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Convex Hull Property and Exclosure Theorems for H-Minimal Hypersurfaces in Carnot Groups

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Abstract: In this paper, we generalize to sub-Riemannian Carnot groups some classical results in the theory of minimal submanifolds. Our main results are for step 2 Carnot groups. In this case, we will prove the convex hull property and some "exclosure theorems" for H-minimal hypersurfaces of class $\mathbf{C}^2$ satisfying a Hörmander-type condition.

Keywords: Carnot groups; Sub-Riemannian geometry; H-minimal hypersurfaces; convex hull property; exclosure theorems

MSC: 49Q15, 46E99, 43A80

1 Introduction

During the last years a theory of both minimal and constant mean curvature (hyper)surfaces in *sub-Riemannian Carnot groups* has been gradually, but only partially developed, even if mainly for the case of Heisenberg groups $\mathbb{H}^n$. For some results and perspectives concerning minimal or constant horizontal mean curvature hypersurfaces in Carnot groups, we refer the reader to [1], [4], [10], [13], [14, 15], [16], [19], [23, 24], [29], [30], [37, 38], [42], [45], [46], but this list is far from being complete.

In this paper we extend to Carnot groups some qualitative (and quantitative) results of the Euclidean theory of minimal surfaces. To be more precise, we will prove suitable versions of the classical enclosure and existence/non-existence theorems for minimal surfaces. We refer the reader to Chapter 6 of the book [21] for a detailed account on this topic; see also [28], [20]. A key feature of all these theorems is that they can be obtained as a straight consequence of the classical *strong maximum principle* for 2nd order elliptic operators.

Let us give a quick survey of the classical results.

Let $x : S_0 \to \mathbb{R}^n$ be an immersion of an m-dimensional $\mathbf{C}^2$ smooth manifold S_0 into the Euclidean n-dimensional space and set $S := x(S_0) \subset \mathbb{R}^n$. By definition, S has the *convex hull property* if, for every domain (that is, open connected) $\mathcal{D} \subset S_0$ such that x maps $\mathcal{D}$ into a bounded subset of $\mathbb{R}^n$, the image of $\mathcal{D}$ lies in the convex hull of its boundary values. It is a classical and well-known result that minimal submanifolds of $\mathbb{R}^n$ satisfy this property; we refer the reader to Osserman's book [40] (see Lemma 7.1, p. 53) and also to the paper [41], where a geometric characterization of this property is given based on the sign of the normal curvatures of the submanifold. It is worth observing that the convex-hull property has several geometric consequences. For example, it implies a sort of "monotonicity" of topology of minimal submanifolds; see [17].

Now, we would like to recall some stronger enclosure theorems that somehow indicate the saddle-surface character of non-flat minimal surfaces. To this end, we begin with the so-called "Hyperboloid theorem", stated in its simplest form:

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Theorem A. *Let $S \subset \mathbb{R}^3$ be a compact minimal surface and assume that ∂S is contained in a solid body which is congruent to the hyperboloid* $\mathcal{H}yp(\epsilon) := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 \leq \epsilon^2\}$, *then* $S \subset \mathcal{H}yp(\epsilon)$.

A straight consequence of this result is the "Cone theorem":

Theorem B. *Let $\mathcal{C}$ be a solid cone congruent to $\mathcal{H}yp(0)$ which consists of the two half-cones $\mathcal{C}^+$ and $\mathcal{C}^-$ corresponding to the two sheets $\mathcal{H}yp^+(0)$ and $\mathcal{H}yp^-(0)$ of $\mathcal{H}yp(0)$. Then there is no connected minimal surface the boundary of which lies in $\mathcal{C}$ and intersects both $\mathcal{C}^+$ and $\mathcal{C}^-$.*

Note that the "test cone" $\mathcal{H}yp(0)$ for non-existence may even be replaced by a slightly larger set; see [20]. The Cone theorem can be used to prove nonexistence results for Plateau problems, or for free (or partially free) boundary value problems.

At this point, let us say a few words about the proofs and their generalizations.

Let $S \subset \mathbb{R}^n$ be an m-dimensional, compact minimal submanifold with boundary. For what concerns the convex-hull property of minimal submanifolds in the Euclidean case, one begins with a well-known fact: the *coordinates functions* $\{x_i : i = 1, \dots, n\}$ *are Δ_{TS}-harmonic,* where Δ_{TS} denotes the Laplace-Beltrami operator on S. From the Δ_{TS}-harmonicity of the coordinate functions it follows that every affine function is Δ_{TS}-harmonic (that is, for every $f : \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = \langle \overline{a}, x\rangle_{\mathbb{R}^n} + b$ ($\overline{a} \in \mathbb{R}^n, b \in \mathbb{R}$), one has $\Delta_{TS} f = 0$). Hence, by the maximum principle for the Laplace-Beltrami operator Δ_{TS}, $f(x)$ reaches its maximum on the boundary ∂S. Then, the convex-hull property follows because any closed convex set is the intersection of its supporting half-planes.

We stress that the aforementioned Hyperboloid and Cone theorems, either the ones above or their generalized n-dimensional versions (see, for instance, [20], [21]), can be proved by using similar arguments mainly based on the maximum principle. In fact, the core of the matter is somehow to find (or "construct") Δ_{TS}-subharmonic "test functions": in the Euclidean framework, these functions are suitable quadratic functions, see [20].

In this paper, we shall try to adapt these ideas to Carnot groups.

Let $\mathbb{G}$ be an n-dimensional Carnot group and let $S \subset \mathbb{G}$ be a non-characteristic hypersurface of class $\mathbf{C}^2$ (for precise definitions concerning Carnot groups and hypersurfaces, we refer the reader to Sections 1.1, 1.2). In this framework, the *HS-Laplacian* Δ_{HS} is a 2nd order differential operator playing the role of the Laplace-Beltrami operator Δ_{TS} in Riemannian geometry. More precisely, let HS be the subbundle of TS generated by horizontal tangential vector fields on S (that is, $H_xS = H_x \cap T_xS$ for every $x \in S$, where H_x denotes the fiber at x of the horizontal subbundle H of $T\mathbb{G}$). If we fix an orthonormal frame $\{Z_1, \dots, Z_{h-1}\}$ for HS, then it follows that $\Delta_{HS} = \sum_{j=1}^{h-1} Z_i^{(2)}$, which is an operator "sum of squares" of vector fields on S.

Our starting point is an elementary formula for Δ_{HS} (see formula (2.1), Section 2.1) that is used to show the Δ_{HS}-subharmonicity of some simple monomial functions of degree 1 and 2 (with respect to the usual dilations in $\mathbb{R}^n$). As a direct consequence, we will find some quadratic, Δ_{HS}-subharmonic "test functions", which are similar to the classical ones; see Section 2.1. It is worth to observe that these calculations hold for the case of step 2 Carnot groups only. The reason is a technical one. For instance, in step k Carnot groups, with $k > 2$, it is not true that all degree 1 (Euclidean) monomials are Δ_{HS}-subharmonic: hence, we cannot apply the same strategy to prove the convex-hull property. It is an open problem to find new classes of test functions for arbitrary step k Carnot groups, when $k > 2$.

Here we would like to stress another key aspect of this paper: the validity of the maximum principle for the HS-Laplacian Δ_{HS}. Basically, in order to prove such a result, we shall apply a generalized version of the "Bony's maximum principle" (see Corollary 3.1 in [8]). More precisely, we will use a theorem by Bonfiglioli and Uguzzoni which holds true under weak regularity assumptions; see [7]. We remark that this result can be applied to our setting by assuming a *Hörmander-type condition for the subbundle HS*; see Definition 2.16.

The validity of this condition seems to be deeply connected with the algebraic features of the underlying Lie algebra $\mathfrak{g}$ of the group $\mathbb{G}$. As an example, the condition holds true in Heisenberg groups $\mathbb{H}^n$ only if $n > 1$; see Example 1.7 and Remark 2.17. For brevity reasons, we do not consider this problem here. Rather, we address the following question:

under which algebraic conditions on $\mathbb{G}$ does the subbundle $HS \subsetneq TS$ of any (smooth enough) non-characteristic hypersurface $S \subset \mathbb{G}$ satisfy the Hörmander condition?

Concerning the proof of the maximum principle (see Theorem 2.21) we also remark that one needs to apply a suitable version of Chow's theorem (with less regularity assumptions): in fact, we will use either a result by Rampazzo and Sussman (see [44]), in the case of step 2 Carnot groups, or a more recent one by Feleqi and Rampazzo (see [22]), for the step k case.

The organization of this paper is as follows.

In Section 1.1 we recall notation, basic definitions and preliminaries on Carnot groups.

In Section 1.2 we briefly introduce the theory of (smooth) hypersurfaces in Carnot groups and describe the main geometric and analytic structures which are needed in the sequel. In particular, we define the *HS*-Laplacian Δ_{HS}.

Section 2.1 contains some explicit calculations for step 2 Carnot groups. More precisely, we compute the *HS*-Laplacian of some simple degree 1 or 2 monomials. In this way we find some important (and at the same time simple) examples of quadratic Δ_{HS}-subharmonic functions: this is a key point of this paper, exactly as it happens in the classical case; see [28], [20], [21].

In Section 2.2 we discuss a suitable version of the strong maximum principle for $\mathbf{C}^2$ solutions of the differential inequality $\Delta_{HS}\phi \geq 0$, under the validity of a Hörmander-type condition for the subbundle *HS*; see Definition 2.16 and Theorem 2.21.

In Section 3 we present our main results for the case of step 2 Carnot groups. In particular, we prove the convex-hull property (see Theorem 3.1) together with suitable versions of the Hyperboloid theorem and of the Cone theorem; see Theorems 3.2 and 3.3. We stress that the axis of the “test hyperboloid/cone” is here assumed to be a horizontal direction. In addition, we prove a (quantitative) consequence of the Cone theorem (see Corollary 3.4) and some inclusion properties for paraboloids and cylinders with axis a vertical direction; see Theorem 3.5.

In Section 3.1 we discuss the case of Heisenberg groups $\mathbb{H}^n$. If $n = 1$ our strategy cannot be applied. Still it can be seen that the convex-hull property for *H*-minimal surfaces of class $\mathbf{C}^2$ follows easily from a classical theorem by Osserman; see Remark 3.6. On the contrary, the case $n > 1$ fits with our previous results and we are able to state a further version of the Hyperboloid/Cone theorem for suitable truncated hyperboloids and cones with axis the T-axis of $\mathbb{H}^n$; see Theorem 3.8 and Corollaries 3.9 and 3.10.

In Section 4 we make a few remarks on the case of step k Carnot groups (with $k \geq 3$). In particular, we have here to say that our only result for the step k setting is a weak version of the convex-hull property; see Definition 4.2 and Theorem 4.3. As already observed, the problem is that much of the calculations for step 2 groups *do not hold* in this general context so that further studies are needed to find new, and hopefully luckier, classes of Δ_{HS}-subharmonic functions.

In the Appendix we prove a technical lemma which states that the *HS*-Laplacian commutes with isometries; see Proposition A.4.

1.1 Carnot groups

A *step k Carnot group* $(\mathbb{G}, \bullet)$ is an n-dimensional connected, simply connected, nilpotent and stratified Lie group with respect to a polynomial group law $\bullet$. We denote by 0 the identity of $\mathbb{G}$ and assume that $\mathfrak{g} = T_0\mathbb{G}$, where $\mathfrak{g}$ denotes the Lie algebra of $\mathbb{G}$. It follows from definition that $\mathfrak{g}$ fulfills the following conditions: $\mathfrak{g} = H_1 \oplus \ldots \oplus H_k$, $[H_1, H_{i-1}] = H_i$ for all $i = 2, \ldots, k+1$, and $H_{k+1} = \{0\}$, where $[\cdot,\cdot]$ denotes the Lie bracket and each H_i is a vector subspace of $\mathfrak{g}$. We set $h_i := \dim H_i\,(i = 1, \ldots, k)$, $n_0 := 0$, and $n_i := \sum_{j=1}^{i} h_j\,(i = 1, \ldots, k)$. Hence $n_1 = h_1$, $n_2 = h_1 + h_2$,..., and $n_k = n$. Note that $H_i \cong \mathbb{R}^{h_i}$ for any $i = 1, \ldots, k$; thus $\mathfrak{g} \cong \oplus_{i=1}^{k}\mathbb{R}^{h_i} = \mathbb{R}^n$. Below, we will often use the notation $H := H_1$ and $V := H_2 \oplus \ldots \oplus H_k$.

Notation 1.1. *Throughout this paper, the differential of a map f is denoted either as df or as $f_\star$ and the pull-back by f is denoted as $f^\star$. Let E be a smooth subbundle of $T\mathbb{G}$, with fiber at $x \in \mathbb{G}$ denoted as E_x. The space of $\mathbf{C}^r$-smooth sections of E is denoted as $\mathfrak{X}^r(E)\,(r \in \mathbb{N} \cup \{0\})$; if $r = \infty$, then we simply write $\mathfrak{X}(E)$. We use the* *following sets of indices:* $\mathfrak{I}_{H_i} := \{n_{i-1}+1, \dots, n_i\}$ *for any* $i = 1, \dots, k$, $\mathfrak{I}_V := \{h_1+1, \dots, n\}$*; in particular, we set* $\mathfrak{I}_H = \mathfrak{I}_{H_1}$. *We use either capital* $I, J, K, \dots$ *or small* $i, j, k, \dots$ *Latin letters for indices belonging to* $\{1, \dots, n\}$ *and Greek letters* $\alpha, \beta, \gamma, \dots$ *for indices belonging to* $\mathfrak{I}_V$. *Finally, we set* $h := h_1$ *and* $v := n - h$. *Any further index notation will be clear from the context.*

Each element $X_0 \in \mathfrak{g}$ induces a left-invariant vector field $X \in \mathfrak{X}(T\mathbb{G})$ such that $X(x) = (L_x)_\star X_0$ and $X(0) = X_0$ for every $x \in \mathbb{G}$. In fact, the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ turns out to be isomorphic to the set $\mathrm{Lie}(\mathbb{G})$ of all left-invariant vector fields of the group; see [32], [49]. In particular, the subspaces H and V of $\mathfrak{g}$ can naturally be viewed as smooth subbundles of the tangent bundle $T\mathbb{G}$ of the group (the fibers of H and V are given, respectively, by $H_x = (L_x)_\star H$ and $V_x = (L_x)_\star V$ for every $x \in \mathbb{G}$). The subbundles H and V of $T\mathbb{G}$ are called, respectively, *horizontal bundle* and *vertical bundle*. We have $\mathrm{rank}(H) = h$ and $\mathrm{rank}(V) = v$.

From now on, we suppose that the horizontal bundle H is generated by a frame $\{X_1, \dots, X_h\}$ of left-invariant vector fields. This frame can be completed to a global graded, left-invariant frame $\{X_1, \dots, X_n\}$ for $T\mathbb{G}$. With no loss of generality, we assume that $X_i(x) = L_{x\star}\mathrm{e}_i$ $(i = 1, \dots, n)$, where $\mathrm{e}_i = \underbrace{(0, \dots, 0, 1, 0, \dots, 0)}_{i-th\ place}$ denotes the i-th vector of the canonical basis of $\mathbb{R}^n (= T_0\mathbb{G})$. We further assume that $\{\mathrm{e}_i : i = 1, \dots, n\}$ of $\mathbb{R}^n$ is *graded*, in the sense that $H_i = \mathrm{span}_{\mathbb{R}}\{\mathrm{e}_i : i = n_{i-1}+1, \dots, n_i\}$ for any $i = 1, \dots, k$. By construction, one has $X_i(0) = \mathrm{e}_i$ for every $i = 1, \dots, n$ (such a frame is sometimes called the *Jacobian basis* of $\mathbb{G}$; see [6]).

Let *exp* $: \mathfrak{g} \to \mathbb{G}$ be the (Lie group) exponential map and denote by *log* $: \mathbb{G} \to \mathfrak{g}$ its inverse. Hereafter, we will use *exponential coordinates of the 1st kind*; see [49], Ch. 2, p. 88.

As for the case of nilpotent Lie groups, the multiplication $\bullet$ of $\mathbb{G}$ is uniquely determined by the "structure" of its Lie algebra $\mathfrak{g}$: this is the content of the *Baker-Campbell-Hausdorff formula*; see [6]. Note that $0 = exp\,(0, \dots, 0)$ and that the inverse of $x = exp\,(x_1, \dots, x_n) \in \mathbb{G}$ is given by $x^{-1} = exp\,(-x_1, \dots, -x_n)$.

A *sub-Riemannian metric* $g_H : H \times H \to \mathbb{R}_+ \cup \{0\}$ is a symmetric positive bilinear form on H. Without loss of generality, we also define a metric $g : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}_+ \cup \{0\}$ on $\mathfrak{g}$ by declaring that $\{\mathrm{e}_i : i = 1, \dots, n\}$ is an orthonormal basis; hence, in particular, the subspaces H_i are automatically orthogonal. The metrics g_H and g, hereafter denoted as $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle$, respectively, extend to the whole group $\mathbb{G}$ by means of left-translations. In this way $(\mathbb{G}, g)$ is a Riemannian manifold. Below, for simplicity, we shall assume that $g_H := g_{|H}$.

The *Carnot-Carathéodory-distance* $d_{CC}(x, y)$ between $x, y \in \mathbb{G}$ is defined as

$$d_{CC}(x, y) := \inf \int \sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle_H}\, dt,$$

where the infimum is taken over all absolutely continuous horizontal curves γ joining x to y. As a matter of fact, by virtue of Chow's connectivity theorem, this is a distance, which makes $(\mathbb{G}, d_{CC})$ a complete geodesic metric space; see [39]. Moreover, we recall that Carnot groups are *homogeneous groups*, that is, they admit a 1-parameter family of automorphisms (usually called *Carnot dilations*) $\delta_t : \mathbb{G} \longrightarrow \mathbb{G}$ $(t \geq 0)$ defined as $\delta_t x := exp\left(\sum_{j=1}^{k} \sum_{i_j=n_{j-1}+1}^{n_j} t^j\, x_{i_j} \mathrm{e}_{i_j}\right)$ for every $x = exp\left(\sum_{j=1}^{k} \sum_{i_j=n_{j-1}+1}^{n_j} x_{i_j} \mathrm{e}_{i_j}\right) \in \mathbb{G}$.

The *structural constants* of $\mathfrak{g}$ associated with the frame $\{X_1, \dots, X_n\}$ are defined by $C_{ij}^r := \langle [X_i, X_j], X_r \rangle$ for all $i, j, r = 1, \dots, n$. They are skew-symmetric and satisfy Jacobi's identity. We mention that the stratification of the Lie algebra $\mathfrak{g}$ implies the following "structural" property: if $i \in \mathfrak{I}_{H_s}$ and $j \in \mathfrak{I}_{H_r}$, then

$$C_{ij}^m \neq 0 \Rightarrow m \in \mathfrak{I}_{H_{s+r}}. \tag{1.1}$$

Notation 1.2. *Let* $\mathbb{G}$ *be a step* 2 *Carnot group (hence* $V = H_2$*). From now on, we will set*

$$C_H^\alpha := [C_{ij}^\alpha]_{i,j=1,\dots,h} \in \mathcal{M}_{h\times h}(\mathbb{R}) \qquad \forall\ \alpha \in \mathfrak{I}_V = \{h+1, \dots, n\}.$$

Notation 1.3. *Let* $\bigwedge^r(T^\star\mathbb{G})$ *be the vector bundle of alternating left-invariant* r*-tensors of* $\mathbb{G}$ *and let* $\mathcal{A}^r(\mathbb{G})$ *be the vector space of left-invariant sections of* $\bigwedge^r(T^\star\mathbb{G})$*, that is, the set of all left-invariant differential* r*-forms. We also denote by* $\mathcal{A}_H^r(\mathbb{G})$ *the vector space of horizontal left-invariant sections of* $\bigwedge^r(H^\star)$*, that is, the set of all horizontal left-invariant* r*-forms.*

Let us define the left-invariant co-frame $\{\omega_i : i = 1, \dots, n\}$ dual to the frame $\{X_i : i = 1, \dots, n\}$, where $\omega_i = X_i^* \in \mathcal{A}^1(\mathbb{G})$ for every $i = 1, \dots, n$. The *left-invariant 1-forms* $\{\omega_i : i = 1, \dots, n\}$ are uniquely determined by the condition $\omega_i(X_j) = \langle X_i, X_j \rangle = \delta_i^j$ for all $i, j = 1, \dots, n$, where δ_i^j denotes Kronecker delta. From now on, we will set $\mathrm{vol}_H := \bigwedge_{i=1}^h \omega_i$ and $\mathrm{vol}_V := \bigwedge_{\alpha=h+1}^n \omega_\alpha$. The (Riemannian) left-invariant volume form of $\mathbb{G}$ is defined as $\sigma_R^n := \bigwedge_{i=1}^n \omega_i = \mathrm{vol}_H \wedge \mathrm{vol}_V$.

Notation 1.4. *We shall denote by $\mathcal{P}_{E_x} : T_x\mathbb{G} \to E_x$ the orthogonal projection map from $T_x\mathbb{G}$ onto E_x. In particular, if the subbundle E is defined by left-translation of a vector subspace E of $\mathfrak{g}$, then we shall simply write $\mathcal{P}_E$ rather than $\mathcal{P}_{E_x}$.*

Definition 1.5. *Let ∇ be the unique left-invariant Levi-Civita connection on $\mathbb{G}$ associated with the fixed left-invariant metric $g = \langle \cdot, \cdot \rangle$. In addition, for any $X, Y \in \mathfrak{X}(H) = \mathbf{C}^\infty(\mathbb{G}, H)$ we define a "partial connection" on $\mathbb{G}$ by setting $\nabla_X^H Y := \mathcal{P}_H(\nabla_X Y)$.*

Let $\{X_1, \dots, X_n\}$ be a global left-invariant frame for $\mathbb{G}$. Then, it turns out that

$$\nabla_{X_i} X_j = \frac{1}{2} \sum_{r=1}^n \left(C_{ij}^r - C_{jr}^i + C_{ri}^j \right) X_r \qquad \forall\ i,\ j = 1, \dots, n; \tag{1.2}$$

see, for instance, Milnor's paper [35], Section 5, pp. 310-311. It is not difficult to check that ∇^H is flat, compatible with the sub-Riemannian metric g_H and torsion-free; see [37, 38]. Concerning the partial connection ∇^H, also called *H-connection*, we refer to [26]; see also [37, 38].

Definition 1.6. *The* horizontal gradient *of $\phi \in \mathbf{C}^1(\mathbb{G})$, say $\mathrm{grad}_H \phi$, is the unique continuous horizontal vector field such that $\langle \mathrm{grad}_H \phi, X \rangle = X\phi$ for all $X \in \mathfrak{X}(H)$. The horizontal divergence of $X \in \mathfrak{X}^1(H)$, denoted as $\mathrm{div}_H X$, is defined at each $x \in \mathbb{G}$ by $\mathrm{div}_H X(x) := \mathrm{Trace}\left(Y \longrightarrow \nabla_Y^H X\right)(x)$ $(Y \in H_x)$. The H-*Laplacian *Δ_H is the 2nd order operator given by $\Delta_H \phi := \mathrm{div}_H(\mathrm{grad}_H \phi)$ for all $\phi \in \mathbf{C}^2(S)$. For any $Y = \sum_{j\in I_H} y_j X_j \in \mathfrak{X}^1(H)$, we denote by $\mathcal{J}_H Y$ the horizontal Jacobian matrix of Y, that is, $\mathcal{J}_H Y := \left[X_i(y_j)\right]_{j,i\in I_H}$. The* horizontal Hessian matrix *of $\phi \in \mathbf{C}^2(\mathbb{G})$ is defined as $\mathrm{Hess}_H \phi := \mathcal{J}_H(\mathrm{grad}_H \phi) = \left[X_i(X_j\phi)\right]_{i,j\in I_H}$. Note that $\Delta_H \phi = \mathrm{Tr}\,(\mathrm{Hess}_H \phi)$ for every $\phi \in \mathbf{C}^2(\mathbb{G})$.*

Example 1.7 (Heisenberg groups $\mathbb{H}^n$)**.** *The base manifold of $\mathbb{H}^n$ is $\mathbb{R}^{2n+1}$ and every $p \in \mathbb{H}^n$ is represented as $p = \exp(z_H, t) \in \mathbb{H}^n$, where $z_H := (x_1, y_1, x_2, y_2, \dots, x_n, y_n)$. The Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}^n$ is described by means of the global left-invariant frame $\{X_1, Y_1, \dots, X_i, Y_i, \dots, X_n, Y_n, T\}$, where $X_i(p) := \frac{\partial}{\partial x_i} - \frac{y_i}{2}\frac{\partial}{\partial t}$, $Y_i(p) := \frac{\partial}{\partial y_i} + \frac{x_i}{2}\frac{\partial}{\partial t}$ for any $i = 1, \dots, n$, and $T(p) := \frac{\partial}{\partial t}$. One has $[X_i, Y_i] = T$ for any $i = 1, \dots, n$, and all other commutators vanish. In other words, $\mathfrak{h}_n$ is a nilpotent and stratified Lie algebra with step 2 and center $\mathrm{span}_\mathbb{R}\{T\}$, that is, $\mathfrak{h}_n = H \oplus H_2$, where $H = \mathrm{span}_\mathbb{R}\{X_1, Y_1, \dots, X_i, Y_i, \dots, X_n, Y_n\}$ and $H_2 = \mathrm{span}_\mathbb{R}\{T\}$. The structural constants of $\mathfrak{h}_n$ are described by the following skew-symmetric $(2n \times 2n)$-matrix*

$$C_H^{2n+1} := \begin{vmatrix} 0 & 1 & \cdot & 0 & 0 \\ -1 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & 0 & 1 \\ 0 & 0 & \cdot & -1 & 0 \end{vmatrix}.$$

1.2 Hypersurfaces

Let $S \subset \mathbb{G}$ be an orientable hypersurface (that is, a codimension 1 submanifold of $\mathbb{G}$) of class $\mathbf{C}^r$ $(r \geq 1)$ and let ν be the (Riemannian) unit normal vector along S. By definition, we say that $x \in S$ is a *characteristic point* if, and only if, $\dim H_x = \dim(H_x \cap T_xS)$. The *characteristic set* of S is given by $C_S := \{x \in S : \dim H_x = \dim(H_x \cap T_xS)\}$. In other words, a point $x \in S$ is non-characteristic if, and only if, H_x is transversal to T_xS. Hence, it turns out that $C_S = \{x \in S : |\mathcal{P}_H \nu(x)| = 0\}$, where $\mathcal{P}_H$ is the orthogonal projection map onto H; see Notation 1.4. We say that a hypersurface $S \subset \mathbb{G}$ is *non-characteristic* if its characteristic set is empty (that is, $|\mathcal{P}_H \nu(x)| \neq 0$ for all $x \in S$). In the theory of $\mathbf{C}^2$ hypersurfaces immersed in Carnot groups, it is of fundamental importance that the Riemannian measure of the characteristic set C_S vanishes: precise estimates of the Riemannian Hausdorff dimension of C_S can be found in [3]; see also [2] for the case of Heisenberg groups. At each non-characteristic point $x \in S \setminus C_S$, we define the *unit H-normal* as $\nu_H(x) := \frac{\mathcal{P}_H \nu(x)}{|\mathcal{P}_H \nu(x)|}$. The *horizontal tangent space* $H_xS := H_x \cap T_xS$ and the *horizontal normal space* $\mathrm{span}_{\mathbb{R}}\{\nu_H(x)\} \subset H_x$ split the horizontal space H_x into an orthogonal direct sum, that is, $H_x = \mathrm{span}_{\mathbb{R}}\{\nu_H(x)\} \oplus H_xS$.

Notation 1.8. *Let $x \in S \setminus C_S$. Throughout this paper we shall denote by $\mathcal{P}_{H_xS} : T_xS \to H_xS$ the orthogonal projection map from T_xS onto H_xS. When the point $x \in S \setminus C_S$ is clear from the context or irrelevant, we shall simply write $\mathcal{P}_{HS}$ instead of $\mathcal{P}_{H_xS}$.*

Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$ non-characteristic hypersurface and denote by ∇^{TS} the induced connection on S from ∇. The tangential connection ∇^{TS} induces a partial connection ∇^{HS} on HS given by

$$\nabla^{HS}_X Y := \mathcal{P}_{HS}\left(\nabla^{TS}_X Y\right) \qquad \forall\, X, Y \in \mathfrak{X}^1(HS) := \mathbf{C}^1(S, HS).$$

In particular, it turns out that $\nabla^{HS}_X Y = \nabla^{H}_X Y - \left\langle \nabla^{H}_X Y, \nu_H \right\rangle \nu_H$ for all $X, Y \in \mathfrak{X}^1(HS)$; see [37, 38].

Definition 1.9 (see [37]). *Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$ non-characteristic hypersurface. The HS-*gradient *of $\psi \in \mathbf{C}^1(S)$, say $\mathrm{grad}_{HS}\,\psi$, is the unique continuous horizontal tangent vector field such that $\langle \mathrm{grad}_{HS}\,\psi, X\rangle = X\psi$ for all $X \in \mathfrak{X}^1(HS)$. The HS-*divergence operator *is defined, for $X \in \mathfrak{X}^1(HS)$ and $x \in S$, by setting $\mathrm{div}_{HS} X(x) :=$* Trace $\left(Y \longrightarrow \nabla^{HS}_Y X\right)(x)$ $(Y \in H_xS)$. *The HS-*Laplacian *Δ_{HS} is the 2nd order differential operator defined as $\Delta_{HS}\,\psi := \mathrm{div}_{HS}(\mathrm{grad}_{HS}\,\psi)$ for all $\psi \in \mathbf{C}^2(S)$. By definition, the* horizontal mean curvature *$\mathcal{H}_H$ of S is given by $\mathcal{H}_H := -\mathrm{div}_H\, \nu_H$.*

Definition 1.10. *Let $S \subset \mathbb{G}$ be a hypersurface of class $\mathbf{C}^r$ $(r \geq 1)$. We call* adapted horizontal frame to *S any horizontal orthonormal frame $\{Z_1, \dots, Z_h\}$ for H such that $H_xS = \mathrm{span}_{\mathbb{R}}\{Z_1(x), \dots, Z_{h-1}(x)\}$ and $Z_h(x) = \nu_H(x)$ for every $x \in S \setminus C_S$. Note in particular that $\{Z_1, \dots, Z_{h-1}\}$ is a horizontal orthonormal frame for $HS|_{S\setminus C_S}$. Furthermore, let $\{\zeta_1, \dots, \zeta_{h-1}\}$ be its dual coframe, which is uniquely defined by the condition $\zeta_i(Z_j) = \delta_i^j$ for all $i, j = 1, \dots, h-1$. We also set $\mathrm{vol}_{HS} := \bigwedge_{i=1}^{h-1} \zeta_i$ to denote the natural volume form on HS. In the sequel, we shall often use the notation $\mathfrak{I}_{HS} := \{1, \dots, h-1\}$.*

Remark 1.11. *Let S be a $\mathbf{C}^2$ non-characteristic hypersurface, with or without boundary. The HS-Laplacian Δ_{HS} is a 2nd order degenerate elliptic operator of the form "sum of squares", which acts on smooth functions defined on S (these operators are called "sub-Laplacians"; see Stein's book [48], or the recent monograph [6]). Precisely, starting from Definition 1.9 and using an adapted horizontal frame $\{Z_i : i \in \mathfrak{I}_{HS}\}$ for HS, we get that*

$$\Delta_{HS}\,\phi = \sum_{i=1}^{h-1} Z_i^{(2)}\phi \qquad \forall\, \phi \in \mathbf{C}^2(S).$$

2 Technical preliminaries and main calculations

2.1 Some calculations for step 2 Carnot groups

This section contains all the calculations needed to prove our main results. Below, we will assume that $\mathbb{G}$ is a step 2 Carnot group and, accordingly, we will set $V := H_2$. In this case, we have the the following explicit formulas for the horizontal frame $\{X_1, \dots, X_h\}$ introduced in Section 1.1:

$$X_i(x) := \mathrm{e}_i + \frac{1}{2}\sum_{\alpha \in \mathfrak{I}_V} \left\langle C^{\alpha}_{H}\mathrm{e}_i, x_H \right\rangle_{\mathbb{R}^h} \mathrm{e}_\alpha, \quad \forall\, i \in \mathfrak{I}_H = \{1, \dots, h\}. \quad X_\alpha(x) := \mathrm{e}_\alpha, \quad \forall\, \alpha \in \mathfrak{I}_V = \{h+1, \dots, n\}.$$

Recall that, in exponential coordinates, any $x \in \mathbb{G}$ is written as $x = exp(x_H, x_V)$ and that $\mathrm{e}_I \equiv \partial_{x_I}$ for every $I = 1, \dots, n$. Concerning the above formulas, which can be obtained by direct calculations, we refer the reader to Chapter 3 of the book [6].

Let S be a hypersurface of class $\mathbf{C}^r$ ($r \geq 2$). We shall make repeatedly use of the formula

$$\Delta_{HS}\,\phi = \Delta_H\,\phi + \mathcal{H}_H\,\frac{\partial \phi}{\partial \nu_H} - \langle \mathrm{Hess}_H\,\phi\,\nu_H, \nu_H\rangle\,; \tag{2.1}$$

see [38]. From now on, we will assume that S is an H-minimal hypersurface, that is, $\mathcal{H}_H = 0$.

We start by studying (Euclidean) degree 1 monomials.

Lemma 2.1. *Let $\mathbb{G}$ be a step 2 Carnot group. The coordinate functions $\{x_I : I = 1, \dots, n\}$ are Δ_H-harmonic. More precisely, we have:*

(i) $\Delta_H\,x_i = 0$ *for any* $i \in \mathfrak{I}_H$*;*
(ii) $\Delta_H\,x_\alpha = 0$ *for any* $\alpha \in \mathfrak{I}_V$.

Proof. We have $grad_H\,x_i = X_i$ and hence $div_H(X_i) = \sum_{j\in\mathfrak{I}_H}\langle \nabla^H_{X_j} X_i, X_j\rangle = 0$ for every $i \in \mathfrak{I}_H$, since $\nabla^H_{X_j} X_i = 0$ for all $i, j \in \mathfrak{I}_H$ (this easily follows from (1.2) and Definition 1.5; see Section 1.1). Moreover, since

$$grad_H\,x_\alpha = \frac{1}{2}\sum_{i\in\mathfrak{I}_H}\left\langle C^\alpha_H \mathrm{e}_i, x_H\right\rangle_{\mathbb{R}^h} X_i = -\frac{1}{2}C^\alpha_H x_H\,, \tag{2.2}$$

it follows that $\Delta_H\,x_\alpha = div_H\,(grad_H\,x_\alpha) = div_H\left(\frac{1}{2}\sum_{i\in\mathfrak{I}_H}\left\langle C^\alpha_H \mathrm{e}_i, x_H\right\rangle_{\mathbb{R}^h} X_i\right)$. Since $C^\alpha_H \mathrm{e}_i$ is a constant vector, we get that $X_j\left(\left\langle C^\alpha_H \mathrm{e}_i, x_H\right\rangle_{\mathbb{R}^h}\right) = \frac{\partial}{\partial x_j}\left(\left\langle C^\alpha_H \mathrm{e}_i, x_H\right\rangle_{\mathbb{R}^h}\right) = \left\langle C^\alpha_H \mathrm{e}_i, \mathrm{e}_j\right\rangle_{\mathbb{R}^h}$. Hence using again the fact that $\nabla^H_{X_j} X_i = 0$ yields

$$\Delta_H\,x_\alpha = \frac{1}{2}\sum_{i,j\in\mathfrak{I}_H}\left\langle C^\alpha_H \mathrm{e}_i, \mathrm{e}_j\right\rangle_{\mathbb{R}^h}\langle X_i, X_j\rangle = \frac{1}{2}\sum_{i,j\in\mathfrak{I}_H}\left\langle C^\alpha_H \mathrm{e}_i, \mathrm{e}_j\right\rangle_{\mathbb{R}^h}\delta^j_i = 0,$$

where the last equality follows from the skew-symmetry of the matrices C^α_H.

□

Now we consider (Euclidean) degree 2 monomials.

Lemma 2.2. *Let $\mathbb{G}$ be a step 2 Carnot group. The following formulas hold:*

(i) $\Delta_H\left(x_i^2\right) = 2$ *for any* $i \in \mathfrak{I}_H$*;*
(ii) $\Delta_H\left(x_\alpha^2\right) = \frac{1}{2}\left|C^\alpha_H x_H\right|^2$ *for any* $\alpha \in \mathfrak{I}_V$.

Proof. For any $i \in \mathfrak{I}_H$, we have

$$\Delta_H\left(x_i^2\right) = div_H\Big(2x_i\underbrace{grad_H\,x_i}_{=X_i}\Big) = 2\Big(\Big\langle\underbrace{grad_H\,x_i}_{=X_i}, X_i\Big\rangle + x_i\underbrace{\Delta_H\,x_i}_{=0}\Big) = 2,$$

where we have used (i) of Lemma 2.1. Moreover, for any $\alpha \in \mathfrak{I}_V$ we have

$$\Delta_H\left(x_\alpha^2\right) = 2\Big(\langle grad_H\,x_\alpha, grad_H\,x_\alpha\rangle + x_\alpha\underbrace{\Delta_H\,x_\alpha}_{=0}\Big) = 2\left|grad_H\,x_\alpha\right|^2 = \frac{1}{2}\left|C^\alpha_H x_H\right|^2,$$

where we have used (ii) of Lemma 2.1 and (2.2). □

Lemma 2.3. *Let $\mathbb{G}$ be a step 2 Carnot group. Then:*

(i) $\mathrm{Hess}_H(x_i) = 0_{h\times h} \in \mathcal{M}_{h\times h}(\mathbb{R})$ *for any* $i \in \mathfrak{I}_H$*;*
(ii) $\mathrm{Hess}_H(x_\alpha) = -\frac{1}{2}C^\alpha_H \in \mathcal{M}_{h\times h}(\mathbb{R})$ *for any* $\alpha \in \mathfrak{I}_V$.

Proof. Since $grad_H x_i = X_i$ for any $i \in \mathfrak{I}_H$, the proof of (i) it is an immediate consequence of the fact that $\nabla^H_{X_j} X_i = 0$ for all $i, j \in \mathfrak{I}_H$. Let $\alpha \in \mathfrak{I}_V$; in order to prove (ii), we first note that

$$\mathrm{Hess}_H(x_\alpha) = \mathcal{J}_H\,(grad_H\, x_\alpha) = \mathcal{J}_H\left(-\frac{1}{2}C^\alpha_H x_H\right).$$

Since $\mathcal{J}_H\, x_H = \mathrm{Id}_{h\times h} \in \mathcal{M}_{h\times h}(\mathbb{R})$, the proof easily follows. □

Lemma 2.4. *Let $\mathbb{G}$ be a step 2 Carnot group. Then:*

(i) $\mathrm{Hess}_H\left(x_i^2\right) = 2\,(X_i \otimes X_i) \in \mathcal{M}_{h\times h}(\mathbb{R})$ *for any $i \in \mathfrak{I}_H$;*
(ii) $\mathrm{Hess}_H\left(x_\alpha^2\right) = \frac{1}{2}\left(C^\alpha_H x_H \otimes C^\alpha_H x_H\right) - x_\alpha C^\alpha_H$ *for any $\alpha \in \mathfrak{I}_V$.*

Proof. We have

$$\mathrm{Hess}_H\left(x_i^2\right) = \mathcal{J}_H\,(2x_i grad_H\, x_i) = 2\,(grad_H\, x_i \otimes grad_H\, x_i) = 2\,(X_i \otimes X_i) \qquad \forall\, i \in \mathfrak{I}_H,$$

where we have used also (i) of Lemma 2.3. Moreover, we have

$$\begin{aligned}\mathrm{Hess}_H\left(x_\alpha^2\right) &= \mathcal{J}_H\,(2x_\alpha grad_H\, x_\alpha) = 2\left(grad_H\, x_\alpha \otimes grad_H\, x_\alpha + x_\alpha \mathrm{Hess}_H(x_\alpha)\right)\\ &= 2\left(-\frac{1}{2}C^\alpha_H x_H\right) \otimes \left(-\frac{1}{2}C^\alpha_H x_H\right) - x_\alpha C^\alpha_H \qquad \forall\, \alpha \in \mathfrak{I}_V,\end{aligned}$$

where we have used also (ii) of Lemma 2.3. □

Remark 2.5. *Let $S \subset \mathbb{G}$ be an H-minimal $\mathbf{C}^2$ hypersurface and let ν_H be its horizontal unit normal vector. By applying Lemma 2.3, it follows that $\left\langle \left(\mathrm{Hess}_H(x_i)\right)\nu_H, \nu_H\right\rangle = 0$ for any $i \in \mathfrak{I}_H$. By skew-symmetry of the matrices C^α_H ($\alpha \in \mathfrak{I}_V$), we get $\left\langle \mathrm{Hess}_H(x_\alpha)\,\nu_H, \nu_H\right\rangle = -\frac{1}{2}\left\langle C^\alpha_H \nu_H, \nu_H\right\rangle = 0$ for any $\alpha \in \mathfrak{I}_V$. Furthermore, by applying* (i) *of Lemma 2.4, it follows that*

$$\left\langle \mathrm{Hess}_H\left(x_i^2\right)\nu_H, \nu_H\right\rangle = 2\,\langle (X_i \otimes X_i)\,\nu_H, \nu_H\rangle = 2\left(\nu_H^i\right)^2 \qquad \forall\, i \in \mathfrak{I}_H.$$

Finally, by using (ii) *of Lemma 2.4 (and again the skew-symmetry of C^α_H) we get that*

$$\left\langle \mathrm{Hess}_H\left(x_\alpha^2\right)\nu_H, \nu_H\right\rangle = \left\langle \left[\frac{1}{2}\left(C^\alpha_H x_H \otimes C^\alpha_H x_H\right) - x_\alpha C^\alpha_H\right]\nu_H, \nu_H\right\rangle = \frac{1}{2}\left\langle C^\alpha_H x_H, \nu_H\right\rangle^2 \qquad \forall\, \alpha \in \mathfrak{I}_V.$$

We are now in a position to state two propositions, which will be important in the following:

Proposition 2.6. *Let $\mathbb{G}$ be a step 2 Carnot group and let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$ non-characteristic H-minimal hypersurface. Then, the standard coordinate functions $\{x_I : I = 1, \dots, n\}$ of $\mathbb{G}$ are Δ_{HS}-harmonic on S. More precisely, the following equations hold:*

(i) $\Delta_{HS}\, x_i = 0$ *for any $i \in \mathfrak{I}_H$;*
(ii) $\Delta_{HS}\, x_\alpha = 0$ *for any $\alpha \in \mathfrak{I}_V$.*

Thus, for any $\overline{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ the function $f_{\overline{\alpha}}(x) := \sum_{I=1}^n \alpha_I x_I$ is Δ_{HS}-harmonic on S.

Proof. Since $\mathcal{H}_H = 0$, the proof of (i) and (ii) follows easily from formula (2.1), by applying Lemma 2.1 and Lemma 2.3.

□

Proposition 2.7. *Let $\mathbb{G}$ be a step 2 Carnot group and let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$ non-characteristic H-minimal hypersurface. Then, the following hold:*

(i) $\Delta_{HS}\left(x_i^2\right) = 2\left(1 - \left(\nu_H^i\right)^2\right) \geq 0$ *for any $i \in \mathfrak{I}_H$;*

(ii) $\Delta_{HS}\left(x_\alpha^2\right) = \frac{1}{2}\left(\left|C_H^\alpha x_H\right|^2 - \left\langle C_H^\alpha x_H, \nu_H\right\rangle^2\right) \geq 0$ *for any* $\alpha \in \mathfrak{I}_V$.

In particular, the monomial functions $\{x_I^2 : I = 1, \ldots, n\}$ *turn out to be* Δ_{HS}*-subharmonic on* S.

Proof. Similarly to Proposition 2.6, since $\mathcal{H}_H = 0$, the proof of (i) and (ii) follows from (2.1), by Lemma 2.2 and Lemma 2.4; see also Remark 2.5. □

Notation 2.8. *Let us set* $g_h(x_H) := \sum_{i=1}^{h-1} x_i^2 - (h-2)x_h^2$. *Let* $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$ *and let* $\overline{\beta}$ *be any* ν*-tuple of strictly positive numbers, that is,* $\overline{\beta} := (\beta_{h+1}, \ldots, \beta_n) \in \mathbb{R}_+^\nu := \mathbb{R}_+ \times \ldots . \times \mathbb{R}_+ \subsetneq \mathbb{R}^\nu$. *We set* $g_{\overline{\beta}}(x_V) := \sum_{\alpha=h+1}^n \beta_\alpha x_\alpha^2$ *and, accordingly,* $g_{\overline{1}}(x_V) := \sum_{\alpha=h+1}^n x_\alpha^2$, *where* $\overline{1} = (1, \ldots, 1) \in \mathbb{R}_+^\nu$. *We also define a quadratic function on* $\mathbb{G}$ *by setting* $g^{(h,\overline{\beta})}(x) := g_h(x_H) + g_{\overline{\beta}}(x_V)$.

Definition 2.9 (Hyperboloids and Cones with horizontal axis). *Let* $\epsilon > 0$ *and* $\overline{\beta} \in \mathbb{R}_+^\nu$. *We define a solid hyperboloid by setting*

$$\mathcal{H}yp(0, \epsilon, X_h, \overline{\beta}) := \left\{x \in \mathbb{G} : g^{(h,\overline{\beta})}(x) = g_h(x_H) + g_{\overline{\beta}}(x_V) \leq \epsilon^2\right\} \tag{2.3}$$

(note that the axis of this hyperboloid is a Euclidean line passing through $0 \in \mathbb{G}$ *and having horizontal direction* X_h*). Also, we denote by* $\mathcal{H}yp(\epsilon, \overline{\beta})$ *any element of the congruence class of* $\mathcal{H}yp(0, \epsilon, X_h, \overline{\beta})$. *In addition, suppose that* $\epsilon = 0$ *in the previous definitions. In this case, we set* $\mathcal{C}(0, X_h, \overline{\beta}) := \mathcal{H}yp(0, 0, X_h, \overline{\beta})$ *to denote the solid cone*

$$\mathcal{C}(0, X_h, \overline{\beta}) := \left\{x \in \mathbb{G} : g^{(h,\overline{\beta})}(x) = g_h(x_H) + g_{\overline{\beta}}(x_V) \leq 0\right\}. \tag{2.4}$$

The upper and lower parts of the cone $\mathcal{C}(0, X_h, \overline{\beta})$ *(with respect to the axis* X_h*) are denoted by* $\mathcal{C}^+(0, X_h, \overline{\beta})$ *and* $\mathcal{C}^-(0, X_h, \overline{\beta})$*, respectively. That is, we set*

$$\mathcal{C}^+(0, X_h, \overline{\beta}) := \mathcal{C}(0, X_h, \overline{\beta}) \cap \{x \in \mathbb{G} : x_h \geq 0\}, \quad \mathcal{C}^-(0, X_h, \overline{\beta}) := \mathcal{C}(0, X_h, \overline{\beta}) \cap \{x \in \mathbb{G} : x_h \leq 0\}.$$

Finally, we denote by $\mathcal{C}(\overline{\beta})$ *any element of the congruence class of* $\mathcal{C}(0, X_h, \overline{\beta})$ *and by* $\mathcal{C}^\pm(\overline{\beta})$ *its upper/lower parts.*

Concerning the notion of "congruence" we refer the reader to Definition A.2 in Section A.

Corollary 2.10. *Let* $\mathbb{G}$ *be a step* 2 *Carnot group, let* $S \subset \mathbb{G}$ *be a* $\mathbf{C}^2$ *non-characteristic H-minimal hypersurface. Then, the functions* $g_h(x_H)$ *and* $g_{\overline{\beta}}(x_V)$ *turn out to be both* Δ_{HS}*-subharmonic on* S. *As a consequence, the quadratic function* $g^{(h,\overline{\beta})}(x) := g_h(x_H) + g_{\overline{\beta}}(x_V)$ *is* Δ_{HS}*-subharmonic on* S.

Proof. In order to prove the first claim, let us calculate the *HS*-Laplacian of the function $g_h(x_H)$ by using (i) in Proposition 2.7. We have

$$\Delta_{HS}\, g_h(x_H) = 2\left(\sum_{i=1}^{h-1}\left(1 - \left(\nu_H^i\right)^2\right) - (h-2)\left(1 - \left(\nu_H^h\right)^2\right)\right) = 2(h-1)\left(\nu_H^h\right)^2 \geq 0,$$

where we have used the identity $|\nu_H|^2 = \sum_{i=1}^h \left(\nu_H^i\right)^2 = 1$. The fact that $g_{\overline{\beta}}(x_V)$ is Δ_{HS}-subharmonic follows from (ii) in Proposition 2.7. The last claim follows from the previous ones. □

Notation 2.11. *Let* $\overline{a} := (a_1, \ldots, a_h) \in \mathbb{R}_+^h := \mathbb{R}_+ \times \ldots . \times \mathbb{R}_+ \subsetneq \mathbb{R}^h$ *and set* $g_{\overline{a}}(x_H) := \sum_{i=1}^h a_i x_i^2$. *Let* $\alpha \in \mathfrak{I}_V$ *and let* V' *be the* $(\nu-1)$*-dimensional subspace of* V *such that* $V = V' \oplus \mathrm{span}_{\mathbb{R}}\{X_\alpha\}$. *We accordingly set* $x_{V'} = (x_{h+1}, \ldots, x_{\alpha-1}, x_{\alpha+1}, \ldots, x_n)$. *Let* $\overline{\beta}' := (\beta_{h+1}, \ldots, \beta_{\alpha-1}, \beta_{\alpha+1}, \ldots, \beta_n) \in \mathbb{R}_+^{\nu-1}$ *be any* $(\nu-1)$*-tuple of strictly positive numbers, where* $\mathbb{R}_+^{\nu-1} := \mathbb{R}_+ \times \ldots \times \mathbb{R}_+ \subsetneq \mathbb{R}^{\nu-1}$. *Finally, set*

$$g_{\overline{\beta}'}(x_{V'}) := \sum_{\substack{h+1 \leqslant \gamma < \alpha \\ \alpha < \gamma \leqslant n}} \beta_\gamma x_\gamma^2, \qquad g^{(\overline{a},\overline{\beta}')} := g_{\overline{a}}(x_H) + g_{\overline{\beta}'}(x_{V'}).$$

Definition 2.12 (Cylinders and Paraboloids with vertical axis)**.** *Let $\overline{a} \in \mathbb{R}_+^h$ and $\overline{\beta} \in \mathbb{R}_+^{\mathrm{v}}$. We define a solid paraboloid by setting*

$$\mathcal{P}ar(0, X_\alpha, \overline{a}, \overline{\beta}') := \left\{x \in \mathbb{G} : g^{(\overline{a}, \overline{\beta}')}(x) = g_{\overline{a}}(x_H) + g_{\overline{\beta}'}(x_{V'}) \le x_\alpha\right\} \tag{2.5}$$

(note that the axis of $\mathcal{P}ar(0, X_\alpha, \overline{a}, \overline{\beta}')$ is a Euclidean line passing through $0 \in \mathbb{G}$ with vertical direction X_α). We denote by $\mathcal{P}ar(\overline{a}, \overline{\beta}')$ any element of the congruence class of $\mathcal{P}ar(0, X_\alpha, \overline{a}, \overline{\beta}')$. Furthermore, let $\epsilon \in \mathbb{R}_+$ and denote by $\mathcal{C}yl(0, X_\alpha, \epsilon, \overline{a}, \overline{\beta}')$ the solid cylinder given by

$$\mathcal{C}yl(0, \epsilon, X_\alpha, \overline{a}, \overline{\beta}') := \left\{x \in \mathbb{G} : g^{(\overline{a}, \overline{\beta}')}(x) \le \epsilon^2\right\}. \tag{2.6}$$

Finally, we denote by $\mathcal{C}yl(\epsilon, \overline{a}, \overline{\beta}')$ any element of the congruence class of $\mathcal{C}yl(0, \epsilon, X_\alpha, \overline{a}, \overline{\beta}')$.

Corollary 2.13. *Let $\mathbb{G}$ be a step 2 Carnot group, let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$ non-characteristic H-minimal hypersurface. Then, the functions $g_{\overline{a}}(x_H)$ and $g_{\overline{\beta}'}(x_{V'})$ turn out to be Δ_{HS}-subharmonic on S. As a consequence, the quadratic functions $g^{(\overline{a}, \overline{\beta}')}(x) := g_{\overline{a}}(x_H) + g_{\overline{\beta}'}(x_{V'})$ and $g_\alpha^{(\overline{a}, \overline{\beta}')}(x) := g^{(\overline{a}, \overline{\beta}')}(x) - x_\alpha$ are both Δ_{HS}-subharmonic on S.*

Proof. The proof follows by applying (ii) of Proposition 2.6 and (i) and (ii) of Proposition 2.7. □

The next result is an immediate consequence of the above calculations and will be used in the proof of Theorem 2.21; see Section 2.2. Below, we will set $r_H := \sqrt{\sum_{i=1}^h x_i^2}$.

Lemma 2.14. *Let $C > 0$ and set $\varphi(x) := e^{-(C/2)\cdot r_H^2}$. Then, we have $(\Delta_{HS}\varphi)(x) < 0$ for every $x \in \mathbb{G}$ such that $r_H < \sqrt{\frac{h-1}{C}}$. Furthermore, let us set $\Omega_C := \left\{x \in \mathbb{G} : r_H < \sqrt{\frac{h-1}{C}}\right\} \subset \mathbb{G}$. Then, for every $\mathbf{C}^2$ compact non-characteristic hypersurface $S \subset \Omega_C$ (with or without boundary), there exists a function $\varphi \in \mathbf{C}^2(S)$ such that $\Delta_{HS}\varphi < 0$ and $\varphi > 0$.*

Proof. First, note that $\Delta_{HS} e^{f(x)} = e^{f(x)}\left(\Delta_{HS} f(x) + |grad_{HS} f|^2\right)$ for every $f \in \mathbf{C}^2(S)$. Now, let $f(x) = -(C/2)\cdot r_H^2$. By Proposition 2.7 we have $\Delta_{HS} f(x) = -\frac{C}{2}\Delta_{HS} r_H^2 = -C(h-1)$. Moreover

$$|grad_{HS} f|^2 = C^2 |x_{HS}|^2 = C^2 |x_H - \langle x_H, \nu_H\rangle|^2$$

and hence

$$\Delta_{HS}\varphi = -C\varphi\left((h-1) - C r_H^2 \left|\frac{x_H}{r_H} - \left\langle \frac{x_H}{r_H}, \nu_H\right\rangle\right|^2\right).$$

The conclusion of the lemma follows from the last formula. □

2.2 Strong maximum principle

First, let us recall a fundamental result in Analysis: Bony's Maximum Principle; see [8]. To this end, let us consider a real 2nd order differential operator $\mathcal{L}$, defined in a connected open set $\Omega \subset \mathbb{R}^N$, which is an operator "sum of squares" of vector fields with $\mathbf{C}^\infty$ coefficients. Precisely, let

$$\mathcal{L}\phi := \sum_{i=1}^r Z_i^{(2)}\phi \qquad \forall\, \phi \in \mathbf{C}^\infty(\Omega), \tag{2.7}$$

$(r < N)$ and assume the following well-known "Hörmander condition":

- $\{Z_1, \ldots, Z_r\}$ is a family of vector fields of class $\mathbf{C}^\infty$ in Ω;
- the rank of the Lie algebra spanned by $\{Z_1, \ldots, Z_r\}$ is equal to N at each point of Ω, that is, $\operatorname{rank}\left(\mathcal{L}ie\{Z_1, \ldots, Z_r\}(x)\right) = N$ for all $x \in \Omega$.

Theorem 2.15 (see Corollary 3.1 in [8]). *Under the above assumptions, let $\psi \in \mathbf{C}^2(\Omega)$ be such that $\mathcal{L}\psi \geq 0$. If ψ has a positive maximum at a point $x_0 \in \Omega$, then ψ has to be constant in Ω, that is $\psi(x) = \psi(x_0)$ for all $x \in \Omega$.*

Roughly speaking this means that sub-Laplacians satisfy an elliptic type strong maximum principle. Let us formulate a key assumption for the sequel.

Definition 2.16 (Hörmander condition for HS). *Let $\mathbb{G}$ be a k-step Carnot group and let $S \subset \mathbb{G}$ be a non-characteristic hypersurface (with or without boundary) of class $\mathbf{C}^r$, with $r \geq k$. We say that the subbundle HS satisfies the* Hörmander condition *if there is an adapted orthonormal frame $\{Z_1, ..., Z_{h-1}\}$ for HS such that*

$$\operatorname{rank}\left(\mathcal{L}ie\{Z_1, ..., Z_{h-1}\}(x)\right) = n - 1 \qquad \forall\, x \in S. \tag{2.8}$$

Remark 2.17 (The Heisenberg groups $\mathbb{H}^n$ satisfy (2.8) iff $n > 1$). *Let $S \subset \mathbb{H}^n$ be a $\mathbf{C}^2$ hypersurface and assume that $n > 1$. Then, we claim that condition (2.8) holds at each non-characteristic point $p \in S \setminus C_S$. To prove this claim, let $\{Z_1, ..., Z_{2n-1}\}$ be an orthonormal frame for $HS|_{S\setminus C_S}$. This frame can be completed to an orthonormal frame for $TS|_{S\setminus C_S}$ by adding the vector field $U := |\mathcal{P}_H \nu|T - \langle \nu, T\rangle \nu_H$. In other words $\{Z_1, ..., Z_{2n-1}, U\}$ is an orthonormal frame for $TS|_{S\setminus C_S}$. For simplicity, set $\varpi := \langle \nu, T\rangle / |\mathcal{P}_H \nu|$. Now, we observe that*[1]

$$\frac{1}{|\mathcal{P}_H \nu|}\langle [Z_i, Z_j], U\rangle = \langle [Z_i, Z_j], (T - \varpi\nu_H)\rangle = (1 + \varpi^2)\langle [Z_i, Z_j], T\rangle \qquad \forall\, i, j \in \mathfrak{I}_{HS}. \tag{2.9}$$

Since $\langle [Z_i, Z_j], T\rangle = \langle C_H^{2n+1} Z_j, Z_i\rangle$, using (2.9) together with the identity $1 + \varpi^2 = 1/|\mathcal{P}_H \nu|^2$ yields

$$\langle [Z_i, Z_j], U\rangle = \frac{1}{|\mathcal{P}_H \nu|}\langle C_H^{2n+1} Z_j, Z_i\rangle \qquad \forall\, i, j \in \mathfrak{I}_{HS}. \tag{2.10}$$

One also verifies that $\ker(C_H^{2n+1}|_{H_pS}) = \operatorname{span}_{\mathbb{R}}\{(C_H^{2n+1}\nu_H)(p)\}$, which is a 1-dimensional subspace of H_pS. Since $n > 1$, it follows that U belongs to the linear $\mathbb{R}$-span of the set $\{Z_l, [Z_i, Z_j] : i, j, l \in \mathfrak{I}_{HS}\}$. Therefore $\operatorname{rank}\left(\mathcal{L}ie\{Z_1, ..., Z_{2n-1}\}(p)\right) = 2n$, as wished. Finally, if $n = 1$, then $HS = \operatorname{span}_{\mathbb{R}}\{C_H^3 \nu_H\}$ is 1-dimensional and condition (2.8) cannot be satisfied.

A first consequence of Theorem 2.15 is contained in the next:

Corollary 2.18 (Strong Maximum Principle: 1st version). *Let $\mathbb{G}$ be a step k Carnot group. Let $S \subset \mathbb{G}$ be a connected, non-characteristic hypersurface (with or without boundary) of class $\mathbf{C}^\infty$ and assume that HS satisfies the Hörmander condition (2.8). Then the HS-Laplacian satisfies the strong maximum principle on S. More precisely, let $\psi \in \mathbf{C}^2(S)$ be such that $\Delta_{HS}\psi \geq 0$. If ψ has a positive maximum at an interior point $x_0 \in \operatorname{Int}(S)$, then ψ has to be constant in S, that is $\psi(x) = \psi(x_0)$ for all $x \in S$.*

Proof. The HS-Laplacian $\Delta_{HS} = \sum_{i=1}^{h-1} Z_i^{(2)}$ (=: $\mathcal{L}$) is a sub-Laplacian on S and the assumptions in Theorem 2.15 are satisfied by the set of vector fields $\{Z_i : i \in \mathfrak{I}_{HS}\}$. More precisely, let $\mathcal{A}$ be a smooth atlas for S and let $(U, \zeta) \in \mathcal{A}$ be such that $x_0 \in U$, where $\zeta : U \to \mathbb{R}^{n-1}$. Let us set $\widetilde{Z}_i := \zeta_\star Z_i$ for any $i \in \mathfrak{I}_{HS}$. Accordingly, we define a 2nd order operator on $\zeta(U)$ by setting $\widetilde{\mathcal{L}} := \sum_{i=1}^{h-1} \widetilde{Z}_i^{(2)}$. By the naturality of Lie brackets (see, for instance, Proposition 13.3 in [32]) one has $[\widetilde{Z}_i, \widetilde{Z}_i] = \zeta_\star[Z_i, Z_j]$ for every $i, j \in \mathfrak{I}_{HS}$. By repeated use of this formula, it follows that $\{\widetilde{Z}_i : i \in \mathfrak{I}_{HS}\}$ is a family of $\mathbf{C}^\infty$ vector fields on $\zeta(U)$ satisfying the Hörmander condition. Furthermore, by using the ζ-relatedness of Z_i and $\widetilde{Z}_i$, it follows that $\mathcal{L}(\phi \circ \zeta) = (\widetilde{\mathcal{L}}\phi)(\zeta)$ for every $\mathbf{C}^2$ function $\phi : \zeta(U) \subseteq \mathbb{R}^{n-1} \to \mathbb{R}$. So let $\psi : S \to \mathbb{R}$ (and $x_0 \in U \subseteq \operatorname{Int}(S)$) be as in the statement of the corollary. In addition, set $\phi := \psi \circ \zeta^{-1}$ and $y_0 := \zeta(x_0)$. Then $\mathcal{L}\psi(x) = \widetilde{\mathcal{L}}\phi(y)$ and $\psi(x_0) = \phi(y_0)$. By applying Theorem 2.15 (with the following obvious modifications: replace Ω by $\zeta(U)$; replace N by $n - 1$; replace $\mathcal{L}$ by $\widetilde{\mathcal{L}}$), we get that ψ has to be constant on U. If $U = S$, this achieves the proof. Otherwise, let $(U', \zeta') \in \mathcal{A}$ be

1 Since the Lie bracket of tangent vector fields is tangent, it follows that $\langle [Z_i, Z_j], \nu\rangle = 0$. This in turn implies that $\langle [Z_i, Z_j], \nu_H\rangle = -\varpi\langle [Z_i, Z_j], T\rangle$ for every $i, j \in \mathfrak{I}_{HS}$.

such that $U \cap U' \neq \emptyset$ and fix $x_0' \in U \cap U'$. Since x_0' must be a positive maximum of ψ, we can use the previous arguments (with x_0 replaced by x_0') in the new chart (U', ζ') and as S is connected, the thesis follows.

□

In order to use less restrictive regularity assumptions, we shall apply to our framework the results of a paper by Bonfiglioli and Uguzzoni; see [7].

Let $\Omega \subset \mathbb{R}^N$ be open and $Z_1, \ldots, Z_r \in Lip_{loc}(\Omega, \mathbb{R}^N)$. Below we write $\phi \in \Gamma^2(\Omega)$ if $\phi : \Omega \to \mathbb{R}$ is a continuous function with continuous Lie-derivatives along $Z_1, \ldots, Z_r$ up to 2nd order. Let us state a simplified version of their result, for the sub-Laplacian $\mathcal{L}$ defined by (2.7).

Theorem 2.19 (see Theorem 1.2 in [7]). *Let $\Omega \subset \mathbb{R}^N$ be open and $Z_1, \ldots, Z_r \in \mathbf{C}^1(\Omega, \mathbb{R}^N)$. Then:*

- *If Ω is bounded and there exists $\varphi \in \Gamma^2(\Omega)$ such that $\mathcal{L}\phi < 0$ and $\varphi > 0$ in Ω, then $\mathcal{L}$ satisfies the Γ^2-Weak Maximum Principle (abbreviated as Γ^2-WMP) on Ω, that is, for every $\phi \in \Gamma^2(\Omega)$ satisfying $\mathcal{L}\phi \geq 0$ in Ω and $\limsup_{x \to x_0} \phi(x) \leq 0$ for any $x_0 \in \partial\Omega$, there holds $\phi \leq 0$ in Ω.*
- *If $\mathcal{L}$ locally satisfies the Γ^2-WMP on Ω, then, for every $\phi \in \Gamma^2(\Omega)$ satisfying $\mathcal{L}\phi \geq 0$ and $\phi \leq 0$ in Ω, the set $F = \{x \in \Omega : \phi(x) = 0\}$ contains (the closure of) the set of points connected to any $x \in F$ by trajectories of $Z_1, \ldots, Z_r$, backward and forward in time.*

Remark 2.20 (Hörmander condition and Chow's Theorem). *Given a family of $\mathbf{C}^\infty$ vector fields on $\mathbb{R}^N$ satisfying the Hörmander condition, Chow's Theorem asserts that any two points of $\mathbb{R}^N$ can be joined by an absolutely continuous curve tangent a.e. to the distribution generated by these vector fields. This result has had many recent generalizations in which the regularity of the vector fields is weakened; see [9], [31], [36] and bibliographic references therein. Among them we would like to mention the paper by Rampazzo and Sussmann [44]. Their result is a nonsmooth version of Chow's Theorem valid for step 2 tangent distributions in $\mathbb{R}^N$ associated with Lipschitz vector fields satisfying (an appropriate version of) the Hörmander condition; see Theorem 2.1 in [44]. Furthermore, an extended version of this result to step k distributions has been recently proved by Feleqi and Rampazzo; see Theorem 4.4 in [22].*

Theorem 2.21 (Consequence of Theorem 1.2 in [7]). *Let $\mathbb{G}$ be a step k Carnot group. Let $S \subset \mathbb{G}$ be a $\mathbf{C}^k$ compact, connected, non-characteristic hypersurface (with or without boundary) and assume that HS satisfies the Hörmander condition* (2.8)*. Then, the HS-Laplacian satisfies the strong maximum principle on S. More precisely, let $\psi \in \mathbf{C}^2(S)$ be such that $\Delta_{HS}\psi \geq 0$. If ψ has a positive maximum at an interior point $x_0 \in \mathrm{Int}(S)$, then ψ has to be constant in S, that is $\psi(x) = \psi(x_0)$ for all $x \in S$.*

Proof. Observe preliminarily that Theorem 2.19 can be applied to our situation by arguing exactly as in the proof of Corollary 2.18. Thus, using Lemma 2.14 yields the existence of a strictly positive function $\varphi \in \mathbf{C}^2(S)$ such that $\Delta_{HS}\varphi < 0$. As a consequence, we can use (the first part of) Theorem 2.19, which ensures the validity of the Γ^2-WMP. This, in turn, makes applicable the second part of the same theorem. Precisely, let $\psi \in \mathbf{C}^2(S)$ be such that $\Delta_{HS}\psi \geq 0$ and $\psi \leq c$ in S, for some $c \in \mathbb{R}$. Under our assumptions, Chow's connectivity property for S follows by applying either Theorem 2.1 in [44], in the step 2 case, or alternatively, Theorem 4.4 in [22], in the case $k > 2$. In other words, any two points in S can be joined by an absolutely continuous curve tangent a.e. to the fibers of the subbundle HS, which is $\mathbb{R}$-linearly generated by the vector fields $\{Z_i : i \in \mathfrak{I}_{HS}\}$. This fact jointly with (the second part of) Theorem 2.19 implies that the closure of $F = \{x \in S : \psi(x) = c\}$ coincides with S. Thus, if ψ reaches its maximum at an interior point of S, it must be everywhere constant.

□

3 Main results in the step 2 case.

Theorem 3.1 (Convex hull property). *Let $\mathbb{G}$ be a step 2 Carnot group. Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$ compact, connected, non-characteristic H-minimal hypersurface with boundary and assume that HS satisfies the Hörmander con-*

dition (2.8). *Then S is contained in the convex hull* $\mathbf{c.h.}(\partial S)$ *of its boundary* ∂S*. Furthermore, if S touches the set* $\mathbf{c.h.}(\partial S)$ *at some interior point, then S is part of a hyperplane; in particular, there is no compact H-minimal hypersurface S without boundary.*

Proof. Let $\overline{\alpha} = (\alpha_1, ..., \alpha_n) \in \mathbb{R}^n$ be a constant vector and set $f_{\overline{\alpha}}(x) := \sum_{I=1}^{n} \alpha_I x_I$. It follows from Proposition 2.6 that the linear function $f_{\overline{\alpha}}$ is Δ_{HS}-harmonic on S. Thus, we can apply the strong maximum principle (see Theorem 2.21) to $f_{\overline{\alpha}}$. Thus, if for some number $K \in \mathbb{R}$, the inequality $f_{\overline{\alpha}}(x) \leq K$ holds true for all $x \in \partial S$, it is also satisfied for all $x \in S$. Since any closed convex set is the intersection of its supporting half-spaces, the first assertion easily follows. Suppose now that $f_{\overline{\alpha}}(x_0) = K$ holds for some $x_0 \in \mathrm{Int}(S)$ in addition to the inequality $f_{\overline{\alpha}}(x) \leq K$ for all $x \in \partial S$. Applying again the strong maximum principle we get that $f_{\overline{\alpha}}(x) = K$ for any $x \in \overline{S} = S$, as wished. □

Theorem 3.2 (Inclusion Property for the Hyperboloid $\mathcal{H}yp(\epsilon, \overline{\beta})$). *Let $\mathbb{G}$ be a step 2 Carnot group. Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$ compact, connected, non-characteristic H-minimal hypersurface with boundary and assume that HS satisfies the Hörmander condition* (2.8)*. Then, the following inclusion property holds: if $\partial S \subset \mathcal{H}yp(\epsilon, \overline{\beta})$, then $S \subset \mathcal{H}yp(\epsilon, \overline{\beta})$.*

Here $\mathcal{H}yp(\epsilon, \overline{\beta})$ is a solid hyperboloid congruent to $\mathcal{H}yp(x_0, \epsilon, X_h, \overline{\beta})$; see (2.3) in Definition 2.9.

Proof. Starting from the invariance of the HS-Laplacian under isometries (see Proposition A.4) we can assume, without loss of generality, that $\mathcal{H}yp(\epsilon, \overline{\beta}) = \mathcal{H}yp(0, \epsilon, X_h, \overline{\beta})$. By Corollary 2.10, the function $g^{(h,\overline{\beta})}(x)$ is Δ_{HS}-subharmonic on S and, by the hypothesis that $\partial S \subset \mathcal{H}yp(0, \epsilon, X_h, \overline{\beta})$, we see that $g^{(h,\overline{\beta})}(x) \leq \epsilon^2$ for every $x \in \partial S$. Therefore, by applying the strong maximum principle (see Theorem 2.21), we get that $g^{(h,\overline{\beta})}(x) \leq \epsilon^2$ for every $x \in S$, which is equivalent to the inclusion property, that is, $S \subset \mathcal{H}yp(0, \epsilon, X_h, \overline{\beta})$. This achieves the proof. □

The following theorem is one of the main results of this paper.

Theorem 3.3 (Non-Existence result for the Cone $\mathcal{C}(\overline{\beta})$). *Let $\mathbb{G}$ be a step 2 Carnot group. Let $\mathcal{C}(\overline{\beta})$ be a solid cone with vertex $x_0 \in \mathbb{G}$ which is congruent to the cone $\mathcal{C}(0, X_h, \overline{\beta})$; see* (2.4) *in Definition 2.9. Let $\mathcal{C}(\overline{\beta})^{\pm}$ be the two disjoint parts of $\mathcal{C}(\overline{\beta})$ corresponding to $\mathcal{C}^{\pm}(0, X_h, \overline{\beta})$. Then, there exists no $\mathbf{C}^2$ compact, connected, non-characteristic H-minimal hypersurface $S \subset \mathbb{G}$ satisfying the Hörmander condition* (2.8) *and with $\partial S \subset \mathcal{C}(\overline{\beta})$ such that $\partial S \cap \mathcal{C}(\overline{\beta})^{+} \neq \emptyset$ and $\partial S \cap \mathcal{C}(\overline{\beta})^{-} \neq \emptyset$.*

Proof. We argue by contradiction. Suppose that such an S exists. Using the invariance of Δ_{HS} under isometries, we can assume that $\mathcal{C}(\overline{\beta}) = \mathcal{C}(0, X_h, \overline{\beta})$. Moreover, by Corollary 2.10 the function $g^{(h,\overline{\beta})}(x)$ is Δ_{HS}-subharmonic on S and, since we are assuming that $\partial S \subset \mathcal{C}(0, X_h, \overline{\beta})$, the inequality $g^{(h,\overline{\beta})}(x) \leq 0$ must hold for every $x \in \partial S$. Hence, by the strong maximum principle (see Theorem 2.21) we get that $g^{(h,\overline{\beta})}(x) \leq 0$ for every $x \in S$, which is equivalent to the fact that $S \subset \mathcal{C}(0, X_h, \overline{\beta})$. By hypothesis, S is connected and $\partial S \cap \mathcal{C}(\overline{\beta})^{\pm} \neq \emptyset$. This implies that S must contain the vertex 0 of the cone $\mathcal{C}(0, X_h, \overline{\beta})$, that is a contradiction to the fact that S is (everywhere) a $\mathbf{C}^2$ hypersurface. This concludes the proof. □

The next result, which is in the spirit of Corollary 3 in [20], explains how to apply the preceding "non-existence theorem" to get quantitative estimates for H-minimal hypersurfaces; see also Chapter 6 in [21].

Corollary 3.4 (Consequence of the Non-Existence result for the Cone $\mathcal{C}(\overline{\beta})$). *Let $\mathbb{G}$ be a step 2 Carnot group. Let $W \in H$, $|W| = 1$, and let $\gamma_W := \exp(\mathbb{R}W)$ be the line through $0 \in \mathbb{G}$ with direction W. Let $t_i \in \mathbb{R}_+$ $(i = 1, 2)$ and set $x_1 := \exp(t_1 W)$, $x_2 := \exp(-t_2 W)$. Let $B_{\mathrm{Eu}}(x_i, \delta_i)$ denote the Euclidean ball centered at $x_i \in \mathbb{G}$ and with radius $0 < \delta_i \leq t_i$ $(i = 1, 2)$. In addition, let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$ compact, connected, non-characteristic H-minimal hypersurface satisfying the Hörmander condition* (2.8)*. Assume that $\partial S \subset B_{\mathrm{Eu}}(x_1, \delta_1) \cup B_{\mathrm{Eu}}(x_2, \delta_2)$ and $\partial S \cap B_{\mathrm{Eu}}(x_i, \delta_i) \neq \emptyset$ for every $i = 1, 2$. Finally, let $R := d_{\mathrm{Eu}}(x_1, x_2)$ be the Euclidean distance between the* *centers of the two balls. Then*

$$R \leq \sqrt{\frac{h-1}{h-2}}(\delta_1+\delta_2).$$

Proof. By contradiction; assume that $R^2 > \frac{h-1}{h-2}(\delta_1+\delta_2)^2$. With no loss of generality, let us suppose that $W = X_h$ and that

$$x_1 = exp\left(0,\dots,0,\underbrace{\frac{R\delta_1}{\delta_1+\delta_2}}_{h-th\ place},0,\dots,0\right), \qquad x_2 = exp\left(0,\dots,0,-\underbrace{\frac{R\delta_2}{\delta_1+\delta_2}}_{h-th\ place},0,\dots,0\right).$$

The fact that $x \in \mathbb{G}$ belongs either to $B_{\mathrm{Eu}}(x_1,\delta_1)$, or to $B_{\mathrm{Eu}}(x_2,\delta_2)$, is expressed by one of the following inequalities:

$$\sum_{i=1}^{h-1} x_i^2 + \left(x_h - \frac{R\delta_1}{\delta_1+\delta_2}\right)^2 + \sum_{\alpha\in\mathfrak{I}_V} x_\alpha^2 < \delta_1^2, \qquad \sum_{i=1}^{h-1} x_i^2 + \left(x_h + \frac{R\delta_2}{\delta_1+\delta_2}\right)^2 + \sum_{\alpha\in\mathfrak{I}_V} x_\alpha^2 < \delta_2^2.$$

By subtracting to both sides of these inequalities the quantity $(h-2)x_h^2$, we get that the function $g^{(h,\overline{1})}(x) = g_h(x_H) + g_{\overline{1}}(x_V)$ (recall that $g_{\overline{1}}(x_V) = \sum_{\alpha\in\mathfrak{I}_V} x_\alpha^2$; see Notation 2.11) satisfies either of the inequalities below:

$$g^{(h,\overline{1})}(x) < \delta_1^2 - (h-2)x_h^2 - \left(x_h - \frac{R\delta_1}{\delta_1+\delta_2}\right)^2, \qquad g^{(h,\overline{1})}(x) < \delta_2^2 - (h-2)x_h^2 - \left(x_h + \frac{R\delta_2}{\delta_1+\delta_2}\right)^2.$$

Denote by RHS the right hand side of the first inequality above. This is a polynomial of degree 2 in the indeterminate x_h. Precisely, we have

$$\mathrm{RHS} = -(h-1)x_h^2 + 2\frac{R\delta_1}{\delta_1+\delta_2}x_h + \delta_1^2\left(1 - \frac{R^2}{(\delta_1+\delta_2)^2}\right).$$

It is easy to check that the discriminant of this polynomial is negative. Therefore, one has $g^{(h,\overline{1})}(x) \leq 0$, and the same happens in the other case. If $x \in B_{\mathrm{Eu}}(x_1,\delta_1)$, then $x \in \mathcal{C}^+(0, X_h, \overline{1})$. Analogously, if $x \in B_{\mathrm{Eu}}(x_2,\delta_2)$, then $x \in \mathcal{C}^-(0, X_h, \overline{1})$. But the fact that the balls $B_{\mathrm{Eu}}(x_1,\delta_1)$ and $B_{\mathrm{Eu}}(x_2,\delta_2)$ are contained, respectively, in the upper and lower cones $\mathcal{C}^+(0, X_h, \overline{1})$ and $\mathcal{C}^-(0, X_h, \overline{1})$, contradicts Theorem 3.3. This achieves the proof. □

Theorem 3.5 (Inclusion Property for Cylinders and Paraboloids with vertical axis)**.** *Let $\mathbb{G}$ be a step 2 Carnot group. Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$ compact, connected, non-characteristic H-minimal hypersurface with boundary and assume that HS satisfies the Hörmander condition* (2.8)*. Then, the following inclusion properties hold:*

- *if $\partial S \subset \mathcal{C}yl(\epsilon, \overline{a}, \overline{\beta}')$, then $S \subset \mathcal{C}yl(\epsilon, \overline{a}, \overline{\beta}')$;*
- *if $\partial S \subset \mathcal{P}ar(\overline{a}, \overline{\beta}')$, then $S \subset \mathcal{P}ar(\overline{a}, \overline{\beta}')$.*

Recall that $\mathcal{C}yl(\epsilon, \overline{a}, \overline{\beta}')$ is a solid cylinder congruent to $\mathcal{C}yl(0, \epsilon, X_\alpha, \overline{a}, \overline{\beta}')$ and $\mathcal{P}ar(\overline{a}, \overline{\beta}')$ is a solid paraboloid congruent to $\mathcal{P}ar(0, X_\alpha, \overline{a}, \overline{\beta}')$; for more details, see Definition 2.12.

Proof. By invariance of Δ_{HS} under isometries (see Proposition A.4) we can clearly assume that $\mathcal{C}yl(\epsilon, \overline{a}, \overline{\beta}') = \mathcal{C}yl(0, \epsilon, X_\alpha, \overline{a}, \overline{\beta}')$ and that $\mathcal{P}ar(\overline{a}, \overline{\beta}') = \mathcal{P}ar(0, X_\alpha, \overline{a}, \overline{\beta}')$. By Corollary 2.13, the functions $g^{(\overline{a},\overline{\beta}')}(x)$ and $g_\alpha^{(\overline{a},\overline{\beta}')}(x)$ are both Δ_{HS}-subharmonic functions on S. Hence, using the fact that either $\partial S \subset \mathcal{C}yl(0, \epsilon, X_\alpha, \overline{a}, \overline{\beta}')$, or $\partial S \subset \mathcal{P}ar(0, X_\alpha, \overline{a}, \overline{\beta}')$, we get either $g^{(\overline{a},\overline{\beta}')}(x) \leq \epsilon^2$, or $g_\alpha^{(\overline{a},\overline{\beta}')}(x) \leq 0$, for every $x \in \partial S$. The proof follows by applying Theorem 2.21. □

3.1 The case of Heisenberg groups $\mathbb{H}^n$.

We already know from Remark 2.17 that our method does not apply to the 1st Heisenberg group $\mathbb{H}^1$. However, we have the following:

Remark 3.6 (Convex hull property for H-minimal surfaces in $\mathbb{H}^1$)**.** *Let $S \subset \mathbb{H}^1$ be a $\mathbf{C}^2$ non-characteristic H-minimal surface. It is well-known that S is a ruled surface; more precisely, S turns out to be ruled by horizontal lines; see [43], [12], [46]. As a consequence, the classical Gaussian curvature of S, seen as a surface in $\mathbb{R}^3$, is everywhere nonpositive and using the main theorem in Osserman's paper [41] one gets that S has the convex-hull property. We also observe that the same holds for complete* "area-stationary surfaces" *of class $\mathbf{C}^2$; see [46]. More precisely, it follows from Theorem 6.15 in [46] that these are ruled surfaces, so that Osserman's result still applies, as claimed.*

Finally, concerning the 1st Heisenberg group $\mathbb{H}^1$, we would like to mention an interesting, and somehow related, "half-space" theorem by Cheng and Hwang; see Theorem D in [13].

For the Heisenberg groups $\mathbb{H}^n$ $(n > 1)$, we are going to prove an inclusion property for a truncated hyperboloid (with axis the T-axis) and a related non-existence result for a suitable truncated cone. Although possible, we do not generalize these last results to general step 2 Carnot groups. Before to start, let us collect some further remarks:

Remark 3.7 (Validity of Theorems 3.1, 3.2, 3.3 and 3.5 in the Heisenberg group $\mathbb{H}^n$, with $n > 1$)**.** *It is worth observing that, by Remark 2.17, all of our step 2 results apply to the Heisenberg groups $\mathbb{H}^n$ $(n > 1)$. In particular, the following facts hold true:*

- *Convex Hull Property; see Theorem 3.1.*
- *Inclusion Property and the related Non-Existence result for (the class of congruence of) a suitable hyperboloid with horizontal axis; see Theorems 3.2 and 3.3 .*
- *Inclusion Property for (the class of congruence of) suitable cylinders and paraboloids with axis the T-axis; see Theorem 3.5.*

For the notation used in this section, see Example 1.7. Recall that $p = exp\,(z_H, t)$, where $z_H = (x_1, y_1, ..., x_n, y_n) \in \mathbb{R}^{2n}$. We also set $r_H := \sqrt{\sum_{i=1}^n (x_i^2 + y_i^2)}$. Now let us consider the function $g_\beta : \mathbb{H}^n \to \mathbb{R}$ given by $g_\beta(z_H, t) := r_H^2 - \beta t^2$, where $\beta \in \mathbb{R}_+$. The set of points satisfying the inequality $g_\beta(z_H, t) \leq \epsilon^2$ is a solid hyperboloid, hereafter denoted as $\mathcal{H}yp(0, \epsilon, T, \beta)$, with axis the T-axis and with (tangent of the angle of) slope given by $\beta > 0$. Precisely, we set

$$\mathcal{H}yp(0, \epsilon, T, \beta) := \{p = exp\,(z_H, t) \in \mathbb{H}^n : g_\beta(z_H, t) \leq \epsilon^2\}.$$

If $\epsilon = 0$, then this region becomes a solid cone, hereafter denoted as $\mathcal{C}(0, T, \beta)$, with the same axis as $\mathcal{H}yp(0, \epsilon, T, \beta)$, and with slope β; also, the upper and lower parts of this cone (with respect to the T-axis) are denoted as $\mathcal{C}^\pm(0, T, \beta)$. In other words, we set

$$\mathcal{C}(0, T, \beta) := \mathcal{H}yp(0, 0, T, \beta)$$

and $\mathcal{C}^\pm(0, T, \beta) := \mathcal{C}(0, T, \beta) \cap \{p = exp\,(z_H, t) \in \mathbb{H}^n : \pm t \geq 0\}$.

Theorem 3.8. *Set $r_H^\star := \sqrt{2(2n-1)/\beta}$. Let $S \subset \mathbb{H}^n$ be a $\mathbf{C}^2$ compact non-characteristic H-minimal hypersurface with boundary and assume that S is contained in the solid cylinder $\mathcal{C}yl(0, r_H^\star, T)$, with axis the T-axis passing through $0 \in \mathbb{H}^n$, defined as*

$$\mathcal{C}yl(0, r_H^\star, T) := \left\{p = exp\,(z_H, t) \in \mathbb{H}^n : r_H \leq r_H^\star\right\}.$$

Then, the function $g_\beta(z_H, t)$ is Δ_{HS}-subharmonic on S.

Proof. By using Proposition 2.7 we get that $\Delta_{HS}\, r_H^2 = 2(2n-1)$ and $\Delta_{HS}\, t^2 = \frac{1}{2}\left(r_H^2 - \left\langle C_H^{2n+1} z_H, \nu_H\right\rangle^2\right)$. From these calculations, we immediately get that

$$\Delta_{HS}\, g_\beta(z_H, t) = (2n-1) - \frac{\beta\, r_H^2}{2}\left(1 - \left\langle \frac{C_H^{2n+1} z_H}{r_H}, \nu_H\right\rangle^2\right).$$

Thus, if $r_H \leq \sqrt{\frac{2(2n-1)}{\beta}} = r_H^\star$, then it follows that $\Delta_{HS}\, g_\beta(z_H, t) \geq 0$, as wished. □

Following the arguments in the proof of Theorem 3.2 with the help of Theorem 3.8 we get:

Corollary 3.9 (Inclusion Property for the Truncated Hyperboloid $\mathcal{H}yp(0, \epsilon, T, \beta) \cap \mathcal{C}yl(0, r_H^\star, T)$). *Let $S \subset \mathbb{H}^n$ $(n > 1)$ be a $\mathbf{C}^2$ compact, connected, non-characteristic H-minimal hypersurface with boundary. Furthermore, let us denote by $\mathcal{H}yp_{trunc}$ any truncated hyperboloid which is congruent to $\mathcal{H}yp(0, \epsilon, T, \beta) \cap \mathcal{C}yl(0, r_H^\star, T)$. Then, the following inclusion property holds: if $\partial S \subset \mathcal{H}yp_{trunc}$, then $S \subset \mathcal{H}yp_{trunc}$.*

Finally, arguing as in the proof of Theorem 3.3 and using Theorem 3.8, we get the following:

Corollary 3.10 (Non-Existence result for the Truncated Cone $\mathcal{C}(0, T, \beta) \cap \mathcal{C}yl(0, r_H^\star, T)$). *Denote by $\mathcal{C}_{trunc}$ any truncated cone which is congruent to $\mathcal{C}(0, T, \beta) \cap \mathcal{C}yl(0, r_H^\star, T)$. Moreover, let $\mathcal{C}^\pm_{trunc}$ be the two disjoint parts of $\mathcal{C}_{trunc}$ corresponding to $\mathcal{C}^\pm(0, T, \beta) \cap \mathcal{C}yl(0, r_H^\star, T)$. Then, there exists no $\mathbf{C}^2$ compact, connected, non-characteristic H-minimal hypersurface $S \subset \mathbb{H}^n$, with $\partial S \subset \mathcal{C}_{trunc}$, such that $\partial S \cap \mathcal{C}^+_{trunc} \neq \emptyset$ and $\partial S \cap \mathcal{C}^-_{trunc} \neq \emptyset$.*

4 Remarks about the case of step k Carnot groups.

Let $\mathbb{G}$ be a step k Carnot group $(k \geq 3)$. In this case, the elements of the horizontal left-invariant frame $\{X_1, \ldots, X_h\}$ have the following general polynomial expression

$$X_j(x) = \mathrm{e}_j + \sum_{i=2}^{k} \sum_{\alpha_i=1}^{h_i} a_{j,\alpha_i}(x_H, x_{H_2}, \ldots, x_{H_{i-1}})\, \mathrm{e}_{\alpha_i} \qquad \forall x \in \mathbb{G} \quad \forall i \in \mathfrak{I}_H, \tag{4.1}$$

where $a_{j,\alpha_i}(x_H, x_{H_2}, \ldots, x_{H_{i-1}})$ is a homogeneous polynomial function of degree $i-1$ (with respect to Carnot dilations); see, for instance, [6], page 59, formula (1.8.1). However, apart from the case of step 2 Carnot groups, the functions $a_{j,\alpha_i}(x_H, x_{H_2}, \ldots, x_{H_{i-1}})$ have a complicated expression, which depends on the structure constants of the Lie algebra. For instance, for step 3 Carnot groups, we remark that the monomial functions $\{x_{\alpha_3} : \alpha_3 \in \mathfrak{I}_{H_3}\}$, where $\mathfrak{I}_{H_3} = \{n_2 + 1, \ldots, n\}$, are not in general Δ_{HS}-harmonic (and not even Δ_{HS}-subharmonic).

Example 4.1. *To give an example, consider the step 3 Carnot group $\mathbb{G}$ on $\mathbb{R}^6$ with 3 generating horizontal vector fields X_1, X_2, X_3 given by*

$$\begin{aligned} X_1 &= \mathrm{e}_1 - \frac{1}{2}x_2\mathrm{e}_4 + \left(-\frac{1}{2}x_5 + \frac{1}{12}x_2x_3\right)\mathrm{e}_6 \\ X_2 &= \mathrm{e}_2 + \frac{1}{2}x_1\mathrm{e}_4 - \frac{1}{2}x_3\mathrm{e}_5 - \frac{1}{6}x_1x_3\mathrm{e}_6 \\ X_3 &= \mathrm{e}_3 + \frac{1}{2}x_2\mathrm{e}_5 + \left(\frac{1}{2}x_4 + \frac{1}{12}x_1x_2\right)\mathrm{e}_6; \end{aligned}$$

see [6], page 226. These vector fields satisfy the following algebraic rules

$$[X_1, X_2] = \mathrm{e}_4 - \frac{1}{2}x_3\mathrm{e}_6, \quad [X_2, X_3] = \mathrm{e}_5 + \frac{1}{2}x_1\mathrm{e}_6, \qquad [X_1, [X_2, X_3]] = \mathrm{e}_6 = -[X_3, [X_1, X_2]],$$

with all other commutators zero. It is elementary to check that $\Delta_H x_6 = 0$ and that

$$\mathrm{Hess}_H(x_6) = \begin{vmatrix} 0 & \frac{x_3}{3} & -\frac{x_2}{6} \\ -\frac{x_3}{6} & 0 & -\frac{x_1}{6} \\ -\frac{x_2}{6} & \frac{x_1}{3} & 0 \end{vmatrix}.$$

Thus, using (2.1) yields $\Delta_{HS}(x_6) = \frac{1}{6}\left\{x_3 \nu_H^1 \nu_H^2 - 2x_2 \nu_H^1 \nu_H^3 + x_1 \nu_H^2 \nu_H^3\right\}$, which is not a positive function.

Nevertheless, we have to stress that for any step k Carnot group the *monomials* $\{x_j : j \in \mathfrak{I}_H\}$ *and* $\{x_{\alpha_2} : \alpha_2 \in \mathfrak{I}_{H_2}\}$ *are* Δ_{HS}*-harmonic*. The last claim follows from (4.1). In fact, it turns out that $a_{j,\alpha_2}(x_H) = \frac{1}{2}\left\langle C_H^{\alpha_2}\mathrm{e}_j, x_H\right\rangle_{\mathbb{R}^h}$ for every $j \in \mathfrak{I}_H$. Hence, if one considers (smooth) functions of the variables (x_H, x_{H_2}) such as $\varphi(x_H, x_{H_2})$, the generating vector fields will act exactly as in the case of step 2 Carnot groups.

Definition 4.2 (Partial Convex Hull). *Let $\mathbb{G}$ be a step k Carnot group and let $D \subset \mathbb{G}$. Moreover, we denote by $\overline{\alpha}^{(1,2)} = (\alpha_1, ..., \alpha_{n_2}) \in \mathbb{R}^{n_2}$ any constant vector and we set $f_{\overline{\alpha}^{(1,2)}}(x) := \sum_{i=1}^{n_2} \alpha_i x_i$. By definition, the* partial convex hull *of D, denoted as* $\mathbf{p.c.h.}(D)$*, is the intersection of all half-spaces $\mathfrak{I}_{K,\overline{\alpha}^{(1,2)}} := \{x \in \mathbb{G} : f_{\overline{\alpha}^{(1,2)}}(x) \leq K\}$ containing D, that is*

$$\mathbf{p.c.h.}(D) := \bigcap_{D\subseteq \mathfrak{I}_{K,\overline{\alpha}^{(1,2)}}} \mathfrak{I}_{K,\overline{\alpha}^{(1,2)}}.$$

Following the arguments in the proof of Theorem 3.1 with the above linear functions we get:

Theorem 4.3 (Partial convex hull property in Carnot groups of step k). *Let $\mathbb{G}$ be a step k Carnot group. Let $S \subset \mathbb{G}$ be a $\mathbf{C}^k$ compact, connected, non-characteristic H-minimal hypersurface with boundary and assume that HS satisfies the Hörmander condition* (2.8)*. Then S is contained in the* partial convex hull $\mathbf{p.c.h.}(\partial S)$ *of its boundary ∂S. Furthermore, if S touches the set* $\mathbf{p.c.h.}(\partial S)$ *at some interior point, then S is part of a hyperplane.*

As a consequence of the previous theorem we can state the following weaker property:

Remark 4.4 (Horizontal convex hull property). *Let $\mathbb{G}$ be a step k Carnot group. For any $X \in \mathfrak{g}$ let us denote by $\mathfrak{I}_K(X) := \{x \in \mathbb{G} : \langle x_H, X\rangle_H \leq K\}$ the "vertical half-space"*[2] *orthogonal to X. We define the* horizontal convex hull **h.c.h.** *of a bounded set $D \subset \mathbb{G}$ as the intersection of all vertical half-spaces $\mathfrak{I}_K(X)$ containing D. Furthermore, let $S \subset \mathbb{G}$ be a $\mathbf{C}^k$ compact, connected, non-characteristic H-minimal hypersurface with boundary and assume that HS satisfies the Hörmander condition* (2.8)*. Then, S is contained in the horizontal convex hull of ∂S.*

A Appendix: Δ_{HS} commutes with isometries

What are "congruences" in Carnot groups? To answer this question, below we will briefly recall some results concerning isometries. Then, we will show that the HS-Laplacian Δ_{HS} commutes with isometries.

Let $\Omega \subset \mathbb{G}$ be an open set and let $f : \Omega \to \mathbb{G}$ be a map of class $\mathbf{C}^1$. By definition, f is an *isometry* if its "Pansu differential" $df(x)$ (see [34]) is an isometry for every $x \in \Omega$. Moreover, one can show that f is distance-preserving if, and only if, $df(x)$ is an isometry for all $x \in \Omega$; see Lemma 2.10 in [18]. Hence, we can always identify distance-preserving maps with isometries.

Recall that an isometry of $\mathbb{G}$ (equipped with a left-invariant distance) is called *affine* if it is the composition of a left translation with a graded automorphism; see [33].

For step k Carnot groups it is known that isometries are affine transformations; see [27]. Recently, this result has been generalized for the case of sub-Finsler distances and isometries defined only on open subsets of the group:

Theorem A.1 (see [33]). *Let $(\mathbb{G}, d_{CC})$ be a step k sub-Riemannian Carnot group. Let $\Omega_1, \Omega_2 \subseteq \mathbb{G}$ be two connected open sets. Let $f : \Omega_1 \to \Omega_2$ be an isometry. If $f(0) = 0$, then f is the restriction to Ω_1 of a graded automorphism of $\mathbb{G}$.*

2 Notice that its boundary is the left-coset of a maximal subgroup of $\mathbb{G}$.

Definition A.2. *We say that two subsets of* $\mathbb{G}$ *are* congruent *if there is an isometry of* $\mathbb{G}$ *carrying one to the other. In particular, if* S_1 *and* S_2 *are two given hypersurfaces of* $\mathbb{G}$*, then* S_1 *and* S_2 *are congruent if, and only if, there exists an isometry* Φ *of* $\mathbb{G}$ *such that* $\Phi|_{S_1}$ *is an isometry from* S_1 *to* S_2.

The last definition allows us to speak of the "congruence class" of a given hypersurface.

Remark A.3 (Horizontal divergence operators)**.** *We make the following remarks:*

(i) *Let* $d_H : \mathcal{A}^r_H(\mathbb{G}) \to \mathcal{A}^{r+1}_H(\mathbb{G})$ *be the horizontal exterior derivative, defined as restriction to* H *of the exterior derivative* $d : \mathcal{A}^r(\mathbb{G}) \to \mathcal{A}^{r+1}(\mathbb{G})$*. Then, the* H*-divergence* div_H *can equivalently be defined by the formula* $d_H(X \lrcorner \mathrm{vol}_H) = div_H X\, \mathrm{vol}_H$*, where* $\lrcorner$ *denotes the "contraction" (or interior product) on differential forms; see, for instance, [32] or [25]. The proof of this fact, which is elementary, can be given as in the Riemannian case; see [47], Lemma 56 in Addendum 1 of Chapter 7.*

(ii) *Let* $d_{HS} : \mathcal{A}^r_{HS}(S) \to \mathcal{A}^{r+1}_{HS}(S)$ *be the horizontal tangent exterior derivative, that is the restriction to* HS *of the tangential exterior derivative* $d_{TS} : \mathcal{A}^r(S) \to \mathcal{A}^{r+1}(S)$*. Note that the* HS*-divergence* div_{HS} *can be defined by the formula* $d_{HS}(X \lrcorner \mathrm{vol}_{HS}) = div_{HS} X\, \mathrm{vol}_{HS}$*; see Definition 1.10. This formula can be proved again as in [47].*

Now, let us analyze the behavior of the HS-Laplacian under isometries.

Proposition A.4 (Δ_{HS} commutes with isometries)**.** *Let* $\Psi : \mathbb{G} \to \mathbb{G}$ *be an isometry,* $S \subset \mathbb{G}$ *a non-characteristic hypersurface of class* $\mathbf{C}^r$ $(r \geq 2)$*, and set* $\widetilde{S} := \Psi(S)$*. Then, one has* $\Delta_{HS}(f \circ \Psi) = \left(\Delta_{H\widetilde{S}} f\right) \circ \Psi$ *for every* $f \in \mathbf{C}^2(\widetilde{S})$*, or equivalently,* $\Delta_{HS}\Psi^\star = \Psi^\star \Delta_{H\widetilde{S}}$.

Proof. The proof is an adaptation of the classical one valid for the Laplace-Beltrami operator; see [5] or [11]. We have here to remark that since Ψ is an isometry (and hence a graded automorphism), the differential $\Psi_\star$ restricted to the horizontal tangent space H_xS at $x \in S$ turns out to be an isometry between H_xS and $H_{\Psi(x)}\widetilde{S}$. Let us prove the following two claims.

Claim 1. *We have* $\Psi_\star grad_{HS} \Psi^\star = grad_{H\widetilde{S}}$.

Proof. Let $f \in \mathbf{C}^1(\widetilde{S})$ and take $Y \in \mathfrak{X}^1(H\widetilde{S})$. Then

$$\begin{aligned}\left\langle \Psi_\star grad_{HS} \Psi^\star(f), Y \right\rangle_{H\widetilde{S}} &= \left\langle \Psi_\star grad_{HS} \Psi^\star(f), \Psi_\star\Psi_\star^{-1} Y \right\rangle_{H\widetilde{S}} = \left\langle grad_{HS} \Psi^\star(f), \Psi_\star^{-1} Y \right\rangle_{HS}\\ &= d(\Psi^\star f)\left[\Psi_\star^{-1} Y\right] = f_\star \Psi_\star \left[\Psi_\star^{-1} Y\right] = df(Y) = \langle grad_{H\widetilde{S}} f, Y\rangle_{H\widetilde{S}}.\end{aligned}$$

□

Claim 2. *We have* $\Psi^\star\left(div_{H\widetilde{S}} \Psi_\star\right) = div_{HS}$.

Proof. By (ii) in Remark A.3, the claim to be proved turns out to be equivalent to

$$d_{HS}\,(X \lrcorner \mathrm{vol}_{HS}) = \Psi^\star div_{H\widetilde{S}}\,(\Psi_\star X) \cdot \mathrm{vol}_{HS}.$$

This formula, in turn, is equivalent to

$$\left(\Psi^{-1}\right)^\star (d_{HS}\,(X \lrcorner \mathrm{vol}_{HS})) = div_{H\widetilde{S}}\,(\Psi_\star X) \cdot \left(\left(\Psi^{-1}\right)^\star \mathrm{vol}_{HS}\right),$$

or also to

$$d_{H\widetilde{S}}\left(\left(\Psi^{-1}\right)^\star (X \lrcorner \mathrm{vol}_{HS})\right) = div_{H\widetilde{S}}\,(\Psi_\star X) \cdot \mathrm{vol}_{H\widetilde{S}},$$

where[3] we have used the fact that $\mathrm{vol}_{HS} = \left(\Psi^{-1}\right)^\star \mathrm{vol}_{H\widetilde{S}}$ (note that $\widetilde{\zeta}_i = \Psi_\star\zeta_i$ $(i = 1, \dots, h-1)$ and that $\{\widetilde{\zeta}_1, \dots, \widetilde{\zeta}_{h-1}\}$ is an orthonormal frame for $H\widetilde{S}$; hence, one has $\mathrm{vol}_{H\widetilde{S}} = \widetilde{\zeta}_1 \wedge \dots \wedge \widetilde{\zeta}_{h-1}$). But the last equality

3 Indeed, one has $\left(\Psi^{-1}\right)^\star d_{HS} = d_{H\widetilde{S}}\left(\Psi^{-1}\right)^\star$.

follows (from the standard definition of divergence operator) since

$$\left(\Psi^{-1}\right)^{\star}(X \lrcorner \mathrm{vol}_{HS}) = (\Psi_{\star}X) \lrcorner \left(\left(\Psi^{-1}\right)^{\star} \mathrm{vol}_{HS}\right) = (\Psi_{\star}X) \lrcorner \mathrm{vol}_{H\widetilde{S}}.$$

□

The proof can now be achieved as follows. By using Claims 1 and 2, we get that

$$\begin{aligned}\Delta_{HS}\Psi^{\star} &= div_{HS}\left(grad_{HS}\Psi^{\star}\right) = div_{HS}\left((\Psi_{\star})^{-1}(\Psi_{\star})grad_{HS}\Psi^{\star}\right)\\ &= div_{HS}\left((\Psi_{\star})^{-1}grad_{H\widetilde{S}}\right) = \Psi^{\star}\left(\Psi^{-1}\right)^{\star}div_{HS}\left((\Psi_{\star})^{-1}grad_{H\widetilde{S}}\right)\\ &= \Psi^{\star}div_{H\widetilde{S}}(grad_{H\widetilde{S}})\\ &= \Psi^{\star}\Delta_{H\widetilde{S}},\end{aligned}$$

where we have used the fact that $div_{H\widetilde{S}} = \left(\Psi^{-1}\right)^{\star} div_{HS}\,(\Psi_{\star})^{-1}$.

□

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References

[1] L. Ambrosio, F. Serra Cassano, D. Vittone, *Intrinsic regular hypersurfaces in Heisenberg groups*, J. Geom. Anal. 16, no. 2, 187–232 (2006).

[2] Z.M. Balogh, *Size of characteristic sets and functions with prescribed gradient,* J. Reine Angew. Math. 564 (2003).

[3] Z.M. Balogh, C. Pintea, H. Rohner, *Size of tangencies to non-involutive distributions,* Indiana Univ. Math. J. 60 , no. 6, 2061-2092 (2011).

[4] V. Barone Adesi, F. Serra Cassano, D.Vittone, *The Bernstein problem for intrinsic graphs in Heisenberg groups and calibrations,* Calc. Var. Partial Differential Equations 30, no. 1, 17-49 (2007).

[5] M. Berger, P. Gauduchon, E. Mazet, *“Le spectre d’une variété riemannienne”,* Lecture Notes in Mathematics, 194, Springer, Berlin-New York, 1971.

[6] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, *“Stratified Lie Groups and Potential Theory for their Sub-Laplacians”*, Springer Monographs in Mathematics, 26. New York, NY: Springer-Verlag, 800 pages (2007).

[7] A. Bonfiglioli, F. Uguzzoni, *Maximum principle and propagation for intrinsicly regular solutions of differential inequalities structured on vector fields* , J. Math. Anal. Appl., 322, 886–900 (2006).

[8] J.M. Bony, *Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés,* Ann. Inst. Fourier (Grenoble), 19, vol. 1, 277-304 (1969).

[9] M. Bramanti, L. Brandolini, M. Pedroni , *Basic properties of nonsmooth Hörmander’s vector fields and Poincaré’s inequality,* Forum Mathematicum, Vol. 25, Issue 4, 703–769 (2013).

[10] L. Capogna, G. Citti, M. Manfredini, *Smoothness of Lipschitz minimal intrinsic graphs in Heisenberg groups H^n, $n > 1$,* Indiana University Mathematics Journal (2008).

[11] Y. Canzani, *Analysis on manifolds via the Laplacian*, Lecture Notes available at: http://www.math.harvard.edu/ canzani/docs/Laplacian.pdf.

[12] L. Capogna, D. Danielli, S. D. Pauls, J. Tyson, *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, Progr. Math., 259. Birkhäuser, Basel, 2007.

[13] J.J Cheng, J.F. Hwang, *Properly embedded and immersed minimal surfaces in the Heisenberg group,* Bull. Austral. Math. Soc., 70 (3), 507–520 (2004).

[14] J.J Cheng, J.F. Hwang, A. Malchiodi, P. Yang, *Minimal surfaces in pseudohermitian geometry,* Ann. Sc. Norm. Sup. Pisa Cl. Sci., 5, IV, 129-179 (2005).

[15] ———, *A Codazzi-like equation and the singular set for C^1 smooth surfaces in the Heisenberg group*, J. Reine Angew. Math., 671:131–198 (2012).

[16] J.J Cheng, J.F. Hwang, P. Yang, *Regularity of C^1 smooth surfaces with prescribed p-mean curvature in the Heisenberg group,* Math. Ann. 344, No. 1, 1-35 (2009).

[17] T.H. Colding W.P. Minicozzi II, *“A course in minimal surfaces”*, Graduate Studies in Mathematics, vol. 121, American Mathematical Society, Providence, RI (2011).
[18] M.G. Cowling, A. Ottazzi, *Conformal maps of Carnot groups*, Ann. Acad. Sci. Fenn. Math. 40, 203-213 (2015).
[19] D. Danielli, N. Garofalo, D.M. Nhieu, *Sub-Riemannian Calculus on Hypersurfaces in Carnot groups*, Adv. Math. 215, no. 1, pp. 292-378 (2007).
[20] U. Dierkes, *Maximum principles and nonexistence results for minimal submanifolds*, Manuscr. Math. 69, 203-218 (1990).
[21] U. Dierkes, S. Hildebrandt, A. Kuster, O. Wohlrab, *“Minimal Surfaces I: Boundary Value Problems”*, Grundlehren der Mathematischen Wissenschaften, Vol. 295, Springer-Verlag, Berlin (1992).
[22] E. Feleqi, F. Rampazzo, *Integral representations for bracket-generating multi-flows*, Discrete Contin. Dyn. Syst. Ser. A., 35, No. 9, 4345–4366 (2015).
[23] B. Franchi, R. Serapioni, F.S. Cassano, *Rectifiability and Perimeter in the Heisenberg Group*, Math. Ann., 321 (2001).
[24] ———, *Regular submanifolds, graphs and area formula in Heisenberg groups.*, Adv. Math. 211, no. 1 (2007).
[25] H. Federer, *“Geometric Measure Theory”*, Springer Verlag (1969).
[26] Z. Ge, *Betti numbers, characteristic classes and sub-Riemannian geometry,* Illinois Jour. of Math., 36, no. 3 (1992).
[27] U. Hamenstädt, *Some regularity theorems for Carnot-Carathéodory metrics*, J. Differential Geom. 32, no. 3, 819–850 (1990).
[28] S. Hildebrandt, *Maximum principles for minimal surfaces and for surfaces of continuous mean curvature*, Math. Z. 128, 253-269 (1972).
[29] R.H. Hladky, S.D. Pauls, *Constant mean curvature surfaces in sub-Riemannian geometry*, J. Diff. Geom. 79, no. 1 (2008).
[30] A. Hurtado, M. Ritoré, C. Rosales, *The classification of complete stable area-stationary surfaces in the Heisenberg group*, Adv. Math. 224, no. 2, 561-600 (2010).
[31] M. Karmanova, S. K. Vodopyanov, *Geometry of Carnot-Carathéodory Spaces, Differentiability, Coarea and Area Formulas*, Analysis and Mathematical Physics. Trends in Mathematics, Birkhauser, Basel, 233-335 (2009).
[32] J.M. Lee, *“ Introduction to Smooth Manifolds”*, Springer Verlag (2003).
[33] E. Le Donne, A. Ottazzi, *Isometries between open sets of Carnot groups and global isometries of homogeneous manifolds*, JGA, DOI: 10.1007/s12220-014 9552-8.
[34] V. Magnani *“Elements of Geometric Measure Theory on sub-Riemannian groups”*, PHD Thesis, SNS, Pisa, (2002).
[35] J.W. Milnor, *Curvatures of left invariant metrics on Lie groups*, Adv. Math. 21, pp. 293-329 (1976).
[36] A. Montanari, D. Morbidelli, *Nonsmooth Hormander vector fields and their control balls*, Trans. Amer. Math. Soc. 364, (2012), 2339-2375, http://arxiv.org/abs/0812.2369.
[37] F. Montefalcone, *Hypersurfaces and variational formulas in sub-Riemannian Carnot groups*, JMPA, 87 (2007).
[38] ———, *Stable H-minimal hypersurfaces*, J. Geom. Anal. 25, 820–870 (2015).
[39] R. Montgomery, *“A Tour of Subriemannian Geometries, Their Geodesics and Applications”*, AMS, Math. Surveys and Monographs, 91 (2002).
[40] R. Osserman, *“A survey of Minimal Surfaces”*, Dover Publications, INC. Mineola, New York (1969).
[41] ———, *The convex hull property of immersed manifolds,* J. Diff. Geom., 6, 267-270 (1971).
[42] S.D. Pauls, *Minimal surfaces in the Heisenberg group*, Geom. Dedicata, 104 201-231 (2004).
[43] P. Pansu, *“Gèometrie du Group d’Heisenberg”*, Thèse pour le titre de Docteur, 3ème cycle, Université Paris VII (1982).
[44] F. Rampazzo, H. J. Sussmann, *Set-valued differential and a nonsmooth version of Chow’s theorem,* Proceedings of the 40th IEEE Conference on Decision and Control; Orlando, Florida, 2001.
[45] M. Ritoré, *Examples of area-minimizing surfaces in the sub-Riemannian Heisenberg group $\mathbb{H}^1$ with low regularity*, Calc. Var. Partial Differ. Equ. 34, No. 2, 179-192 (2009).
[46] M. Ritoré, C. Rosales, *Area stationary surfaces in the Heisenberg group $\mathbb{H}^1$*, Adv. Math. 219, No. 2, pp. 633-671 (2008).
[47] M. Spivak, *“A comprehensive introduction to differential geometry”*, Vol.4, Publish or Perish (1999).
[48] E.M. Stein, *“Harmonic Analysis”*, Princeton University Press (1993).
[49] V.S. Varadarajan, *Lie groups, Lie algebras and their representations*, Reprint of the 1974 edition. Graduate Texts in Mathematics. 102. New York, NY: Springer. xiii, 430 pages (1984).