarging n, u factors through the open subscheme $\mathsf{G} = \mathcal{G}^0$ of $\mathcal{G}$. Since the formation of lft Néron models is compatible with étale base change we may assume that T is a split torus over S and $\mathsf{L} \simeq \mathbb{Z}^r$. Fix a basis of $\mathbb{Z}^r$ and let $e_i\colon S \to \mathbb{Z}^r, i = 1, \dots, r$, denote the corresponding morphisms. It is sufficient to check that up to enlarging n each map $\mathsf{u} \circ e_i$ (which corresponds to a K-rational point of G_K) factors through G. This follows from [17, IV, 8.8.1.1]. □

3.2. **The Ogus realization.** For a 1-motive $\mathsf{M}_K = [\mathsf{u}_K\colon \mathsf{L}_K \to \mathsf{G}_K]$ over K consider its model $\mathsf{M} = [\mathsf{u}\colon \mathsf{L} \to \mathsf{G}]$ over a $S = \mathrm{Spec}(\mathcal{O}_K[1/n])$ given in Lemma 3.1.2. For any $v \in \mathcal{P}_n$ let $\mathsf{M}_{\mathcal{O}_{K_v}}$ be the base change to $\mathcal{O}_{K_v}$ of the model M and let M_{k_v} be the reduction of $\mathsf{M}_{\mathcal{O}_{K_v}}$ modulo $\mathfrak{p}_v$.

3.2.1. **Definition.** Let $V := T_{\mathrm{dR}}(\mathsf{M}_K)$ be the de Rham realization (see Definition 2.3.1) as a K-vector space. Let $V_v := T_{\mathrm{dR}}(\mathsf{M}_{\mathcal{O}_{K_v}}) \otimes_{\mathcal{O}_{K_v}} K_v$ together with the induced isomorphism obtained by $g_v\colon V \otimes_K K_v \to V_v$ and compatibility of T_{dR} with base change.

Consider a model M over $S = \mathrm{Spec}(\mathcal{O}_K[1/n])$ as in Lemma 3.1.2. For every $v \in \mathcal{P}_n$ consider the canonical isomorphism $T_{\mathrm{dR}}(\mathsf{M}_{\mathcal{O}_{K_v}}) \simeq T_{\mathrm{cris}}(\mathsf{M}_{k_v})$ as stated in [2, Cor. 4.2.1]. Via this identification $T_{\mathrm{dR}}(\mathsf{M}_{\mathcal{O}_{K_v}})$ is endowed with a σ_v^{-1}-semilinear endomorphism Φ_v (Verschiebung). Let F_v be the σ_v-semilinear endomorphism $(\Phi_v \otimes \mathrm{id}_{K_v})^{-1}$ on V_v. In this way, we have associated to M_K an object $\mathcal{V} := (V, (V_v, F_v)_{v\in\mathcal{P}_n}, (g_v)_{v\in\mathcal{P}_n})$ in $\mathbf{Og}(K)$. The usual weight filtration on 1-motives induces a weight filtration $W_\bullet \mathcal{V}$ on $\mathcal{V}$ in $\mathbf{Og}(K)$. We shall denote

$$T_{\mathrm{Og}}(\mathsf{M}_K) := (\mathcal{V}, W_\bullet \mathcal{V}).$$

Note that Φ_v is, in general, not invertible on $T_{\mathrm{dR}}(\mathsf{M}_{\mathcal{O}_{K_v}})$ and this F_v is the Frobenius of [2, §4.1] divided by p_v (see [2, §4.3 (4.b)]), *i.e.*, $p_v F_v$ is the map associated to the Verschiebung $\mathsf{M}_{\mathcal{O}_{K_v}}^{(p_v)} \to \mathsf{M}_{\mathcal{O}_{K_v}}$. This choice is made so that the weight filtration on 1-motives and on the isocrystals are compatible. We have:

3.2.2. **Lemma.** *$T_{\mathrm{Og}}(\mathsf{M}_K) \in \mathbf{FOg}(K)$.*

Proof. In fact $W_0\mathcal{V}/W_{-1}\mathcal{V} = T_{\mathrm{Og}}([\mathsf{L}_K \to 0])$ so that the underlying K-vector space is $T_{\mathrm{dR}}([\mathsf{L}_K \to 0]) = \mathsf{L}_K \otimes K$ and similarly for $W_{-2}\mathcal{V} = T_{\mathrm{Og}}([0 \to \mathsf{T}_K])$ with underlying vector space $Y_K \otimes K$ where Y_K the cocharacter group of the torus T_K. For T the S-torus which is a model of T_K let Y be the group of cocharacters of T. According to [2, §4.1] for almost all unramified places v the σ_v-semilinear homomorphism F_v on $T_{\mathrm{cris}}([\mathsf{L}_{k_v} \to 0]) = \mathsf{L} \otimes W(k_v)$ (respectively on $T_{\mathrm{cris}}([0 \to \mathsf{T}_{k_v}])) = Y \otimes W(k_v))$ is $1 \otimes \sigma_v$ (respectively the map $1 \mapsto 1 \otimes p_v^{-1}\sigma_v$). Hence $W_0\mathcal{V}/W_{-1}\mathcal{V}$ and $W_{-2}\mathcal{V}$ are pure of weight 0 and -2 respectively (in the sense of §1.3).

On the other hand $W_{-1}\mathcal{V}/W_{-2}\mathcal{V} = T_{\mathrm{Og}}(\mathsf{A}_K)$ and A is a model over S of the abelian variety A_K. Thanks to [2, Thm. A, p. 111] for every unramified place v of good reduction for A_K we can identify $T_{\mathrm{cris}}([0 \to \mathsf{A}_{k_v}])$ and the σ_v-semilinear homomorphism F_v with the covariant Dieudonné module or equivalently with the crystalline homology of the reduction A_{k_v} of A. The latter defines a pure F-K_v-isocrystal of weight -1 thanks to a homological version of [19, Cor. 1 (2)]. □

Recall that a functor between abelian categories with a weight filtration respects the splittings (in the sense of [6, Def. D.2.2]) if it takes pure objects to pure objects of the same weight.

3.2.3. **Proposition.** *There is a functor*

$$T_{\mathrm{Og}} \colon \mathcal{M}_{1,\mathbb{Q}} \to \mathbf{FOg}(K)$$

which associates to a 1-motive M_K the object $T_{\mathrm{Og}}(\mathsf{M}_K)$ in $\mathbf{FOg}(K)$ provided by Lemma 3.2.2. This functor respects the splittings and its essential image is contained in $\mathbf{FOg}(K)_{(1)}$ (see Definition 1.4.4).

Proof. It follows from the proof of the Lemma 3.2.2 and Remark 1.4.3 (b). In fact, $T_{\mathrm{Og}}(\mathsf{M}_K)$ is l-effective and e-effective in weight 0 and Artin-Lefschetz in weight -2. Moreover, it is e-effective in weight -1 by [19, Cor. 1 (2)]. Let $L := T_{\mathrm{dR}}(\mathsf{M})$ be the Lie algebra of the universal extension of the model M over $\mathrm{Spec}(\mathcal{O}_K[1/n])$ as in Lemma 3.1.2. It is a $\mathcal{O}_K[1/n]$-lattice in $V = T_{\mathrm{dR}}(\mathsf{M}_K)$ and $L \otimes \mathcal{O}_{K_v} \cong T_{\mathrm{dR}}(\mathsf{M}_{\mathcal{O}_{K_v}})$ is preserved by $p_v F_v$ as remarked above. Hence $T_{\mathrm{Og}}(\mathsf{M}_K)(-1) \in \mathbf{FOg}(K)^{\mathrm{eff}}$. □

3.2.4. **Definition.** The Bost-Ogus realization

$$T_{\mathrm{BOg}}\colon \mathcal{M}_{1,\mathbb{Q}} \to \mathbf{BOg}(K)$$

associates to a 1-motive M_K an object $T_{\mathrm{BOg}}(\mathsf{M}_K) := \Psi T_{\mathrm{Og}}(\mathsf{M}_K)(-1)$ in $\mathbf{BOg}(K)$ where Ψ is the functor defined in (1.1).

Note that if $\mathsf{M}_K = [\mathsf{L}_K \to \mathsf{G}_K]$, then $T_{\mathrm{BOg}}(\mathsf{M}_K) = \mathbf{Lie}(\mathsf{G}_K^\natural)$ with **Lie** as in (3.1).

3.3. **The main Theorem.** For a 1-motive $\mathsf{M}_K = [\mathsf{u}_K\colon \mathsf{L}_K \to \mathsf{G}_K]$ over K consider the universal extension $\mathsf{M}^\natural$ of a model M over $S = \mathrm{Spec}(\mathcal{O}_K[1/n])$ and the morphism τ in (2.6). For every unramified place v in $\mathcal{P}_n$ let $\mathsf{M}^\natural_{\mathcal{O}_{K_v}} = [\mathsf{L}_{\mathcal{O}_{K_v}} \to \mathsf{G}^\natural_{\mathcal{O}_{K_v}}]$ be the base change of $\mathsf{M}^\natural$ to $\mathcal{O}_{K_v}$. Recall the K_v-linear map

$$\delta_v\colon \mathrm{Lie}(\mathsf{L}_{\mathcal{O}_{K_v}} \otimes \mathbb{G}_a) \otimes_{\mathcal{O}_{K_v}} K_v \to \mathrm{Lie}(\mathsf{G}^\natural_{\mathcal{O}_{K_v}}) \otimes_{\mathcal{O}_{K_v}} K_v$$

considered in Remark 2.3.2.

3.3.1. **Lemma.** *The homomorphism δ_v is the unique section of* $\mathrm{Lie}(\tau_{K_v})$ *in the category of F-K_v-isocrystals.*

Proof. By Remark 2.3.2 δ_v is a K_v-linear section of $\mathrm{Lie}(\tau_{K_v})$. Since p_vF_v on $T_{\mathrm{dR}}(\mathsf{M}_{K_v})$ is the K_v-linear extension of the Frobenius $\mathfrak{f}_{\mathcal{O}_{K_v}}$ on $T_{\mathrm{cris}}(\mathsf{M}_{k_v})$ discussed in [2, §4.1], it suffices to show that δ_v is Frobenius equivariant and, hence, provides a splitting of F-K_v-isocrystals. Now Lemma 1.2.1(iii) and the fact that morphisms of pure F-K_v-isocrystals of different weight are trivial imply that such a splitting is unique.

Recall that $\mathfrak{f}_{\mathcal{O}_{K_v}}$ on $T_{\mathrm{cris}}(\mathsf{M}_{k_v})$ is defined by the Verschiebung $\mathsf{M}^{(p)}_{k_v} \to \mathsf{M}_{k_v}$ thanks to the crystalline nature of universal extensions (see [2, §4], [3, Thm. 2.1]). More precisely, for any $n \in \mathbb{N}$ there is a canonical lift $V^\natural_n\colon \sigma_v^*\mathsf{M}^\natural_{W_n(k_v)} \to \mathsf{M}^\natural_{W_n(k_v)}$ of the Verschiebung on $\mathsf{M}^\natural_{k_v}$ where σ_v also denotes the Frobenius on $\mathrm{Spec}(W_n(k_v))$, and the homomorphism $\mathrm{Lie}(V^\natural_n)\colon T_{\mathrm{dR}}(\mathsf{M}_{W_n(k_v)}) \otimes_{\sigma_v} W_n(k_v) \to T_{\mathrm{dR}}(\mathsf{M}_{W_n(k_v)})$ defines the Frobenius $\mathfrak{f}_{W_n(k_v)}$ on $T_{\mathrm{dR}}(\mathsf{M}_{W_n(k_v)})$. Now, the construction is compatible with the truncation maps. Hence the morphisms $V^\natural_n$, $n \geq 1$, provide a morphism $V^\natural$ of complexes of formal schemes over $\mathcal{O}_{K_v} = W(k_v)$

$$\begin{array}{ccc} \sigma_v^*\mathsf{L}_{\mathcal{O}_{K_v}} & \xrightarrow{(V^\natural)_{-1}} & \mathsf{L}_{\mathcal{O}_{K_v}} \\ \downarrow{\scriptstyle \sigma_v^*\widehat{\mathsf{u}}} & & \downarrow{\scriptstyle \widehat{\mathsf{u}}} \\ \sigma_v^*\widehat{\mathsf{G}}^\natural_{\mathcal{O}_{K_v}} & \xrightarrow{(V^\natural)_0} & \widehat{\mathsf{G}}^\natural_{\mathcal{O}_{K_v}}. \end{array}$$

Note that since the Verschiebung on L_{k_v} can be identified with the multiplication by p_v, the morphism $(V^\natural)_{-1}$ maps $x \in \mathsf{L}_{\mathcal{O}_{K_v}} \times_{\mathcal{O}_{K_v},\sigma_v} \mathrm{Spf}(\mathcal{O}_{K_v})$ to $p_vx \in \mathsf{L}_{\mathcal{O}_{K_v}}$. By the functoriality of (the formal) δ_v in Remark 2.2.5 we then get a commutative diagram

$$\begin{array}{ccc} \mathrm{Lie}(\mathsf{L}_{\mathcal{O}_{K_v}} \otimes \mathbb{G}_a) \otimes_{\mathcal{O}_{K_v}} K_v \otimes_{\sigma_v} K_v & \xrightarrow{x\otimes 1\otimes 1 \mapsto p_vx\otimes 1} & \mathrm{Lie}(\mathsf{L}_{\mathcal{O}_{K_v}} \otimes \mathbb{G}_a) \otimes_{\mathcal{O}_{K_v}} K_v \\ \downarrow{\scriptstyle \delta_v\otimes \mathrm{id}} & & \downarrow{\scriptstyle \delta_v} \\ \mathrm{Lie}(\mathsf{G}^\natural_{\mathcal{O}_{K_v}}) \otimes_{\mathcal{O}_{K_v}} K_v \otimes_{\sigma_v} K_v & \longrightarrow & \mathrm{Lie}(\mathsf{G}^\natural_{\mathcal{O}_{K_v}}) \otimes_{\mathcal{O}_{K_v}} K_v. \end{array}$$

Since the upper (respectively, lower) horizontal arrow defines the Frobenius $\mathfrak{f}_{\mathcal{O}_{K_v}}$ on $T_{\mathrm{dR}}([\mathsf{L}_{K_v} \to 0])$ (respectively, on $T_{\mathrm{dR}}(\mathsf{M}_{K_v})$), the result follows. $\square$

3.3.2. **Lemma.** *The functor T_{Og} is faithful.*

Proof. Let $\varphi\colon \mathsf{M}_K \to \mathsf{N}_K$ be a morphism of 1-motives such that $T_{\mathrm{Og}}(\varphi) = 0$. In particular $T_{\mathrm{dR}}(\varphi) = 0$. Then $n\varphi = 0$ for a suitable n. Indeed, let $\varphi_{\mathbb{C}}$ be the base change of φ to $\mathbb{C}$. Then $T_{\mathrm{dR}}(\varphi_{\mathbb{C}}) = 0$ implies $T_{\mathbb{C}}(\varphi_{\mathbb{C}})\colon T_{\mathbb{Z}}(\mathsf{M}_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{C} \to T_{\mathbb{Z}}(\mathsf{N}_{\mathbb{C}}) \otimes_{\mathbb{Z}} \mathbb{C}$ is the zero map by [15, 10.1.8]. Hence $nT_{\mathbb{Z}}(\varphi_{\mathbb{C}}) = 0$ for a suitable n. Then one concludes that $n\varphi_{\mathbb{C}} = 0$ by [15, 10.1.3] and hence $n\varphi = 0$. □

3.3.3. **Remark.** One could provide an alternative proof of Lemma 3.3.2 using an argument similar to the one adopted for the faithfulness in Theorem 3.1.1.

3.3.4. **Lemma.** *The functor T_{Og} restricted to $\mathcal{M}_{0,\mathbb{Q}}$ is full.*

Proof. First consider two 1-motives $\mathsf{M}_K = [\mathbb{Z}^r \to 0]$, $\mathsf{N}_K = [\mathbb{Z}^s \to 0]$. Write $e_1, \dots, e_r$ for the standard basis of $\mathbb{Z}^r$ and similarly for $\mathbb{Z}^s$. We use the same letters for the induced bases on the de Rham realizations. Any morphism $\psi\colon T_{\mathrm{dR}}(\mathsf{M}_K) \to T_{\mathrm{dR}}(\mathsf{N}_K)$ corresponds to an $s \times r$ matrix $C = (c_{ij}) \in M_{s,r}(K)$, *i.e.*, $\psi(e_j) = \sum_i c_{ij} e_i$. Observe that $c_{ij} \in \mathcal{O}[1/n]$ for n sufficiently divisible and hence $C \in M_{s,r}(\mathcal{O}_{K_v})$ for $v \in \mathcal{P}_n$. If ψ is a morphism in $\mathbf{Og}(K)$, the compatibility with the F_v's implies that $\sigma_v(C) = C$ where $\sigma_v(C) = (\sigma_v(c_{ij}))$. Indeed

$$F_v(\psi(e_j)) = F_v(\sum_i c_{ij} e_i) = \sum_i \sigma_v(c_{ij}) e_i,$$

and

$$\psi(F_v(e_j)) = \psi(e_j) = \sum_i c_{ij} e_i.$$

Now $\sigma_v(c_{ij}) = c_{ij}$ for all $v \in \mathcal{P}_n$, is equivalent to $c_{ij}^{p_v} \equiv c_{ij} \pmod{p_v}$ for all $v \in \mathcal{P}_n$. We conclude by Kronecker's theorem that $c_{ij} \in \mathbb{Q}$; hence $C \in M_{s,r}(\mathbb{Q})$. Let m be the a positive integer such that $mC \in M_{s,r}(\mathbb{Z})$. Then $m\psi = \mathrm{Lie}(\varphi \otimes \mathrm{id})$ with $\varphi \in \mathrm{Hom}(\mathsf{M}_K, \mathsf{N}_K) \simeq M_{s,r}(\mathbb{Z})$ the homomorphism which maps e_j to $\sum_i mc_{ij} e_i$ in degree -1.

We conclude the proof of the fullness by Galois descent. Let $\mathsf{M}_K = [\mathsf{L}_K \to 0], \mathsf{N}_K = [\mathsf{F}_K \to 0]$ be 1-motives in $\mathcal{M}_{0,\mathbb{Q}}$ and let $\psi\colon T_{\mathrm{dR}}(\mathsf{M}_K) \to T_{\mathrm{dR}}(\mathsf{N}_K)$ be a morphism in $\mathbf{FOg}(K)$; in particular the base change of ψ to K_v is compatible with the F_v's for almost all places v. Let K' be a finite Galois extension of K such that $\mathsf{L}_{K'}$ and $\mathsf{F}_{K'}$ are split. Then, any unramified place v' of K' is above an unramified place v of K and $K'_{v'}/K_v$ is a finite unramified extension. Then by [2, Corollary 4.2.1] and the fact that the de Rham realization and the Verschiebung morphism are compatible with extension of the base, the formation of $T_{\mathrm{Og}}(\mathsf{M}_K)$ behaves well with respect to base field extension. Let $\psi_{K'}\colon T_{\mathrm{dR}}(\mathsf{M}_{K'}) \to T_{\mathrm{dR}}(\mathsf{N}_{K'})$ be the morphisms in $\mathbf{FOg}(K')$ induced by ψ. Denote by the same letter also the associated morphism of vector groups $\mathsf{L}_{K'} \otimes \mathbb{G}_a \to \mathsf{F}_{K'} \otimes \mathbb{G}_a$. By above discussions we may assume that $\psi_{K'}$ comes from a morphism $\varphi_{K'}\colon \mathsf{L}_{K'} \to \mathsf{F}_{K'}$. Let us denote by ζ both an element in $\mathrm{Gal}(K'/K)$ and the induced morphism on 1-motives. In order to check that $\varphi_{K'}$ descends to K, it suffices to check that $\zeta \circ \varphi_{K'} = \varphi_{K'} \circ \zeta$. Since we work over fields of characteristic 0, the morphism $\iota_{\mathsf{F}}\colon \mathsf{F}_{K'} \to \mathsf{F}_{K'} \otimes \mathbb{G}_a, x \mapsto x \otimes 1$ has trivial kernel. Hence it is sufficient to check that $\iota_{\mathsf{F}} \circ \zeta \circ \varphi_{K'} = \iota_{\mathsf{F}} \circ \varphi_{K'} \circ \zeta$. Now,

$$\iota_{\mathsf{F}} \circ \zeta \circ \varphi_{K'} = \zeta \circ \iota_{\mathsf{F}} \circ \varphi_{K'} = \zeta \circ \psi_{K'} \circ \iota_{\mathsf{L}} = \psi_{K'} \circ \zeta \circ \iota_{\mathsf{L}} = \psi_{K'} \circ \iota_{\mathsf{L}} \circ \zeta = \iota_{\mathsf{F}} \circ \varphi_{K'} \circ \zeta.$$

This concludes the proof. □

3.3.5. **Theorem.** *The functor* $T_{\mathrm{Og}}\colon \mathcal{M}_{1,\mathbb{Q}} \to \mathbf{FOg}(K)$ *is fully faithful.*

Proof. The faithfulness was proved in Lemma 3.3.2. For the fullness, let $\mathsf{M}_K = [\mathsf{u}_K\colon \mathsf{L}_K \to \mathsf{G}_K]$ and $\mathsf{N}_K = [\mathsf{v}_K\colon \mathsf{F}_K \to \mathsf{H}_K]$ be 1-motives. Suppose given a morphism $\psi\colon T_{\mathrm{Og}}(\mathsf{M}_K) \to T_{\mathrm{Og}}(\mathsf{N}_K)$ in $\mathbf{FOg}(K)$. By Definition 3.2.4 we also get a morphism $\psi'\colon T_{\mathrm{BOg}}(\mathsf{M}_K) \to T_{\mathrm{BOg}}(\mathsf{N}_K)$ in $\mathbf{BOg}(K)$. Using Theorem 3.1.1 the morphism ψ' comes from a morphism $\widetilde{g}_K\colon \mathsf{G}_K^\natural \to \mathsf{H}_K^\natural$, *i.e.*, $\mathbf{Lie}(\widetilde{g}_K) = m\psi'$ in $\mathbf{BOg}(K)$ for a suitable $m \in \mathbb{N}$. We may assume $m = 1$. By Chevalley theorem (see [12, Lemma 2.3]) $\widetilde{g}_K$ yields a morphism on the semi-abelian quotients $g_K\colon \mathsf{G}_K \to \mathsf{H}_K$. Now consider the morphism $T_{\mathrm{Og}}(g_K)\colon T_{\mathrm{Og}}(\mathsf{G}_K) \to T_{\mathrm{Og}}(\mathsf{H}_K)$ and compare with the morphism induced by ψ on weight -1 parts as displayed in the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{\mathrm{Og}}(\mathsf{G}_K) & \longrightarrow & T_{\mathrm{Og}}(\mathsf{M}_K) & \longrightarrow & T_{\mathrm{Og}}(\mathsf{L}_K[1]) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_{-1}} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi_0} & & \\
0 & \longrightarrow & T_{\mathrm{Og}}(\mathsf{H}_K) & \longrightarrow & T_{\mathrm{Og}}(\mathsf{N}_K) & \longrightarrow & T_{\mathrm{Og}}(\mathsf{F}_K[1]) & \longrightarrow & 0
\end{array}$$

Since by construction $T_{\mathrm{BOg}}(g_K) = \psi'_{-1}$ in $\mathbf{BOg}(K)$ we deduce that $T_{\mathrm{Og}}(g_K) = \psi_{-1}$ as well. In fact, $T_{\mathrm{dR}}(g_K) = T_{\mathrm{BOg}}(g_K) = T_{\mathrm{Og}}(g_K)$ coincide on the underlying K-vector spaces via T_{dR}.

It follows from the Lemma 3.3.4 that there exists a morphism $f_K\colon \mathsf{L}_K \to \mathsf{F}_K$ such that $T_{\mathrm{Og}}(f_K) = m\psi_0\colon T_{\mathrm{Og}}(\mathsf{L}_K[1]) \to T_{\mathrm{Og}}(\mathsf{F}_K[1])$ for an $m \in \mathbb{N}$. As above, we may assume $m = 1$.

Note that if the pair (f_K, g_K) gives a morphism of 1-motives $\varphi_K\colon \mathsf{M}_K \to \mathsf{N}_K$, i.e., if $g_K \circ \mathsf{u}_K = \mathsf{v}_K \circ f_K$, then $T_{\mathrm{BOg}}(\varphi_K) = \psi'$ by construction. As the functor Ψ of (1.1) is faithful, the fact that $T_{\mathrm{Og}}(\varphi_K) - \psi$ induces the zero morphism between the Bost-Ogus realizations implies that $T_{\mathrm{Og}}(\varphi_K) = \psi$ in $\mathbf{FOg}(K)$.

It is then sufficient to check that, up to multiplication by a positive integer, we have $\widetilde{g}_K \circ \mathsf{u}_K^\natural = \mathsf{v}_K^\natural \circ f_K$. After replacing K with a finite extension we may further assume that L and F are constant. Take v an unramified place of good reduction both for M_K and N_K such that $\widetilde{g}_K$ extends to a morphism $\widetilde{g}\colon \mathsf{G}^\natural \to \mathsf{H}^\natural$ over $W(k_v) = \mathcal{O}_{K_v}$. In particular, $\mathsf{u}_K^\natural$ and $\mathsf{v}_K^\natural$ extend to morphisms $\mathsf{u}^\natural\colon \mathsf{L} \to \mathsf{G}^\natural, \mathsf{v}^\natural\colon \mathsf{F} \to \mathsf{H}^\natural$ over $\mathcal{O}_{K_v}$ and hence are determined by the induced homomorphisms between the $\mathcal{O}_{K_v}$-rational points. It suffices then to prove that the following diagram

$$\begin{array}{ccc}
\mathsf{L}(\mathcal{O}_{K_v}) & \xrightarrow{\mathsf{u}^\natural} & \mathsf{G}^\natural(\mathcal{O}_{K_v}) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \widetilde{g}} \\
\mathsf{F}(\mathcal{O}_{K_v}) & \xrightarrow{\mathsf{v}^\natural} & \mathsf{H}^\natural(\mathcal{O}_{K_v})
\end{array} \tag{3.2}$$

commutes, up to multiplication by a positive integer. Consider the following diagram

$$\begin{array}{ccccccc}
\mathsf{L}(\mathcal{O}_{K_v}) & \xrightarrow{\alpha_{\mathsf{L}}} & \mathrm{Lie}(\mathsf{L} \otimes \mathbb{G}_a) \otimes K_v & \xrightarrow{\delta_v^{\mathsf{M}}} & \mathrm{Lie}(\mathsf{G}^\natural) \otimes K_v & \xrightarrow[\simeq]{\log^{-1}_{\mathsf{G}^\natural,\mathbb{Q}}} & \mathsf{G}^\natural(\mathcal{O}_{K_v}) \otimes_{\mathbb{Z}} \mathbb{Q} \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \psi_0 \otimes K_v} & & \downarrow{\scriptstyle \psi \otimes K_v} & & \downarrow{\scriptstyle \widetilde{g} \otimes \mathbb{Q}} \\
\mathsf{F}(\mathcal{O}_{K_v}) & \xrightarrow{\alpha_{\mathsf{F}}} & \mathrm{Lie}(\mathsf{F} \otimes \mathbb{G}_a) \otimes K_v & \xrightarrow{\delta_v^{\mathsf{N}}} & \mathrm{Lie}(\mathsf{H}^\natural) \otimes K_v & \xrightarrow[\simeq]{\log^{-1}_{\mathsf{H}^\natural,\mathbb{Q}}} & \mathsf{H}^\natural(\mathcal{O}_{K_v}) \otimes_{\mathbb{Z}} \mathbb{Q}
\end{array}$$

where the map α_{L} is the composition of the canonical map $\mathsf{L}(\mathcal{O}_{K_v}) \to (L\otimes\mathbb{G}_a)(\mathcal{O}_{K_v})\otimes_{\mathbb{Z}} \mathbb{Q}, x \mapsto (x\otimes 1)\otimes 1$ with $\log_{L\otimes\mathbb{G}_a,\mathbb{Q}}$ and similarly for α_{F}. Note that we here identify $T_{\mathrm{dR}}(\mathsf{L}_K[1])_v$ with $\mathrm{Lie}(\mathsf{L}\otimes\mathbb{G}_a)\otimes K_v$ and $T_{\mathrm{dR}}(\mathsf{F}_K[1])_v$ with $\mathrm{Lie}(\mathsf{F}\otimes\mathbb{G}_a)\otimes K_v$; we also identify $T_{\mathrm{dR}}(\mathsf{M}_K)_v$ with $\mathrm{Lie}(\mathsf{G}^\natural)\otimes K_v$ and $T_{\mathrm{dR}}(\mathsf{N}_K)_v$ with $\mathrm{Lie}(\mathsf{H}^\natural)\otimes K_v$. Hence the most left square commutes since by definition $T_{\mathrm{Og}}(f_K) = \psi_0$. The commutativity of the square in the middle follows from Lemma 3.3.1 as ψ and ψ_0 are morphisms in $\mathbf{Og}(K)$ so that $\psi\otimes K_v$ and $\psi_0\otimes K_v$ commutes with F_v. The last square on the right commutes by functoriality of the logarithm as $\psi\otimes K_v = \psi'\otimes K_v = \mathrm{Lie}(\widetilde{g}_K\otimes K_v)$ on the underlying K_v-vector spaces. Finally, note that the composition of the upper (respectively, lower) horizontal arrows is $\mathsf{u}^\natural\otimes\mathrm{id}_{\mathbb{Q}}$ (respectively, $\mathsf{v}^\natural\otimes\mathrm{id}_{\mathbb{Q}}$) by definition of δ_v in (2.5). Hence (3.2) commutes up to multiplication by a positive integer. □

3.3.6. **Example.** Let $\mathsf{M}_K = [\mathsf{u}_K\colon \mathbb{Z}\to\mathbb{G}_{m,K}]$ and $\mathsf{N}_K = [\mathsf{v}_K\colon \mathbb{Z}\to\mathbb{G}_{m,K}]$ be two 1-motives over K. Set $a := \mathsf{u}_K(1)\in K^*$ and $b := \mathsf{v}_K(1)\in K^*$. Note that any morphism $(f_K, g_K)\colon \mathsf{M}_K\to\mathsf{N}_K$ is of the type $f_K = m$, $g_K = r$ with $a^r = b^m$. In particular, for general $a, b\in K^*$ the unique morphism between M_K and N_K is the zero morphism. By Definition 2.3.1, it follows from (2.6) that

$$T_{\mathrm{dR}}(\mathsf{M}_K) = T_{\mathrm{dR}}(\mathsf{N}_K) = \mathrm{Lie}(\mathbb{G}_{m,K})\oplus\mathrm{Lie}(\mathbb{G}_{a,K}) = K\oplus K.$$

The de Rham realisation of the two 1-motives yields objects in $\mathbf{Og}(K)$ with the same underlying structure of filtered K-vector spaces: the filtration being induced by the weight filtration $W_{-1}\mathsf{M}_K = [0\to\mathbb{G}_{m,K}]$ and $\mathrm{gr}_0^W\mathsf{M}_K = [\mathbb{Z}\to 0]$. As described in Definition 3.2.1 (*cf.* the proof of Lemma 3.2.2), for any unramified place v of K, $T_{\mathrm{dR}}(\mathsf{M}_{K_v}) = K_v\oplus K_v$ is endowed with the σ_v-semilinear operator F_v such that for $(x,y)\in K_v\oplus K_v$ we have that $F_v(x,0) = (p_v^{-1}\sigma_v(x), 0)$ and $\overline{F_v((x,y))} = \sigma_v(y)$ where the $\overline{(\ \ )}$ stands for the class in the weight 0 quotient K_v; the same holds for N_K.

Note that, in general, we have several K-linear homomorphisms $T_{\mathrm{dR}}(\mathsf{M}_K)\to T_{\mathrm{dR}}(\mathsf{N}_K)$ that preserve the filtration and commute with the F_v's on the graded pieces of the filtration but do not arise from morphisms of 1-motives (even up to isogeny). For example, take $a = 1$ and $b = 2$ and the identity map on $K\oplus K$. This example shows that knowing fullness for the pure weight parts is not enough to deduce the Theorem 3.3.5.

For general $a, b\in K^*$, knowing F_v on $T_{\mathrm{dR}}(\mathsf{M}_K)\otimes K_v = K_v\oplus K_v$ is equivalent to give a Frobenius equivariant splitting $\delta_v^{\mathsf{M}}\colon K_v\to K_v\oplus K_v$ of the weight 0 quotient K_v thanks to Lemma 3.3.1. Now, assume $a, b\in\mathcal{O}_{K_v}$. Then, by Examples 2.2.2, we can write $\delta_v^{\mathsf{M}}(1) = (\log(a^{p_v(p_v^{n_v}-1)}))/p_v(p_v^{n_v}-1)\oplus 1$. As a consequence, the identity map on $K\oplus K$ commutes with the sections $\delta_v^{\mathsf{M}}, \delta_v^{\mathsf{N}}$ if and only if $a^{p_v(p_v^{n_v}-1)} = b^{p_v(p_v^{n_v}-1)}$, thus if, and only if, $a^{p_v^{n_v}-1} = b^{p_v^{n_v}-1}$ (as K_v does not contain non-trivial p_v-roots of unity being absolutely unramified). We conclude that there exists a morphism $T_{\mathrm{Og}}(\mathsf{M}_K)\to T_{\mathrm{Og}}(\mathsf{N}_K)$ in $\mathbf{FOg}(K)$ which is identity on the underlying vector spaces if and only if $\mathsf{M}_K = \mathsf{N}_K$.

3.3.7. **Remark.** If we work with $\mathbf{BOg}(K)$ and even with the filtered analogue, it is not true that the functor T_{BOg} is full on 1-motives, in general. For example, take $\mathsf{M} = \mathbb{Z}[1]$ and note that $\mathrm{End}_{\mathcal{M}_{1,\mathbb{Q}}}(\mathsf{M}) = \mathbb{Q}$ while $\mathrm{End}_{\mathbf{BOg}(K)}(T_{\mathrm{BOg}}(\mathsf{M})) = K$.

References

[1] Y. André: *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)* Panoramas et Synthèses **17**, Société Mathématique de France, 2004.

[2] F. Andreatta & L. Barbieri-Viale: Crystalline realizations of 1-motives, *Math. Ann.* **331** N. 1 (2005) 111-172.

[3] F. Andreatta & A. Bertapelle: Universal extension crystals of 1-motives and applications, *Journal of Pure and Appl. Algebra* **215** (2011) 1919-1944.

[4] L. Barbieri-Viale & A. Bertapelle: Sharp de Rham realization, *Adv. Math.* **222** (2009) 1308-1338.

[5] A. Bertapelle: Deligne's duality for de Rham realizations of 1-motives, *Math. Nachr.* **282** (2009) 1637-1655.

[6] L. Barbieri-Viale & B. Kahn: *On the derived category of* 1-*motives* Astérisque **381** Société Mathématique de France, Paris, 2016.

[7] P. Berthelot, L. Breen & W. Messing: *Théorie de Dieudonné cristalline II* Lect. Notes in Math. **930**, Springer, New York, 1982.

[8] S. Bosch, W. Lütkebohmert & M. Raynaud: *Néron models* Erg. der Math. Grenz. **21**, Springer-Verlag, Berlin, 1990.

[9] J.-B. Bost: Algebraic leaves of algebraic foliations over number fields *Publ. Math.* IHES **93** (2001) 161-221.

[10] A. Chambert-Loir: Théorèmes d'algébricité en géométrie diophantienne d'après J.-B. Bost, Y. André, D. & G. Chudnovsky, Séminaire Bourbaki Mars 2001, n. 886.

[11] B. Chiarellotto: Weights in rigid cohomology applications to unipotent F-isocrystals, *Annales Scientifiques* ENS **31** Issue 5 (1998) 683-715.

[12] B. Conrad: A modern proof of Chevalley's theorem on algebraic groups, *J. Ramanujam Math. Soc.* **17** (2002) 1-18.

[13] M. Demazure & P. Gabriel: *Groupes algébriques* Tome I: Géométrie algébrique, généralités, groupes commutatifs, Masson & Cie, Éditeur, Paris, 1970.

[14] M. Demazure & A. Grothendieck (Eds.): *Schémas en groupes* Séminaire de Géométrie Algébrique du Bois Marie 1962-64 (SGA 3). Lect. Notes in Math. **151**, Springer-Verlag, Berlin, 1970.

[15] P. Deligne: Théorie de Hodge III. *Publ. Math.* IHES **44** (1974) 5-78.

[16] B. Edixhoven, G. van der Geer & B. Moonen: *Abelian varieties* Preprint: `http://www.math.ru.nl/~bmoonen/research.html`.

[17] A. Grothendieck (with J. Dieudonné): Eléments de géométrie algébrique. *Publ. Math.* IHES **11** (III, 1-5), **28** (IV, 8-15) and **32** (IV, 16-21) (1961-1967).

[18] U. Jannsen: Mixed motives, motivic cohomology and Ext-groups, in the *Proceedings of the International Congress of Mathematicians* Vol. 1, 2 (Zurich, 1994), 667-679, Birkhäuser, Basel, 1995.

[19] N. Katz & W. Messing: Some consequences of the Riemann hypothesis for varieties over finite fields, *Invent. Math.* **23** (1974), 73-77.

[20] M. Kisin & S. Wortmann: A note on Artin motives, *Math. Research Letters* **10** (2003) 375-389.

[21] W. Messing: *The crystals associated to Barsotti-Tate groups: with applications to abelian schemes* Lect. Notes in Math. **264**, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

[22] D. Mumford: *Abelian varieties* Tata Institute of Fundamental Research Studies in Mathematics, 5, Bombay, 1970.

[23] J. T. Tate: p-divisible groups, in Proc. Conf. Local Fields (Driebergen, 1966) 158-183 Springer, Berlin 1967.

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